Advanced Studies in Pure Mathematics 12, 1987
Galois Representations and Arithmetic Algebraic Geometry
pp. 343–373

Class Field Theory for Two Dimensional Local Rings

Shuji Saito

Contents

Introduction

Let A be an excellent normal two-dimensional henselian local ring with finite residue field F and quotient field K. The purpose of this paper is to construct the class field theory for K, using the method and results in [S–1]. Let P be the set of all prime ideals of height one in A. For each $\mathfrak{p} \in P$, let $A_\mathfrak{p}$ be the henselization of A at $\mathfrak{p}$ and let $K_\mathfrak{p}$ (resp. $\kappa(\mathfrak{p})$) be its quotient (resp. residue) field. Then, $K_\mathfrak{p}$ is a henselian two-dimensional local field in the sense of [K–1] (cf. also [S–1] (2.2)). For such a field, K. Kato constructed the class field theory in [K–1] and [K–2]. Then, our method is to put together these local theories, which is a standard technique in the classical class field theory. To state our main results, we recall briefly some results in [K–1] and [K–2]: In general, for a noetherian scheme Z, Put $H^1(Z) = H^1(Z_{et}, \boldsymbol{Q}/\boldsymbol{Z})$ which is identified with the Pontrijagin dual of the abelian fundamental group $\pi_1^{ab}(Z)$. For a noetherian ring R, we put $H^1(R) = H^1(\mathrm{Spec}(R))$. For each $\mathfrak{p} \in P$, Kato constructed a canonical homomorphism

$$\Psi_{K_\mathfrak{p}}: H^1(K_\mathfrak{p}) \longrightarrow (K_2(K_\mathfrak{p}))^*_{\mathrm{tor}},$$

Received April 2, 1986.

where for a group G, $(G)^*_{\text{tor}}$ denotes the group of all homomorphisms χ: $G \to \boldsymbol{Q}/\boldsymbol{Z}$ of finite orders. Moreover, we have the following commutative diagram

$$
\tag{0.1}
\begin{array}{ccc}
H^1(K_{\mathfrak{p}}) & \longrightarrow & (K_2(K_{\mathfrak{p}}))^*_{\text{tor}} \\
\uparrow & & \uparrow \\
H^1(A_{\mathfrak{p}}) \simeq H^1(\kappa(\mathfrak{p})) & \xrightarrow{\ \alpha\ } & (\kappa(\mathfrak{p})^\times)^*_{\text{tor}},
\end{array}
$$

where α is the dual of the classical reciprocity map for the henselian discrete valuation field $\kappa(\mathfrak{p})$ with finite residue field (cf. [Se]), and the right vertical arrow is the dual of the tame symbol

$$\partial_{\mathfrak{p}}: K_2(K_{\mathfrak{p}}) \longrightarrow \kappa(\mathfrak{p})^\times;\ \{f, g\} \longrightarrow (-1)^{\alpha\beta} f^\beta g^{-\alpha} |_{\mathfrak{p}},$$

where we put $\alpha = \operatorname{ord}_{K_{\mathfrak{p}}}(f)$ and $\beta = \operatorname{ord}_{K_{\mathfrak{p}}}(g)$. We note that $\operatorname{Ker}(\partial_{\mathfrak{p}})$ coincides with $K_2(A_{\mathfrak{p}})$. Then results of [K–1] and [K–2] state that $\Psi_{K_{\mathfrak{p}}}$ is almost isomorphic (for a precise statement, see (1.5)).

Starting with this local result, we put together all maps $\Psi_{K_{\mathfrak{p}}}$ for $\mathfrak{p} \in P$ to obtain a description of $H^1(K)$. Here, we give the following corollary of our main result (1.10).

Theorem (0.2). (1) *There exists an integer $r(A) \geqq 0$ such that the kernel of the natural map*

$$\iota_K: H^1(K) \longrightarrow \prod_{\mathfrak{p} \in P} H^1(K_{\mathfrak{p}})$$

is isomorphic to $(\boldsymbol{Q}/\boldsymbol{Z})^{r(A)}$.

(2) *An element $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in P} \in \prod_{\mathfrak{p} \in P} H^1(K_{\mathfrak{p}})$ lies in the image of ι_K if and only if the following two conditions are satisfied.*

(A) *For almost all $\mathfrak{p} \in P$, $\chi_{\mathfrak{p}}$ lies in the subgroup $H^1(A_{\mathfrak{p}}) \simeq H^1(\kappa(\mathfrak{p}))$.*

(B) *For any $a \in K_2(K)$, we have*

$$\sum_{\mathfrak{p} \in P} \Psi_{K_{\mathfrak{p}}}(\chi_{\mathfrak{p}})(a_{\mathfrak{p}}) = 0,$$

where $a_{\mathfrak{p}}$ is the image of a in $K_2(K_{\mathfrak{p}})$ (Note that $a_{\mathfrak{p}}$ lies in $K_2(A_{\mathfrak{p}})$ for almost all $\mathfrak{p} \in P$ so that (A) *and* (0.1) *implies that the sum is a finite sum).*

Moreover, the integer $r(A)$ in (0.2.1) is calculated as the $\boldsymbol{Z}$-rank of the graph of the exceptional fiber of a resolution of Spec (A) (cf. §2).

(0.2.2) may be desirable and familiar in its form for those who knows the classical class field theory. On the other hand, the existence of the non-trivial kernel of ι_K is a new phenomenon in our class field theory. In fact, for $\chi \in \operatorname{Ker}(\iota_K)$, $\operatorname{Ker}(\chi)$ corresponds to a cyclic extension L of K in which any $\mathfrak{p} \in P$ splits completely, namely $L \otimes_K K_{\mathfrak{p}}$ is isomorphic to a finite number of copies of $K_{\mathfrak{p}}$ (cf. §2).

Lastly, to establish the unramified class field theory in our context, we first recall that there exists a complex coming from the localization theory for the K-theory on $\mathrm{Spec}(A)$;

$$(0.3) \qquad K_2(A) \longrightarrow K_2(K) \xrightarrow{\partial_2} \bigoplus_{\mathfrak{p}\in P} \kappa(\mathfrak{p})^\times \xrightarrow{\partial_1} \boldsymbol{Z} \longrightarrow 0.$$

Here $K_2(K) \to \kappa(\mathfrak{p})^\times$ is given by the tame symbol $\partial_\mathfrak{p}$, and $\kappa(\mathfrak{p})^\times \to \boldsymbol{Z}$ is the composite map

$$\kappa(\mathfrak{p})^\times \xrightarrow[\mathrm{ord}_{\kappa(\mathfrak{p})}]{} \boldsymbol{Z} \xrightarrow[d_\mathfrak{p}]{} \boldsymbol{Z}$$

where $d_\mathfrak{p} = [F_\mathfrak{p} : F]$ with the residue field $F_\mathfrak{p}$ of $\kappa(\mathfrak{p})$ and $F = A/\mathfrak{m}_A$ ($\mathfrak{m}_A$ is the maximal ideal of A). We define

$$(0.4) \quad SK_1(A) = \mathrm{Coker}(\partial_2) \quad \text{and} \quad SK_1(A)^0 = \mathrm{Ker}(\partial_1)/\mathrm{Im}(\partial_2) = \mathrm{Ker}(\partial),$$

where we put $\partial : SK_1(A) \to \boldsymbol{Z}$ to be the map induced by ∂_1. When A is regular, (0.3) is exact by [Q] and [B] so that we have

$$SK_1(A)^0 = 0 \quad \text{and} \quad SK_1(A) \xrightarrow[\partial]{\sim} \boldsymbol{Z}.$$

Let $X = \mathrm{Spec}(A) - \{\mathfrak{m}_A\}$. By definition, $\pi_1^{\mathrm{ab}}(X)$ coincides with the $\mathrm{Gal}(K^{u.r.}/K)$, where $K^{u.r.}$ is the maximum abelian extension of K in which any $\mathfrak{p} \in P$ is unramified. Let

$$\tilde{\psi}_A : \bigoplus_{\mathfrak{p}\in P} \kappa(\mathfrak{p})^\times \longrightarrow \pi_1^{\mathrm{ab}}(X)$$

be the sum of the composite maps for all $\mathfrak{p} \in P$

$$\kappa(\mathfrak{p})^\times \longrightarrow \mathrm{Gal}(\kappa(\mathfrak{p})^{\mathrm{ab}}/\kappa(\mathfrak{p})) \longrightarrow \pi_1^{\mathrm{ab}}(X),$$

where the first map is the reciprocity map for $\kappa(\mathfrak{p})$ (cf. [Se]).

Theorem (0.5). *The map $\tilde{\psi}_A$ induces a canonical homomorphism*

$$\psi_A : SK_1(A) \longrightarrow \pi_1^{\mathrm{ab}}(X),$$

and we have the following properties.

(1) *We have the commutative diagram*

$$\begin{array}{ccc} SK_1(X) & \xrightarrow{\partial} & \boldsymbol{Z} \\ \downarrow{\scriptstyle \psi_A} & & \downarrow \\ \pi_1^{\mathrm{ab}}(X) & \xrightarrow{\beta} & \mathrm{Gal}(F^{\mathrm{ab}}/F), \end{array}$$

where the right vertical arrow sends $1 \in \boldsymbol{Z}$ to the Frobenius over F, and the map β is the composite $\pi_1^{\mathrm{ab}}(X) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(A)) \simeq \mathrm{Gal}(F^{\mathrm{ab}}/F)$.

(2) *We have an isomorphism*

$$\pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\psi_A)} \simeq (\hat{\boldsymbol{Z}})^{r(A)},$$

where $\overline{\mathrm{Im}(\psi_A)}$ *is the closure of the image of* ψ_A.

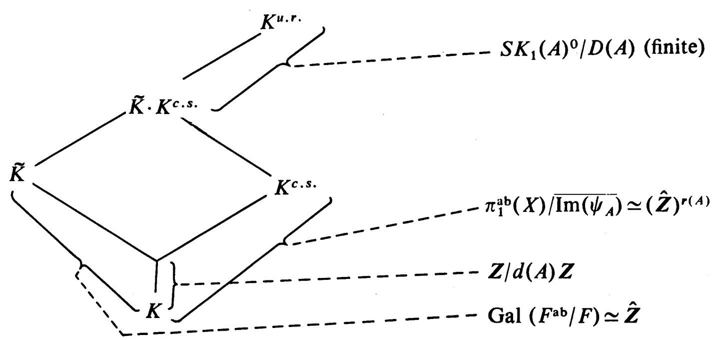
(3) *The kernel of* ψ_A *is equal to the maximum divisible subgroup* $D(A)$ *of* $SK_1(A)$.

(4) *The image of* $SK_1(A)^0$ *under* ψ_A *is finite and coincides with the torsion part* $\pi_1^{\mathrm{ab}}(X)_{\mathrm{tor}}$ *of* $\pi_1^{\mathrm{ab}}(X)$, *namely we have an isomorphism of finite abelian groups*

$$SK_1(A)^0/D(A) \simeq \pi_1^{\mathrm{ab}}(X)_{\mathrm{tor}}.$$

The Galois theoretic interpretation of our results is explained in the following diagram in which each extension is galois.

Extensions Galois groups



where $K^{\mathrm{u.r.}}$ is defined as before, $K^{\mathrm{c.s.}}$ is the maximum subextension of $K^{\mathrm{u.r.}}/K$ in which any $\mathfrak{p} \in P$ splits completely, $\tilde{K}$ is the quotient field of the strict henselization of A, and $d(A)$ is the greatest common divisor of $d_{\mathfrak{p}}$ for all $\mathfrak{p} \in P$.

Finally, the author expresses his hearty thanks to Professor K. Kato, who suggested to make these researches and gave him many ideas.

Notations

For a field k, k^{ab}; a maximum abelian extension of k.

For a discrete valuation field k, $\mathcal{O}_k$; the ring of integers, $\mathfrak{m}_k$; its maximal ideal, $\mathrm{ord}_k\colon k^\times \to \boldsymbol{Z}$; the normalized additive valuation of k.

For an abelian group G, $(G)^*$ (resp. $(G)^*_{\mathrm{tor}}$); the set of all homomorphism $\chi\colon G \to \boldsymbol{Q}/\boldsymbol{Z}$ (resp. of finite orders).

For an abelian group M and integer $n>0$,

$$M_n := \mathrm{Ker}\,(M \xrightarrow{n} M) \quad \text{and} \quad M/n := \mathrm{Coker}\,(M \xrightarrow{n} M).$$

For an abelian group M and a prime number l,

$$M(l) = \varinjlim_{\nu} M_{l^\nu}.$$

For a commutative ring R and an integer $i \geqq 0$, $K_i(R)$; the i-th K-group of R in the sense of [Q].

For a field k and an integer $i \geqq 0$, $K_i^M(k)$; the i-th Milnor K-group of k (cf. [Mil]). It is well-known that $K_i^M(k) = K_i(k)$ for $i \leqq 2$.

§ 1. The statements of the main results

First, we introduce some notations and results in [K–1] and [K–2] (cf. also [S–1] §2).

For a field Λ and a prime number l, we define an l-primary torsion group $H^i(\Lambda)(l)$ by;

$$H^i(\Lambda)(l) = \varinjlim_{n} H^i(\Lambda, \mu_{l^n}^{\otimes(i-1)}) \qquad \text{if} \quad l \neq \mathrm{ch}\,(\Lambda),$$

$$H^i(\Lambda)(l) = \varinjlim_{n} H^1(\Lambda, W_n\Omega_{\Lambda,\log}^{i-1}) \quad \text{if} \quad l = p := \mathrm{ch}\,(\Lambda) > 0,$$

where $W_n\Omega_{\Lambda,\log}^{i-1}$ is the logarithmic part of the De Rham-Witt complex on $\mathrm{Spec}\,(\Lambda)_{\mathrm{et}}$ (cf. [I]) and the transition maps are induced by the multiplication by p. We put

$$H^i(\Lambda) = \bigoplus_{l} H^i(\Lambda)(l).$$

We can see that

$$H^1(\Lambda) \simeq \mathrm{Hom}_{\mathrm{cont}}\,(\mathrm{Gal}\,(\Lambda^{\mathrm{ab}}/\Lambda), \boldsymbol{Q}/\boldsymbol{Z}) \quad \text{and} \quad H^2(\Lambda) \simeq \mathrm{Br}\,(\Lambda).$$

On the other hand, in case $l \neq \mathrm{ch}\,(\Lambda)$, we have the Tate's Galois symbol

$$K_j^M(\Lambda) \longrightarrow H^j(\Lambda, \mu_{l^n}^{\otimes j}),$$

and in case $l = \mathrm{ch}\,(\Lambda)$, we have the differential symbol

$$K_j^M(\Lambda) \longrightarrow H^0(\Lambda, W_n\Omega_{\Lambda,\log}^j).$$

Hence, using cup products on cohomologies, we have a pairing

$$H^i(\Lambda) \times K_j^M(\Lambda) \longrightarrow H^{i+j}(\Lambda).$$

In case Λ is a two-dimensional henselian local field in the sense of [K–1] and [K–2] (cf. also [S–1] (2.2)), there is a canonical isomorphism (cf. [K–1] §5 and [K–2] §3)

$$\text{inv}_A\colon H^3(A) \xrightarrow{\sim} \boldsymbol{Q}/\boldsymbol{Z}. \tag{1.1}$$

Hence, we obtain a canonical pairing

$$\langle\ ,\ \rangle_A\colon H^1(A)\times K_2(A) \longrightarrow H^3(A)\simeq \boldsymbol{Q}/\boldsymbol{Z}, \tag{1.2}$$

and a canonical homomorphism

$$\Psi_A\colon H^1(A) \longrightarrow (K_2(A))^*_{\text{tor}}. \tag{1.3}$$

Moreover, we see that the following diagram is commutative

$$\begin{array}{ccc} H^1(A) & \xrightarrow{\Psi_A} & (K_2(A))^*_{\text{tor}} \\ \uparrow & & \uparrow \\ H^1(\mathcal{O}_A)\simeq H^1(k) & \xrightarrow{\alpha} & (k^\times)^*_{\text{tor}}, \end{array} \tag{1.4}$$

where k is the residue field of $\mathcal{O}_A$ which is a henselian discrete valuation field with finite residue field, and α is the dual of the classical reciprocity map for k (cf. [Se]).

Theorem (1.5) (cf. [K–1] and [K–2]). (1) *Ψ_A is injective.*

(2) *If $\chi\in(K_2(A))^*_{\text{tor}}$ has an order prime to* ch(k), *it lies in the image of Ψ_A.*

(3) *The image of Ψ_A lies in the subgroup $D(K_2(A))$ of $(K_2(A))^*_{\text{tor}}$ consisting of all elements which annihilate some open subgroup of $K_2(A)$. Here the topology on $K_2(A)$ is defined in* [K–1] §7 (*cf. also* [S–1] (2.3)). *Moreover, if k is complete, we have*

$$\Psi_A\colon H^1(A)\simeq D(K_2(A)).$$

Now, fix A as the introduction. We define a pairing

$$\langle\ ,\ \rangle_K\colon H^1(K)\times I_K \longrightarrow \boldsymbol{Q}/\boldsymbol{Z};\ \langle\chi, a\rangle_K=\sum_{\mathfrak{p}\in P}\langle\chi_{\mathfrak{p}}, a_{\mathfrak{p}}\rangle_{K_{\mathfrak{p}}}, \tag{1.6}$$

where I_K is the idele group defined to be the restricted product of $K_2(K_{\mathfrak{p}})$ for $\mathfrak{p}\in P$ with respect to the subgroups $K_2(A_{\mathfrak{p}})$. Note that for each $\chi\in H^1(K)$ its image $\chi_{\mathfrak{p}}$ in $H^1(K_{\mathfrak{p}})$ lies in the subgroup $H^1(A_{\mathfrak{p}})$ for almost all $\mathfrak{p}\in P$. Hence, by (1.4), $\langle\chi_{\mathfrak{p}}, a_{\mathfrak{p}}\rangle=0$ for almost all $\mathfrak{p}\in P$ so that (1.6) is well defined. Moreover, by the reciprocity law for A (cf. [S–1] (2.9) and [S–2] Ch. I), we see that $\langle\ ,\ \rangle_K$ induces a canonical pairing

$$\langle\ ,\ \rangle_K\colon H^1(K)\times C_K \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}. \tag{1.7}$$

Here we put $C_K=\text{Coker}\,(K_2(K)\to I_K)$ and $K_2(K)$ is mapped diagonally to I_K. Thus we obtain a natural homomorphism

$$\Psi_K\colon H^1(K) \longrightarrow (C_K)^*_{\text{tor}}. \tag{1.8}$$

Let S be a finite (may be empty) subset of P and put

$$C_S = \mathrm{Coker}\,(\prod_{\mathfrak{p}\in P-S} K_2(A_\mathfrak{p}) \to C_K)$$
$$\simeq \mathrm{Coker}\,(K_2(K) \to (\bigoplus_{\mathfrak{p}\in P-S} \kappa(\mathfrak{p})^\times) \oplus (\bigoplus_{\mathfrak{p}\in S} K_2(K_\mathfrak{p}))).$$

We call C_S *the idele class group for A with the modulus S*. When S is empty, C_S is nothing other than $SK_1(A)$ defined in the introduction. Note that S is considered a closed subset of $X = \mathrm{Spec}\,(A) - \{\mathfrak{m}_A\}$ and put $U_S = X - S$. Then $H^1(U_S)$ is the Pontrijagin dual of $\mathrm{Gal}\,(K_S^{\mathrm{u.r.}}/K)$, where $K_S^{\mathrm{u.r.}}$ is the maximum abelian extension of K where every $\mathfrak{p}\in P-S$ is unramified. Thus image of $H^1(U_S)$ in $H^1(K_\mathfrak{p})$ is contained in $H^1(A_\mathfrak{p})$ for every $\mathfrak{p}\in P-S$. Hence, by (1.4) Ψ_K induces

$$\Psi_S\colon H^1(U_S) \longrightarrow (C_S)^*_{\mathrm{tor}}. \tag{1.9}$$

In particular, we have obtained the map ψ_A in (0.5) as the dual of Ψ_S for $S=\phi$.

Now, our main result is the following

Theorem (1.10). (1) *There exists an integer $r(A)\geqq 0$ such that*

$$\mathrm{Ker}\,(\Psi_K) = \mathrm{Ker}\,(\Psi_S) \simeq (\boldsymbol{Q}/\boldsymbol{Z})^{r(A)}.$$

(2) *An element $\omega\in(C_S)^*_{\mathrm{tor}}$ lies in the image of Ψ_S if and only if $\omega_\mathfrak{p}$ lies in the image of $\Psi_{K_\mathfrak{p}}$ for every $\mathfrak{p}\in S$, where $\omega_\mathfrak{p}$ is the restriction of ω to the $\mathfrak{p}$-component of C_S.*

(3) *An element $\omega\in(C_K)^*_{\mathrm{tor}}$ lies in the image of Ψ_K if and only if the following conditions are satisfied*:

(A) *For almost all $\mathfrak{p}\in P$, $\omega_\mathfrak{p}$ annihilates the subgroup $K_2(A_\mathfrak{p})$.*

(B) *For any $\mathfrak{p}\in P$, $\omega_\mathfrak{p}$ lies in the image of $\Psi_{K_\mathfrak{p}}$.*

Note that (1.10.3) is an immediate consequence of (1.10.2).

Definition (1.11). An element $\omega\in(C_S)^*_{\mathrm{tor}}$ is *continuous* if for any $\mathfrak{p}\in S$, the restriction of ω to $K_2(K_\mathfrak{p})$ annihilates some open subgroup (cf. (1.5.3)). We denote by $D(C_S)$ the subgroup of $(C_S)^*_{\mathrm{tor}}$ consisting of all continuous elements.

By (1.5.3), the image of Ψ_S lies in $D(C_S)$.

Remark (1.12). By the definition of the topology of $K_2(K_\mathfrak{p})$ (cf. [K–1] §7, also [S–1] §2), we see that any $\omega\in(C_S)^*_{\mathrm{tor}}$ of the order prime to $\mathrm{ch}\,(K)$ is continuous.

Corollary (1.13). *Let $\omega\in(C_S)^*_{\mathrm{tor}}$.*

(1) *If* ω *has an order prime to* $\mathrm{ch}(\kappa(\mathfrak{p}))$ *for every* $\mathfrak{p} \in S$, *it lies in the image of* Ψ_S.

(2) *If* $\kappa(\mathfrak{p})$ *is complete for every* $\mathfrak{p} \in S$, ω *lies in the image of* Ψ_S *if and only if it lies in* $D(C_S)$.

Corollary (1.14). *If A is complete, we have an exact sequence*

$$0 \longrightarrow (\boldsymbol{Q}/\boldsymbol{Z})^{r(A)} \longrightarrow H^1(U_S) \xrightarrow{\Psi_S} D(C_S) \longrightarrow 0.$$

(1.13) *and* (1.14) *follow at once from* (1.10) *and* (1.5).

Corollary (1.15). *Let S be a non-empty finite subset of P, and assume that A is regular. Then,* $(\chi_{\mathfrak{p}})_{\mathfrak{p} \in S} \in \bigoplus_{\mathfrak{p} \in S} H^1(K_{\mathfrak{p}})$ *comes from* $H^1(U_S)$ *if and only if*

$$\sum_{\mathfrak{p} \in S} \Psi_{K_{\mathfrak{p}}}(\chi_{\mathfrak{p}}) \colon \bigoplus_{\mathfrak{p} \in S} K_2(K_{\mathfrak{p}}) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

annihilates the diagonal image of $K_2(A_S)$, *where* A_S *is the affine ring of* U_S.

(1.15) follows from (1.10), (1.4) and the following

Lemma (1.16). *Let the assumption be as* (1.15). *Then the sequence*

$$K_2(A_S) \longrightarrow K_2(K) \xrightarrow{\partial} \bigoplus_{\mathfrak{p} \in P-S} \kappa(\mathfrak{p})^{\times}.$$

is exact. Moreover, $\mathrm{Coker}(\partial)$ *is finite and isomorphic to the cokernel of the sum of the following maps for* $\mathfrak{p} \in S$;

$$\kappa(\mathfrak{p})^{\times} \xrightarrow[\mathrm{ord}_{\kappa(\mathfrak{p})}]{} \boldsymbol{Z} \xrightarrow[d_{\mathfrak{p}}]{} \boldsymbol{Z} \text{ (cf. (0.3))}.$$

The first assertion of (1.16) follows from the localization theory for the K-theory on $U_S = \mathrm{Spec}(A_S)$. The second follows from the exactness of the sequence (0.3) when A is regular (cf. [Q] and [B]).

The proof of (1.10.1) will be given in Section 2, that of (1.10.2) will be given in Section 7 and Section 8, and finally (0.5) will be deduced from (1.10) and the finiteness theorem (4.1) (cf. the end of §8).

§2. The determination of $\mathrm{Ker}(\Psi_K)$

First, we recall the following

Definition (2.1) (cf. [S–2] Ch. II (2.1)). Let Z be a noetherian scheme. A finite etale covering $f\colon W \to Z$ is called a c.s. covering, if for any closed point z of Z, $z \times_Z W$ is isomorphic to a finite sum of z. We denote by $\pi_1^{\mathrm{c.s.}}(Z)$ the quotient group of $\pi_1^{\mathrm{ab}}(Z)$ which classifies abelian c.s. coverings of Z. In other words, $\pi_1^{c.s.}(Z)$ is the dual of the intersection of the kernels of the maps

$$H^1(Z, \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow H^1(z, \boldsymbol{Q}/\boldsymbol{Z})$$

for all closed points z of Z.

Let A be as before and $X = \mathrm{Spec}(A) - x$, where x is the unique closed point of $\mathrm{Spec}(A)$. By definition, $\pi_1^{c.s.}(X)$ coincides with $\mathrm{Gal}(K^{c.s.}/K)$, where $K^{c.s.}$ is defined in the introduction, and by the injectivity of $\Psi_{K_\mathfrak{p}}$ for $\mathfrak{p} \in P$, we have an isomorphism

$$\mathrm{Ker}(\Psi_K) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{c.s.}(X), \boldsymbol{Q}/\boldsymbol{Z}).$$

Thus, we are reduced to determine the structure of $\pi_1^{c.s.}(X)$. This is done by the same method in [S–2] Ch. II §2: Choose a resolution of $\mathrm{Spec}(A)$, by which we mean a proper birational morphism

$$f: \mathfrak{X} \longrightarrow \mathrm{Spec}(A)$$

which satisfies the following conditions: $\mathfrak{X}$ is a two-dimensional regular scheme, $Y := (f^{-1}(x))_{\mathrm{red}}$ is a geometrically connected one dimensional scheme over $F = A/\mathfrak{m}_A$ such that any irreducible component of Y is regular and it has only ordinary double points as singularities, and f induces an isomorphism $\mathfrak{X} \backslash Y \simeq X$. Then, we have the following two basic facts.

Proposition (2.2). *The specialization map*

$$\pi_1^{ab}(X) \longrightarrow \pi_1^{ab}(\mathfrak{X}) \simeq \pi_1^{ab}(Y)$$

induces an isomorphism

$$\pi_1^{c.s.}(X) \simeq \pi_1^{c.s.}(Y).$$

Proposition (2.3). *Let Γ be the graph of Y (cf.* [S–2] *Ch. II* (2.3)). *Then, there exists an isomorphism*

$$\pi_1^{c.s.}(Y) \simeq H_1(\Gamma, \boldsymbol{Z}) \otimes_{\boldsymbol{Z}} \hat{\boldsymbol{Z}} \simeq \hat{\boldsymbol{Z}}^r,$$

where r is the rank of $H_1(\Gamma, \boldsymbol{Z})$.

Definition (2.4). The rank $r(A)$ of A is the rank of $H_1(\Gamma, \boldsymbol{Z})$.

Clearly, $r(A)$ does not depend on the choice of $\mathfrak{X}$.

The proof of (2.3) is given in [S–2] Ch. II (2.4), and (2.2) is proved in the same argument as the proof of [S–2] Ch. II (2.2).

Consequently, we obtained the desired isomorphism

$$\mathrm{Ker}(\Psi_K) \simeq H^1(\Gamma, \boldsymbol{Q}/\boldsymbol{Z}) \simeq (\boldsymbol{Q}/\boldsymbol{Z})^{r(A)}.$$

§ 3. The ramification along the special fiber

Let A, K and P be as before and let S be a finite subset of P. In this section, we fix a resolution of $\mathrm{Spec}(A)$ as in Section 2,

$$f\colon \mathfrak{X} \longrightarrow \mathrm{Spec}(A),$$

and define a new idele class group T_S which takes an account of the ramification along special fiber of f. For this, we introduce some notations (cf. [S–1] §1). Let $X=\mathrm{Spec}(A)-x$, where x is the unique closed point of $\mathrm{Spec}(A)$. By definition, P is identified with the set of all closed points of X. Let $Y=(f^{-1}(x))_{\mathrm{red}}$ and let Y_0 (resp. Y_1) be the set of all closed (resp. generic) points of Y. For $\eta \in Y_1$, let K_η be the quotient field of the henselization of the local ring of $\mathfrak{X}$ at η and let η_0 be the set of all $x \in Y_0$ lying on the closure of η in $\mathfrak{X}$. For $x \in Y_0$, let A_x be the henselization of the local ring of $\mathfrak{X}$ at x. It is an excellent regular two-dimensional henselian local ring. Let K_x be the quotient field of A_x and P_x be the set of all prime ideals of height one in A_x. For $x \in Y_0$, let Y_1^x be the subset of P_x consisting of all elements lying over some element of Y_1. For $\eta \in Y_1$ and $x \in \eta_0$, there is a unique $\eta_x \in Y_1^x$ lying over η, and $\kappa(\eta_x)$ is the henselization of $\kappa(\eta)$ at x. We denote by K_{η_x} the henselization of K_x at η_x. For each $x \in Y_0$, the cardinality of Y_1^x is one or two, according as x is a regular point of Y or not. For $\mathfrak{p} \in P$, its closure in $\mathfrak{X}$ contains a unique $x \in Y_0$ when we denote "$\mathfrak{p}\to x$". For $x \in Y_0$, let P^x be the subset of P consisting of all $\mathfrak{p}$ such that $\mathfrak{p}\to x$. For $\mathfrak{p} \in P^x$, $\mathfrak{p}_x := \mathfrak{p}A_x$ is an element of P_x and $A_\mathfrak{p} \simeq (A_x)_{\mathfrak{p}_x}$. Thus, P^x is identified with a subset of P_x, and P_x is the disjoint union of P^x and Y_1^x.

To define T_S, we first put for each $x \in Y_0$,

$$T_{S.x}=\mathrm{Coker}\,(K_2(R_{x.S})\to(\bigoplus_{\mathfrak{p}\in S^x} K_2(K_\mathfrak{p}))\oplus(\bigoplus_{\eta_x\in Y_1^x} K_2(K_{\eta_x}))),$$

where $S^x=S\cap P^x$ and $R_{x.S}$ is the affine ring of $\mathrm{Spec}(A_x)\times_{\mathfrak{X}} U_S$ with $U_S=X-S$. We see easily that for $\eta \in Y_1$, each $a \in K_2(K_\eta)$ has a trivial image under the map

$$K_2(K_\eta) \longrightarrow K_2(K_{\eta_x}) \longrightarrow T_{S.x}$$

for almost all $x \in \eta_0$. Thus, we define

$$T_S=\mathrm{Coker}\,(\bigoplus_{\eta\in Y_1} K_2(K_\eta)\to \bigoplus_{x\in Y_0} T_{S.x}). \tag{3.1}$$

In view of the exactness of the following sequence (cf. (1.16))

$$K_2(R_{x.S}) \longrightarrow K_2(K_x) \longrightarrow \bigoplus_{\mathfrak{p}\in U_S^x} \kappa(\mathfrak{p})^\times \longrightarrow 0 \quad (U_S^x=P^x-S^x),$$

we have the following isomorphism

(3.2) $T_{S.x} \simeq$ the cokernel of the map

$$K_2(K_x) \longrightarrow (\bigoplus_{\mathfrak{p}\in U_S^x} \kappa(\mathfrak{p})^\times) \oplus (\bigoplus_{\mathfrak{p}\in S^x} K_2(K_\mathfrak{p})) \oplus (\bigoplus_{\eta_x \in Y_1^x} K_2(K_{\eta_x})).$$

In particular, $T_{S.x}$ coincides with the idele class group for A_x with the modulus $S^x \cup Y_1^x$ defined in Section 1. Hence, we have the natural map

$$\varphi_x \colon T_{S.x} \longrightarrow \pi_1^{ab}(\mathrm{Spec}\,(R_{x.S})) \longrightarrow \pi_1^{ab}(U_S)$$

for each $x \in Y_0$, where the first map is the dual of the map (1.9) for $R_{x.S}$. Then, by the class field theory for K_η ($\eta \in Y_1$) (cf. [K–3]), the sum of φ_x for $x \in Y_0$ induces a natural homomorphism

$$\varphi_S \colon T_S \longrightarrow \pi_1^{ab}(U_S),$$

and we get a natural homomorphism

(3.3) $$\Phi_S \colon H^1(U_S) \longrightarrow (T_S)^*_{\mathrm{tor}}.$$

On the other hand, by (3.2), we have a natural map

$$\rho_x \colon (\bigoplus_{\mathfrak{p}\in U_S^x} \kappa(\mathfrak{p})^\times) \oplus (\bigoplus_{\mathfrak{p}\in S^x} K_2(K_\mathfrak{p})) \longrightarrow T_{S.x}$$

for each $x \in Y_0$. Then, we can easily see that $\bigoplus_{x\in Y_0} \rho_x$ induces a canonical map

(3.4) $$\rho_S \colon C_S \longrightarrow T_S.$$

Moreover, by the reciprocity law for K_x (cf. [S–1] (2.9) and [S–2] Ch. I), we see that the following diagram is commutative

(3.5) $$\begin{array}{ccc} & H^1(U_S) & \\ {\scriptstyle \Phi_S}\swarrow & & \searrow{\scriptstyle \Psi_S} \\ (T_S)^*_{\mathrm{tor}} & \xrightarrow{(\rho_S)^*} & (C_S)^*_{\mathrm{tor}}. \end{array}$$

Now we will define a descending filtration on T_S which controls the ramification along the special fiber Y. For this, let $\mathscr{I}$ be the set of all coherent ideals I of $\mathcal{O}_{\mathfrak{X}}$ such that $\mathrm{Supp}\,(\mathcal{O}_{\mathfrak{X}}/I) = Y$. For $I \in \mathscr{I}$ and an integer $n \geqq 1$, let $U_I^n K_2(K_{\eta_x})$ be the subgroup of $K_2(K_{\eta_x})$ generated by all elements of the form $\{a, b\}$ with $a \in (K_{\eta_x})^\times$ and $b \in 1 + I^n \mathcal{O}_{K_{\eta_x}}$. Then, for $n \geqq 1$, we define $U_I^n T_S$ to be the image in T_S of

$$\bigoplus_{x\in Y_0} \bigoplus_{\eta_x \in Y_1^x} U_I^n K_2(K_{\eta_x}).$$

Remark (3.6). By definition, for any I, $J\in\mathscr{I}$, there exists an integer $e>0$ such that $I^e\subset J$ so that $U_I^{ne}T\subset U_J^nT$ for any $n\geqq 1$.

Definition (3.7). We define $D(T_S)$ to be the subgroup of $(T_S)^*_{\mathrm{tor}}$ consisting of all χ which annihilate $U_I^nT_S$ for some n and whose restriction to $K_2(K_{\mathfrak{p}})$ annihilate some open subgroup for every $\mathfrak{p}\in S$. By (3.6), this definition does not depend on the choice of I.

Remark (3.8). Fix any $I\in\mathscr{I}$. Then, for any positive integer r prime to $\mathrm{ch}(K)$, taking n large enough, we have $rK_2(K_{\eta_x})\supset U_I^nK_2(K_{\eta_x})$. In particular, any $\omega\in(T_S)^*_{\mathrm{tor}}$ of the order prime to $\mathrm{ch}(K)$ lies in $D(T_S)$ (cf. (1.12)).

By the class field theory for K_η (cf. [K–3]), the image of Φ_S lies in $D(T_S)$. Moreover, we have

Lemma (3.9). *Let $\omega\in D(T_S)$, and for $x\in Y_0$ and $\eta_x\in Y_1^x$ (resp. $\mathfrak{p}\in S$), let ω_{η_x} (resp. $\omega_{\mathfrak{p}}$) be the restriction of ω to $K_2(K_{\eta_x})$ (resp. $K_2(K_{\mathfrak{p}})$). Then there exist elements $(\chi_\eta)_{\eta\in Y_1}$ of $\bigoplus_{\eta\in Y_1} H^1(K_\eta)$ such that $\omega_{\eta_x}=\Psi_{K_{\eta_x}}(\chi_{\eta_x})$ for $\eta\in Y_1$ and $x\in\eta_0$. Here χ_{η_x} is the image of χ_η in $H^1(K_{\eta_x})$. Moreover, for each $x\in Y_0$ and $a\in K_2(R_{x.S})$, we have*

$$(*)\qquad \sum_{\eta_x\in Y_1^x}\omega_{\eta_x}(a)+\sum_{\mathfrak{p}\in S^x}\omega_{\mathfrak{p}}(a)=0.$$

This follows at once from the definition of T_S and the class field theory for K_η (cf. [K–3] §1).

§4. A finiteness theorem

Let the notations be as before and fix a resolution $\mathfrak{X}$ of $\mathrm{Spec}(A)$ as Section 2. The main purpose of this section is to prove the following theorem, which is viewed as an analogue of the finiteness of the Hilbert class field of an algebraic number field in the classical case.

Theorem (4.1). *The kernel of the specialization homomorphism*

$$\pi_1^{\mathrm{ab}}(X)\longrightarrow\pi_1^{\mathrm{ab}}(\mathfrak{X})\simeq\pi_1^{\mathrm{ab}}(Y)$$

is finite.

Corollary (4.2). *The image of $SK_1(A)^0$ under the map* (*cf.* (0.5))

$$\psi_A\colon SK_1(A)\longrightarrow\pi_1^{\mathrm{ab}}(X)$$

is finite and equal to the torsion part $\pi_1^{\mathrm{ab}}(X)_{\mathrm{tor}}$ of $\pi_1^{\mathrm{ab}}(X)$. Moreover, we have an isomorphism

$$\pi_1^{ab}(X)/\pi_1^{ab}(X)_{tor} \simeq (\hat{\boldsymbol{Z}})^{r(A)+1},$$

where $r(A)$ is the rank of A (cf. (2.4)).

This follows from (4.1) and the results of Section 2, together with the finiteness of the kernel of the map

$$\pi_1^{ab}(Y) \longrightarrow (\pi_1^{c.s.}(Y) \oplus \mathrm{Gal}(F^{ab}/F)) \simeq (\hat{\boldsymbol{Z}})^{r(A)+1}.$$

Let $T = T_\phi$ and $\Phi = \Phi_\phi$: $H^1(X) \to (T)^*_{tor}$ be defined as in (3.1) and (3.3). For $I \in \mathscr{I}$, define a descending filtration $U_I^n T$ ($n \geqq 1$) on T as Section 3. Then, (4.1) follows from the following

Proposition (4.3). (1) *For each $n \geqq 1$, put* $\mathrm{Gr}_I^n T = U_I^n T / U_I^{n+1} T$. *Then* $\mathrm{Gr}_I^n T$ *is finite for any n and is trivial for any sufficient large n.*

(2) *Let $J \in \mathscr{I}$ be the ideal of definition for Y. Then, $U_J^1 T / U_I^1 T$ is finite (By definition, $J \subset I$).*

First, we deduce (4.1) from (4.3). We have to show that $H^1(X)/H^1(\mathfrak{X})$ is finite. For each $\eta \in Y_1$, let $H^1(K_\eta)^t$ (resp. $H^1(\mathcal{O}_{K_\eta})$) be the subgroup of $H^1(K_\eta)$ consisting of all χ corresponding to tamely ramified (resp. unramified) cyclic extensions of K_η. We have

$$H^1(K_\eta)^t / H^1(\mathcal{O}_{K_\eta}) \simeq \mathrm{Hom}\,(\mu(\kappa(\eta)), \boldsymbol{Q}/\boldsymbol{Z}),$$

where $\mu(\kappa(\eta))$ is the group of all roots of unity in $\kappa(\eta)$ and it is finite. On the other hand, by SGA1X (1.8), we have

$$H^1(\mathfrak{X}) = \mathrm{Ker}\,(H^1(X) \to \bigoplus_{\eta \in Y_1} H^1(K_\eta)/H^1(\mathcal{O}_{K_\eta})).$$

Hence it suffices to show that $H^1(X)/H^1(X)^t$ is finite, where we put

$$H^1(X)^t = \mathrm{Ker}\,(H^1(X) \to \bigoplus_{\eta \in Y_1} H^1(K_\eta)/H^1(K_\eta)^t).$$

Let $\chi \in H^1(X)$ and assume that $\Phi(\chi)$ annihilates $U_J^1 T$. By the class field theory for K_η (cf. [K–3]), this implies that the image of χ in $H^1(K_\eta)$ belongs to $H^1(K_\eta)^t$ for any $\eta \in Y_1$. This proves that $H^1(X)/H^1(X)^t$ injects into $D(U_J^1 T)$ which is the subgroup of $(U_J^1 T)^*_{tor}$ consisting of all elements annihilating $U_I^n T$ for some $n \geqq 1$. Consequently, (4.1) follows at once from (4.3).

To prove (4.3), we introduce some notations. Let I and J be as (4.3) and let Z be the closed subscheme of $\mathfrak{X}$ defined by I. For each integer $n \geqq 0$, put

$$\Omega_Z(n) = \Omega_Z \otimes_{\mathcal{O}_Z} I^n/I^{n+1} \quad \text{and} \quad \mathcal{O}_Z(n) = I^n/I^{n+1}.$$

Lemma (4.4). (1) *For $n \geqq 1$, there exist homomorphisms*

$$\alpha^n: H^1(Z, \Omega_Z(n)) \longrightarrow \mathrm{Gr}_I^n T,$$

$$\beta^n: H^1(Z, \mathcal{O}_Z(n)) \longrightarrow \mathrm{Gr}_I^n T/\mathrm{Im}(\alpha^n).$$

Moreover, β^n is surjective.

(2) *There exist homomorphisms*

$$\alpha: H^1(Z, \Omega_Z \otimes_{\mathcal{O}_Z} J/I) \longrightarrow \Gamma := U_J^1 T/U_I^1 T,$$

$$\beta: H^1(Z, J/I) \longrightarrow \Gamma/\mathrm{Im}(\alpha).$$

Moreover, β is surjective.

Since Z is projective scheme over a finite ring, (4.3.2) and the first assertion of (4.3.1) follows from (4.4) and EGAIII (2.2.2). As for the second assertion of (4.3.1), by (3.6) we may replace I with any other $I' \in \mathscr{I}$. Then it follows from (4.4) and EGAIII (2.2.2) together with the following

Lemma (4.5). *There exists $I \in \mathscr{I}$ which is invertible and ample.*

Proof. It suffices to find a positive divisor

$$\Theta = \sum_{\eta \in Y_1} n_\eta \cdot E_\eta \quad (n_\eta > 0 \text{ for any } \eta \in Y_1)$$

such that $\mathcal{O}_{\mathfrak{X}}(-\Theta)$ is ample, where for $\eta \in Y_1$ E_η is its closure in $\mathfrak{X}$. For this, by [K1] it suffices to find Θ such that $(\Theta, E_\eta) < 0$ for any $\eta \in Y_1$ and it is possible by the negative definiteness of the intersection pairing on $\bigoplus_{\eta \in Y_1} \mathbf{Z}$ (cf. [Mum]).

We start the proof of (4.4.1). The proof of (4.4.2) is similar and left to the readers. In the following, for a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathscr{F}$ and for $\eta \in Y_1$, $x \in Y_0$ and $\eta_x \in Y_1^x$, we put

$$\mathscr{F}_\eta = \mathscr{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{K_\eta}, \quad \mathscr{F}_x = \mathscr{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} A_x \quad \text{and} \quad \mathscr{F}_{\eta_x} = \mathscr{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{K_{\eta_x}}.$$

First, we have a homomorphism (cf. [K–2])

$$\alpha_{\eta_x}^n: \Omega_Z(n)_{\eta_x} \longrightarrow \mathrm{Gr}_I^n K_2(K_{\eta_x}); \frac{da}{a} \otimes b \longrightarrow \{1+b, a\} \bmod U_I^{n+1} K_2(K_{\eta_x}).$$

$$(a \in (\mathcal{O}_{K_{\eta_x}}/I_{\eta_x})^\times \quad \text{and} \quad b \in I_{\eta_x}^n)$$

Secondly, fix an element $\pi \in K$ such that $\mathrm{ord}_{K_\eta}(\pi) = 1$ for any $\eta \in Y_1$. Then, we have a homomorphism

$$\beta_{\eta_x}^n: \mathcal{O}_Z(n)_{\eta_x} \longrightarrow \mathrm{Gr}_I^n K_2(K_{\eta_x})/\mathrm{Im}(\alpha_{\eta_x}^n); \; b \longrightarrow \{1+b, \pi\} \bmod \mathrm{Im}(\alpha_{\eta_x}^n).$$

$$(b \in I_{\eta_x}^n)$$

It is easily seen that $\beta^n_{\eta_x}$ does not depend on the choice of π and that it is surjective. On the other hand, by the localization theories on Z and on the henselization of Z at x, we have the following exact sequences for $\mathscr{F}=\Omega_Z(n)$ and $\mathcal{O}_Z(n)$,

$$\bigoplus_{\eta\in Y_1}\mathscr{F}_\eta \longrightarrow \bigoplus_{x\in Y_0} H^1_x(Z,\mathscr{F}) \longrightarrow H^1(Z,\mathscr{F}) \longrightarrow 0,$$

$$\mathscr{F}_x \longrightarrow \bigoplus_{\eta_x\in Y_1^x}\mathscr{F}_{\eta_x} \longrightarrow H^1_x(Z,F) \longrightarrow 0.$$

From this and the definition of T, we can easily see that $\alpha^n_{\eta_x}$ and $\beta^n_{\eta_x}$ for $x\in Y_0$ and $\eta_x\in Y_1^x$ induce the desired homomorphisms α^n and β^n.

§ 5. The Hasse principle for A

Let k be an algebraic number field or an algebraic function field of one variable over a finite field. For simplicity, we assume that k has no real place. Then, by the classical Hasse principle and Artin's reciprocity law for k, we have an exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \xrightarrow{\ \iota\ } \bigoplus_{x\in P_k} \boldsymbol{Q}/\boldsymbol{Z} \xrightarrow{\text{addition}} \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0.$$

Here P_k denotes the set of all finite places of k and ι sends $c\in H^2(k)$ to $(\mathrm{inv}_x(c_x))_{x\in P_k}$, where c_x is the image of c in $\mathrm{Br}(k_x)$ (k_x is the henselization of k at x) and

$$\mathrm{inv}_x\colon \mathrm{Br}(k_x)\simeq \boldsymbol{Q}/\boldsymbol{Z}$$

is the classical invariant map. The main purpose of this section is to give the analogue of this result in our context. So let A, K and P be as before. As K. Kato has pointed out in [K–4], it is not $\mathrm{Br}(K)$ but $H^3(K)$ defined in Section 1 that plays a central role in our context.

The starting point is the homomorphism

$$\iota_K\colon H^3(K) \longrightarrow \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Q}/\boldsymbol{Z};\ a \longrightarrow (\mathrm{inv}_{K_\mathfrak{p}}(a_\mathfrak{p}))_{\mathfrak{p}\in P},$$

where $a_\mathfrak{p}$ is the image of a in $H^3(K_\mathfrak{p})$ and

$$\mathrm{inv}_{K_\mathfrak{p}}\colon H^3(K_\mathfrak{p})\simeq \boldsymbol{Q}/\boldsymbol{Z}$$

is the canonical isomorphism defined in [K–1] §5 (cf. (1.1)). By the reciprocity law for A (cf. [S–1] (2.9) and [S–2] Ch. I), the sequence

$$(5.1)\qquad H^3(K) \xrightarrow{\ \iota_K\ } \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Q}/\boldsymbol{Z} \xrightarrow{\text{addition}} \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0$$

is a complex. Now our main result is the following

Theorem (5.2). *The sequence* (5.1) *is exact and we have a natural isomorphism*

$$\mathrm{Ker}(\iota_K) \simeq (\boldsymbol{Q}/\boldsymbol{Z})^{r(A)},$$

where $r(A)$ *is the rank of* A (*cf.* (2.4)). *Namely, we have an exact sequence*

$$0 \longrightarrow (\boldsymbol{Q}/\boldsymbol{Z})^{r(A)} \longrightarrow H^3(K) \xrightarrow{\iota_K} \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Q}/\boldsymbol{Z} \xrightarrow{\text{addition}} \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0.$$

In this section, we prove the following prime-to-ch(K) part of (5.2) and the ch(K)-primary part will be proved in the next section.

Theorem (5.3). *Let* n *be a positive integer prime to* ch(K). *Then there is an exact sequence*

$$0 \longrightarrow (\boldsymbol{Z}/n\boldsymbol{Z})^{r(A)} \longrightarrow H^3(K, \mu_n^{\otimes 2}) \xrightarrow{\iota_K} \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Z}/n\boldsymbol{Z} \xrightarrow{\text{addition}} \boldsymbol{Z}/n\boldsymbol{Z} \longrightarrow 0.$$

Let S be a finite subset of P and $\lambda_S\colon U_S = X\backslash S \to X$ be the inclusion map. Assume that if n is not prime to ch(F) S contains any $\mathfrak{p}\in P$ such that ch$(\kappa(\mathfrak{p}))=$ ch(F). Then the localization theory on X gives the following long exact sequence

$$\cdots \longrightarrow H^i(K, \mu_n^{\otimes 2}) \xrightarrow{\partial^i} \bigoplus_{\mathfrak{p}\in P} H_{\mathfrak{p}}^{i+1}(X, \lambda_{S!}\mu_n^{\otimes 2}) \longrightarrow H^{i+1}(X, \lambda_{S!}\mu_n^{\otimes 2}) \longrightarrow \cdots$$

By the localization theory on Spec$(A_{\mathfrak{p}})$, we have an isomorphism for $i \geq 2$

$$H_{\mathfrak{p}}^{i+1}(X, \lambda_{S!}\mu_n^{\otimes 2}) \simeq \begin{cases} H^i(K_{\mathfrak{p}}, \mu_n^{\otimes 2}) & \text{if } \mathfrak{p}\in S, \\ H^{i-1}(\kappa(\mathfrak{p}), \mu_n) & \text{if } \mathfrak{p}\in P-S. \end{cases}$$

Moreover, we have the following natural isomorphisms

$$H^2(K_{\mathfrak{p}}, \mu_n^{\otimes 2}) \simeq K_2(K_{\mathfrak{p}})/n \quad (\text{cf. [M–S]}),$$

$$H^3(K_{\mathfrak{p}}, \mu_n^{\otimes 2}) \simeq \boldsymbol{Z}/n\boldsymbol{Z} \quad (\text{cf. [K–1] §5}),$$

$$H^1(\kappa(\mathfrak{p}), \mu_n) \simeq \kappa(\mathfrak{p})^{\times}/n \quad \text{and} \quad H^2(\kappa(\mathfrak{p}), \mu_n) \simeq \boldsymbol{Z}/n\boldsymbol{Z}.$$

Hence, using the isomorphism $H^2(K, \mu_n^{\otimes 2}) \simeq K_2(K)/n$ (cf. [M–S]), we can see Coker$(\partial^2) \simeq C_S/n$, and this gives a natural injection

$$\alpha_S\colon C_S/n \longrightarrow H^3(X, \lambda_{S!}\mu_n^{\otimes 2}),$$

and an exact sequence

$$0 \longrightarrow \mathrm{Coker}(\alpha_S) \longrightarrow H^3(K, \mu_n^{\otimes 2}) \xrightarrow{\iota_K} \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Z}/n\boldsymbol{Z}$$

$$\longrightarrow H^4(X, \lambda_{S!}\mu_n^{\otimes 2}) \longrightarrow 0,$$

where the surjectivity of the last map follows from the fact $cd_l(K)=3$ by [S–1] (5.1). Since we know that the image of ι_K lies in the kernel of the addition map, we get a natural surjection

$$\beta_S\colon H^4(X,\ \lambda_{S!}\mu_n^{\otimes 2}) \longrightarrow \boldsymbol{Z}/n\boldsymbol{Z}.$$

Thus, for the proof of (5.3), it suffices to show the following

Lemma (5.4). *The map β_S is an isomorphism and there exists a natural isomorphism* $\mathrm{Coker}(\alpha_S)\simeq(\boldsymbol{Z}/n\boldsymbol{Z})^{r(A)}$.

First, we show the following

Claim (5.5). (5.4) is true if we assume the following.

(*) There is a resolution $\mathfrak{X}$ of Spec (A) (cf. §2) such that for any $x\in Y_0$ the following sequence is exact

$$0\longrightarrow H^3(K_x,\ \mu_n^{\otimes 2})\xrightarrow{\iota_{K_x}}\bigoplus_{\mathfrak{p}\in P_x}\boldsymbol{Z}/n\boldsymbol{Z}\xrightarrow{\text{addition}}\boldsymbol{Z}/n\boldsymbol{Z}\longrightarrow 0.$$

Let $j\colon X\to\mathfrak{X}$ and $i\colon Y\to\mathfrak{X}$ be the inclusion maps and put $\mathscr{F}=i^*Rj_*(\lambda_{S!}\mu_n^{\otimes 2})$. By the proper base change theorem, we have

$$(5.6)\qquad H^i(X,\ \lambda_{S!}\mu_n^{\otimes 2})\simeq H^i(Y,\ \mathscr{F})\qquad\text{for any}\quad i\geqq 0.$$

The localization theories on Y and on the henselization of Y at $x\in Y_0$ give the following long exact sequences

$$(5.7)\qquad\cdots\longrightarrow\bigoplus_{x\in Y_0}H_x^i(Y,\ \mathscr{F})\longrightarrow H^i(Y,\ \mathscr{F})\longrightarrow\bigoplus_{\eta\in Y_1}H^i(\eta,\ \mathscr{F})\longrightarrow\cdots,$$

$$(5.8)\qquad\cdots\longrightarrow H_x^i(Y,\ \mathscr{F})\longrightarrow H^i(x,\ i_x^*\mathscr{F})\longrightarrow\bigoplus_{\eta_x\in Y_1^x}H^i(\eta_x,\ \mathscr{F})\longrightarrow\cdots,$$

where $i_x\colon x\to Y$ is the inclusion. Moreover, we have isomorphisms

$$(5.9)\qquad\begin{aligned}&H^i(\eta,\ \mathscr{F})\simeq H^i(K_\eta,\ \mu_n^{\otimes 2}),\\&H^i(\eta_x,\ \mathscr{F})\simeq H^i(K_{\eta_x},\ \mu_n^{\otimes 2}),\\&H^i(x,\ i_x^*\mathscr{F})\simeq\Lambda^i:=H^i(\mathrm{Spec}\,(R_x),\ (\lambda_{x.S})_!\mu_n^{\otimes 2}),\end{aligned}$$

where R_x is the affine ring of $\mathrm{Spec}\,(A_x)\times_{\mathfrak{X}}X$ and we put

$$\lambda_{x.S}=\lambda_S\times_X\mathrm{Spec}\,(R_x)\colon\ \mathrm{Spec}\,(R_{x.S})\longrightarrow\mathrm{Spec}\,(R_x).$$

To compute Λ^i, the localization theory on $\mathrm{Spec}\,(R_x)$ gives the following exact sequence

$$\Lambda^2 \longrightarrow K_2(K_x)/n \longrightarrow (\bigoplus_{\mathfrak{p}\in U_S^x} \kappa(\mathfrak{p})^\times/n)\oplus(\bigoplus_{\mathfrak{p}\in S^x} K_2(K_\mathfrak{p})/n) \longrightarrow$$

$$\longrightarrow \Lambda^3 \longrightarrow H^3(K_x, \mu_n^{\otimes 2}) \xrightarrow{\gamma} \bigoplus_{\mathfrak{p}\in P^x} H^3(K_\mathfrak{p}, \mu_n^{\otimes 2}) \longrightarrow$$

$$\downarrow \wr\, \mathrm{inv}_{K_\mathfrak{p}}$$

$$\longrightarrow \Lambda^4 \longrightarrow H^4(K_x, \mu_n^{\otimes 2})=0 \qquad\qquad \boldsymbol{Z}/n\boldsymbol{Z}$$

The vanishing of $H^4(K_x, \mu_n^{\otimes 2})$ follows from [S–1] (5.1). On the other hand, by (5.5) (*), we see

$$\mathrm{Coker}(\gamma)=0 \quad \text{and } \mathrm{Ker}(\gamma)\simeq \mathrm{Ker}(\bigoplus_{\eta_x\in Y_1^x} \boldsymbol{Z}/n\boldsymbol{Z} \xrightarrow{\sigma} \boldsymbol{Z}/n\boldsymbol{Z}),$$

where σ is the addition map or the identity according as $\# Y_1^x=2$ or 1. In particular, we have $\Lambda^4=0$, and from these computations and (5.8), we see isomorphisms

$$H_x^4(Y, \mathscr{F})\simeq \boldsymbol{Z}/n\boldsymbol{Z},$$

$$H_x^3(Y, \mathscr{F})\simeq T_{S.x}/n \quad (\text{cf. §3}),$$

Hence, by (5.7), we get an exact sequence

(5.10) $$0 \longrightarrow T_S/n \longrightarrow H^3(Y, \mathscr{F}) \longrightarrow \mathrm{Ker}(\theta) \longrightarrow 0,$$

and an isomorphism

(5.11) $$H^4(Y, \mathscr{F})\simeq \mathrm{Coker}(\theta),$$

where

$$\theta\colon \bigoplus_{\eta\in Y_1} H^3(K_\eta, \mu_n^{\otimes 2}) \longrightarrow \bigoplus_{x\in Y_0} \boldsymbol{Z}/n\boldsymbol{Z}$$

is described as follows: Recall that there is a canonical isomorphism for each $\eta\in Y_1$ (cf. [K–1] §5)

$$H^3(K_\eta, \mu_n^{\otimes 2})\simeq \mathrm{Br}(\kappa(\eta))_n.$$

Then, the composite map

$$\mathrm{Br}(\kappa(\eta))_n\simeq H^3(K_\eta, \mu_n^{\otimes 2}) \xrightarrow{\theta} \bigoplus_{x\in Y_0} \boldsymbol{Z}/n\boldsymbol{Z} \xrightarrow{\rho_x} \boldsymbol{Z}/n\boldsymbol{Z}$$

is the zero map if $x\notin \eta_0$, and if $x\in\eta_0$, it coincides with the map

$$\mathrm{Br}(\kappa(\eta))_n \longrightarrow \mathrm{Br}(\kappa(\eta_x))_n \xrightarrow[\mathrm{inv}_x]{\sim} \boldsymbol{Z}/n\boldsymbol{Z}.$$

Here ρ_x is the projection to the x-component. Hence, by the classical exact sequence

$$0 \longrightarrow \mathrm{Br}(\kappa(\eta)) \longrightarrow \bigoplus_{x\in\eta_0} \mathrm{Br}(\kappa(\eta_x)) \xrightarrow{\text{addition}} \boldsymbol{Q}/\boldsymbol{Z} \longrightarrow 0,$$
$$\downarrow \wr\, \mathrm{inv}K_{\mathfrak{p}}$$
$$\boldsymbol{Q}/\boldsymbol{Z}$$

we have isomorphisms

$$\mathrm{Ker}(\theta) \simeq (\boldsymbol{Z}/n\boldsymbol{Z})^{r(A)} \quad \text{and} \quad \mathrm{Coker}(\theta) \simeq \boldsymbol{Z}/n\boldsymbol{Z}.$$

Furthermore, we see that the following diagram is commutative

(5.12)
$$\begin{array}{ccccc} 0 & \longrightarrow & C_S/n & \xrightarrow{\alpha_S} & H^3(X, \lambda_{S!}\mu_n^{\otimes 2}) \\ & & \downarrow \rho_S & & \downarrow \wr\, (5.6) \\ 0 & \longrightarrow & T_S/n & \longrightarrow & H^3(Y, \mathscr{F}). \end{array}$$

Thus, the proof of (5.5) is reduced to the following

Lemma (5.13). *Let S be any finite subset of P and let r be any positive integer prime to* ch(K). *Then ρ_S induces an isomorphism*

$$\rho_S\colon C_S/r \simeq T_S/r.$$

Proof. For a finite subset S' of P containing S, we have a commutative diagram

$$\begin{array}{ccccccc} \prod_{\mathfrak{p}\in S'-S} K_2(A_{\mathfrak{p}}) & \longrightarrow & C_{S'} & \longrightarrow & C_S & \longrightarrow & 0 \\ \| & & \downarrow \rho_{S'} & & \downarrow \rho_S & & \\ \prod_{\mathfrak{p}\in S'-S} K_2(A_{\mathfrak{p}}) & \longrightarrow & T_{S'} & \longrightarrow & T_S & \longrightarrow & 0, \end{array}$$

where the horizontal sequences are exact. Hence, it suffices to show (5.13) replacing S with S', so we may suppose that r is invertible on U_S. Then, the injectivity follows at once from (5.12). As for the surjectivity, fix $I \in \mathscr{I}$ as Section 3. Since r is prime to ch(K), if we take m large enough, we have $rK_2(K_{\eta_x}) \supset U_I^m K_2(K_{\eta_x})$ for any $x \in Y_0$ and $\eta_x \in Y_1^x$. Then our assertion follows from the surjectivity of

$$K_2(K_x) \longrightarrow \prod_{\eta_x \in Y_1^x} K_2(K_{\eta_x})/U_I^m K_2(K_{\eta_x}),$$

which follows from the approximation theorem for a finite number of discrete valuations on K_x.

Now we complete the proof of (5.4). We may suppose $n = l^m$ for some prime number $l \neq$ ch(K) and an integer $m > 0$. Then, if $l \neq$ ch(F), it follows at once from (5.5) and [S-1] (6.2.1). If $l = p :=$ ch(F) and ch$(K) = 0$, we see that there exists a resolution $\mathfrak{X}$ of Spec(A) such that for each $x \in Y_0$, A_x satisfies [S–1] (4.2.1) or (4.2.2). Hence, if K contains a primitive p-th root of unity, it follows from (5.5) and [S–1] (6.2.2). In general, it suffices to show (5.4) assuming that A satisfies either [S–1] (4.2.1) or (4.2.2). Let A' be the integral closure of A in $K'=K(\zeta_p)$ and let S' be the set of prime ideals of A' lying over S. By an easy computation, we see that $r(A')=0$. Since we have proved (5.4) for A', we have

$$\text{Coker}(\alpha_{S'})=0 \quad \text{and} \quad \beta_{S'}\colon H^4(X', \lambda_{S'!}\mu_n^{\otimes 2}) \simeq \boldsymbol{Z}/n\boldsymbol{Z}. \tag{5.14}$$

On the other hand, we have the norm maps

$$N\colon C_{S'}/n \longrightarrow C_S/n \quad \text{and} \quad N\colon H^i(X', \lambda_{S'!}\mu_n^{\otimes 2}) \longrightarrow H^i(X, \lambda_{S!}\mu_n^{\otimes 2}),$$

and the natural maps

$$R\colon C_S/n \longrightarrow C_{S'}/n \quad \text{and} \quad R\colon H^i(X, \lambda_{S!}\mu_n^{\otimes 2}) \longrightarrow H^i(X', \lambda_{S'!}\mu_n^{\otimes 2}),$$

and the composite maps $N \cdot R$ are the multiplication by $[K'\colon K]$ which is prime to n. Hence the norm maps N are surjective. Consequently, (5.4) for A follows from (5.14) and the commutative diagrams

$$\begin{array}{ccc} C_{S'} & \xrightarrow{\alpha_{S'}} & H^3(X', \lambda_{S'!}\mu_n^{\otimes 2}) \\ \big\downarrow N & & \big\downarrow N \\ C_S & \xrightarrow{\alpha_S} & H^3(X, \lambda_{S!}\mu_n^{\otimes 2}) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^4(X', \lambda_{S'!}\mu_n^{\otimes 2}) & \xrightarrow{\beta_{S'}} & \boldsymbol{Z}/n\boldsymbol{Z} \\ \big\downarrow N & & \big\| \\ H^4(X, \lambda_{S!}\mu_n^{\otimes 2}) & \xrightarrow{\beta_S} & \boldsymbol{Z}/n\boldsymbol{Z}. \end{array}$$

§6. A duality theorem on the p-primary part

Let A and K be as before and assume $\text{ch}(K)=p>0$. In this section, we give the proof of the p-primary part of (5.2), namely we prove

Theorem (6.1). *Let $\nu_m = W_m\Omega^2_{X,\log}$ be the logarithmic part of the De Rham-Witt complex on X_{et}. Then, there is an exact sequence*

$$0 \longrightarrow (\boldsymbol{Z}/p^m\boldsymbol{Z})^{r(A)} \longrightarrow H^1(K, \nu_m) \xrightarrow{\iota_K} \bigoplus_{\mathfrak{p}\in P} \boldsymbol{Z}/p^m\boldsymbol{Z} \xrightarrow{\text{addition}} \boldsymbol{Z}/p^m\boldsymbol{Z} \longrightarrow 0.$$

Proof. Consider the localization sequence on X

$$\cdots \longrightarrow H^i(K, \nu_m) \xrightarrow{\partial^i} \bigoplus_{\mathfrak{p}\in P} H^{i+1}_{\mathfrak{p}}(X, \nu_m) \longrightarrow H^{i+1}(X, \nu_m) \longrightarrow \cdots$$

We have natural isomorphisms

$$\begin{aligned} H^1_{\mathfrak{p}}(X, \nu_m) &\simeq \text{Coker}(H^0(A_{\mathfrak{p}}, \nu_m) \to H^0(K_{\mathfrak{p}}, \nu_m)) \\ &\simeq \text{Coker}(K_2(A_{\mathfrak{p}})/p^m \to K_2(K_{\mathfrak{p}})/p^m) \\ &\simeq \kappa(\mathfrak{p})^\times/p^m \quad (\text{cf. [B–K] §2}), \end{aligned}$$

$$H^2_{\mathfrak{p}}(X, \nu_m) \simeq H^1(K_{\mathfrak{p}}, \nu_m) \simeq \boldsymbol{Z}/p^m\boldsymbol{Z} \quad \text{(cf. [K–2] §3)},$$

$$H^0(K, \nu_m) \simeq K_2(K)/p^m \text{ (cf. [B–K] §2).}$$

Hence we see Coker $(\partial^0) \simeq SK_1(A)/p^m$ (cf. the introduction) and this gives a natural injective homomorphism

$$\text{(6.2)} \qquad \alpha\colon SK_1(A)/p^m \longrightarrow H^1(X, \nu_m),$$

and an exact sequence

$$0 \longrightarrow \operatorname{Coker}(\alpha) \longrightarrow H^1(K, \nu_m) \xrightarrow{\iota_K} \bigoplus_{\mathfrak{p} \in P} \boldsymbol{Z}/p^m\boldsymbol{Z} \longrightarrow H^2(X, \nu_m) \longrightarrow 0,$$

where we used the fact $\mathrm{cd}_p(K)=1$ (cf. SGA4X). Since we know that the image of ι_K lies in the kernel of the addition map (cf. [S–1] (2.9) or [S–2] Ch. I), we get a natural surjective homomorphism

$$\text{(6.3)} \qquad \beta\colon H^2(X, \nu_m) \longrightarrow \boldsymbol{Z}/p^m\boldsymbol{Z}.$$

Now (6.1) follows from the following

Lemma (6.4). *The map β is an isomorphism and there is a natural isomorphism* Coker $(\alpha) \simeq (\boldsymbol{Z}/p^m\boldsymbol{Z})^{r(A)}$.

To prove (6.4), we give a duality theorem for the p-torsion cohomology groups for X. For simplicity, we write

$$H^i(X, \boldsymbol{Z}/p^m\boldsymbol{Z}) = H^i(\boldsymbol{Z}/p^m\boldsymbol{Z}) \quad \text{and} \quad H^i(X, \nu_m) = H^i(\nu_m)$$

By SGA4X, we have

$$H^i(\nu_m) = H^i(\boldsymbol{Z}/p^m\boldsymbol{Z}) = 0 \qquad \text{for} \quad i \geqq 3.$$

On the other hand, by the pairings on the cohomology groups

$$H^i(\nu_m) \times H^j(\boldsymbol{Z}/p^m\boldsymbol{Z}) \longrightarrow H^{i+j}(\nu_m)$$

combined with (6.3), we get a canonical pairing

$$H^i(\nu_m) \times H^{2-i}(\boldsymbol{Z}/p^m\boldsymbol{Z}) \longrightarrow \boldsymbol{Z}/p^m\boldsymbol{Z},$$

and a canonical homomorphism

$$\psi^i_m\colon H^i(\nu_m) \longrightarrow H^{2-i}(\boldsymbol{Z}/p^m\boldsymbol{Z})^*.$$

Now the key result is the following

Theorem (6.5). (1) *The map ψ^i_m is an isomorphism of finite abelian groups for $i=1$ and 2.*

(2) *The map* ψ_m^0 *is injective and has a dense image. If* A *is complete, it is an isomorphism.*

First, assuming (6.5), we complete the proof of (6.4). The first assertion of (6.4) follows at once from (6.5.1) for $i=2$. As for the second isomorphism of (6.4), (6.5.1) for $i=1$ implies that we have an isomorphism

$$\gamma\colon H^1(X, \nu_m)\simeq H^1(X, \boldsymbol{Z}/p^m\boldsymbol{Z})^* = \pi_1^{ab}(X)/p^m.$$

Moreover, we can see that the following diagram is commutative

$$(6.6)\qquad \begin{array}{ccc} SK_1(A)/p^m & \xrightarrow{\alpha} & H^1(X, \nu_m) \\ & \searrow^{\psi_A} & \downarrow \wr \gamma \\ & & \pi_1^{ab}(X)/p^m, \end{array}$$

where ψ_A is the reciprocity map for A (cf. (0.5) and §1). Consequently, our assertion follows at once from the result of Section 2.

Proof of (6.5). First we have already known the finiteness of $H^1(\boldsymbol{Z}/p^m\boldsymbol{Z})$ by the results of Section 4. On the other hand, we have an exact sequences

$$(6.7)\qquad 0 \longrightarrow \boldsymbol{Z}/p^{m-1}\boldsymbol{Z} \longrightarrow \boldsymbol{Z}/p^m\boldsymbol{Z} \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z} \longrightarrow 0,$$

$$0 \longrightarrow \nu_1 \longrightarrow \nu_m \longrightarrow \nu_{m-1} \longrightarrow 0 \quad \text{(cf. [C–S–S] §1 Lemma 3)},$$

which give rise to the long exact sequences of cohomology groups compatible with the maps ψ_m^i. Hence, by a usual sort of argument using the induction on m, we are reduced to the case $m=1$. Only difficulty is that in the proof of the injectivity of ψ_m^1, we use the following fact: Consider the exact sequence arising from (6.7)

$$H^2(\boldsymbol{Z}/p^{m-1}\boldsymbol{Z})^* \longrightarrow H^1(\boldsymbol{Z}/p\boldsymbol{Z})^* \xrightarrow{\iota} H^1(\boldsymbol{Z}/p^m\boldsymbol{Z})^*$$

Then the image of

$$\psi_{m-1}^0\colon H^0(\nu_{m-1}) \longrightarrow H^2(\boldsymbol{Z}/p^{m-1}\boldsymbol{Z})^*$$

maps surjectively to Ker(ι). Indeed this fact follows from the density of the image of ψ_{m-1}^0 which is an induction hypothesis and from the finiteness of $H^1(\boldsymbol{Z}/p\boldsymbol{Z})$.

Now, to prove (6.5) for $m=1$, we recall the theory of local dualities on Spec(A) (cf. [H–1] and [H–2]), by which we know the following fact: Let $\omega_A = H^0(X, \Omega_X^2)$, where Ω_X^2 is the second exterior power of Ω_X over $\mathcal{O}_X$. Then ω_A is a dualizing module on Spec(A) in the sense of [H–2] and it is a finite A-module without torsions of rank one. There exists a natural homomorphism called the residue homomorphism

$$\text{res}: H^2_x(\text{Spec}(A), \omega_A) \longrightarrow F = A/\mathfrak{m}_A,$$

where for a finite A-module M, $H^i_x(\text{Spec}(A), M)$ denotes the i-th local cohomology group with support at the unique closed point x of $\text{Spec}(A)$. Since F is a finite field in our situation, we get the map

$$\text{Tr}_{F/F_p} \cdot \text{res}: H^2_x(\text{Spec}(A), \omega_A) \longrightarrow \boldsymbol{F}_p = \boldsymbol{Z}/p\boldsymbol{Z}.$$

By this map and the Yoneda pairing, we get a canonical pairing for any finite A-module M,

(6.8) $$H^i_x(\text{Spec}(A), M) \times \text{Ext}^{2-i}_A(M, \omega_A) \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z}.$$

Then, by the local duality theorem (cf. [H–1]), we get the following

Theorem (6.9). *For each integer* $i \geqq 0$, *the pairing* (6.8) *induces isomorphisms*

$$\text{Ext}^{2-i}_A(M, \omega_A) \otimes_A \hat{A} \simeq H^i_x(\text{Spec}(A), M)^*,$$

$$H^i_x(\text{Spec}(A), M) \simeq D(\text{Ext}^{2-i}_A(M, \omega_A)),$$

where $\hat{A}$ *denotes the completion of* A, $D(*)$ *denotes the Pontrijagin dual of* $*$, *and* $\text{Ext}^j_A(M, \omega_A)$ *is endowed with the* $\mathfrak{m}_A$*-adic topology on a finite* A*-module.*

Now, using the localization sequence

$$\cdots \longrightarrow H^i_x(\text{Spec}(A), M) \longrightarrow H^i(\text{Spec}(A), M) \longrightarrow H^i_{\text{et}}(X, M|_X) \longrightarrow \cdots,$$

and the fact (cf. [H–1] and [H–2])

$$H^i_x(\text{Spec}(A), A) = H^i_x(\text{Spec}(A), \omega_A) = 0 \qquad \text{for} \quad i \neq 2,$$

we get a canonical pairing

(6.10) $$H^i(X, \mathcal{O}_X) \times H^{1-i}(X, \Omega^2_X) \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z},$$

which induces isomorphisms

(6.11) $$H^1(X, \Omega^2_X) \simeq D(A) \quad \text{and} \quad H^1(X, \mathcal{O}_X) \simeq D(\omega_A),$$

$$\hat{A} \simeq D(H^1(X, \Omega^2_X)) \quad \text{and} \quad \omega_A \otimes_A \hat{A} \simeq D(H^1(X, \mathcal{O}_X)),$$

where we should note

$$H^0(X, \mathcal{O}_X) = A \quad \text{and} \quad H^0(X, \Omega^2_X) = \omega_A.$$

On the other hand, we have exact sequences of sheaves on X_{et},

$$0 \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z} \longrightarrow \mathcal{O}_X \xrightarrow{1-f} \mathcal{O}_X \longrightarrow 0$$

$$0 \longrightarrow \nu_1 \longrightarrow \Omega_X^2 \xrightarrow{1-\gamma} \Omega_X^2 \longrightarrow 0,$$

where f is the Frobenius and γ is the Cartier operator. Moreover, the map induced on $H^i(X, \mathcal{O}_X)$ by f is the dual of the map induced on $H^{1-i}(X, \Omega_X^2)$ by γ with respect to the pairing (6.10). Thus, (6.5) for $m=1$ follows from (6.11).

§7. The proof of the existence theorem (the prime-to-p part)

In the last two sections, we complete the proof of (1.10.2), which is viewed as the existence theorem in our class field theory.

In this section, we prove the following prime-to-ch(K) part.

Theorem (7.1). *Let $\omega \in (C_S)^*_{\mathrm{tor}}$ have an order of a power of a prime number $l \neq \mathrm{ch}(K)$. Assume that the restriction of ω to $(K_2(K_{\mathfrak{p}}))^*$ comes from $H^1(K_{\mathfrak{p}})$ for every $\mathfrak{p} \in S$. Then, ω lies in the image of Ψ_S.*

First, we have the following special case of (7.1).

Proposition (7.2). *Let $\omega \in (C_K)^*_{\mathrm{tor}}$ have an order of a power of a prime number $l \neq \mathrm{ch}(K)$ and assume that it satisfies* (1.10.3) (A) *and* (B). *Assume further the following condition*:
(*) *A is regular, K contains a primitive l-th root of unity and if* ch$(K)=0$ *and $l=\mathrm{ch}(F)$, A satisfies either* [S–1] (4.3.2) (A) *or* (B).
Then, ω lies in the image of Ψ_K.

The proof of (7.2) will be given in the end of this section.

Corollary (7.3). *Let l be a prime number different from* ch(K) *and let S be a non-empty finite subset of P. Assume* (7.2) (*). *Then, $(\chi_{\mathfrak{p}})_{\mathfrak{p}\in S} \in \bigoplus_{\mathfrak{p}\in S} H^1(K_{\mathfrak{p}})(l)$ lies in the image of $H^1(U_S)$ if and only if*

$$\sum_{\mathfrak{p}\in S} \Psi_{K_{\mathfrak{p}}}(\chi_{\mathfrak{p}}) \colon \bigoplus_{\mathfrak{p}\in S} K_2(K_{\mathfrak{p}}) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

annihilates the diagonal image of $K_2(A_S)$, where A_S is the affine ring of U_S.

(7.3) follows from (7.2) and (1.16).

Now we give the proof of (7.1) assuming (7.2). By a usual sort of norm argument, we may assume that K contains a primitive l-th root of unity. Take a resolution $\mathfrak{X}$ of Spec(A) as Section 2. In case ch$(K)=0$ and

$l = \mathrm{ch}(F)$, we blow up in advance $\mathfrak{X}$ successively at some closed points so that A_x satisfies (7.2) (*) for any $x \in Y_0$. By (5.13), ω is viewed as an element of $(T_S)^*_{\mathrm{tor}}$ through the map ρ_S. Moreover, since ω has an order prime to $\mathrm{ch}(K)$, it lies in the subgroup $D(T_S)$ (cf. (3.7) and the proof of (5.13)). Hence, by (3.9), we can find

$$(\chi_\eta, \chi_\mathfrak{p})_{\eta \in Y_1, \mathfrak{p} \in S} \in (\bigoplus_{\eta \in Y_1} H^1(K_\eta)) \oplus (\bigoplus_{\mathfrak{p} \in S} H^1(K_\mathfrak{p}))$$

such that $\chi_\mathfrak{p}$ (resp. χ_{η_x}) is mapped to $\omega_\mathfrak{p}$ for $\mathfrak{p} \in S$ (resp. ω_{η_x} for $x \in Y_0$ and $\eta_x \in Y_1^x$) under the map (1.3) for $K_\mathfrak{p}$ (resp. K_{η_x}), where the notations are as (3.9). Furthermore, since (7.2) (*) is satisfied for A_x, (7.3) implies that the condition (3.9) (*) is replaced by the following condition:
(7.4) For each $x \in Y_0$, the element

$$(\chi_{\eta_x}, \chi_\mathfrak{p})_{\eta_x \in Y_1^x, \mathfrak{p} \in S^x} \in (\bigoplus_{\eta_x \in Y_1^x} H^1(K_{\eta_x})) \oplus (\bigoplus_{\mathfrak{p} \in S^x} H^1(K_\mathfrak{p}))$$

lies in the image of $H^1(R_{x,S})$.

Therefore, in view of the commutative diagram (3.5), we are reduced to show the following

Lemma (7.5). *The image of the natural map*

$$H^1(U_S) \longrightarrow (\bigoplus_{\eta \in Y_1} H^1(K_\eta)) \oplus (\bigoplus_{\mathfrak{p} \in S} H^1(K_\mathfrak{p})) \quad (U_S = X - S)$$

consists of all elements $(\chi_\eta, \chi_\mathfrak{p})_{\eta \in Y_1, \mathfrak{p} \in S}$ *which satisfy* (7.4).

Proof. Let $j_S: U_S \to \mathfrak{X}$ and $i_S: Y_S = \mathfrak{X} \backslash U_S \to \mathfrak{X}$ be the natural inclusion maps. Using the proper base change theorem for etale cohomology, we can see that the natural homomorphism

$$H^1(U_S) \simeq H^1(\mathfrak{X}, Rj_{S*}\mathbf{Q}/\mathbf{Z}) \longrightarrow H^1(Y_S, i_S^* Rj_{S*}\mathbf{Q}/\mathbf{Z})$$

is an isomorphism. Put $\mathscr{F} = i_S^* Rj_{S*}\mathbf{Q}/\mathbf{Z}$. Then, using the localization theory on Y_S, we get an exact sequence

$$H^1(U_S) \longrightarrow (\bigoplus_{\eta \in Y_1} H^1(K_\eta)) \oplus (\bigoplus_{\mathfrak{p} \in S} H^1(K_\mathfrak{p})) \longrightarrow \bigoplus_{x \in Y_0} H_x^2(Y_S, \mathscr{F}).$$

On the other hand, the localization theory on the henselization of Y_S at $x \in Y_0$ gives an exact sequence

$$H^1(R_{x,S}) \longrightarrow (\bigoplus_{\eta_x \in Y_1^x} H^1(K_{\eta_x})) \oplus (\bigoplus_{\mathfrak{p} \in S^x} H^1(K_\mathfrak{p})) \longrightarrow H_x^2(Y_S, \mathscr{F}).$$

Thus (7.5) follows from these exact sequences and this completes the proof of (7.1) modulo the proof of (7.2)

Proof of (7.2). In the first step, we prove (7.2) assuming that $l\omega=0$. First, we suppose that $l\neq \mathrm{ch}(F)$. By the regularity of A,

$$H^1(X)_l \simeq H^1(A)_l \simeq \boldsymbol{Z}/l\boldsymbol{Z} \quad \text{and} \quad H^2(X, \boldsymbol{Z}/l\boldsymbol{Z})=0 \quad \text{(cf. [S–1] (7.6))}.$$

Hence, by the localization theory on X, we have an exact sequence

$$(7.2.1) \qquad 0 \longrightarrow \boldsymbol{Z}/l\boldsymbol{Z} \longrightarrow H^1(K)_l \longrightarrow \bigoplus_{\mathfrak{p}\in \boldsymbol{P}} H^0(\kappa(\mathfrak{p}), \mu_l^{\otimes -1}) \longrightarrow 0.$$

On the other hand, since (0.3) is exact by the regularity of A (cf. [B] and [Q]), we have an exact sequence

$$0 \longrightarrow \mathrm{Coker}\,(K_2(A) \rightarrow \prod_{\mathfrak{p}\in \boldsymbol{P}} K_2(A_{\mathfrak{p}})) \longrightarrow C_K \longrightarrow \boldsymbol{Z} \longrightarrow 0.$$

Hence, noting the canonical isomorphisms

$$K_2(A_{\mathfrak{p}})/l \simeq K_2(\kappa(\mathfrak{p}))/l \simeq \mu_l(\kappa(\mathfrak{p})),$$
$$K_2(A)/l \simeq K_2(F)/l = 0,$$

we have an exact sequence

$$0 \longrightarrow \prod_{\mathfrak{p}\in \boldsymbol{P}} \mu_l(\kappa(\mathfrak{p})) \longrightarrow C_K/l \longrightarrow \boldsymbol{Z}/l\boldsymbol{Z} \longrightarrow 0.$$

This proves our assertion in view of (7.2.1).

Next, we suppose $\mathrm{ch}(K)=0$ and $l=p:=\mathrm{ch}(F)$. In the following, we will give the proof assuming that A satisfies [S–1] (4.3.2) (B). The proof for the other case is similar and we omit it. Put $R=A\left[\frac{1}{p}\right]$ and $P'=P\backslash\{\mathfrak{p}_\alpha, \mathfrak{p}_\beta\}$, and let K_ν be the henselization of K at $\mathfrak{p}_\nu$ for $\nu=\alpha$ and β. Using the localization theory on $\mathrm{Spec}\,(R)$, we have an exact sequence

$$(7.2.2) \qquad 0 \longrightarrow H^1(R)_p \longrightarrow H^1(K)_p \longrightarrow \bigoplus_{\mathfrak{p}\in \boldsymbol{P}'} H^0(\kappa(\mathfrak{p}), \mu_p^{\otimes -1}) \longrightarrow 0.$$

On the other hand, define

$$I_1 = \mathrm{Coker}\,(K_2(R) \rightarrow \prod_{\nu=\alpha,\beta} K_2(K_\nu)) \quad \text{and} \quad I_0 = \prod_{\mathfrak{p}\in \boldsymbol{P}'} K_2(A_{\mathfrak{p}}).$$

We have an exact sequence and canonical isomorphisms

$$I_0 \longrightarrow C_K \longrightarrow I_1 \longrightarrow 0,$$
$$I_0/p = \prod_{\mathfrak{p}\in \boldsymbol{P}'} K_2(A_{\mathfrak{p}})/p \simeq \prod_{\mathfrak{p}\in \boldsymbol{P}'} K_2(\kappa(\mathfrak{p}))/p \simeq \prod_{\mathfrak{p}\in \boldsymbol{P}'} \mu_p(\kappa(\mathfrak{p})).$$

Combining these with (7.2.2), we are reduced to prove

Claim (7.2.3). The induced map

$$\mathrm{Coker}\,(H^1(R)_p \to \bigoplus_{\nu=\alpha,\beta} H^1(K_\nu)_p) \longrightarrow (I_1)^*$$

is injective.

From now on, fix a primitive p-th root of unity and put

$$\bar{K}_i(L) = K_i(L)/p \text{ for a field } L \text{ and } i=1,\ 2.$$

Then, by the Kummer theory, we have isomorphisms

$$H^1(K_\nu)_p \simeq \bar{K}_1(K_\nu)\ (\nu=\alpha,\ \beta) \quad \text{and} \quad H^1(K)_p \simeq \bar{K}_1(K).$$

Combining this with (1.2), we have a pairing

$$\langle\ ,\ \rangle_\nu : \bar{K}_2(K_\nu) \otimes \bar{K}_1(K_\nu) \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z} \quad (\nu=\alpha,\ \beta),$$

Since the image of the map

$$H^1(R)_p \longrightarrow \prod_{\nu=\alpha,\beta} H^1(K_\nu)_p \simeq \prod_{\nu=\alpha,\beta} \bar{K}_1(K_\nu)$$

is equal to the image of $R^\times$, (7.2.3) is reduced to the following

Claim (7.2.4). For $i=1$ and 2, define

$$\Delta^i = \mathrm{Im}\,(K_i(R) \to \prod_{\nu=\alpha,\beta} \bar{K}_i(K_\nu)).$$

Then, Δ^1 is the annihilator of Δ^2 in the pairing

$$\langle\ ,\ \rangle_\alpha + \langle\ ,\ \rangle_\beta : \prod_{\nu=\alpha,\beta} \bar{K}_2(K_\nu) \otimes \prod_{\nu=\alpha,\beta} \bar{K}_1(K_\nu) \longrightarrow \boldsymbol{Z}/p\boldsymbol{Z}.$$

This claim is proved in the same argument as [S–1] Section 4.

Finally, we prove (7.2) in general case. We consider the following commutative diagram

$$\begin{array}{ccccc} & H^1(K)(l) & \longrightarrow & \bigoplus_{\mathfrak{p}\in\boldsymbol{P}} H^2_{\mathfrak{p}}(X,\ \boldsymbol{Q}_l/\boldsymbol{Z}_l) & \xrightarrow{\lambda} H^2(X,\ \boldsymbol{Q}_l/\boldsymbol{Z}_l) \\ & \downarrow{\scriptstyle \Psi_K} & & \downarrow{\scriptstyle \tau} & \\ 0 \longrightarrow & (C_K)^*(l) & \xrightarrow{\sigma} & \bigoplus_{\mathfrak{p}\in\boldsymbol{P}} (K_2(A_{\mathfrak{p}}))^*(l). & \end{array}$$

The assumption (1.10.3) (B) implies $\sigma(\omega) \in \mathrm{Im}\,(\tau)$. Hence, we are reduced to prove the injectivity of the induced map

$$\rho : \mathrm{Im}\,(\lambda) \longrightarrow \mathrm{Coker}\,(\sigma).$$

By what we have proved, this is reduced to the following

Claim. The map λ induces a surjection

$$\bigoplus_{\mathfrak{p}\in P} H^2_{\mathfrak{p}}(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l)_l \longrightarrow\!\!\!\!\rightarrow H^2(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l)_l .$$

To prove the claim, we consider the localization sequence

$$\bigoplus_{\mathfrak{p}\in P} H^2_{\mathfrak{p}}(X, \boldsymbol{Z}/l\boldsymbol{Z}) \longrightarrow H^2(X, \boldsymbol{Z}/l\boldsymbol{Z}) \longrightarrow H^2(K, \boldsymbol{Z}/l\boldsymbol{Z}) \stackrel{\iota}{\longrightarrow} \bigoplus_{\mathfrak{p}\in P} H^3_{\mathfrak{p}}(X, \boldsymbol{Z}/l\boldsymbol{Z}).$$

We have isomorphisms

$$H^2(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l)_l \simeq H^2(X, \boldsymbol{Z}/l\boldsymbol{Z}) \quad \text{and} \quad H^2_{\mathfrak{p}}(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l)_l \simeq H^2_{\mathfrak{p}}(X, \boldsymbol{Z}/l\boldsymbol{Z}),$$

which follow from the isomorphisms

$$H^1(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l) \simeq H^1(A)(l) \simeq \boldsymbol{Q}_l/\boldsymbol{Z}_l \quad \text{and} \quad H^1_{\mathfrak{p}}(X, \boldsymbol{Q}_l/\boldsymbol{Z}_l) = 0.$$

Hence, it suffices to show the injectivity of ι. Since K contains a primitive l-th root of unity, we have isomorphisms

$$H^2(K, \boldsymbol{Z}/l\boldsymbol{Z}) \simeq \mathrm{Br}(K)_l,$$

$$H^3_{\mathfrak{p}}(X, \boldsymbol{Z}/l\boldsymbol{Z}) \simeq \begin{cases} \mathrm{Coker}(\mathrm{Br}(A_{\mathfrak{p}})_l \to \mathrm{Br}(K_{\mathfrak{p}})_l) & \text{if} \quad l \neq \mathrm{ch}(\kappa(\mathfrak{p})), \\ \mathrm{Br}(K_{\mathfrak{p}})_l \quad \text{if} \quad l = \mathrm{ch}(\kappa(\mathfrak{p})). \end{cases}$$

Consequently, our assertion follows from the fact that

$$\mathrm{Ker}(\mathrm{Br}(K) \to \bigoplus_{\mathfrak{p}\in P} \mathrm{Br}(K_{\mathfrak{p}})/\mathrm{Br}(A_{\mathfrak{p}}))$$

is contained in $\mathrm{Br}(X)$, which is trivial by the regularity of A.

§8. The proof of the existence theorem (the p-primary part)

Let the notations be as before and assume $p = \mathrm{ch}(K) > 0$. In this section, we give the proof of the p-primary part of (1.10.2), namely

Theorem (8.1). *Let $\omega \in (C_S)^*_{\mathrm{tor}}$ have an order p^m for an integer $m > 0$. Assume that the restriction of ω to $(K_2(K_{\mathfrak{p}}))^*$ comes from $H^1(K_{\mathfrak{p}})$ for every $\mathfrak{p} \in S$. Then ω lies in the image of Ψ_S.*

Proof. Let $\lambda_S \colon U_S = X \backslash S \to X$ be the inclusion map. Using the same notations as Section 6, there is a long exact sequence

$$\cdots \longrightarrow H^i(X, \lambda_{S!}\nu_m) \longrightarrow H^i(X, \nu_m) \longrightarrow \bigoplus_{\mathfrak{p}\in S} H^i(\mathrm{Spec}(A_{\mathfrak{p}}), \nu_m) \longrightarrow \cdots,$$

and we have the following natural isomorphisms

$$H^0(\mathrm{Spec}\,(A_{\mathfrak{p}}),\, \nu_m) \simeq K_2(A_{\mathfrak{p}})/p^m \quad \text{(cf. [B–K] §2)},$$

$$H^i(\mathrm{Spec}\,(A_{\mathfrak{p}}),\, \nu_m) = 0 \qquad \text{for} \quad i \geqq 1 \quad \text{(cf. [K–2] §3)}.$$

Hence we have an isomorphism

$$H^2(X,\, \lambda_{S!}\nu_m) \simeq H^2(X,\, \nu_m) \simeq \boldsymbol{Z}/p^m\boldsymbol{Z} \quad \text{(cf. §6)} \tag{8.2}$$

and an exact sequence

$$H^0(X,\, \nu_m) \longrightarrow \bigoplus_{\mathfrak{p}\in S} K_2(A_{\mathfrak{p}})/p^m \longrightarrow H^1(X, \lambda_{S!}\nu_m) \longrightarrow H^1(X,\, \nu_m) \longrightarrow 0. \tag{8.3}$$

On the other hand, we have the localization sequence on X

$$H^0(K,\, \nu_m) \longrightarrow \bigoplus_{\mathfrak{p}\in P} H^1_{\mathfrak{p}}(X,\, \lambda_{S!}\nu_m) \longrightarrow H^1(X,\, \lambda_{S!}\nu_m),$$

and we have the following isomorphisms

$$H^0(K,\, \nu_m) \simeq K_2(K)/p^m \quad \text{(cf. [B–K] §2)},$$

$$H^1_{\mathfrak{p}}(X,\, \lambda_{S!}\nu_m) \simeq \begin{cases} K_2(K_{\mathfrak{p}})/p^m & \text{if} \quad \mathfrak{p} \in S. \\ \mathrm{Coker}\,(K_2(A_{\mathfrak{p}})/p^m \to K_2(K_{\mathfrak{p}})/p^m) \simeq \kappa(\mathfrak{p})^\times/p^m & \text{if} \quad \mathfrak{p} \notin S. \end{cases}$$

Hence, we get a natural injective homomorphism

$$\alpha_S\colon C_S/p^m \longrightarrow H^1(X,\, \lambda_{S!}\nu_m). \tag{8.4}$$

Then, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^1(X,\, \boldsymbol{Z}/p^m\boldsymbol{Z}) & \longrightarrow & H^1(U_S,\, \boldsymbol{Z}/p^m\boldsymbol{Z}) & \longrightarrow \\
& \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & \\
0 \longrightarrow & H^1(X,\, \nu_m)^* & \longrightarrow & H^1(X,\, \lambda_{S!}\nu_m)^* & \longrightarrow \\
& & & \downarrow{\scriptstyle \gamma_5} & \\
& & & (C_S)^* &
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \bigoplus_{\mathfrak{p}\in S} H^2_{\mathfrak{p}}(X,\, \boldsymbol{Z}/p^m\boldsymbol{Z}) & \longrightarrow & H^2(X,\, \boldsymbol{Z}/p^m\boldsymbol{Z}) \\
& \downarrow{\scriptstyle \gamma_3} & & \downarrow{\scriptstyle \gamma_4} \\
\longrightarrow & \bigoplus_{\mathfrak{p}\in S} (K_2(A_{\mathfrak{p}})/p^m)^* & \longrightarrow & H^0(X,\, \nu_m)^*
\end{array}$$

with the exact horizontal sequences. Here the upper sequence comes from the localization theory on X and the lower one is the dual of (8.3). The maps γ_1, γ_2, and γ_4 come from the pairings

$$H^i(X, Z/p^m Z)\times H^{2-i}(X, \nu_m) \longrightarrow H^2(X, \nu_m)\simeq Z/p^m Z,$$

$$H^i(U_S, Z/p^m Z)\times H^{2-i}(X, \lambda_{S!}\nu_m) \longrightarrow H^2(X, \lambda_{S!}\nu_m)\simeq Z/p^m Z \text{ (cf. (8.2))}$$

and γ_3 comes from the map (1.3) $\Psi_{K_\mathfrak{p}}$ for $K_\mathfrak{p}$ noting (0.1) and the fact

$$H^2_\mathfrak{p}(X, Z/p^m Z)\simeq \mathrm{Coker}\,(H^1(\mathrm{Spec}\,(A_\mathfrak{p}), Z/p^m Z)\to H^1(K_\mathfrak{p}, Z/p^m Z)).$$

The map γ_5 is the dual of the map (8.4) and we see that the composite map $\gamma_5\cdot\gamma_2$ is nothing other than the map Ψ_S. Since (8.4) is injective, γ_5 is surjective. Hence we can find $\tilde{\omega}$ in $H^1(X, \lambda_{S!}\nu_m)^*$ which maps to ω under γ_5. Then the assumption on ω in (8.1) implies that the image of $\tilde{\omega}$ in $\bigoplus_{\mathfrak{p}\in S}(K_2(A_\mathfrak{p})/p^m)^*$ lies in the image of γ_3. On the other hand, by (6.5) we know that γ_1 is an isomorphism and γ_4 is injective. Thus, by an easy diagram chasing, we can see that $\tilde{\omega}$ lies in the image of γ_2 and this completes the proof.

Finally, we complete the proof of (0.5). Now that we have obtained (1.10) and (4.2), there remains only the proof of (0.5.3).

Lemma (8.5). *For any integer $n>0$, Ψ_A induces an injection*

$$SK_1(A)/n \hookrightarrow \pi_1^{ab}(X)/n.$$

Proof. It suffices to show (8.5) assuming either that n is a power of $\mathrm{ch}\,(K)$ or that n is prime to $\mathrm{ch}\,(K)$. In the former case, it follows from (6.2) and (6.6). In the latter case, we note that the dual Ψ_A of ψ_A factors as follows (cf. (3.5)),

$$\Psi_A\colon H^1(X) \xrightarrow{\Phi} D(T) \hookrightarrow (T)^*_{\mathrm{tor}} \xrightarrow{\rho^*} (SK_1(A))^*_{\mathrm{tor}},$$

where $T=T_S$, $\Phi=\Phi_S$ and $\rho=\rho_S$ for $S=\phi$ (cf. §3). By (1.10.2) and (5.13), it is surjective on the prime-to-ch$\,(K)$ part. On the other hand, by the results in Section 2, its kernel is divisible. Now (8.5) follows by taking the dual of Ψ_A.

Let $D(A)$ be the maximum divisible subgroup of $SK_1(A)$. Since $\pi_1^{ab}(X)$ has no divisible element by (4.1), $D(A)\hookrightarrow \mathrm{Ker}\,(\psi_A)$. Hence, it suffices to show that $\mathrm{Ker}\,(\psi_A)$ is divisible. By (8.5), we have

$$\mathrm{Ker}\,(\psi_A)=\bigcap_n n\cdot SK_1(A)=\bigcap_n n\cdot SK_1(A)^0,$$

where n ranges over all positive integer. On the other hand, if we put N the order of $\pi_1^{ab}(X)_{\mathrm{tor}}$ (cf. (4.2)), $N\cdot SK_1(A)\hookrightarrow \mathrm{Ker}\,(\psi_A)$. Clearly, this proves our assertion.

References

[B] S. Bloch, K_2 and algebraic cycles, Ann. of Math. **99** (1974), 349–379.

[B–K] S. Bloch and K. Kato, p-adic etale cohomology, to appear in Publ. Math. IHES.

[C–S–S] J.-L. Colliot-Thélène, J.-J. Sansuc and C. Soulé, Torsion dans le groupe de Chow de codimension deux, Duke Math. J. **50** (1983), 763–801.

[H–1] R. Hartshorne, Local cohomology, Lecture Notes in Math. 41, Springer-Verlag (1967).

[H–2] R. Hartshorne, Residues and duality, Lecture Notes in Math. 20, Springer-Verlag (1966).

[I] L. Illusie, Complexe de De Rham-Witt et cohomologie cristalline, Ann. Sci. Ec. Norm. Sup. **12** (1979), 501–661.

[K–1] K. Kato, A generalization of local class field theory by using K-groups I, J. Fac. Sci. Univ. of Tokyo, Sec. IA, **26** (1979), 303–376.

[K–2] K. Kato, ——— II, ibid. **27** (1980), 602–683.

[K–3] K. Kato, ——— III, ibid. **29** (1982), 31–43.

[K–4] K. Kato, A Hasse principle for two dimensional global fields, J. für die reine und angewandte Math. **366** (1986), 142–183.

[Kl] S. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. **84** (1966), 293–344.

[Mil] J. Milnor, Algebraic K-theory and quadratic forms, Invent. Math. **9** (1970), 318–344.

[Mum] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.E.S. **9** (1961), 5–22.

[M–S] A. S. Mercuriev and A. A. Suslin, K-cohomology of Severi-Brauer varieties and norm residue homomorphism, preprint.

[Q] D. Quillen, Higher algebraic K-theory I, Lecture Notes in Math. 341, Springer-Verlag (1973), 85–147.

[S–1] S. Saito, Arithmetic on two dimensional local rings, Invent. Math. **85** (1986), 379–414.

[S–2] S. Saito, Class field theory for curves over local fields, Journal of Number Theory **21** (1985), 44–80.

[Se] J.-P. Serre, Corps Locaux, Hermann (1962).

Department of Mathematics
Faculty of Science
University of Tokyo
Tokyo, 113 Japan