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On Kronecker's Limit Formula for Certain Biquadratic Fields

Shuji Konno

§ 1. Introduction

In [1], Asai studied the Kronecker's limit formula for the Eisenstein series associated to an algebraic number field of class number one. He obtained the function $h(\xi)$ as an analogy of $\log|\eta(z)|$ and he showed that $h(\xi)$ satisfies certain differential equation and transformation formula. Recently, Elstrodt, Grunewald and Mennicke [2], generalized the above limit formula for the case of imaginary quadratic fields with arbitrary class number. They also showed many interesting formulas for the function associated to $h(\xi)$.

Our aim in this paper is to consider the Kronecker's limit formula for the zeta-function of certain biquadratic fields in connection with $h(\xi)$. To be more precise, let L be the composite of two imaginary quadratic fields k and K. We assume that the class number of k is one and that the discriminants of k and K have no common factor. Let C be any absolute ideal class of L and let $\zeta_L(s, C)$ be the zeta-function of the class C. We show that the limit formula for $\zeta_L(s, C)$ can be written by means of the curvilinear integral of the function $h(\xi)$. Here the curve is a semi-circle in $\boldsymbol{R}^3$ cannonically associated to the ideal class C (§ 4, Theorem 1). Since the Fourier coefficients of $h(\xi)$ are given by modified Bessel function, each term of the integral decreases rapidly. In Section 5, we give an approximation formula for each term appearing in the limit formula (§ 5, Theorems 2, 3). Finally using the table for the modified Bessel function ([6]), we give the approximate values for the integrals in the case $L=\boldsymbol{Q}(\sqrt{-4}, \sqrt{-3})$.

Notations. We denote by $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, the rational number field, the real number field, and the complex number field. For an associative ring A with an identity, $A^\times$ denote the group of invertible elements. For $z \in \boldsymbol{C}$, $z \to \bar{z}$ denotes the complex conjugation, $S(z)=z+\bar{z}$ and $|z|^2=z\bar{z}$. For an algebraic extension X of Y, $N_{X/Y}$ means the relative norm.

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§2. Preliminaries

Let k and K be imaginary quadratic fields of discriminants $-d_1$ and $-d_2$, respectively. Let us define $L=kK$. Throughout this paper we assume that the class number of k is one and $(d_1, d_2)=1$, $(d_1>0, d_2>0)$. Hence L is a biquadratic extension of $\boldsymbol{Q}$ whose galois group is abelian of type (2, 2).

Let $\mathfrak{g}$ denote the ring of integers in k and w_k be the number of roots of unity in k. Let $M=\boldsymbol{Q}(\sqrt{d_1d_2})$ be the real quadratic field. Let $\varepsilon_1>1$ denote the fundamental unit of M. We define $\varepsilon=\varepsilon_1$ or ε_1^2, according as $N_{M/\boldsymbol{Q}}(\varepsilon_1)=+1$ or -1. Let ε' denote the conjugate of ε over $\boldsymbol{Q}$. We see easily that $\varepsilon>1>\varepsilon'>0$. Let σ be the generator of $\mathrm{Gal}(L/k)$. The restriction of σ on K also generates $\mathrm{Gal}(K/\boldsymbol{Q})$. In what follows, we write $\tilde{x}=x^\sigma$ for $x \in L$.

Let C be any absolute ideal class of L. As usual, we denote by $\zeta_L(s, C)=\sum_{\mathfrak{a}\in C} N_{L/\boldsymbol{Q}}(\mathfrak{a})^{-s}$, $(\mathrm{Re}(s)>1)$, the zeta function of the ideal class C. Let C^{-1} be the inverse class of C. Since the class number of k is one, we may choose an ideal $\mathfrak{q} \in C^{-1}$ such that $\mathfrak{q}=\mathfrak{g}+\mathfrak{g}\omega$, $\omega \in L$ $(\omega \notin \mathfrak{g})$.

Lemma 1. *With the notation above, we can find* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathfrak{g})$ *such that* $\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}$. *In particular, we have* $\varepsilon=c\omega+d$, $\varepsilon'=c\tilde{\omega}+d$ *and* $\varepsilon-\varepsilon'=c(\omega-\tilde{\omega})$.

Proof. Recall that $(\sqrt{d_1d_2})^\sigma=((\sqrt{-d_1})(-\sqrt{-d_2}))^\sigma=\sqrt{-d_1}\sqrt{-d_2}=-\sqrt{d_1d_2}$. Hence we see $\varepsilon^\sigma=\varepsilon'$. Since $\varepsilon\mathfrak{q}=\mathfrak{q}$, we have $\varepsilon\omega=a\omega+b$ and $\varepsilon=c\omega+d$ for some $a, b, c, d \in \mathfrak{g}$. The action of σ on these relations implies $\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega & \tilde{\omega} \\ 1 & 1 \end{pmatrix}$. Consequently, we obtain $\varepsilon\varepsilon'=1=ad-bc$.

Let $B=\boldsymbol{C}+\boldsymbol{C}j$ denote the quaternion division algebra over $\boldsymbol{R}$, where j satisfies $j^2=-1$ and $zj=j\bar{z}$ for any $z \in \boldsymbol{C}$. Any element $\xi \in B$ can be written as $\xi=x+yi+uj+vij$ $(x, y, u, v \in \boldsymbol{R})$. Let $\xi\rightarrow\bar{\xi}=x-yi-ui-vij$ denote the quaternionic conjugation of B and let $N(\xi)=\xi\bar{\xi}$ denote the norm of ξ. (The quaternionic involution of B coincides with the complex conjugation on $\boldsymbol{C}$. Therefore we use the same notation $\xi\rightarrow\bar{\xi}$ for $\xi \in B$.)

By the three-dimensional hyperbolic space we mean the subset of B consisting of elements $\xi=x+yi+vj$ with $x, y, v \in \boldsymbol{R}$ and $v>0$. Let H denote the three-dimensional hyperbolic space. We denote the point of H by $\xi=z+vj$ $(z \in \boldsymbol{C}, v>0)$. The group $SL_2(\boldsymbol{C})$ act on H by

$$(1) \qquad SL_2(\boldsymbol{C})\times H \ni \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \xi\right)\longrightarrow(\alpha\xi+\beta)(\gamma\xi+\delta)^{-1} \in H.$$

Here the quotient $(\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}$ is taken in B. To be more precise, it is given by

$$(2)\qquad (\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}=\frac{(\alpha z+\beta)(\overline{\gamma z+\delta})+\alpha\bar{\gamma}v^2}{|\gamma z+\delta|^2+|\gamma|^2\cdot v^2}+\frac{v}{|\gamma z+\delta|^2+|\gamma|^2\cdot v^2}j.$$

The hyperbolic metric on H is defined by

$$(3)\qquad ds^2=\frac{1}{v^2}(dx^2+dy^2+dv^2).$$

The group $PSL_2(C)=SL_2(C)/\{\pm 1\}$ act on H as the group of isometries. Moreover, the discrete subgroup $PSL_2(\mathfrak{g})=SL_2(\mathfrak{g})/\{\pm 1\}$ of $PSL_2(C)$ act on H properly discontinuously.

§ 3. Eisenstein series and semi-circles in H

We now introduce the Eisenstein series. Let $\xi=z+vj\in H$. Consider the series

$$(4)\qquad E(\xi, s)=\sum_{\{m,n\}}{}' \frac{v^{2s}}{N(m-\xi n)^{2s}}=\sum_{\{m,n\}}{}' \frac{v^{2s}}{(|m-nz|^2+|n|^2v^2)^{2s}}\qquad (\mathrm{Re}(s)>1).$$

The summation is taken over all non-associated pairs $\{m, n\}$ for $(m, n)\in \mathfrak{g}\times\mathfrak{g}-\{(0, 0)\}$. We call two pairs (m_ν, n_ν) $(\nu=1, 2)$ are associated if $m_2=\eta m_1$, $n_2=\eta n_1$ with $\eta\in\mathfrak{g}^\times$ are satisfied. The series (4) converges uniformly on any compact set in $\mathrm{Re}(s)>1$. It satisfies

$$(5)\qquad E((\alpha\xi+\beta)(\gamma\xi+\delta)^{-1}, s)=E(\xi, s)$$

for any $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL_2(\mathfrak{g})$. The Eisenstein series $E(\xi, s)$ is the special case of the function studied in Asai [1], Elstrodt, Grunewald and Mennicke [2]. The following Lemma is proved in [1], [2].

Lemma 2. *The function $s\to E(\xi, s)$ has a meromorphic continuation to the whole s-plane. At $s=1$ the continued function has expansion*

$$E(\xi, s)=\frac{2\pi^2}{w_k d_1}\left(\frac{1}{s-1}+\alpha_0-2-\log v^2+h(\xi)\right)+O(|s-1|).$$

The constant α_0 and the function $h(\xi)$ are given by

$$\alpha_0=2\lim_{s\to 1}\left(\frac{w_k\sqrt{d_1}}{2\pi}\zeta_k(s)-\frac{1}{s-1}\right),$$

$$h(\xi)=\frac{w_k d_1}{2\pi^2}\zeta_k(2)\cdot v^2+4w_k \sum_{0\neq mn,\{m,n\}'}\left|\frac{m}{n}\right| K_1\left(4\pi\frac{|mn|}{\sqrt{d_1}}v\right) v e^{2\pi i S(mnz/\sqrt{-d_1})}$$
$$(\xi=z+vj\in H).$$

The summation $\sum_{0\neq mn,\{m,n\}'}$ *is taken over all non-equivalent classes* $\{m, n\}'$ *with respect to the equivalence relation* $(m, n)\sim(m\eta, n\eta^{-1})$ *for* $m, n\in\mathfrak{g}$ *and* $\eta\in\mathfrak{g}^{\times}$. In Lemma 2, K_s is the modified Bessel function ([6], p. 183)

$$(6)\qquad 2K_s(X)=\left(\frac{X}{2}\right)^s\int_0^\infty e^{-(t+(X/2)^2t^{-1})}t^{-s-1}dt \qquad (X>0).$$

Notations C, $\mathfrak{q}$, ω, $\tilde{\omega}$, ε, ε' and $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ being the same as in Lemma 1. Let us denote by Γ_ω the half-circle in H, which is perpendicular to the complex z-plane at the points ω and $\tilde{\omega}$. For any $\xi\in H$, we define $u=c(\xi-\tilde{\omega})(\omega-\xi)^{-1}$. Observe that u is an element of B, but not always lie in H.

Lemma 3. *The point* ξ *is on* Γ_ω *if and only if* u *is written as* $u_1 j$ *for some* $u_1>0$. *To be more precise, if* $\xi=z+vj$ *is on* Γ_ω *then* u_1 *is given by* $u_1=(\varepsilon-\varepsilon')v/N(\omega-\xi)$. *Conversely, for any* $u_1>0$ *the point* $\xi=(u_1j+c)^{-1}(u_1j\omega+c\tilde{\omega})$ *is on* Γ_ω *and* ξ *corresponds to* $u=u_1j$.

Proof. For any $\xi=z+vj\in H$, we see that

$$\begin{aligned} u&=\frac{c}{N(\omega-\xi)}(\xi-\tilde{\omega})\overline{(\omega-\xi)}\\ &=\frac{c}{N(\omega-\xi)}\{(z-\tilde{\omega})\overline{(\omega-z)}-v^2\}+\frac{(\varepsilon-\varepsilon')v}{N(\omega-\xi)}j.\end{aligned}$$

Since $c(\omega-\tilde{\omega})=\varepsilon-\varepsilon'>0$. Our aim is to show that ξ is on Γ_ω if and only if $A=0$, where we denoted $A=(z-\tilde{\omega})\overline{(\omega-z)}-v^2$. Put $z=x+yi$, $\omega=\omega_1+\omega_2 i$ and $\tilde{\omega}=\tilde{\omega}_1+\tilde{\omega}_2 i$ ($x, y, \omega_j, \tilde{\omega}_j\in\boldsymbol{R}; j=1, 2$). We have

$$\begin{aligned} A=&\{(x-\tilde{\omega}_1)(\omega_1-x)+(y-\tilde{\omega}_2)(\omega_2-y)-v^2\}\\ &+\{(y-\tilde{\omega}_2)(\omega_1-x)-(\omega_2-y)(x-\tilde{\omega}_1)\}i.\end{aligned}$$

We now identify the point $x+yi+vj$ in H with the point (x, y, v) in $\boldsymbol{R}^3$. Suppose ξ is on Γ_ω then the vectors $\overrightarrow{\tilde{\omega}\xi}$ and $\overrightarrow{\xi\omega}$ are orthogonal. This means that $\mathrm{Re}(A)=0$. Since the point $z=x+yi$ lies on a segment $\omega\tilde{\omega}$, it satisfies $(y-\tilde{\omega}_2)/(\omega_2-y)=(x-\tilde{\omega}_1)/(\omega_1-x)$. Hence we have $\mathrm{Im}(A)=0$. Thus $\xi\in\Gamma_\omega$ implies $A=0$. The converse is obvious. Finally, for a given

$u_1>0$ the point ξ satisfying $u=u_1 j=c(\xi-\tilde{\omega})(\omega-\xi)^{-1}$ is on Γ_ω by the above. Solving for ξ, we obtain $\xi=(u+c)^{-1}(u\omega+c\tilde{\omega})$.

Lemma 4. *For any $\xi \in H$, let u be as in Lemma 3. Then the transformation $\xi \to \xi^*=(a\xi+b)(c\xi+d)^{-1}$ induces $u \to u^*=\varepsilon^2 u$. In particular, the curve Γ_ω goes onto itself.*

Proof. We extend the transformation $\xi \to \xi^*$ of H to the transformation of $H \cup C$ in an obvious way. Then the points ω and $\tilde{\omega}$ are fixed points of this transformation. Thus u goes to $u^*=c(\xi^*-\tilde{\omega})(\omega-\xi^*)^{-1}$. Now we have

$$\begin{aligned}\xi^*-\tilde{\omega}&=(a\xi+b)(c\xi+d)^{-1}-(c\tilde{\omega}+d)^{-1}(a\tilde{\omega}+b)\\&=(c\tilde{\omega}+d)^{-1}\{(c\tilde{\omega}+d)(a\xi+b)-(a\tilde{\omega}+b)(c\xi+d)\}(c\xi+d)^{-1}\\&=(c\tilde{\omega}+d)^{-1}(\xi-\tilde{\omega})(c\xi+d)^{-1}.\end{aligned}$$

Similarly, we have

$$\omega-\xi^*=(c\omega+d)^{-1}(\omega-\xi)(c\xi+d)^{-1}.$$

Consequently, by Lemma 1, we obtain

$$u^*=c(c\tilde{\omega}+d)^{-1}(\xi-\tilde{\omega})(\omega-\xi)^{-1}(c\omega+d)=\varepsilon^2 u.$$

By Lemmas 3 and 4, we can view u_1 as a positive parameter for the curve Γ_ω. But in what follows, we sometimes choose $\mu=u_1|c|^{-1}$ as the positive parameter for the curve Γ_ω. For $\xi \in \Gamma_\omega$, they are given by

$$(7) \qquad \xi=(u+c)^{-1}(u\omega+c\tilde{\omega}), \quad u=u_1 j, \quad 0<u_1<+\infty$$

or recalling that $c(\omega-\tilde{\omega})=\varepsilon-\varepsilon'=|c||\omega-\tilde{\omega}|$,

$$(8) \qquad \xi=\frac{\mu^2\omega+\tilde{\omega}}{\mu^2+1}+\frac{|\omega-\tilde{\omega}|\mu}{\mu^2+1}j, \qquad 0<\mu<+\infty.$$

The transformation $\xi \to \xi^*$ also induces $\mu \to \mu\varepsilon^2$.

Let us now consider the restriction of $E(\xi, s)$ to $\xi \in \Gamma_\omega$. Put

$$(9) \qquad \lambda=\frac{1}{2\log\varepsilon}\log\mu.$$

We consider the restricted function to be a function of λ. We shall denote it by $g(\lambda)$. Obviously, g is a C^1-class function. The transformation $\xi \to \xi^*$ induces $\lambda \to \lambda+1$. In view of (5), we see that $g(\lambda+1)=g(\lambda)$. Thus g has Fourier expansion

(10) $$g(\lambda)=\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k\lambda},$$

where c_k are given by

(11) $$c_k=\int_{\lambda_0}^{\lambda_0+1} g(\lambda)e^{-2\pi i k\lambda}d\lambda \qquad (k=0, \pm 1, \pm 2, \cdots).$$

We should remark that the constant λ_0 can be chosen arbitrarily, since the integrand is periodic with period 1.

§ 4. Proof of Theorem 1

Let C be any absolute ideal class of L. It is well known that the function $\zeta_L(s, C)$ is continued to the whole s-plane meromorphically. It has simple pole only at $s=1$. The aim of this section is to prove the

Theorem 1. *With the notations above. We choose an ideal* $\mathfrak{q} \in C^{-1}$ *such that* $\mathfrak{q}=\mathfrak{g}+\mathfrak{g}\omega$ *with* $\omega \in L$. *Then* $\zeta_L(s, C)$ *has Laurent expansion*

$$\zeta_L(s, C)=\frac{2\pi^2 R_L}{d_1 d_2}\left[\frac{1}{s-1}+\alpha_0-\log d_2-\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2-h(\xi))\frac{d\mu}{\mu}\right] +O(|s-1|),$$

where R_L *is the regulator of* L. *The integral is taken along the curve*

$$\xi=z+vj=\frac{\mu^2\omega+\tilde{\omega}}{\mu^2+1}+\frac{|\omega-\tilde{\omega}|\mu}{\mu^2+1}j, \qquad \mu_0\leqq\mu\leqq\mu_0\varepsilon^2,$$

with an arbitrarily chosen $\mu_0>0$. *The constant* α_0 *and the function* $h(\xi)$ *are given in Lemma* 2.

Proof. To prove this assertion, we compute the constant term of the Fourier expansion of $g(\lambda)$. By (9) and (11), we have

(12) $$c_0=\int_{\lambda_0}^{\lambda_0+1} g(\lambda)d\lambda=\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}\sum_{\{m,n\}}{}' \frac{v^{2s}}{N(m-\xi n)^{2s}}\frac{d\mu}{\mu},$$

where we denoted $\mu_0=\varepsilon^{2\lambda_0}$. Replacing ξ by $(u+c)^{-1}(u\omega+c\tilde{\omega})$, we have

$$m-\xi n=(u+c)^{-1}\{u(m-\omega n)+c(m-\tilde{\omega}n)\}.$$

Put $\beta=m-\omega n$. Then we see $\beta \in \mathfrak{q}$ and $\tilde{\beta}=m-\tilde{\omega}n$. Since $u=\mu|c|j$ and $v=|\omega-\tilde{\omega}|\mu(\mu^2+1)^{-1}$, we have

$$\frac{v^{2s}}{N(m-\xi n)^{2s}}=\frac{(|\omega-\tilde{\omega}|\mu)^{2s}}{(\mu^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}. \tag{13}$$

Let ε_0 denote the fundamental unit of L satisfying $|\varepsilon_0|>1$. Let $\langle\varepsilon_0\rangle$ and $\langle\varepsilon\rangle$ be the cyclic groups generated by ε_0 and ε, respectively. We can view $\langle\varepsilon\rangle$ to be a subgroup of $\langle\varepsilon_0\rangle$. Put $q=[\langle\varepsilon_0\rangle:\langle\varepsilon\rangle]$. Recall that $\varepsilon_0\bar{\varepsilon}_0$ is a unit of M of norm 1. Hence we can write $\varepsilon_0\bar{\varepsilon}_0=\varepsilon^r$ with $r=1$ or 2. In fact, if $r\geqq 3$ then we see $|\varepsilon_0\varepsilon^{-1}|=\varepsilon^{(r-2)/2}>1$. But the fact $\varepsilon_0^q=\varepsilon$ implies $|\varepsilon_0\varepsilon^{-1}|^q=\varepsilon^{1-q}>1$. This is a contradiction.

Note that $\varepsilon_0\tilde{\varepsilon}_0$ is a root of unity in k. Thus if β is replaced by $\beta\varepsilon_0^j$ $(j=0, \pm 1, \pm 2, \cdots)$ in (13), we see that

$$\frac{(|\omega-\tilde{\omega}|\mu)^{2s}}{(\mu^2|\beta\varepsilon_0^j|^2+|\tilde{\beta}\tilde{\varepsilon}_0^j|^2)^{2s}}=\frac{(|\omega-\tilde{\omega}|\mu)^{2s}}{(\mu^2\varepsilon^{jr}|\beta|^2+\varepsilon^{-jr}|\tilde{\beta}|^2)^{2s}}=\frac{(|\omega-\tilde{\omega}|(\mu\varepsilon^{jr}))^{2s}}{((\mu\varepsilon^{jr})^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}. \tag{14}$$

In (12) the integrand converges uniformly on any compact set in $R(s)>1$. We can interchange the summation and the integration.

$$\begin{aligned} c_0&=\frac{1}{2\log\varepsilon}\sum_{\{m,n\}}{}'\int_{\mu_0}^{\mu_0\varepsilon^2}\frac{v^{2s}}{N(m-\xi n)^{2s}}\frac{d\mu}{\mu} \\ &=\frac{1}{2w_k\log\varepsilon}\sum_{\substack{m,n\in\mathfrak{g}\\(m,n)\neq(0,0)}}{}'\int_{\mu_0}^{\mu_0\varepsilon^2}\frac{v^{2s}}{N(m-\xi n)^{2s}}\frac{d\mu}{\mu}. \end{aligned}$$

By definition, when (m, n) runs over all pairs in $\mathfrak{g}\times\mathfrak{g}-\{(0, 0)\}$, β runs over all non-zero numbers in $\mathfrak{q}$. Let $\beta_1, \beta_2\in\mathfrak{q}$. We see that $(\beta_1)=(\beta_2)$ if and only if $\beta_2=\pm\varepsilon_0^j\beta_1$ $(j=0, \pm 1, \pm 2, \cdots)$. Hence by (13) and (14), we have

$$\begin{aligned} c_0&=\frac{|\omega-\tilde{\omega}|^{2s}}{2w_k\log\varepsilon}\sum_{0\neq\beta\in\mathfrak{q}}{}'\int_{\mu_0}^{\mu_0\varepsilon^2}\frac{\mu^{2s}}{(\mu^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}\frac{d\mu}{\mu} \\ &=\frac{|\omega-\tilde{\omega}|^{2s}}{w_k\log\varepsilon}\sum_{0\neq(\beta)\subset\mathfrak{q}}{}'\sum_{j=-\infty}^{+\infty}\int_{\mu_0}^{\mu_0\varepsilon^2}\frac{(\mu\varepsilon^{jr})^{2s}}{((\mu\varepsilon^{jr})^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}\frac{d\mu}{\mu} \\ &=\frac{|\omega-\tilde{\omega}|^{2s}}{w_k\log\varepsilon}\sum_{0\neq(\beta)\subset\mathfrak{q}}{}'\sum_{j=-\infty}^{+\infty}\int_{\mu_0\varepsilon^{jr}}^{\mu_0\varepsilon^{jr+2}}\frac{\mu^{2s}}{(\mu^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}\frac{d\mu}{\mu}. \end{aligned}$$

Since $r=1$ or $r=2$, we have

$$\sum_{j=-\infty}^{+\infty}\int_{\mu_0\varepsilon^{jr}}^{\mu_0\varepsilon^{jr+2}}\cdots\frac{d\mu}{\mu}=\frac{2}{r}\int_0^\infty\cdots\frac{d\mu}{\mu}.$$

Thus we can write

$$c_0=\frac{2|\omega-\tilde{\omega}|^{2s}}{w_k r\log\varepsilon}\sum_{0\neq(\beta)\subset\mathfrak{q}}{}'\int_0^\infty\frac{\mu^{2s}}{(\mu^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}\frac{d\mu}{\mu}. \tag{15}$$

Put $\mu_1=(|\beta/\tilde{\beta}|^2)\mu^2$. Then we have

$$\int_0^\infty \frac{\mu^{2s}}{(\mu^2|\beta|^2+|\tilde{\beta}|^2)^{2s}}\frac{d\mu}{\mu}=\frac{1}{2|\beta\tilde{\beta}|^{2s}}\int_0^\infty \frac{\mu_1^{2s}}{(\mu_1+1)^{2s}}\frac{d\mu_1}{\mu_1}$$
$$=\frac{1}{2N_{L/Q}(\beta)^s}\frac{\Gamma(s)^2}{\Gamma(2s)}.$$

Recall that $|\omega-\tilde{\omega}|=|N_{L/k}(\mathfrak{q})|\sqrt{d_2}=\sqrt{N_{L/Q}(\mathfrak{q})d_2}$ and $R_L=2\log|\varepsilon_0|=r\log\varepsilon$. Thus (15) can be written as

$$c_0=\frac{d_2^s}{w_k R_L}\frac{\Gamma(s)^2}{\Gamma(2s)}\zeta_L(s, C). \tag{16}$$

On the other hand, by Lemma 2 and (12), we have

$$c_0=\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}\frac{2\pi^2}{w_k d_1}\left(\frac{1}{s-1}+\alpha_0-2-\log v^2+h(\xi)\right)\frac{d\mu}{\mu}+O(|s-1|)$$
$$=\frac{2\pi^2}{w_k d_1}\left(\frac{1}{s-1}+\alpha_0-2-\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2-h(\xi))\frac{d\mu}{\mu}\right)+O(|s-1|).$$

In view of (16), we can write

$$\zeta_L(s, C)=\frac{2\pi^2 R_L}{d_1 d_2}d_2^{1-s}\frac{\Gamma(2s)}{\Gamma(s)^2}\left(\frac{1}{s-1}+\alpha_0-2-\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2-h(\xi))\frac{d\mu}{\mu}\right)$$
$$+O(|s-1|).$$

At $s=1$, $\Gamma(s)$ has expansion $\Gamma(s)=1+a_1(s-1)+\cdots$. Hence, we have, $\Gamma(s)^2=1+2a_1(s-1)+\cdots$, $\Gamma(2s-1)=1+2a_1(s-1)+\cdots$, and $\Gamma(2s)/\Gamma(s)^2=(2s-1)\Gamma(2s-1)/\Gamma(s)^2=\{1+2(s-1)\}\{1+b_1(s-1)^2+\cdots\}$. Consequently we obtain

$$\zeta_L(s, C)=\frac{2\pi^2 R_L}{d_1 d_2}\{1-\log d_2\cdot(s-1)+\cdots\}\{1+2(s-1)\}\{1+b_1(s-1)^2+\cdots\}$$
$$\times\left\{\frac{1}{s-1}+\alpha_0-2-\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}\cdots\frac{d\mu}{\mu}+O(|s-1|)\right\}$$
$$=\frac{2\pi^2 R_L}{d_1 d_2}\left(\frac{1}{s-1}+\alpha_0-\log d_2-\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2-h(\xi))\frac{d\mu}{\mu}\right)$$
$$+O(|s-1|).$$

This completes the proof.

§ 5. Approximation formula

By Theorem 1, the constant term of the expansion of $\zeta_L(s, C)$ was

$$\alpha_0 - \log d_2 - \frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2 - h(\xi))\frac{d\mu}{\mu}.$$

Observe that only the term

$$I = \frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}(\log v^2 - h(\xi))\frac{d\mu}{\mu} \tag{17}$$

depends on the choice of C. Therefore our aim in this section is to consider an approximation formula for I.

Throughout this section, we always assume that

$$O < \mu_0 < \mu_0\varepsilon^2 < 1 \tag{18}$$

and μ runs through the interval $[\mu_0, \mu_0\varepsilon^2]$. For $m \in \mathfrak{g}$ $(m \neq 0)$, we define ([2], (2, 18))

$$\sigma_1(m) = \sum_{0\neq(n)|m} |n|^2,$$

where the summation is taken over all non-zero ideals (n) dividing m. Then the function $h(\xi)$ of Lemma 2 can be written as

$$\begin{aligned} h(\xi) &= \frac{w_k d_1}{2\pi^2}\zeta_k(2)v^2 \\ &+ 4w_k \sum_{0\neq m\in\mathfrak{g}} \frac{\sigma_1(m)}{|m|} K_1(4\pi|m|v/\sqrt{d_1})v\cos(4\pi(m_1y + m_2x)/\sqrt{d_1}), \end{aligned} \tag{19}$$

where we denoted $m = m_1 + m_2 i$ and $\xi = x + yi + vj$.

In (19), $K_1(4\pi|m|v/\sqrt{d_1})$ tends to zero with great rapidity when $|m|$ is large. Therefore we are going to compute for each term in I.

Lemma 5. *We have*

(i) $$\frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2} v^2\frac{d\mu}{\mu} = \frac{|\omega-\tilde{\omega}|^2}{4\log\varepsilon}\left(\frac{1}{1+\mu_0^2} - \frac{1}{1+\mu_0^2\varepsilon^4}\right),$$

(ii) $$\begin{aligned} \frac{1}{2\log\varepsilon}\int_{\mu_0}^{\mu_0\varepsilon^2}\log v^2\frac{d\mu}{\mu} &= \log|\omega-\tilde{\omega}|^2 + \log(\mu_0\varepsilon)^2 \\ &+ \frac{1}{2\log\varepsilon}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\{(\mu_0\varepsilon^2)^{2n} - \mu_0^{2n}\}. \end{aligned}$$

In the infinite series of (ii), *the error obtained by stopping at any term is less than the absolute value of the next term.*

Proof. (i) This follows by taking $t=1+\mu^2$.
(ii) By (8), we see

$$\int_{\mu_0}^{\mu_0\varepsilon^2} \log v^2 \frac{d\mu}{\mu} = \log \varepsilon^2 \log|\omega-\tilde{\omega}|^2 + \log \varepsilon^2 \log(\mu_0\varepsilon)^2 - \int_{\mu_0}^{\mu_0\varepsilon^2} \log(1+\mu^2)\frac{d\mu}{\mu}.$$

Applying Taylor's theorem for $\log(1+X)$ with $X=\mu^2$, we have

$$\begin{aligned}\int_{\mu_0}^{\mu_0\varepsilon^2} &\log(1+\mu^2)\frac{d\mu}{\mu}\\ &=\int_{\mu_0}^{\mu_0\varepsilon^2}\left\{\sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n}\mu^{2n}+\frac{(-1)^{N-1}}{N}\mu^{2N}(1+\theta\mu^2)^{-N}\right\}\frac{d\mu}{\mu} \qquad (0<\theta<1)\\ &=\frac{1}{2}\sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n^2}\{(\mu_0\varepsilon^2)^{2n}-\mu_0^{2n}\}+\frac{(-1)^{N-1}}{2N^2}\theta_1\{(\mu_0\varepsilon^2)^{2N}-\mu_0^{2N}\}\\ &\qquad\qquad (0<\theta_1<1).\end{aligned}$$

From this we obtain our assertions.

In what follows, we denote

$$(20)\qquad \begin{cases}\omega=\omega_1+\omega_2\cdot i, \qquad \tilde{\omega}=\tilde{\omega}_1+\tilde{\omega}_2\cdot i,\\ \lambda=\lambda(m)=\dfrac{2\pi|m|}{\sqrt{d_1}}, \qquad m=m_1+m_2\cdot i \quad (m\in\mathfrak{g}),\\ \alpha=\dfrac{4\pi}{\sqrt{d_1}}(m_1\omega_2+m_2\omega_1),\\ \beta=\dfrac{4\pi}{\sqrt{d_1}}\{(m_1\tilde{\omega}_2+m_2\tilde{\omega}_1)-(m_1\omega_2+m_2\omega_1)\},\\ t=\beta(1+\mu^2)^{-1}.\end{cases}$$

$$(21)\qquad I(m)=\int_{\mu_0}^{\mu_0\varepsilon^2} K_1\left(\frac{4\pi|m|v}{\sqrt{d_1}}\right) v\cos\left(\frac{4\pi(m_1y+m_2x)}{\sqrt{d_1}}\right)\frac{d\mu}{\mu}$$
$$(m\in\mathfrak{g},\ m\neq 0).$$

By (8), we see that $x=(\mu^2\omega_1+\tilde{\omega}_1)(1+\mu^2)^{-1}$, $y=(\mu^2\omega_2+\tilde{\omega}_2)(1+\mu^2)^{-1}$, $v=|\omega-\tilde{\omega}|\mu(1+\mu^2)^{-1}$. Hence, by a simple computation we have

$$(22)\qquad I(m)=\int_{\mu_0}^{\mu_0\varepsilon^2} K_1(2\lambda v)v\cos(\alpha+t)\frac{d\mu}{\mu}, \qquad (m\neq 0).$$

Let

$$\cos(\alpha+t)=\sum_{j=0}^{\infty} a_j t^j \tag{23}$$

be the Taylor expansion in t, where

$$a_{2\nu+1}=\frac{(-1)^{\nu+1}}{(2\nu+1)!}\sin\alpha, \quad a_{2\nu}=\frac{(-1)^{\nu}}{(2\nu)!}\cos\alpha \quad (\nu=0, 1, 2, \cdots). \tag{24}$$

It is known that ([6], p. 80, (15))

$$K_1(X)=\frac{1}{X}+\sum_{i=0}^{\infty}\frac{(X/2)^{2i+1}}{i!(i+1)!}\left\{\log\left(\frac{X}{2}\right)-\frac{1}{2}\psi(i+1)-\frac{1}{2}\psi(i+2)\right\} \quad (X>0), \tag{25}$$

where ψ denotes the logarithmic derivative of the Γ-function. For a positive integer i, it is known

$$\psi(1)=-\gamma, \quad \psi(i+1)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{i}-\gamma. \tag{26}$$

Here γ is the Euler's constant. For simplicity we denote

$$\varphi(i)=\tfrac{1}{2}\{\psi(i+1)+\psi(i+2)\} \qquad (i=0, 1, 2, \cdots). \tag{27}$$

Let $X=2\lambda v$ in (25). Then we have

$$K_1(2\lambda v)v=\frac{1}{2\lambda}+\sum_{i=0}^{\infty}\frac{(\lambda v)^{2i+1}v}{i!(i+1)!}\{\log(\lambda v)-\varphi(i)\}. \tag{28}$$

Recall that $|t|\leqq|\beta|$ and the range of v is of the form $[v_0, v_1]$ $(v_1>v_0>0)$. The right hand side of (23) and (28) converges absolutely and uniformly in μ. Thus, for any given $\eta_1>0$, we can find positive integers p and q such that

$$\begin{aligned} &K_1(2\lambda v)v\cos(\alpha+t) \\ &=\left(\frac{1}{2\lambda}+\sum_{i=0}^{p}\frac{(\lambda v)^{2i+1}v}{i!(i+1)!}\{\log(\lambda v)-\varphi(i)\}\right)\left(\sum_{j=0}^{q}a_j t^j\right)+R_{pq}, \end{aligned} \tag{29}$$

$|R_{pq}|\leqq\eta_1$ (uniformly in μ).

In the following we shall fix the numbers p and q. We let the indices i and j run through $\{0, 1, 2, \cdots, p\}$ and $\{0, 1, 2, \cdots, q\}$, respectively. Let us denote

$$B_{ij}=\frac{\lambda^{2i+1}|\omega-\tilde{\omega}|^{2i+2}}{i!(i+1)!}a_j\beta^j, \qquad B'_{ij}=B_{ij}\{\log(\lambda|\omega-\tilde{\omega}|)-\varphi(i)\}.$$

Since $\log(\lambda v) - \varphi(i) = \{\log(\lambda|\omega - \tilde{\omega}|) - \varphi(i)\} + \log\mu - \log(1+\mu^2)$. The integrand of $I(m)$ can be written as

(30) $$K_1(2\lambda v)v\cos(\alpha+t) = \sum_{\nu=1}^{4} I_\nu + R_{pq},$$

where

(31) $$\begin{cases} I_1 = \dfrac{1}{2\lambda}\sum_{j=0}^{q} a_j\beta^j(1+\mu^2)^{-j}, \\ I_2 = \sum_{i,j} B'_{ij}\mu^{2i+2}(1+\mu^2)^{-2i-j-2}, \\ I_3 = \sum_{i,j} B_{ij}\mu^{2i+2}(1+\mu^2)^{-2i-j-2}\log\mu, \\ I_4 = -\sum_{i,j} B_{ij}\mu^{2i+2}(1+\mu^2)^{-2i-j-2}\log(1+\mu^2). \end{cases}$$

Let η_1 be as in (29). Let $\eta_2 = \frac{1}{5}\eta_1$. Then we have

Lemma 6. *We can find positive integers r and r_1 such that*

$$I_1 = \frac{\cos\alpha}{2\lambda} + \frac{1}{2\lambda}\sum_{j=1}^{q}\sum_{k=0}^{r} a_j\beta^j\binom{-j}{k}\mu^{2k} + T_1,$$

$$I_2 = \sum_{i,j}\sum_{k=0}^{r} B'_{ij}\binom{-2i-j-2}{k}\mu^{2i+2k+2} + T_2,$$

$$I_3 = \sum_{i,j}\sum_{k=0}^{r} B_{ij}\binom{-2i-j-2}{k}\mu^{2i+2k+2}\log\mu + T_3,$$

$$I_4 = \sum_{i,j}\sum_{k=0}^{r}\sum_{l=0}^{r_1} B_{ij}\binom{-2i-j-2}{k}\frac{(-1)^l}{l}\mu^{2i+2k+2l+2} + T_4 + T_5$$

where T_ν satisfies $|T_\nu| < \eta_2$ $(1 \leqq \nu \leqq 5)$ uniformly in μ.

Proof. Suppose $\mu \in [\mu_0, \mu_0\varepsilon^2]$ and let $M = j$ $(j \geqq 1)$ or $M = 2i+2+j$ $(0 \leqq i \leqq p,\ 0 \leqq j \leqq q)$. By Taylor's theorem with Lagrange's form of the remainder, we have

$$(1+\mu^2)^{-M} = \sum_{k=0}^{N}\binom{-M}{k}\mu^{2k} + T_N^{(M)},$$

$$|T_N^{(M)}| < \left|\binom{-M}{N+1}\mu^{2N+2}\right| \qquad (N > 1).$$

Since $M < M'$ implies $\left|\binom{-M}{N+1}\mu^{2N+2}\right| < \left|\binom{-M'}{N+1}\mu^{2N+2}\right|$, we see that

$$|T_N^{(M)}|<\left|\binom{-2p-q-2}{N+1}(\mu_0\varepsilon^2)^{2N+2}\right|\longrightarrow 0 \qquad (N\longrightarrow\infty).$$

Hence we can find a positive integer r satisfying the inequalities

$$(32)\qquad \begin{aligned}
&\left(\frac{1}{2\lambda}\sum_{j=1}^{q}|a_j\beta^j|\right)\left|\binom{-q}{r+1}(\mu_0\varepsilon^2)^{2r+2}\right|<\eta_2,\\
&\left(\sum_{i,j}|B'_{ij}\mu^{2i+2}|\right)\left|\binom{-2p-q-2}{r+1}(\mu_0\varepsilon^2)^{2r+2}\right|<\eta_2,\\
&\left(\sum_{i,j}|B_{ij}\mu^{2i+2}\log\mu|\right)\left|\binom{-2p-q-2}{r+1}(\mu_0\varepsilon^2)^{2r+2}\right|<\eta_2,\\
&\left(\sum_{i,j}|B_{ij}\mu^{2i+2}\log(1+\mu^2)|\right)\left|\binom{-2p-q-2}{r+1}(\mu_0\varepsilon^2)^{2r+2}\right|<\eta_2,
\end{aligned}$$

uniformly in μ. Recall that

$$\begin{aligned}
T_1&=\frac{1}{2\lambda}\sum_{j=1}^{q}a_j\beta^jT_r^{(j)}\\
T_2&=\sum_{i,j}B'_{ij}\mu^{2i+2}T_r^{(2i+j+2)},\\
T_3&=\sum_{i,j}B_{ij}\mu^{2i+2}\log\mu\cdot T_r^{(2i+j+2)}\\
T_4&=-\sum_{i,j}B_{ij}\mu^{2i+2}\log(1+\mu^2)\cdot T_r^{(2i+j+2)}.
\end{aligned}$$

We see that $|T_\nu|<\eta_2$ $(1\leqq\nu\leqq 4)$, uniformly in μ. Put

$$\log(1+\mu^2)=\sum_{l=1}^{N}\frac{(-1)^{l-1}}{l}\mu^{2l}+L_N,$$

$$T^{(N)}=-\sum_{i,j}\sum_{k=0}^{r}B_{ij}\binom{-2i-j-2}{k}\mu^{2i+2k+2}L_N.$$

Since $|L_N|<(\mu_0\varepsilon^2)^{2N+2}/(N+1)\to 0\,(N\to\infty)$, we can find a positive integer r_1 such that $|T^{(r_1)}|<\eta_2$. Then $T_5=T^{(r_1)}$ satisfies our assertion.

For any non-negative integers i, j, k we define

$$(33)\qquad \begin{aligned}
A_{ijk}&=\frac{\lambda^{2i+1}|\omega-\tilde{\omega}|^{2i+2}}{i!(i+1)!}a_j\beta^j\binom{-2i-j-2}{k},\\
A'_{ijk}&=A_{ijk}\{\log(\lambda|\omega-\tilde{\omega}|)-\varphi(i)\}.
\end{aligned}$$

Then we have

Theorem 2. *Let $I(m)$ be as in* (21). *For any given $\eta>0$, we can find positive integers p, q, r and r_1 such that*

$$I(m)=\frac{\log\varepsilon}{\lambda}\sum_{j=0}^{q}a_j\beta^j+\frac{1}{4\lambda}\cdot\sum_{j=1}^{q}\sum_{k=1}^{r}a_j\beta^j\binom{-j}{k}\frac{(\mu_0\varepsilon^2)^{2k}-\mu_0^{2k}}{k}$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}A'_{ijk}\frac{(\mu_0\varepsilon^2)^{2(i+k+1)}-\mu_0^{2(i+k+1)}}{i+k+1}$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}A_{ijk}\left[\frac{(\mu_0\varepsilon^2)^{2(i+k+1)}\log(\mu_0\varepsilon^2)-\mu_0^{2(i+k+1)}\log\mu_0}{i+k+1}\right.$$
$$\left.-\frac{(\mu_0\varepsilon^2)^{2(i+k+1)}-\mu_0^{2(i+k+1)}}{2(i+k+1)^2}\right]$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}\sum_{l=1}^{r_1}A_{ijk}\frac{(-1)^l}{l}\frac{(\mu_0\varepsilon^2)^{2(i+k+l+1)}-\mu_0^{2(i+k+l+1)}}{i+k+l+1}+R,$$

where R satisfies $|R|<\eta$.

Proof. Let $\eta_1=\eta/(4\log\varepsilon)$ in (29) and in Lemma 7. Since

$$\int_{\mu_0}^{\mu_0\varepsilon^2}\mu^{2\nu}\log\mu\frac{d\mu}{\mu}=\frac{1}{2\nu}\{(\mu_0\varepsilon^2)^{2\nu}\log(\mu_0\varepsilon^2)-\mu_0^{2\nu}\log\mu_0\}$$
$$-\frac{1}{4\nu^2}\{(\mu_0\varepsilon^2)^{2\nu}-\mu_0^{2\nu}\}\qquad(\nu\geqq 1),$$

we obtain our theorem.

In particular, if we choose $\mu_0\varepsilon^2=\frac{1}{2}$ and $\mu_0=\frac{1}{2}\varepsilon'^2$, we obtain

Theorem 3. *Let $I(m)$ be as in* (21). *For any given $\eta>0$, we can find positive integers p, q, r and r_1 such that*

$$I(m)=\frac{\log\varepsilon}{\lambda}\sum_{j=0}^{q}a_j\beta^j+\frac{1}{4\lambda}\sum_{j=1}^{q}\sum_{k=1}^{r}a_j\beta^j\binom{-j}{k}\frac{1-\varepsilon'^{4k}}{4^k k}$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}A'_{ijk}\frac{1-\varepsilon'^{4(i+k+1)}}{4^{i+k+1}(i+k+1)}$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}A_{ijk}\left[\frac{\log(1/2)-\varepsilon'^{4(i+k+1)}\log(\varepsilon'^2/2)}{4^{i+k+1}(i+k+1)}\right.$$
$$\left.-\frac{1-\varepsilon'^{4(i+k+1)}}{2\cdot 4^{i+k+1}(i+k+1)^2}\right]$$
$$+\frac{1}{2}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}\sum_{l=1}^{r_1}A_{ijk}\frac{(-1)^l}{l}\frac{1-\varepsilon'^{4(i+k+l+1)}}{4^{i+k+l+1}(i+k+l+1)}+R,$$

where $|R|<\eta$.

Example. Let $k=Q(\sqrt{-4})$, $K=Q(\sqrt{-3})$, $L=kK$ and $\mathfrak{q}=\mathfrak{g}+\mathfrak{g}\omega$ with $\omega=\frac{1}{2}(1+\sqrt{-3})$. Then we see $M=Q(\sqrt{3})$ and $\varepsilon=\varepsilon_1=2+\sqrt{3}$. We choose μ_0 such that $\mu_0\varepsilon^2=0.99$. Using the table for $K_1(X)$ in [6], we can compute the approximate values for $I(m)$. (We availed the value $e^x K_1(x)$ and e^x for $x=0.02n$; $n=1, 2, 3, \cdots$). Put $t=2\sqrt{3}\pi|m|\mu/(1+\mu^2)$ $(=2\lambda v)$. Then, by a little computation, we have

$$(34)\qquad I(m)=\frac{(-1)^{m_2}\sqrt{3}}{2}\int_{t_1}^{t_2} K_1(t)\cos\left(\frac{m_1}{|m|}\sqrt{3\pi^2|m|^2-t^2}\right)\frac{dt}{\sqrt{3\pi^2|m|^2-t^2}}$$

where $t_1=0.769647\cdots|m|$, and $t_2=5.4411232\cdots|m|$. Using (34) and the values for $K_1(t)$, we obtain $I(1)=0.040\pm0.001$, $I(1+i)=-0.0215\pm0.0004$, $I(1+2i)=0.0065\pm0.0001$, $I(2+i)=0.0039\pm3\times10^{-5}$, $I(4+2i)=0.0004\pm 10^{-5}$, $I(2+4i)>0$ and $|I(2+4i)|<10^{-5}$, $I(5)>0$ and $|I(5)|<10^{-5}$.

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College of General Education
Osaka University
Toyonaka, Osaka, 560 Japan