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# On Generalized Periods of Cusp Forms

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## Introduction

Manin [1], [2] defined a $p$-adic measure and $p$-adic Hecke series attached to cusp forms with respect to the full modular group.

In the present paper it is our aim to give a few remarks on $p$-adic measures attached to the Bernoulli functions and the cusp forms, and to give a $p$-adic expression of generalized periods of the cusp forms. Namely we here discuss the generalized periods of the cusp forms with any Dirichlet characters.

## § 1. Nasybullin's lemma

We set $q=p$ for any prime $p>2$ and $q=4$ for the prime $p=2$. Let $\bar{f}=[f, q]$ be the least common multiple $f$ and $q$, and $\boldsymbol{Z}$ the rational integer ring.

The ring $\boldsymbol{Z}_{\bar{f}}=\varprojlim_{n} \boldsymbol{Z}/p^n\bar{f}\boldsymbol{Z}$, $n\geqq 0$, the inverse limit with natural homomorphisms, is isomorphic to the direct product of the rational $p$-adic integer ring $\boldsymbol{Z}_p$ and the residue class ring $\boldsymbol{Z}/f_0\boldsymbol{Z}$ with a natural number $f_0$ such that $\bar{f}=p^l f_0$, $(f_0, p)=1$.

Let $\boldsymbol{Z}_{\bar{f}}^*$ be the multiplicative group of $\boldsymbol{Z}_{\bar{f}}$, so that it is isomorphic to the direct product of the unit groups $\boldsymbol{Z}_p^*$ and $(\boldsymbol{Z}/f_0\boldsymbol{Z})^*$.

Let $K$ be a field over the rational $p$-adic number field $\boldsymbol{Q}_p$. Then we call a function $\mu$ a $K$-measure on $\boldsymbol{Z}_{\bar{f}}^*$, if $\mu$ is a finitely additive function defined on open-closed subsets in $\boldsymbol{Z}_{\bar{f}}^*$, whose values are in the field $K$. Any open-closed subset in $\boldsymbol{Z}_{\bar{f}}^*$ is a disjoint union of some finite intervals $I_{a,n}=a+p^n\bar{f}\boldsymbol{Z}_{\bar{f}}$ in $\boldsymbol{Z}_{\bar{f}}^*$, where $a\in\boldsymbol{Z}$ prime to $\bar{f}$, and therefore a $K$-measure $\mu$ is determined by its values on all the intervals in $\boldsymbol{Z}_{\bar{f}}^*$.

Let $\boldsymbol{Q}^{(f)}$ denote the set of such rational numbers, each denominator of which is a divisor of $\bar{f}p^n$ for some $n\geqq 0$.

Then Nasybullin's lemma reads as follows [1].

**Lemma 1.** *Let $R$ be a $K$-valued function defined on $\boldsymbol{Q}^{(f)}$ with a property: There exist two constants $A$, $B\in K$ such that*

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$$R(x+1)=R(x), \qquad \sum_{k=0}^{p-1} R\left(\frac{x+k}{p}\right)=AR(x)+BR(px)$$

*hold for any number* $x \in \boldsymbol{Q}^{(f)}$. *Furthermore, let* $\rho \neq 0$ *be a root of equation* $y^2=Ay+Bp$. *Then there exists a* $K(\rho)$*-measure* $\mu$ *on* $\boldsymbol{Z}_{\bar{f}}^*$, *such that*

$$\mu(I_{a,n})=\rho^{-n}R\left(\frac{a}{p^n\bar{f}}\right)+B\rho^{-(n+1)}R\left(\frac{a}{p^{n-1}\bar{f}}\right)$$

*holds for any interval* $I_{a,n}$.

*Proof.* We have indeed

$$\begin{aligned}
&\sum_{k=0}^{p-1} \mu(I_{a+p^n\bar{f}k,n+1}) \\
&=\sum_{k=0}^{p-1} \rho^{-(n+1)}R\left(\frac{a+p^n\bar{f}k}{p^{n+1}\bar{f}}\right)+\sum_{k=0}^{p-1} B\rho^{-(n+2)}R\left(\frac{a+p^n\bar{f}k}{p^n\bar{f}}\right) \\
&=\sum_{k=0}^{p-1} \rho^{-(n+1)}R\left(\frac{a+p^n\bar{f}k}{p^{n+1}\bar{f}}\right)+\sum_{k=0}^{p-1} B\rho^{-(n+2)}R\left(\frac{a}{p^n\bar{f}}\right) \\
&=\rho^{-(n+1)}\left(AR\left(\frac{a}{p^n\bar{f}}\right)+BR\left(\frac{a}{p^{n-1}\bar{f}}\right)\right)+pB\rho^{-(n+2)}R\left(\frac{a}{p^n\bar{f}}\right) \\
&=\rho^{-(n+2)}(A\rho+pB)R\left(\frac{a}{p^n\bar{f}}\right)+\rho^{-(n+1)}BR\left(\frac{a}{p^{n-1}\bar{f}}\right) \\
&=\rho^{-n}R\left(\frac{a}{p^n\bar{f}}\right)+B\rho^{-(n+1)}R\left(\frac{a}{p^{n-1}\bar{f}}\right) \\
&=\mu(I_{a,n}).
\end{aligned}$$

Namely we see

$$\mu(I_{a,n})=\sum_{\substack{b \bmod p^{n+1}\bar{f} \\ b\equiv a \pmod{p^n\bar{f}}}} \mu(I_{b,n+1}).$$

This proves our assertion, because any open-closed subset is a disjoint union of some finite intervals, as already remarked above.

## § 2. The Bernoulli functions

Let $B_m(x)$ be the $m$-th Bernoulli polynomial and $P_m(x)$ the $m$-th Bernoulli function, namely

$$P_m(x)=B_m(x) \quad \text{for } 0\leqq x<1, \quad P_m(x+1)=P_m(x) \text{ for any real } x,$$

where we take $B_1=-1/2$.

As is well known, we have for any real number $x$ the Fourier expansions

$$P_m(x) = -m! \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n x}}{(2\pi i n)^m} \qquad (m=1, 2, \cdots).$$

Herein the summation means to take sum over all the integers except 0.

Hence we have $P_m(x) \in \boldsymbol{Q}$ for $x \in \boldsymbol{Q}^{(f)}$ and

$$\begin{aligned}\sum_{k=0}^{p-1} P_m\left(\frac{x+k}{p}\right) &= -m! \sum_{n=-\infty}^{\infty}{}' \frac{e^{(2\pi i/p)nx}}{(2\pi i n)^m} \sum_{k=0}^{p-1} e^{(2\pi i n/p)k} \\ &= -m!\, p \sum_{\substack{n=-\infty \\ n\equiv 0 \ (\mathrm{mod}\ p)}}^{\infty}{}' \frac{e^{(2\pi i/p)nx}}{(2\pi i n)^m} \\ &= -m!\, p^{1-m} \sum_{l=-\infty}^{\infty}{}' \frac{e^{2\pi i l x}}{(2\pi i l)^m} \\ &= p^{1-m} P_m(x).\end{aligned}$$

Namely the function $P_m(x)$ satisfies the property of Nasybullin's lemma with the constants $A = p^{1-m}$, $B=0$. Then $\rho \neq 0$ is equal to $p^{1-m}$, as $\rho^2 = A\rho + Bp$ reduces simply to $\rho^2 = p^{1-m}\rho$.

Thus we obtain the following

**Theorem 1.** *The function $\mu_m$ defined on any $I_{a,n}$ by*

$$\mu_m(I_{a,n}) = (p^n \bar{f})^{m-1} P_m\left(\frac{a}{p^n \bar{f}}\right)$$

*gives us a $\boldsymbol{Q}_p$-measure on $\boldsymbol{Z}_{\bar{f}}^*$.*

Now, we state a result on the growth of the measure $\mu_m$ as follows: For any integer $a$ with $(a, \bar{f}) = 1$ we have

$$(p^l \bar{f})^m \frac{1}{m} P_m\left(\frac{a}{p^l \bar{f}}\right)(1 - c^m) \equiv 0 \pmod{p^0},$$

where $c \in \boldsymbol{Z}$ means a parameter which we may take as $c \equiv 1 \pmod{p^l \bar{f}}$ [5].

This yields for the integer $a$ with $0 \leqq a < p^l \bar{f}$, $(a, \bar{f}) = 1$

$$(p^l \bar{f})^{m-1} P_m\left(\frac{a}{p^l \bar{f}}\right) \equiv 0 \pmod{p^{-2l-2\nu_p(\bar{f})}},$$

namely

$$|\mu_m(I_{a,l})|\leqq p^{2l+2\nu_p(\bar{f})}.$$

Herein $|\ |$ means the $p$-adic valuation normalized such as $|p|=p^{-1}$ and $\nu_p(\ )$ denotes the corresponding exponential valuation.

By the way, for any measure $\mu$ on $\boldsymbol{Z}_{\bar{f}}^{*}$ we set

$$\varepsilon_l=\max_b |\mu(I_{b,l})|p^{-l}.$$

We call $\mu$ to be moderate growth if $\varepsilon_l\to 0$ as $l\to\infty$.

Then the measure $\mu_0(I_{a,l})=(p^l\bar{f})^{-1}$ is not moderate growth, because $\varepsilon_l\to|\bar{f}|$ as $l\to\infty$.

We know that, if a measure $\mu$ is moderate growth and a function $f(x)$ on $\boldsymbol{Z}_{\bar{f}}^{*}$ satisfies the Lipschitz condition, then there exists a Riemann integral of $f(x)$ on $\boldsymbol{Z}_{\bar{f}}^{*}$ with respect to the measure $\mu$ [1], [2]. However, we also know that there exists a $p$-adic integral for any uniformly differentiable function with respect to the measure $\mu_0$, even though $\mu_0$ is not moderate growth [5].

## § 3. The $p$-adic Mellin-Mazur transform

Let $\chi$ be a primitive Dirichlet character modulo $f$. Then we can define the $p$-adic $L$-function of Kubota-Leopoldt by the following $p$-adic Mellin-Mazur transform with respect to $\mu_m$:

$$\begin{aligned} L(\mu_m, \chi) &= \int_{\boldsymbol{Z}_{\bar{f}}^{*}} \chi(a)d\mu_m(a) \\ &= \lim_{l\to\infty} \sum_{\substack{a \bmod p^l\bar{f} \\ a\in \boldsymbol{Z},(a,\bar{f})=1}} \chi(a)\mu_m(I_{a,l}). \end{aligned}$$

As the character $\chi$ is constant on the interval $I_{a,0}$ we see directly

$$\begin{aligned} L(\mu_m, \chi) &= \sum_{a \bmod \bar{f}} \chi(a)\mu_m(I_{a,0}) \\ &= \sum_{a \bmod \bar{f}} \chi(a)\bar{f}^{m-1}P_m\left(\frac{a}{\bar{f}}\right) \\ &= \bar{f}^{m-1} \sum_{a \bmod \bar{f}} \chi(a)P_m\left(\frac{a}{\bar{f}}\right) \\ &= (1-\chi(p)p^{m-1})B_\chi^m, \end{aligned}$$

where $B_\chi^m$ denotes the $m$-th Bernoulli number belonging to $\chi$. Hence we have

$$\begin{aligned} -\frac{1}{m}L(\mu_m, \chi\omega^{-m}) &= -\frac{1}{m}(1-\chi\omega^{-m}(p)p^{m-1})B_{\chi\omega^{-m}}^m \\ &= L_p(1-m, \chi). \end{aligned}$$

## § 4. The Mazur measure

We here consider a special kind of measure $\mu^{(c)}$, called a Mazur measure on $Z_{\bar{f}}^{*}$. Take a positive integer $c$ prime to $\bar{f}$ and assume that $0<a<p^n\bar{f}$, $(a,\bar{f})=1$ for the integer $a$ of an interval $I_{a,n}=a+p^n\bar{f}Z_{\bar{f}}$ always.

Then we set, by making use of $[x]$, the greatest integer not exceeding $x$,

$$\mu^{(c)}(I_{a,n})=\left[\frac{ac}{p^n\bar{f}}\right]-\frac{c-1}{2}\in\frac{1}{2}Z\subset Q_p.$$

The function $\mu^{(c)}$ determines a $Q_p$-measure on $Z_{\bar{f}}^{*}$. Namely it holds that

$$\sum_{k=0}^{p-1}\mu^{(c)}(I_{a+p^n\bar{f}k,n+1})=\mu^{(c)}(I_{a,n}).$$

It is easily seen that this formula is equivalent to

$$\sum_{k=0}^{p-1}\left[\frac{j+kc}{p}\right]=j+\frac{(p-1)(c-1)}{2}\quad\text{for any non-negative } j\in Z,$$

which can be verified by counting the number of the lattice points in a rectangle with vertices $(0, 0)$, $(p, 0)$, $(p, c)$, $(0, c)$ in the case $j=0$ and then by induction on $j$.

**Remark.** If $\mu(I_{a,n})=[ac/p^n\bar{f}]+d$ with a constant $d$ gives a measure, then the constant $d$ is necessarily equal to $(c-1)/2$.

## § 5. Comparison of the measures

The measure $\mu_0(I_{a,n})=(p^n\bar{f})^{-1}$ is a so-called invariant measure on $Z_{\bar{f}}^{*}$, namely it is independent of $a$.

To calculate the magnitude of the measure $\mu_m$ we see

$$\begin{aligned}\mu_m(I_{a,n})&=(p^n\bar{f})^{m-1}P_m\left(\frac{a}{p^n\bar{f}}\right)\\&=(p^n\bar{f})^{m-1}B_m\left(\left\{\frac{a}{p^n\bar{f}}\right\}\right)\\&=(p^n\bar{f})^{m-1}\left(B+\left\{\frac{a}{p^n\bar{f}}\right\}\right)^m\\&=(p^n\bar{f})^{m-1}\sum_{j=0}^{m}\binom{m}{j}\left\{\frac{a}{p^n\bar{f}}\right\}^j B_{m-j}\end{aligned}$$

$$=(p^n\bar{f})^{m-1}\sum_{j=0}^{m}\binom{m}{j}\left(\frac{a}{p^n\bar{f}}-\left[\frac{a}{p^n\bar{f}}\right]\right)^j B_{m-j}$$

$$=\sum_{j=0}^{m}\binom{m}{j}(p^n\bar{f})^{m-1-j}\left(a-p^n\bar{f}\left[\frac{a}{p^n\bar{f}}\right]\right)^j B_{m-j}.$$

Thus, when $n$ tends to the infinity we have asymptotically

$$\mu_m(I_{a,n})\sim m\left(a-p^n\bar{f}\left[\frac{a}{p^n\bar{f}}\right]\right)^{m-1}B_1+(p^n\bar{f})^{-1}\left(a-p^n\bar{f}\left[\frac{a}{p^n\bar{f}}\right]\right)^m$$

$$\sim(p^n\bar{f})^{-1}a^m-\frac{1}{2}ma^{m-1}-ma^{m-1}\left[\frac{a}{p^n\bar{f}}\right]$$

$$\sim(p^n\bar{f})^{-1}a^m.$$

This shows that we have

$$\mu_m(I_{a,n})\sim a^m\mu_0(I_{a,n}),$$

which we denote by $d\mu_m(a)=a^m d\mu_0(a)$.

For the measure $\mu_m^c(I_{a,n})=\mu_m(I_{ac,n})$ with an integer $c>1$, $(c,\bar{f})=1$, we see quite similarly

$$\mu_m^c(I_{a,n})\sim(ca)^m\mu_0(I_{a,n})-m(ca)^{m-1}\mu^{(c)}(I_{a,n}).$$

Thus we obtain

**Theorem 2.** *We have the formulas*

$$d\mu_m(a)=a^m d\mu_0(a),$$
$$d\mu_m^c(a)=(ca)^m d\mu_0(a)-m(ca)^{m-1}d\mu^{(c)}(a).$$

**Remark.** In the above, if we take the integers $a$ such as $0<a<p^n\bar{f}$, we have $[a/p^n\bar{f}]=0$, and the term $ma^{m-1}/2$ is neglisible, because we have always

$$\lim_{\rho\to\infty}\sum_{j=1}^{p^\rho\bar{f}}{}^{*}\,\chi(j)g(\langle j\rangle)=0$$

for any Dirichlet character $\chi$ with conductor $f$ and any continuous function $g$ on $1+q\mathbf{Z}_{\bar{f}}$. The notations $*$, $\langle\ \rangle$ mean the usual sense [6].

## § 6. An invariant integral

Let $\chi$ be a primitive Dirichlet character with conductor $f$, and take the parameter $c\in\mathbf{Z}$, such as $c>1$, $(c,\bar{f})=1$. Then we have by definition

$$\int_{Z_f^*} \chi(ca)d\mu_m^c(a) = \int_{Z_f^*} \chi(a)d\mu_m(a),$$

whence by Theorem 2 we see

$$\chi(c)c^m \int_{Z_f^*} \chi(a)a^m d\mu_0(a) - m\int_{Z_f^*} \chi(ca)(ca)^{m-1}d\mu^{(c)}(a)$$
$$= \int_{Z_f^*} \chi(a)a^m d\mu_0(a).$$

Hence we have

$$(1-\chi(c)c^m)\int_{Z_f^*} \chi(a)a^m d\mu_0(a) = -m\int_{Z_f^*} \chi(ca)(ca)^{m-1}d\mu^{(c)}(a).$$

Namely we have

**Theorem 3.** *We have a formula*

$$-\frac{1}{m}(1-\chi(c)c^m)(1-\chi(p)p^{m-1})B_\chi^m = \int_{Z_f^*} \chi(ca)(ca)^{m-1}d\mu^{(c)}(a).$$

This formula was the starting point of our earlier investigation on the Bernoulli numbers [4].

## § 7. Generalized periods of cusp forms

Let $\phi(z) = \sum_{n=1}^{\infty} \lambda_n e^{2\pi i n z}$ be a cusp form of weight $w+2$ with respect to the full modular group $SL(2, \boldsymbol{Z})$ and assume that it is a normalized eigenfunction of all the Hecke operators, namely $\phi \mid T_n = \lambda_n \phi$ for any $n \geqq 1$ with $\lambda_1 = 1$.

Then Manin [2] defined a function $Q_m(x)$ on $\boldsymbol{Q}$ for any integer $m$ in $0 \leqq m \leqq w$:

$$Q_m(x) = \int_0^{i\infty} \phi(z+x)z^m dz,$$

and with certain suitable numbers $\omega^+, \omega^- \in \boldsymbol{R}$

$$Q_m^+(x) = \frac{i}{\omega^+} \operatorname{Im} \int_0^{i\infty} \phi(z+x)z^m dz,$$
$$Q_m^-(x) = \frac{1}{\omega^-} \operatorname{Re} \int_0^{i\infty} \phi(z+x)z^m dz,$$

so that $Q_m^+(x)$, $Q_m^-(x)$ with $x \in \boldsymbol{Q}$ take algebraic values [2].

The functions $Q_m(x)$, $Q_m^+(x)$ and $Q_m^-(x)$ are analogues of the Bernoulli functions, which can be seen as follows. We compute the value $Q_m(a/f)$ with $a \in \boldsymbol{Z}$:

$$\begin{aligned}
Q_m\left(\frac{a}{f}\right)&=\int_0^{i\infty}\phi\left(z+\frac{a}{f}\right)z^m dz\\
&=\sum_{n=1}^{\infty}\int_0^{i\infty}\lambda_n e^{2\pi i n(z+a/f)}z^m dz\\
&=\sum_{n=1}^{\infty}\int_0^{\infty}\lambda_n e^{-2\pi n t}e^{2\pi i n(a/f)}i^{m+1}t^m dt\\
&=i^{m+1}\sum_{n=1}^{\infty}\lambda_n\int_0^{\infty}e^{-2\pi n t}t^m e^{2\pi i n(a/f)}dt\\
&=i^{m+1}\sum_{n=1}^{\infty}\lambda_n\frac{e^{2\pi i n(a/f)}}{(2\pi n)^{m+1}}\Gamma(m+1)\\
&=i^{m+1}m!\,\frac{1}{(2\pi)^{m+1}}\sum_{n=1}^{\infty}\lambda_n\frac{1}{n^{m+1}}e^{2\pi i n(a/f)}.
\end{aligned}$$

Namely we have

$$Q_m\left(\frac{a}{f}\right)=i^{m+1}\frac{m!}{(2\pi)^{m+1}}\sum_{n=1}^{\infty}\lambda_n\frac{1}{n^{m+1}}e^{2\pi i n(a/f)}.$$

This formula can be seen as an analogue of the Fourier expansion of the value of the ordinary partial zeta function $\zeta(s, a, f)$ at $s=1-m$.

Especially the number

$$Q_m(0)=\int_0^{i\infty}\phi(z)z^m dz=r^m(\phi)$$

is called a period of the cusp form $\phi$.

In general we have in some half complex plane of $s$

$$\int_0^{i\infty}\phi(z)z^{s-1}dz=i^s\frac{\Gamma(s)}{(2\pi)^s}\sum_{n=1}^{\infty}\lambda_n\frac{1}{n^s},$$

hence we see

$$\sum_{n=1}^{\infty}\lambda_n\frac{1}{n^{m+1}}=\frac{(2\pi)^{m+1}}{m!}\frac{1}{i^{m+1}}r^m(\phi).$$

In the sequel we calculate a $p$-adic expression of $(1/m)Q_m(a/f)$, the values $\zeta(1-m, a, f, \phi)$ of the partial zeta function attached to the cusp form $\phi$ at $s=1-m$.

Let $f$ be any natural number, $\chi$ a primitive Dirichlet character with conductor $f$. Hence $f$ is not equal to 2 necessarily. Then we define generalized periods $r_{\chi}^{m}(\phi)$ with Dirichlet characters $\chi$ of the cusp form $\phi$ by

$$r_{\chi}^{m}(\phi)=\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{a}{f}\right).$$

Then the generalized periods satisfy certain relations, called the Shimura-Eichler relations [1].

We may also define $r_{\chi}^{m}(\phi)$ as

$$f^{m-1}\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{a}{f}\right)$$

from the view point of an exact analogy to the generalized Bernoulli numbers. But we remove the factor $f^{m-1}$ for simplicity.

First we have

$$\sum_{a=1}^{p^n f}\chi(a)Q_m\left(\frac{a}{p^n f}\right)=\sum_{a=1}^{f}\chi(a)\sum_{b=0}^{p^n-1}Q_m\left(\frac{a+bf}{p^n f}\right).$$

Because $Q_m(x)$ is periodic with the period 1, we see

$$\begin{aligned}&\sum_{b \bmod p^n}Q_m\left(\frac{a+bf}{p^n f}\right)\\&=\sum_{b \bmod p^n}Q_m\left(\frac{a}{p^n f}+\frac{b}{p^n}\right)\\&=\sum_{b=0}^{p^{n-1}-1}\sum_{c=0}^{p-1}Q_m\left(\frac{a}{p^n f}+\frac{b}{p^n}+\frac{c}{p}\right)\\&=\sum_{b=0}^{p^{n-1}-1}\left\{AQ_m\left(\frac{a}{p^{n-1}f}+\frac{b}{p^{n-1}}\right)+BQ_m\left(\frac{a}{p^{n-2}f}+\frac{b}{p^{n-2}}\right)\right\},\end{aligned}$$

where the constants $A$, $B$ are equal to $\lambda_p p^{-m}$, $-p^{w-2m}$ respectively.

These constants come from the identity in Nasybullin's lemma, and indeed in our case the equality $\phi\,|\,T_p=\lambda_p\phi$ means

$$\sum_{k=0}^{p-1}Q_m\left(\frac{x+k}{p}\right)=\lambda_p p^{-m}Q_m(x)-p^{w-2m}Q_m(px),$$

namely $A=\lambda_p p^{-m}$, $B=-p^{w-2m}$[2].

Therefore we have for any fixed $a_f$

$$\begin{aligned}&\sum_{\substack{a \bmod p^n f\\ a\equiv a_f \pmod f}}Q_m\left(\frac{a}{p^n f}\right)\\&=\lambda_p p^{-m}\sum_{\substack{a' \bmod p^{n-1}f\\ a'\equiv a_f \pmod f}}Q_m\left(\frac{a'}{p^{n-1}f}\right)-p^{w-2m+1}\sum_{\substack{a'' \bmod p^{n-2}f\\ a''\equiv a_f \pmod f}}Q_m\left(\frac{a''}{p^{n-2}f}\right).\end{aligned}$$

Here we set with $n \geqq 0$

$$S_n^{(m)}(a_f) = \sum_{\substack{a \bmod p^n f \\ a \equiv a_f \pmod{f}}} Q_m\left(\frac{a}{p^n f}\right),$$

which we denote by $S_n$ for simplicity.

Let $f(x) = \sum_{n=0}^{\infty} S_n x^n$ be the generating function for the numbers $S_n$. Then we have

$$f(x) = S_0 + S_1 x + Ax(f(x) - S_0) + pBf(x)x^2,$$

whence we see

$$f(x) = \frac{S_0 + (S_1 - AS_0)x}{1 - Ax - pBx^2}.$$

If $\alpha$, $\beta$ denote the roots of the equation $1 - Ax - pBx^2 = 0$, then we see easily

$$f(x) = \sum_{k=0}^{\infty} (C\alpha^k + D\beta^k)x^k$$

with

$$C = \frac{1}{\alpha - \beta}(\alpha S_0 - AS_0 + S_1), \qquad D = \frac{1}{\alpha - \beta}(-\beta S_0 + AS_0 - S_1)$$

for $\alpha \neq \beta$.

These formulas yield

$$\begin{aligned} S_n &= \frac{1}{\alpha - \beta}\{\alpha^n(\alpha S_0 - AS_0 + S_1) - \beta^n(\beta S_0 - AS_0 + S_1)\} \\ &= S_0 \sum_{j=0}^{n} \alpha^j \beta^{n-j} - AS_0 \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} + S_1 \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j}. \end{aligned}$$

Because the roots $\rho$, $\rho'$ of the equation $p^2 - \lambda_p \rho + p^{1+w} = 0$ are mutually complex conjugate by virtue of the Ramunujan conjecture $|\lambda_p|_\infty \leqq 2p^{(w+1)/2}$, we see $\alpha$, $\beta$ are complex conjugate mutually, and we have $\alpha = p^{-m}\rho$, $\beta = p^{-m}\bar{\rho}$.

Put this fact in the above formula and we obtain easily

$$S_n = S_0 p^{-mn} \sum_{j=0}^{n} \rho^j \bar{\rho}^{n-j} - S_0 \lambda_p p^{-mn} \sum_{j=0}^{n-1} \rho^j \bar{\rho}^{n-1-j} + S_1 p^{-m(n-1)} \sum_{j=0}^{n-1} \rho^j \bar{\rho}^{n-1-j}.$$

Furthermore we have

$$S_1=\sum_{a \bmod pf} Q_m\left(\frac{a}{pf}\right)=AQ_m\left(\frac{a_f}{f}\right)+BQ_m\left(\frac{pa_f}{f}\right)$$
$$=\lambda_p p^{-m}S_0-p^{w-2m}Q_m\left(\frac{pa_f}{f}\right).$$

Thus we obtain finally

$$S_n=S_0p^{-mn}\sum_{j=0}^{n}\rho^j\bar{\rho}^{n-j}-p^{w-m(n+1)}Q_m\left(\frac{pa_f}{f}\right)\sum_{j=0}^{n-1}\rho^j\bar{\rho}^{n-1-j}.$$

Note that this holds even when $\bar{\rho}=\rho$.

On the other hand, we have $\sum_{j=0}^{n}\rho^j\bar{\rho}^{n-j}=\lambda_{p^n}$, which can be proved by considering the $p$-part of the Euler product of the Dirichlet series attached to the cusp form $\phi$.

Hence we see

$$\sum_{a=1}^{p^nf}\chi(a)Q_m\left(\frac{a}{p^nf}\right)=\sum_{a=1}^{f}\chi(a)S_n^{(m)}(a)$$
$$=\sum_{a=1}^{f}\chi(a)S_0^{(m)}(a)p^{-mn}\lambda_{p^n}-\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{pa}{f}\right)p^{w-m(n+1)}\lambda_{p^{n-1}}.$$

Namely we have

$$\sum_{a=1}^{p^nf}\chi(a)p^{mn}Q_m\left(\frac{a}{p^nf}\right)=\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{a}{f}\right)\lambda_{p^n}-\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{pa}{f}\right)p^{w-m}\lambda_{p^{n-1}}.$$

Now we have

$$\sum_{a=1}^{f}\chi(a)Q_m\left(\frac{pa}{f}\right)=0 \qquad \text{for } p\,|\,f,$$
$$\sum_{a=1}^{f}\chi(pa)\,Q_m\left(\frac{pa}{f}\right)=r_\chi^m(\phi) \qquad \text{for } p\nmid f.$$

Thus we obtain the following

**Theorem 4.** *For any natural number $n\geqq 1$ we have*

$$p^{mn}\sum_{a=1}^{p^nf}\chi(a)Q_m\left(\frac{a}{p^nf}\right)=(\lambda_{p^n}-p^{w-m}\lambda_{p^{n-1}}\bar{\chi}(p))r_\chi^m(\phi).$$

Next, we assume $\nu_p(\rho)<(1+w)/2$, and then see $\nu_p(\bar{\rho})=1+w-\nu_p(\rho)$ $>(1+w)/2$, because $\rho\bar{\rho}=p^{1+w}$, $\rho+\bar{\rho}=\lambda_p$.

Because $\nu_p(\bar{\rho}/\rho)>0$ we see $p$-adically

$$\rho^{-n}\lambda_{p^n}=\frac{1-(\bar\rho/\rho)^{n+1}}{1-\bar\rho/\rho}\to\frac{1}{1-\bar\rho/\rho}\quad\text{and}\quad\rho^{-n}\lambda_{p^{n-1}}\to\frac{\rho-1}{1-\bar\rho/\rho}\quad\text{if } n\to\infty.$$

Hence we obtain

**Theorem 5.** *Under the assumption $\nu_p(\rho)<(1+w)/2$ we have*

$$\lim_{n\to\infty}\rho^{-n}p^{mn}\sum_{a=1}^{p^nf}\chi(a)Q_m^{\pm}\left(\frac{a}{p^nf}\right)=\left\{\frac{\rho}{\rho-\bar\rho}-\frac{1}{\rho-\bar\rho}p^{w-m}\bar\chi(p)\right\}r_\chi^m(\phi)^{\pm},$$

*where we mean*

$$r_\chi^m(\phi)^{\pm}=\sum_{a=1}^{f}\chi(a)\,Q_m^{\pm}\left(\frac{a}{f}\right).$$

Similarly we have

$$\lim_{n\to\infty}\rho^{-(n+1)}p^{mn-m+w}\sum_{a=1}^{p^nf}\chi(a)Q_m^{\pm}\left(\frac{a}{p^{n-1}f}\right)$$
$$=\rho^{-2}p^{1+w}\left\{\frac{\rho}{\rho-\bar\rho}-\frac{1}{\rho-\bar\rho}p^{w-m}\bar\chi(p)\right\}r_\chi^m(\phi)^{\pm}.$$

Now, we define a $p$-adic measure $\mu_m^{\pm}$ on $Z_{\bar f}^*$, due to Manin, as follows: By changing the notation for intervals slightly we set

$$\mu_m^{\pm}(I'_{a,n})=\rho^{-n}p^{mn}Q_m^{\pm}\left(\frac{a}{p^nf}\right)-\rho^{-n-1}p^{mn-m+w}Q_m^{\pm}\left(\frac{a}{p^{n-1}f}\right)$$

with $I'_{a,n}=a+p^nfZ_{\bar f}$.

This determines a $p$-adic measure having algebraic values and, if $\nu_p(\rho)<(1+w)/2$, then it is moderate growth.

For any Dirichlet character $\chi$ with conductor $f$ we have by definition

$$\int_{Z_{\bar f}^*}\chi(a)d\mu_m^{\pm}(a)=\lim_{n\to\infty}\sum_{\substack{a \bmod p^nf\\(a,\bar f)=1}}\chi(a)\mu_m^{\pm}(I'_{a,n}).$$

Hence we see, by making use of the property of the measure $\mu_m^{\pm}$,

$$\int_{Z_{\bar f}^*}\chi(a)\,d\mu_m^{\pm}(a)=\sum_{\substack{a \bmod pf\\(a,\bar f)=1}}\chi(a)\mu_m^{\pm}(I'_{a,1})$$
$$=\sum_{a \bmod pf}^{*}\chi(a)\left\{\rho^{-1}p^mQ_m^{\pm}\left(\frac{a}{pf}\right)-\rho^{-2}p^wQ_m^{\pm}\left(\frac{a}{f}\right)\right\},$$

where $*$ means to take sum over all integers prime to $p$ in the given range.

Therefore we have in the case $\bar{f}=f$

$$\int_{Z_f^*} \chi(a)d\mu_m^{\pm}(a)=\rho^{-1}p^m \sum_{a=1}^{f} \chi(a)\left\{\lambda_p p^{-m} Q_m^{\pm}\left(\frac{a}{f}\right)-p^{w-2m} Q_m^{\pm}\left(\frac{pa}{f}\right)\right\}$$
$$-\rho^{-2}p^{w+1}\sum_{a=1}^{f}\chi(a)Q_m^{\pm}\left(\frac{a}{f}\right)$$
$$=r_\chi^m(\phi)^{\pm}.$$

In the case where $\bar{f}\neq f$ the above sum is equal to

$$\rho^{-1}p^m \sum_{a=1}^{pf} \chi(a)Q_m^{\pm}\left(\frac{a}{pf}\right)-\rho^{-1}p^m \sum_{b=1}^{f}\chi(pb)Q_m^{\pm}\left(\frac{b}{f}\right)$$
$$-\rho^{-2}p^w \sum_{a=1}^{pf}\chi(a)Q_m^{\pm}\left(\frac{a}{f}\right)+\rho^{-2}p^w\sum_{b=1}^{f}\chi(pb)Q_m^{\pm}\left(\frac{pb}{f}\right)$$
$$=\rho^{-1}p^m \sum_{a=1}^{f} \chi(a)\left\{\lambda_p p^{-m} Q_m^{\pm}\left(\frac{a}{f}\right)-p^{w-2m} Q_m^{\pm}\left(\frac{pa}{f}\right)\right\}$$
$$-\rho^{-1}p^m\chi(p)r_\chi^m(\phi)^{\pm}-\rho^{-2}p^{w+1}r_\chi^m(\phi)^{\pm}+\rho^{-2}p^w r_\chi^m(\phi)^{\pm}$$
$$=(\rho^{-1}\lambda_p-\rho^{-1}p^{w-m}\chi(p)-\rho^{-1}p^m\chi(p)-\rho^{-2}p^{w+1}+\rho^{-2}p^w)r_\chi^m(\phi)^{\pm}$$
$$=(1-\rho^{-1}p^m\chi(p))(1-\rho^{-1}p^{w-m}\bar{\chi}(p))r_\chi^m(\phi)^{\pm}.$$

Consequently we obtain

**Theorem 6.** *The sama assumption being as in the above we have*

$$\int_{Z_f^*}\chi(a)d\mu_m^{\pm}(a)=(1-\rho^{-1}p^m\chi(p))(1-\rho^{-1}p^{w-m}\bar{\chi}(p))r_\chi^m(\phi)^{\pm}.$$

This formula generalizes the one of Manin, which corresponds to the case $f\equiv 0 \pmod p$.

We may also call the numbers $r_\chi^m(\phi)^{\pm}$ the generalized periods of the cusp form $\phi$. Thus the formula in our theorem gives us a $p$-adic expression of these periods, analogous to the $p$-adic expression for the generalized Bernoulli numbers.

It should be noted that Višik [8] has also investigated $p$-adic measures connected with cusp forms of higher level.

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