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The Lower Central Series of the Pure Braid Group of an Algebraic Curve

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Dedicated to Professor Nagayoshi Iwahori on his 60-th birthday

§ 1. Main results

This short note contains four theorems: two results on the lower central series of the pure braid group of a Riemann surface and two arithmetic analogies of these theorems.

Let us describe the first two results. Let R be a Riemann surface of genus g, let $\prod_{i=1}^{n} R$ denote the n-fold product space of R, and let $F_{0,n}R$ denote the subspace

$$F_{0,n}R=\{(z_1, z_2, \cdots, z_n) \in \prod_{i=1}^{n} R \mid z_i \neq z_j, \text{ if } i \neq j\}.$$

The space $F_{0,n}R$ is a $K(\pi, 1)$-space, and the fundamental group of $F_{0,n}R$ is the pure braid group with n strings of the Riemann surface R (cf. Birman [2], Chap. 2).

In general, for a simplicial complex X we define the *holonomy Lie algebra* of X over $\boldsymbol{Q}$ in the following way (see [12]). Let

$$\eta: H_2(X; \boldsymbol{Q}) \longrightarrow \wedge^2 H_1(X; \boldsymbol{Q})$$

be the dual of the cup product homomorphism. Let $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ be the free Lie algebra generated by $H_1(X; \boldsymbol{Q})$ over $\boldsymbol{Q}$. We identify the homogeneous part of degree 2 in $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ with $\wedge^2 H_1(X; \boldsymbol{Q})$. We denote by J the homogeneous ideal of $\mathscr{L}(H_1(X; \boldsymbol{Q}))$ generated by the image of η. The holonomy Lie algebra $\mathfrak{g}_X$ over $\boldsymbol{Q}$ is defined to be $\mathscr{L}(H_1(X; \boldsymbol{Q}))/J$. Let

$$\Gamma_1\mathfrak{g}_X \supset \cdots \supset \Gamma_m\mathfrak{g}_X \supset \cdots$$

be the lower central series of $\mathfrak{g}_X$ defined recursively by

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$$\Gamma_1\mathfrak{g}_X=\mathfrak{g}_X \quad \text{and} \quad \Gamma_{m+1}\mathfrak{g}_X=[\mathfrak{g}_X, \Gamma_m\mathfrak{g}_X], \qquad (m\geqq 1).$$

We denote by $\mathrm{gr}_m(\mathfrak{g}_X)$ the successive quotient $\Gamma_m\mathfrak{g}_X/\Gamma_{m+1}\mathfrak{g}_X$. We obtain the associated graded Lie algebra

$$\mathrm{gr}(\mathfrak{g}_X)=\bigoplus_{m=1}^{\infty}\mathrm{gr}_m(\mathfrak{g}_X).$$

Let $\mathscr{L}_X$ be the *Malcev Lie algebra* of X over $\boldsymbol{Q}$, which is also called the nilpotent completion of the fundamental group of X over $\boldsymbol{Q}$ (see [17]).

When X is a compact Kähler manifold (see [5], [17]), or X is the complement of a complex hypersurface in the complex projective space (see [11], [17]), we have an isomorphism

$$\hat{\mathfrak{g}}_X\cong\mathscr{L}_X$$

of Lie algebras over $\boldsymbol{Q}$, where $\hat{\mathfrak{g}}_X$ is the nilpotent completion of $\mathfrak{g}_X$.

We shall show that a similar result holds for $F_{0,n}R$.

(1.1) **Theorem.** *Let R be a Riemann surface of genus $g\geqq 1$. Put $X_n=F_{0,n}R$ $(n\geqq 1)$, then we have an isomorphism of Lie algebras over $\boldsymbol{Q}$:*

$$\hat{\mathfrak{g}}_{X_n}\cong\mathscr{L}_{X_n}.$$

Especially this isomorphism induces an isomorphism of graded Lie algebras over $\boldsymbol{Q}$:

$$\mathrm{gr}\,\mathfrak{g}_{X_n}\cong[\mathrm{gr}\,\pi_1(X_n)]\otimes_{\boldsymbol{Z}}\boldsymbol{Q}.$$

Let X be an algebraic variety over $\boldsymbol{C}$, and $\pi_1(X)$ its fundamental group. Let

$$\Gamma_1\pi_1(X)=\pi_1(X)\supset\Gamma_2\pi_1(X)\supset\cdots\supset\Gamma_m\pi_1(X)\supset\cdots$$

be the lower central series of $\pi_1(X)$. Let $\rho(m)$ be the rank of the abelian group $\mathrm{gr}_m(\pi_1(X))=\Gamma_m\pi_1(X)/\Gamma_{m+1}\pi_1(X)$. Let $b_i(X)$ be the i-th Betti number of X.

In [1], Aomoto proposed to find examples of $K(\pi, 1)$ varieties satisfying the identity:

$$\text{(P)} \qquad \prod_{m=1}^{\infty}(1-t^m)^{\rho(m)}=\sum_{i=0}^{2\dim X}(-1)^i b_i(X)t^i$$

as the formal power series in $\boldsymbol{Z}[[t]]$ of variable t with coefficients in $\boldsymbol{Z}$. Several examples of such varieties are known (see [6], [12], [13]).

(1.2) **Theorem.** *Let R be a Riemann surface of genus $g\geqq 1$. Set*

$X_n = F_{0,n}R$ $(n \geqq 1)$. *Then*

(i) $\mathrm{gr}_m(\pi_1(X_n))$ *is a free* $\boldsymbol{Z}$*-module for any* $m \geqq 1$;

and

(ii) *For* $X = X_n$, *the identity* (P) *is true.*

Remark. In this case, the right hand side of (P) is given by

$$(1-2gt+t^2)\prod_{i=0}^{n-2}\{1-(2g+i)t\}.$$

Now let us explain the latter two results. Fix a field K and a prime number l distinct from the characteristic of K. Let X be a smooth algebraic variety defined over K, and $\mathcal{G}$ be the pro-l completion of the algebraic fundamental group $\pi_1(X \times_K \bar{K}, *)$ of $X \times_K \bar{K}$. Then we can consider the "non-abelian" l-adic representation

$$\varphi: \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Aut}(\mathcal{G})/\mathrm{Inn}(\mathcal{G}),$$

where $\mathrm{Aut}(\mathcal{G})$ is the group of bicontinuous automorphisms of $\mathcal{G}$ equipped with compact-open topology. Then $\mathrm{Aut}(\mathcal{G})$ is a profinite group and the normal subgroup $\mathrm{Inn}(\mathcal{G})$ consisting of inner automorphisms is a closed subgroup of $\mathrm{Aut}(\mathcal{G})$ (cf. Ihara [10], Section 1).

Let us consider the *closed higher commutator subgroups* $\{\Gamma_m\mathcal{G}\}_{m\geqq 1}$ defined recursively by

$$\Gamma_1\mathcal{G} = \mathcal{G} \quad \text{and} \quad \Gamma_{m+1}\mathcal{G} = \overline{[\mathcal{G}, \Gamma_m\mathcal{G}]} \qquad (m \geqq 1),$$

where $\overline{}$ means the closure. Since each $\Gamma_m\mathcal{G}$ is fully-invariant under any continuous endomorphism of $\mathcal{G}$, the representation induces an l-adic representation

$$\psi_m: \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Aut}(\mathrm{gr}_m(\mathcal{G}) \otimes_Z \boldsymbol{Q}_l).$$

Here $\mathrm{gr}_m(\mathcal{G}) = \Gamma_m\mathcal{G}/\Gamma_{m+1}\mathcal{G}$ for each $m \geqq 1$. Thus we have a graded Lie algebra over $\boldsymbol{Q}_l$:

$$\mathrm{gr}(\mathcal{G}) \otimes_{Z_l} \boldsymbol{Q}_l = \bigoplus_{m=1}^{\infty} \mathrm{gr}_m(\mathcal{G}) \otimes_{Z_l} \boldsymbol{Q}_l$$

with an action of the Galois group $\mathrm{Gal}(\bar{K}/K)$.

Now let us define the étale analogy of the holonomy Lie algebra. Let $H^i(X \times \bar{K}, \boldsymbol{Q}_l)$ be the i-th l-adic étale cohomology group of $X \times_K \bar{K}$. Let H_i be the dual $\mathrm{Gal}(\bar{K}/K)$-module of $H^i(X \times \bar{K}, \boldsymbol{Q}_l)$, and

$$\eta: H_2 \longrightarrow \wedge^2 H_1$$

be the dual Gal $(\bar{K}/K)$-homomorphism of the cup product

$$\wedge^2 H^1(X\times\bar{K}, \boldsymbol{Q}_l)\longrightarrow H^2(X\times\bar{K}, \boldsymbol{Q}_l).$$

Let $\mathscr{L}(H_1)$ be the free Lie algebra generated by H_1 over $\boldsymbol{Q}_l$. Then $\mathscr{L}(H_1)$ has a natural action of Gal $(\bar{K}/K)$ by universality.

The homogeneous part $\mathscr{L}(H_1)_2$ of degree 2 in $\mathscr{L}(H_1)$ is naturally identified with $\wedge^2 H_1$. The image of η defines a subspace of $\mathscr{L}(H_1)_2$ invariant under Gal $(\bar{K}/K)$. Let J be the ideal in $\mathscr{L}(H_1)$ generated by the image of η. Then we put

$$\mathfrak{g}_{X,l}=\mathscr{L}(H_1)/J,$$

and call it the *étale holonomy Lie algebra.* We define the lower central series $\{\Gamma_m\mathfrak{g}_{X,l}\}_{m\geqq 1}$ in the same way and we denote by $\mathrm{gr}(\mathfrak{g}_{X,l})$ the associated graded Lie algebra over $\boldsymbol{Q}_l$.

Let C be a geometrically connected complete smooth algebraic curve of genus g defined over K. Then we can define an open subvariety $F_{0,n}C$ defined over K in the n-fold product $\prod_{i=1}^n C$ similarly as $F_{0,n}R$ for a Riemann surface R.

(1.3) **Theorem.** *Let C be a geometrically connected complete smooth curve over K. Put $X_n=F_{0,n}C$ $(n\geqq 1)$, and let $\mathscr{G}_n$ be the pro-l completion of the algebraic fundamental group $\pi_1(X_n\times\bar{K}, *)$. Then we have an isomorphism of graded Lie algebras over $\boldsymbol{Q}_l$ with* Gal $(\bar{K}/K)$ *actions*:

$$\mathrm{gr}(\mathfrak{g}_{X_n,l})\cong\mathrm{gr}(\mathscr{G}_n)\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l.$$

Let us formulate an analogy of Aomoto's problem. Recall the l-adic representation

$$\psi_m\colon \mathrm{Gal}(\bar{K}/K)\longrightarrow\mathrm{Aut}(\mathrm{gr}_m(\mathscr{G})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l)$$

defined for an algebraic variety X over K. Then for each element σ of Gal $(\bar{K}/K)$, we can consider the characteristic polynomial of $\psi_m(\sigma)$ with variable t:

$$\det(1-\psi_m(\sigma)t^m).$$

Form an infinite product

$$\prod_{m=1}^{\infty}\det(1-\psi_m(\sigma)t^m)$$

in the formal power series ring $\boldsymbol{Q}_l[[t]]$.

For X an algebraic variety over K, an arithmetic version of the identity (P) may be formulated in the following way:

$$\text{(Q)} \qquad \prod_{m=1}^{\infty} \det(1-\psi_m(\sigma)t^m) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}(\sigma|_{H^i(X\times\bar{K},\mathbf{Q}_l)}^{-1})t^i$$

for $\sigma \in \mathrm{Gal}(\bar{K}/K)$.

(1.4) **Theorem.** *Let C be a geometrically connected smooth complete curve over K of genus $g \geqq 1$. Let $\mathscr{G}_n$ be the pro-l completion of the fundamental group $\pi_1(X_n \times \bar{K}, *)$ of $X_n = F_{0,n}C$. Then*

(i) $\mathrm{gr}_m(\mathscr{G}_n)$ *is a free $\mathbf{Z}_l$-module for any $m \geqq 1$;*

and

(ii) *For $X = X_n$, the identity* (Q) *is true.*

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§ 2. A simple lemma

Let G be a group, F a normal subgroup of G, and $H = G/F$ the quotient group of G by F. Let $[F, F]$ be the commutator subgroup of F. Then we can define an action of H on $F^{ab} = F/[F, F]$:

$$\rho: H \longrightarrow \mathrm{Aut}(F^{ab})$$

by

$$h(f \bmod [F, F]) = gfg^{-1} \bmod [F, F] \qquad (h \in H, f \in F),$$

where g is an element of G such that $h = gF \in G/F$.

(2.1) **Lemma.** *Assume that the action ρ is trivial (i.e. $\rho(H) = \{1\}$) and that F is free group of rank $r \geqq 2$. Let $\{\Gamma_m G\}_{m\geqq 1}$, $\{\Gamma_m F\}_{m\geqq 1}$ and $\{\Gamma_m H\}_{m\geqq 1}$ be the lower central series of G, F and H, respectively. Then we have the following:*

(i) $\Gamma_m F = \Gamma_m G \cap F$ *for any* $m \geqq 1$;

(ii) $\mathrm{gr}_m(G) = \Gamma_m G/\Gamma_{m+1} G$ *is a free abelian group, if* $\mathrm{gr}_m(H) = \Gamma_m H/\Gamma_{m+1} H$ *is a free abelian group*;

(iii) *We have*

$$\mathrm{rank}\ \mathrm{gr}_m(G) = \mathrm{rank}\ \mathrm{gr}_m(F) + \mathrm{rank}\ \mathrm{gr}_m(H) \qquad \textit{for any } m \geqq 1.$$

Proof. It is easy to deduce (ii) and (iii) from (i). In fact, we have an obvious exact sequence of groups:

$$1 \longrightarrow \Gamma_m G \cap F \longrightarrow \Gamma_m G \longrightarrow \Gamma_m H \longrightarrow 1$$

for each $m \geqq 1$. Assume that (1) is true. Then the above exact sequence reads

$$1 \longrightarrow \Gamma_m F \longrightarrow \Gamma_m G \longrightarrow \Gamma_m H \longrightarrow 1 \qquad \text{(for any } m \geqq 1\text{)}.$$

Hence the sequence of modules:

$$1 \longrightarrow \mathrm{gr}_m(F) \longrightarrow \mathrm{gr}_m(G) \longrightarrow \mathrm{gr}_m(H) \longrightarrow 1$$

is exact for any $m \geqq 1$. It is well known that $\mathrm{gr}_m(F)$ is a free abelian group, if F is a free group (cf. [16]). Thus the statement (ii) and (iii) follow immediately from the last sequence.

Let us prove (i). Regard G as an automorphism group of F by conjugation:

$$g(x) = gxg^{-1} \qquad (x \in F,\ g \in G).$$

Then the action of G on F is compatible with the filtration $\{\Gamma_m F\}$ of F, since higher commutators $\Gamma_m F$ are fully invariant subgroups of F. Recall an exercise of Bourbaki [3] on the induced filtration of an automorphism group of a filtered group. For each $x \in F$, set

$$\nu_F(x) = \sup\{m \mid x \in \Gamma_m F\} \qquad \text{(especially } \nu_F(1) = \infty\text{)},$$

and define a function ν on G by

$$\nu(g) = \inf_{x \in F}\{\nu_F(x^{-1}g(x)) - \nu_F(x)\}.$$

(2.2) **Sublemma.** ([3], Chap. 2, § 4, Exercise 9). *For any $m \in \boldsymbol{Z}$, put*

$$G_m = \{g \in G \mid \nu(g) \geqq m\}.$$

Then $\{G_m\}_{m \geqq 1}$ defines a decreasing filtration on G such that $G_0 = G$, and

$$[G_k, G_l] \subset G_{k+l} \qquad \textit{for any } k, l \geqq 0.$$

The proof of the above sublemma is passed to the reader as an exercise.

By assumption that $\rho: H \longrightarrow \mathrm{Aut}(F^{\mathrm{ab}})$ is trivial, the group G also acts trivially on $\mathrm{gr}_1(F)=F^{\mathrm{ab}}$. Since $\mathrm{gr}(F)=\bigoplus_{m=0}^{\infty}\mathrm{gr}_m(F)$ is generated by $\mathrm{gr}_1(F)$ (cf. [16]), G acts trivially on the whole $\mathrm{gr}(F)$. Hence $G_1=G$. Therefore the filtration $\{G_m\}_{m\geqq 1}$ is a central filtration.

Note that the lower central filtration decreases faster than any other central filtration, i.e.

$$\Gamma_m G\subset G_m \qquad \text{for any } m\geqq 1.$$

Then we have

$$\Gamma_m F\subset\Gamma_m G\cap F\subset G_m\cap F \qquad \text{for any } m\geqq 1.$$

Thus, in order to prove (i), it suffices to show that

$$G_m\cap F\subset\Gamma_m F \qquad \text{for any } m\geqq 1.$$

Assume that $x\in G_m\cap F$. Then

$$(\sharp)\qquad \nu_F(y\cdot x(y^{-1}))=\nu_F(yxy^{-1}x^{-1})\geqq m+\nu_F(y), \text{ for any } y\in F.$$

Write $\nu_F(x)=k$ and $\nu_F(y)=l$. Since the rank r of F is at least 2, we can choose y in F such that

$$x \bmod \Gamma_{k+1}F \quad\text{and}\quad y \bmod \Gamma_{l+1}F$$

are linearly independent elements in the graded free Lie algebra $\mathrm{gr}(F)$. Then we have

$$[y,x]=yxy^{-1}x^{-1}\in\Gamma_{k+l}F-\Gamma_{k+1+l}F$$

(cf. Corollary 5.12 (iii) of Section 5.7, p. 342 of [16]). Thus by ($\sharp$), we have

$$k+l\geqq m+l, \qquad \text{i.e. } k\geqq m,$$

which in turn means $x\in\Gamma_k F\subset\Gamma_m F$.

Since x is an arbitrary element in $G_m\cap F$, we have

$$G_m\cap F\subset\Gamma_m F$$

as desired, which completes the proof of our lemma.

§ 3. Proof of Theorem (1.2)

Assume that $n=1$. Then Theorem (1.2) is proved by Labute [15] for $g\geqq 2$, and trivial for $g=1$, since G is abelian in the latter case. Let us show Theorem (1.2) by induction on n.

Consider the projection mapping π: $F_{0,n+1}R=X_{n+1}\rightarrow F_{0,n}R=X_n$ defined by

$$(z_1, \cdots, z_{n+1}) \in F_{0,n+1}R \longmapsto (z_1, \cdots, z_n) \in F_{0,n}R.$$

Then the fibre of π is a punctured Riemann surface

$$R-\{p_1, \cdots, p_n\},$$

and π is a locally trivial fibration (cf. Birman [2], Chapter 1).

Since X_n is a $K(\pi, 1)$ space, and the fibres of π are connected, we have an exact sequence:

$$(3.1) \qquad 1 \longrightarrow \pi_1(R-\{p_1, \cdots, p_n\}) \longrightarrow \pi_1(X_{n+1}) \longrightarrow \pi_1(X_n) \longrightarrow 1.$$

Set $G=\pi_1(X_{n+1})$, $F=\pi_1(R-\{p_1, \cdots, p_n\})$ and $H=\pi_1(X_n)$. Then F is a free group of rank $2g+n-1$, if $n\geqq 1$. Thus in order to apply Lemma (2.1), we want to know the monodromy mapping:

$$\rho\colon H \longrightarrow \mathrm{Aut}\,(F^{\mathrm{ab}})=\mathrm{Aut}\,(H_1(R-\{p_1, \cdots, p_n\}, \boldsymbol{Z}))$$

is trivial. We observe that the exact sequence (3.1) admits a section s: $H\longrightarrow G$ and that G^{ab} is isomorphic to the direct sum $F^{\mathrm{ab}}\oplus H^{\mathrm{ab}}$. The injection $F^{\mathrm{ab}}\rightarrow G^{\mathrm{ab}}$ is compatible with the action of an element of H via s. For $x \in H$, the monodromy $\rho(x)$ can be written in the form

$$\rho(x)\cdot g=s(x)\cdot g\cdot s(x)^{-1}$$

for $g \in F$, which means that H acts trivially on G^{ab}. By using the compatibility it follows that H acts trivially on F^{ab}. Therefore the assumptions of Lemma (2.1) are satisfied for the groups appearing in the sequence (3.1).

Set $\rho_G(m)=\mathrm{rank}\ \mathrm{gr}_m\,(G)$, $\rho_F(m)=\mathrm{rank}\ \mathrm{gr}_m\,(F)$, and $\rho_H(m)=\mathrm{rank}\ \mathrm{gr}_m\,(H)$. Then we have

$$(3.3) \qquad \prod_{m=1}^{\infty}(1-t^m)^{\rho_G(m)}=\prod_{m=1}^{\infty}(1-t^m)^{\rho_F(m)}\prod_{m=1}^{\infty}(1-t^m)^{\rho_H(m)}$$

by (iii) of Lemma (2.1), and $\mathrm{gr}_m(G)$ is a free abelian group if and only if $\mathrm{gr}_m(H)$ is free abelian. Set

$$P(t; X_n)=\sum_{i=1}^{n}(-1)^i b_i(X_n)t^i.$$

Then we have

(3.4) $$P(t; X_{n+1}) = \{1-(2g+n-1)t\}P(t; X_n)$$

for $n \geqq 1$, since the fibration π is locally trivial. By Witt's formula ([16], Chapter 5), we have

(3.5) $$\prod_{m=1}^{\infty} (1-t^m)^{\rho_F(m)} = 1-(2g+n-1)t.$$

Thus Theorem (1.2) is valid for X_{n+1}, if it is valid for X_n. Since the case $n=1$ is settled by Labute [15], we can complete the proof of Theorem (1.2) by induction on n.

(3.6) **Remark.** Fix a prime l. Let F_l, $G_{n+1,l}$, and $G_{n,l}$ be the pro-l completions of the groups $\pi_1(R-\{p_1, \cdots, p_n\})$, $\pi_1(X_{n+1})$, and $\pi_1(X_n)$, respectively. Then we have an exact sequence of groups:

(3.7) $$1 \longrightarrow F_l \longrightarrow G_{n+1,l} \longrightarrow G_{n,l} \longrightarrow 1.$$

In fact, since $G_{n,l}$ acts trivially on $H^1(F, Z/lZ) = \mathrm{Hom}\,(F^{\mathrm{ab}}, Z/lZ)$, this follows immediately from Proposition 4 of Friedlander [7]. Or one can note that the exact sequence:

(3.8) $$\begin{aligned} 1 &\longrightarrow \pi_1(R-\{p_1, \cdots, p_n\})/\Gamma_m \pi_1(R-\{p_1, \cdots p_n\}) \\ &\longrightarrow \pi_1(X_{n+1})/\Gamma_m \pi_1(X_{n+1}) \\ &\longrightarrow \pi_1(X_n)/\Gamma_m \pi_1(X_n) \longrightarrow 1 \qquad \text{(for any } m \geqq 1\text{)}, \end{aligned}$$

which follows from Lemma (2.1). The last exact sequence readily implies (3.7).

§ 4. Proof of Theorem (1.1)

We shall use the same notations as in the previous sections.

(4.1) **Lemma.** *We have an isomorphism:*

$$H^j(\mathfrak{g}_{X_n}; Q) \cong H^j(X_n; Q), \qquad j \geqq 0.$$

Proof. In the case $n=1$, we can show by an elementary computation that the complex $(R(X_1)., \partial.)$ defined in (4.13) is acyclic. Thus we obtain the desired isomorphism in this case. Let $\pi: X_{n+1} \to X_n$ be the natural projection map. The fiber of π is a punctured Riemann surface

$$Z_n = R - \{p_1, \cdots, p_n\}.$$

We have the following exact sequence of holonomy Lie algebras:

(4.2) $$0 \longrightarrow \mathfrak{g}_{Z_n} \longrightarrow \mathfrak{g}_{X_{n+1}} \xrightarrow{\pi_*} \mathfrak{g}_{X_n} \longrightarrow 0$$

which admits a section $\sigma\colon \mathfrak{g}_{X_n} \to \mathfrak{g}_{X_{n+1}}$.

We shall show that the associated Hochschild-Serre spectral sequence

(4.3) $$E_2^{p,q} = H^p(\mathfrak{g}_{X_n}; H^q(\mathfrak{g}_{Z_n}; \boldsymbol{Q})) \Longrightarrow H^{p+q}(\mathfrak{g}_{X_{n+1}}; \boldsymbol{Q})$$

degenerates at the E_2-term. Let us recall that $\mathfrak{g}_{Z_n}$ is a free Lie algebra of rank $2g+n-1$. Hence we have $H^j(\mathfrak{g}_{Z_n}; \boldsymbol{Q})=0$ for $j>1$, therefore we have the following exact sequence:

(4.4) $$\cdots \longrightarrow E_2^{p,0} \xrightarrow{\pi^*} H^p(\mathfrak{g}_{X_{n+1}}; \boldsymbol{Q}) \longrightarrow E_2^{p-1,1} \xrightarrow{d_2} E_2^{p+1,0} \xrightarrow{\pi^*} H^{p+1}(\mathfrak{g}_{X_{n+1}}; \boldsymbol{Q}) \longrightarrow \cdots$$

which admits a section $\sigma^*\colon H^p(\mathfrak{g}_{X_n}; \boldsymbol{Q}) \to E_2^{p,0}$, $p \geqq 0$. Thus we have split short exact sequences:

(4.5) $$0 \longrightarrow E_2^{p,0} \longrightarrow H^p(\mathfrak{g}_{X_{n+1}}; \boldsymbol{Q}) \longrightarrow E_2^{p-1,1} \longrightarrow 0.$$

On the other hand, by using the same argument, we see that the Serre spectral sequence

$$E_2^{p,q} = H^p(X_n; H^q(Z_n; \boldsymbol{Q})) \Longrightarrow H^{p+q}(X_{n+1}; \boldsymbol{Q})$$

associated with the fibration $\pi\colon X_{n+1} \to X_n$ degenerates at the E_2-term, which gives an isomorphism

(4.6) $$H^p(X_{n+1}; \boldsymbol{Q}) \cong H^p(X_n; \boldsymbol{Q}) \oplus [H^{p-1}(X_n; \boldsymbol{Q}) \otimes H^1(Z_n; \boldsymbol{Q})].$$

By means of the hypothesis of the induction and (4.5), (4.6) we obtain the desired isomorphism.

(4.7) *We have*

$$\dim \Gamma_j \mathfrak{g}_{X_n} / \Gamma_{j+1} \mathfrak{g}_{X_n} = \operatorname{rank} \Gamma_j \pi_1(X_n) / \Gamma_{j+1} \pi_1(X_n), \qquad j \geqq 1.$$

Proof. Since the holonomy Lie algebra $\mathfrak{g}_{X_n}$ can be written in the form $\mathfrak{g}_{X_n} = L(H_1(X_n; \boldsymbol{Q}))/J$ with some homogeneous ideal J generated by elements of degree 2, it is clear that we have a free $U(\mathfrak{g}_{X_n})$-resolution of $\boldsymbol{Q}$

(4.8) $$\longrightarrow K_j \xrightarrow{p_j} K_{j-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{p_1} K_0 \xrightarrow{\varepsilon} \boldsymbol{Q} \longrightarrow 0$$

satisfying the following conditions:

(i) $\deg p_j(u) = \deg u + 1$ for any homogeneous element u of K_j.
(ii) K_j is a free $U(\mathfrak{g}_{X_n})$-module of finite rank.

By taking the $U(\mathfrak{g}_{X_n})$-dual sequence of (4.8), we have

$$H^j(\mathfrak{g}_{X_n}; \boldsymbol{Q}) \cong K_j^\vee, \qquad j \geqq 0$$

where $^\vee$ means the $U(\mathfrak{g}_{X_n})$-dual. By means of Lemma 4.2, we obtain an isomorphism

$$H^j(X_n; \boldsymbol{Q}) \cong K_j^\vee, \qquad j \geqq 0.$$

Let K_j^m denote the homogeneous part of degree m of K_j. By considering the exact sequence of $\boldsymbol{Q}$-vector spaces

$$\cdots \longrightarrow K_j^m \xrightarrow{p_j} K_{j-1}^{m+1} \longrightarrow \cdots$$

induced from the sequence (4.8), we obtain the formula

$$\text{(4.9)} \qquad \left(\sum_{p=0}^{\infty} \chi(p) t^p\right)\left(\sum_i (-1)^i b_i(X_n) t^i\right) = 1.$$

where $\chi(p) = \dim U(\mathfrak{g}_{X_n})^p$. We put $\rho(j) = \dim [\Gamma_j \mathfrak{g}_X / \Gamma_{j+1} \mathfrak{g}_X]$. We know by Poincaré-Birkhoff-Witt (see [20]) that

$$\text{(4.10)} \qquad \sum_{p=0}^{\infty} \chi(p) t^p = \prod_{j=1}^{\infty} (1 - t^j)^{-\rho(j)}.$$

It follows from Theorems (1.2), (4.9) and (4.10) that

$$\rho(j) = \operatorname{rank} \Gamma_j \pi_1(X_n) / \Gamma_{j+1} \pi_1(X_n)$$

which completes the proof.

Let X be a polyhedron. By using a theorem of Sullivan ([17] (5.11)) and the definition of the holonomy Lie algebra $\mathfrak{g}_X$ we have a surjective homomorphism

$$\pi_j : \mathfrak{g}_X / \Gamma_j \mathfrak{g}_X \longrightarrow \mathscr{L}_X / \Gamma_j \mathscr{L}_X, \qquad j \geqq 0.$$

In the case $X = X_n$, we have proved that

$$\dim [\Gamma_j \mathfrak{g}_X / \Gamma_{j+1} \mathfrak{g}_X] = \dim [\Gamma_j \mathscr{L}_X / \Gamma_{j+1} \mathscr{L}_X].$$

Hence we obtain that the homomorphism π_j is an isomorphism for any $j \geqq 0$, which completes the proof of Theorem (1.1).

(4.12) **Remark.** The resolution (4.8) may be explicitly given by the complex $(R(X_n)., \partial.)$ defined in (4.13). We can prove it in the same way as in [12]. This resolution will be used in the remark (6.4).

(4.13) **Definition.** Let X be a simplicial complex and let $\mathfrak{g}_X$ be the holonomy Lie algebra of X. We define the complex $(R(X)., \partial.)$ in the following way (cf. [1], [12]). We denote by $U(\mathfrak{g}_X)$ the universal enveloping algebra of $\mathfrak{g}_X$ equipped with the structure of a graded algebra such that $\deg x=1$ for an element $x \in H_1(X; \boldsymbol{Q})$. We put

$$R(X)_k = \mathrm{Hom}_{U(\mathfrak{g}_X)}(U(\mathfrak{g}_X) \otimes H^k(X; \boldsymbol{Q}), U(\mathfrak{g}_X))$$

and we define $\partial_k: R(X)_k \to R(X)_{k-1}$ to be the right $U(\mathfrak{g}_X)$-module homomorphism defined by

$$\partial_k \lambda(1 \otimes \varphi) = \lambda(\sum_j x_j \otimes \omega^j \cup \varphi)$$

for $\lambda \in R(X)_k$ and $\varphi \in H^k(X; \boldsymbol{Q})$, where $\{\omega^j\}$ denotes a basis of $H^1(X; \boldsymbol{Q})$ and $\{x_j\}$ denotes its dual basis of $H_1(X; \boldsymbol{Q})$.

§ 5. Proof of Theorem (1.3)

In the first place, we show that the statement of Theorem (1.3) is also true for arbitrary curves.

(5.1) **Proposition.** *Let X be a smooth geometrically connected algebraic curve defined over K. Let $\mathscr{G}$ be the pro-l completion of the fundamental group $\pi_1(X \times \bar{K}, *)$ of X with $l \neq \mathrm{ch}(K)$. Let $\mathfrak{g}_{X,l}$ be the étale holonomy Lie algebra of X. Then we have an isomorphism of graded Lie algebras over $\boldsymbol{Q}_l$ with action of* $\mathrm{Gal}(\bar{K}/K)$:

$$\mathrm{gr}(\mathfrak{g}_{X,l}) \cong \mathrm{gr}(\mathscr{G}) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l.$$

Proof. Let n be the cardinality of $\tilde{X}-X$, where $\tilde{X}$ is the smooth compactification of X, and let g be the genus of $\tilde{X}$. Assume that the characteristic of K is 0. Then by a standard argument, we see that it suffices to show our proposition when K is a subfield of $\boldsymbol{C}$.

Since $\mathrm{gr}_1(\mathfrak{g}_{X,l}) \cong H^1_{\text{ét}}(X \times \bar{K}, \boldsymbol{Q}_l)^\vee$ as $\mathrm{Gal}(\bar{K}/K)$-modules, and $\mathrm{gr}_1(\mathscr{G}) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l$ is also isomorphic to the dual $\mathrm{Gal}(\bar{K}/K)$-module of $H^1_{\text{ét}}(X \times \bar{K}, \boldsymbol{Q}_l)$, we have a natural homomorphism of graded Lie algebras with $\mathrm{Gal}(\bar{K}/K)$-actions

$$\theta: \mathrm{gr}(\mathscr{L}(H^1_{\text{ét}}(X \times \bar{K}, \boldsymbol{Q}_l)^\vee)) \longrightarrow \mathrm{gr}(\mathscr{G}) \otimes_{\boldsymbol{Z}_l} \boldsymbol{Q}_l.$$

By applying the comparison theorem of Artin [9], we can see that the image of

$$\eta\colon H^2_{\text{ét}}(X\times K, \boldsymbol{Q}_l)^\vee \longrightarrow \wedge^2 \{H^1_{\text{ét}}(X\times \bar{K}, \boldsymbol{Q}_l)^\vee\}$$

is contained in the kernel of θ by Theorem (1.1). Therefore the homomorphism θ factors through

$$\operatorname{gr}(\mathfrak{g}_{X,l}) \longrightarrow \operatorname{gr}(\mathscr{G})\otimes_{Z_l}\boldsymbol{Q}_l.$$

Applying again the comparison theorem, we have the isomorphism of Proposition by Theorem (1.1).

When the characteristic of K is positive, we can lift the quasi-projective curve X to a smooth curve $\tilde{X}$ defined over a field $\tilde{K}$ of characteristic 0. Then $\mathscr{G}$ and the pro-l completion of $\pi_1(\tilde{X}\times\bar{K}, *)$ are isomorphic by Grothendieck [8] or by Popp [18], which completes the proof of our proposition.

The proof of Theorem (1.3) is similar to the proof of Proposition (5.1). Once one can establish an isomorphism of Theorem (1.3) as graded Lie algebras such that it induces the tautological isomorphism

$$\operatorname{gr}_1(\mathfrak{g}_{X,l}) = H^1_{\text{ét}}(X\times\bar{K}, \boldsymbol{Q}_l) \cong \operatorname{gr}_1(\mathscr{G})\otimes_{Z_l}\boldsymbol{Q}_l$$

at degree 1, then the universality shows that this isomorphism of graded Lie algebras is compatible with the action of $\operatorname{Gal}(\bar{K}/K)$.

Thus, the comparison theorem of Artin [9] shows that Theorem (1.3) follows immediately from Theorem (1.1), when K is of characteristic 0.

When K is of positive characteristic, we can lift the curve C to a smooth projective curve $\tilde{C}$ over a field $\tilde{K}$ of characteristic 0. Then we can define $\tilde{X}_n = F_{0,n}\tilde{C}$, which is a lifting of $F_{0,n}C$ to characteristic 0. Then, applying the argument of specialization to the fibring

$$\tilde{V} \longrightarrow \tilde{X}_{n+1} \xrightarrow{\tilde{\pi}} \tilde{X}_n,$$

where $\tilde{V}$ is a geometric fibre of $\tilde{\pi}$, we can show a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{G}_{\tilde{V}} & \longrightarrow & \tilde{\mathscr{G}}_{n+1} & \longrightarrow & \tilde{\mathscr{G}}_n & \longrightarrow & 0 \qquad \text{(exact)} \\ & & \wr\| & & \wr\| & & \wr\| & & \\ 0 & \longrightarrow & \mathscr{G}_V & \longrightarrow & \mathscr{G}_{n+1} & \longrightarrow & \mathscr{G}_n & \longrightarrow & 0 \qquad \text{(exact)} \end{array}$$

by induction on n (cf. Remark (3.6)). Here $\mathscr{G}_V$ (resp. $\mathscr{G}_{\tilde{V}}$) is the pro-l completion of the fundamental group of the geometric fibre V (resp. $\tilde{V}$) of $\pi\colon X_{n+1}\to X_n$ (resp. $\tilde{\pi}$), and $\mathscr{G}_n$ (resp. $\tilde{\mathscr{G}}_n$) is the pro-l completion of $\pi_1(X_n\times\bar{K}, *)$ (resp. $\pi_1(\tilde{X}_n\times\tilde{K}, *)$). Hence we complete the proof of Theorem (1.3).

§ 6. Proof of Theorem (1.4)

We discuss that the statement of Theorem (1.4) is valid for arbitrary curves. This result includes the case $n=1$ in Theorem (1.4) as a special case, since $X_1=C$.

(6.1) **Proposition.** *Let X be a smooth geometrically connected algebraic curve over K with $X\neq \boldsymbol{P}^1_K$, and let $\mathscr{G}$ be the pro-l completion of the fundamental group $\pi_1(X\times \bar{K}, *)$ with $l\neq \mathrm{ch}(K)$. Then we have*

(i) $\mathrm{gr}_m(\mathscr{G})$ *is a free $\boldsymbol{Z}_l$-module for any $m\geqq 1$;*

and

(ii) $\prod_{m=1}^{\infty} \det(1-\psi_m(\sigma)t^m)=\sum_{i=0}^{2}(-1)^i \operatorname{tr}(\sigma^{-1}|_{H^i_{\text{ét}}(X\times\bar{K},\boldsymbol{Q}_l)})t^i$.

Proof. The part (i) follows from a result of Labute [14] by the comparison theorem of Artin, if the characteristic of K is 0. If $\mathrm{ch}(K)>0$, then we can lift X to characteristic 0, and apply the result of [13] or [15].

Let us show the part (ii). By Theorem (1.3), $\psi_m(\sigma)$ are semisimple elements of $GL(\mathrm{gr}_m(\mathscr{G})\otimes_{Z_l}\boldsymbol{Q}_l)$ for all $m\geqq 1$, if $\psi_1(\sigma)$ is semisimple. Since each characteristic polynomial $\det(1-\psi_m(\sigma)t^m)$ depends only on the semisimple part of $\psi_m(\sigma)$ in the Jordan decomposition of $\psi_m(\sigma)$, it suffices to show (ii), when $\psi_1(\sigma)$ is semisimple.

Also we may extend the field of base from $\boldsymbol{Q}_l$ to the algebraic closure $\bar{\boldsymbol{Q}}_l$. Thus we can assume that $\psi_1(\sigma)$ is represented by a diagonal matrix with respect to some basis of $\mathrm{gr}_1(\mathscr{G})\otimes_{Z_l}\bar{\boldsymbol{Q}}_l$.

Let g be the genus of the smooth compactification $\tilde{X}$ of X over K. Put $n=$the cardinality of $\tilde{X}-X$. If $g=0$ and $n\leqq 2$ or $g=1$ and $n=0$, then $\mathscr{G}$ is abelian. There is nothing to prove in this case. Assume that $n>0$ and $\mathscr{G}$ is not abelian. Then $\mathscr{G}$ is a pro-l free group, hence $\mathrm{gr}(\mathscr{G})$ is a free Lie algebra over $\boldsymbol{Z}_l$ generated by $\mathrm{gr}_1(\mathscr{G})$. Thus, when X is noncomplete, our proposition is reduced to show the following.

(6.2) **Lemma** (cf. [3], Chap. 2). *Let $\{X_1, X_2, \cdots, X_q\}$ be a set of q letters, and let L be the free Lie algebra generated by $\{X_1, \cdots, X_q\}$ over a field F. Let $\{S_1, \cdots, S_q\}$ be q elements of $F-\{0\}$, and let $\tau_m(S_1, S_2, \cdots, S_q)$ be the linear mapping $L_m\to L_m$ on the homogeneous part L_m of degree m of L, induced by the mapping*

$$\sum_{i=1}^{q} a_i X_i \longmapsto \sum_{i=1}^{q} a_i S_i X_i \qquad (a_i \in F\ (1\leqq i\leqq q))$$

on L_1. Then we have

$$\prod_{m=1}^{\infty} \det\{1-\tau_m(S_1, \cdots, S_q)t^m\}=1-\left(\sum_{i=1}^{q} S_i\right)t.$$

When X is a complete curve, it is necessary to see the results of Labute [14] more precisely. We have to formulate an analogy of Lemma (6.2) in this case. Note that $\mathscr{G}$ is a Demušhin group of genus g, and the intersection form on $H^1(X\times\bar{K}, \boldsymbol{Q}_l)$ defines a skew symmetric bilinear form

$$\mathrm{gr}_1(\mathscr{G})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l\times \mathrm{gr}_1(\mathscr{G})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l\longrightarrow\boldsymbol{Q}_l(1).$$

Here $\boldsymbol{Q}_l(1)$ is the module $\boldsymbol{Q}_l$ with $\mathrm{Gal}(\bar{K}/K)$ actions via the cyclotomic character. Recall the structure of $\mathrm{gr}(\mathscr{G})$ (cf. [14], [16]). In order to complete the proof of Proposition (6.1), it suffices to show the following.

(6.3) **Lemma.** *Let* $\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$ *be* $2g$ *letters, and let* L *be a free Lie algebra generated by* $\{X_1, \cdots, X_g, Y_1, \cdots, Y_g\}$ *over* F. *Let* J *be the ideal of* L *generated by* $\sum_{i=1}^{g}[X_i, Y_i]$, *and* $\mathfrak{g}=L/J$ *be the quotient graded Lie algebra. Let*

$$\tau_m(S_1, \cdots, S_g)\colon \mathfrak{g}_m\longrightarrow\mathfrak{g}_m$$

be the linear mapping on the homogeneous part $\mathfrak{g}_m$ *of* $U(\mathfrak{g})$ *with degree* m, *induced from the mapping on the set*

$$X_i\longmapsto S_iX_i \quad \text{and} \quad Y_i\longmapsto S_i^{-1}Y_i \qquad (1\leqq i\leqq g).$$

Then we have an identity

$$\prod_{m=1}^{\infty}\det\{1-\tau_m(S_1, \cdots, S_g)t^m\}=1-\left\{\sum_{i=1}^{g}(S_i+S_i^{-1})\right\}t+t^2.$$

Proof. When $S_1=S_2=\cdots=S_g=1$, the identity of Lemma is the last formula in Théorème 2 of Labute [14]. Actually, we can see readily that Théorème 2 of [14] implies immediately our Lemma. We give some multi-degree for the generators. Consider the free module $\bigoplus_{i=0}^{g}\boldsymbol{Z}u_i$ generated by $g+1$ letters u_i $(0\leqq i\leqq g)$. Then we define that the "degree" of X_i is u_0+u_i for each i $(1\leqq i\leqq g)$, and the "degree" of Y_i is u_0-u_i. Then the relator $\sum_{i=1}^{g}[X_i, Y_i]$ has the degree $2u_0$. We extend this multi-degree so that for any element A in $\mathfrak{g}_m$, the equivalence:

$$\tau_m(S_1, \cdots, S_g)\cdot A=S_1^{m_1}\cdots S_g^{m_g}\cdot A$$
$$\Longleftrightarrow A \text{ is of multi-degree } mu_0+\textstyle\sum_{i=1}^{g}m_iu_i, \text{ holds.}$$

Then the formula (1)

$$U(t)=\frac{V(t)}{1+t^dV(t)}$$

in Théorème 2 of [14] reads (after the correction of sign before $t^d V(t)$, which is an obvious mistake)

$$\prod_{m=1}^{\infty} \det\{1-\tau_m(S_1, \cdots, S_g)t^m\}^{-1}=\frac{V(t)}{1+t^2V(t)},$$

with $V(t)=[1-\{\sum_{i=1}^{g}(S_i+S_i^{-1})t\}]^{-1}$, in our setting. This completes the proof of our Lemma.

Now let us start the proof of Theorem (1.4). We have settled the case X_1. Let us proceed by induction on n. Consider the map $\pi: X_{n+1}\to X_n$. Let η be the generic point of X_n, and $\bar{\eta}$ the geometric point over η. Let $V_{\bar{\eta}}$ be the geometric fibre of π over $\bar{\eta}$. Then $V_{\bar{\eta}}$ is a curve which is an open subscheme of $C\times_K k(\bar{\eta})$. Let $\mathcal{F}$ be the pro-l completion of $\pi_1(V_{\bar{\eta}}, *)$. Then $\mathrm{Gal}(k(\bar{\eta})/k(\eta))$ acts on $\mathrm{gr}(\mathcal{F})=\bigoplus_{m=1}^{\infty}\mathrm{gr}_m(\mathcal{F})$.

Let Γ be the subgroup

$$\mathrm{Gal}(\overline{K(X_n)}/(\bar{K}(X_n))$$

of

$$\mathrm{Gal}(k(\bar{\eta})/k(\eta))=\mathrm{Gal}(\overline{K(X_n)}/K(X_n)).$$

Here $K(X_n)$ (resp. $\bar{K}(X_n)$) is the function field of X_n over K (resp. $\bar{K}$). Then the composition:

$$\Gamma\longrightarrow \mathrm{Gal}(k(\bar{\eta})/k(\eta))\longrightarrow \mathrm{Aut}(\mathrm{gr}(\mathcal{F}))$$

factors through $\pi_1(X_n\times\bar{K}, *)$, because π is a smooth morphism. (Grothendieck's specialisation theorem). By an argument similar to that of the proof of Theorem (1.2) (cf. Section 3), we can show that $\pi_1(X_n\times\bar{K}, *)$ acts trivially on $\mathrm{gr}(\mathcal{F})$. *A fortiori*, Γ acts trivially on $\mathrm{gr}(\mathcal{F})$. Therefore $\mathrm{Gal}(k(\bar{\eta})/k(\eta))/\Gamma\cong\mathrm{Gal}(\bar{K}(X_n)/K(X_n))$ acts on $\mathrm{gr}(\mathcal{F})$. Since X_n is geometrically connected, i.e. $K(X_n)$ is a regular extension of K,

$$\mathrm{Gal}(\bar{K}(X_n)/K(X_n))\cong\mathrm{Gal}(\bar{K}/K).$$

Hence $\mathrm{Gal}(\bar{K}/K)$ acts on $\mathrm{gr}(\mathcal{F})$.

The fibration

$$V_{\bar{\eta}}\longrightarrow X_{n+1}\xrightarrow{\ \pi\ } X_n$$

induces an exact sequence

$$1\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}_{n+1}\longrightarrow\mathcal{G}_n\longrightarrow 1$$

as we have already seen in Section 5. If we put $F=\mathscr{F}$, $G=\mathscr{G}_{n+1}$ and $H=\mathscr{G}_n$, then the three groups also satisfy the assumption of Lemma (2.1). Thus we have an exact sequence

$$0\longrightarrow \mathrm{gr}_m(\mathscr{F})\longrightarrow \mathrm{gr}_m(\mathscr{G}_{n+1})\longrightarrow \mathrm{gr}_m(\mathscr{G}_n)\longrightarrow 0,$$

which is easily checked to be an exact sequence of $\mathrm{Gal}(\bar{K}/K)$-modules.

Recall that $H^1_{\text{ét}}(V_{\bar{\eta}}, \boldsymbol{Q}_l)$ is a $\mathrm{Gal}(\bar{K}/K)$-module which is dual to $\mathrm{gr}_1(\mathscr{F})\otimes_{\boldsymbol{Z}_l}\boldsymbol{Q}_l$, i.e. $\pi_1(X_n\times\bar{K}, *)$ acts trivially on it. In other words, the first direct image $R^1\pi_*\boldsymbol{Q}_l$ of $\boldsymbol{Q}_l$ with respect to $\pi\colon X_{n+1}\to X_n$ is a constant local system. Thus, by the spectral sequence of Leray, we have an isomorphism of $\mathrm{Gal}(\bar{K}/K)$-modules

$$H^i(X_{n+1}\times\bar{K}, \boldsymbol{Q}_l)\cong H^i(X_n\times\bar{K}, \boldsymbol{Q}_l)\oplus\{H^{i-1}(X_n\times\bar{K}, \boldsymbol{Q}_l)\otimes H^1(V_{\bar{\eta}}, \boldsymbol{Q}_l)\}.$$

(we omit subscprit "ét" from now on). Therefore

$$\sum_{i=0}^{2n+1}(-1)^i\,\mathrm{tr}\,(\sigma^{-1}|_{H^i(X_{n+1}\times\bar{K},\boldsymbol{Q}_l)})t^i$$
$$=(1-\mathrm{tr}\,(\sigma^{-1}|_{H^1(V_{\bar{\eta}},\boldsymbol{Q}_l)})t)\left\{\sum_{i=0}^{2n}(-1)^i\,\mathrm{tr}\,(\sigma^{-1}|_{H^i(X_n\times\bar{K},\boldsymbol{Q}_l)})t^i\right\}$$

for any $\sigma\in\mathrm{Gal}(\bar{K}/K)$. Thus in order to complete the proof of Theorem (1.4) by induction on n, it suffices to show that for $\{\psi_{m,\mathscr{F}}(\sigma)\}$ with respect to $\mathscr{F}$, the identity

$$\prod_{i=1}^{\infty}\det(1-\psi_{m,\mathscr{F}}(\sigma)t^m)=1-\mathrm{tr}\,(\sigma^{-1}|_{H^1(V_{\bar{\eta}},\boldsymbol{Q}_l)})t$$

is valid. Since the last identity is shown by Proposition (6.1), we can complete the proof of Theorem (1.4).

(6.4) **Remark.** We want to discuss an extension of Theorem (1.4) for automorphisms σ of $X=F_{0,n}C$. Under the same notation and assumption as in Theorem (1.1), we consider the universal enveloping algebra $U(\mathfrak{g}_{X,l})$ of the étale holonomy Lie algebra $\mathfrak{g}_{X,l}$ of X. Any automorphism $\sigma\colon X\times\bar{K}\to X\times\bar{K}$ of $X\times\bar{K}$ induces an automorphism $H^1(\sigma)$ of $H^1(X\times\bar{K}, \boldsymbol{Q}_l)$. Since $H^1(\sigma)$ is compatible with the cup product, $H^1(\sigma)$ induces an automorphism $U(\sigma)$ of $U(\mathfrak{g}_{X,l})$ by the universality of $U(\mathfrak{g}_{X,l})$. Similarly $\mathrm{Gal}(\bar{K}/K)$ acts on $U(\mathfrak{g}_{X,l})$.

Put

$$K_l^i=\mathrm{Hom}_{U(\mathfrak{g}_{X,l})}(U(\mathfrak{g}_{X,l})\otimes_{\boldsymbol{Q}_l}H^1(X\times\bar{K}, \boldsymbol{Q}_l), U(\mathfrak{g}_{X,l})).$$

Then we can define a derivation $\delta\colon K^i\to K^{i-1}$ as in Section (4.13). Then δ

is compatible with the action of the Galois group $\mathrm{Gal}(\bar{K}/K)$ and with any automorphism σ of $X\times\bar{K}$. Moreover, if σ is defined over K, the actions of $\mathrm{Gal}(\bar{K}/K)$ and σ on the complex $\{K_l^i\}$ commute. By the comparison theorem of Artin and the specialization theorem, the complex:

$$\longrightarrow K_l^i \longrightarrow K_l^{i-1} \longrightarrow \cdots \longrightarrow \mathbf{Q}_l \longrightarrow 0$$

is also exact.

Let $\psi_n(\sigma)$ be the action of $\sigma \in \mathrm{Aut}(X)$ on the homogeneous component of degree n in $\mathfrak{g}_{X,l}$. Let us consider the induced graduation on $U(\mathfrak{g}_{X,l})$, and let $\varphi_\rho(\sigma)$ be the restriction of $U(\sigma)$ to the homogeneous part of degree ρ in $U(\mathfrak{g}_{X,l})$. Then the above complex implies an equality:

$$(\sharp) \qquad \left(\sum_{\rho=0}^{\infty} \mathrm{tr}\, \varphi_\rho(\sigma) t^\rho\right)\left(\sum_{i=0}^{2\dim X} \mathrm{tr}\,(\sigma^{-1}|_{H^i(X\times\bar{K},\mathbf{Q}_l)}) t^i\right) = 1.$$

On the other hand, Poincaré-Birkhoff-Witt theorem implies that

$$(\sharp\sharp) \qquad \prod_{n=1}^{\infty} \det(1-\psi_n(\sigma)t^n)^{-1} = \sum_{\rho=0}^{\infty} \mathrm{tr}\, \varphi_\rho(\sigma) t^\rho.$$

In fact, in order to prove the above identity, after extending scalars from $\mathbf{Q}_l$ to $\bar{\mathbf{Q}}_l$, it suffices to show the identity for semisimple automorphisms $H^1(\sigma)$ of $H^1(X\times\bar{K}, \mathbf{Q}_l)\otimes\bar{\mathbf{Q}}_l$. Then we can choose a basis in each homogeneous part of $\mathfrak{g}_{X,l}$, such that $\psi_n(\sigma)$ is represented by a diagonal matrix.

Thus the last two identities show that in Theorem (1.4) we can replace an element σ of $\mathrm{Gal}(\bar{K}/K)$ by an automorphism σ of $X\times\bar{K}$.

(6.5) **Remark.** Let $\mathbf{Z}[G]$ be the group algebra of a group G over $\mathbf{Z}$, and let I be the augmentation ideal of $\mathbf{Z}[G]$, i.e. the kernel of the augmentation map $\mathbf{Z}[G]\to\mathbf{Z}$. Then we can define a filtration F_rG on G by putting $F_rG=\{x \in G \mid x-1 \in I^r\}$. These subgroups F_rG are called dimension subgroups of G. Let $G=\pi_1F_{0,n}R$ for a Riemann surface of genus$\geqq 1$. Then, combining Theorem (1.2) (i) with a result of Quillen (cf. Corollary 4.2 of [19]), we have an equality

$$\Gamma_rG=F_rG$$

for any r.

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