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Some Problems on Three Point Ramifications and Associated Large Galois Representations

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Dedicated to Professor Ichiro Satake for his 60th birthday

Introduction

Let $\boldsymbol{Q}$ be the rational number field, $\bar{\boldsymbol{Q}}$ be its algebraic closure, and l be a fixed prime number. Then the absolute Galois group $G_{\boldsymbol{Q}}=\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ admits a canonical representation

$$\varphi=\varphi_{\boldsymbol{Q}}: G_{\boldsymbol{Q}}\longrightarrow \mathrm{Out}\, \pi_1^{\text{pro-}l}(\boldsymbol{P}^1_{\bar{\boldsymbol{Q}}}\backslash\{0, 1, \infty\}),$$

in the outer automorphism group of the pro-l fundamental group of the punctured projective line, which arises from the exact sequence

$$1\longrightarrow \pi_1^{\text{pro-}l}(\boldsymbol{P}^1_{\bar{\boldsymbol{Q}}}\backslash\{0, 1, \infty\})\longrightarrow \pi_1^{\text{pro-}l}(\boldsymbol{P}^1_{\boldsymbol{Q}}\backslash\{0, 1, \infty\})\longrightarrow G_{\boldsymbol{Q}}\longrightarrow 1.$$

Recently, several authors started (perhaps more or less independently) to work on this type of "large Galois representations"; Belyĭ [3], Grothendieck [7], Deligne [5], [6], the author [9], [10], etc. In this report, we pose and discuss various basic open problems related to this represetation φ and its natural "subrepresentations" ψ.

§ 1. The Galois representation φ

(1–1) First, let us repeat the definition of the Galois representation $\varphi_{\boldsymbol{Q}}$ more precisely in terms of function fields. Let M be the maximum pro-l extension of the rational function field $K=\bar{\boldsymbol{Q}}(t)$ unramified outside $t=0, 1, \infty$. Then $M/\boldsymbol{Q}(t)$ is also a Galois extension. So, identifying the two Galois groups $\mathrm{Gal}(K/\boldsymbol{Q}(t))$ and $G_{\boldsymbol{Q}}=\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ in the obvious way, we obtain an exact sequence of Galois groups

$$1\longrightarrow \mathrm{Gal}(M/K)\longrightarrow \mathrm{Gal}(M/\boldsymbol{Q}(t))\longrightarrow G_{\boldsymbol{Q}}\longrightarrow 1.$$

Put $\mathfrak{F}=\mathrm{Gal}(M/K)$ and $\tilde{\mathfrak{F}}=\mathrm{Gal}(M/\boldsymbol{Q}(t))$. Then the composite of three canonical homomorphisms

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$$\tilde{\mathfrak{F}} \longrightarrow \mathrm{Int}\,\tilde{\mathfrak{F}} \xrightarrow{\mathrm{Res}} \mathrm{Aut}\,\mathfrak{F} \longrightarrow \mathrm{Out}\,\mathfrak{F}$$

factors through G_Q and defines the homomorphism

$$\varphi_Q\colon G_Q \longrightarrow \mathrm{Out}\,\mathfrak{F}; \qquad \mathfrak{F} = \mathrm{Gal}\,(M/K).$$

Here, for any topological group X, Aut X, Int X and Out X=Aut X/Int X denote the groups of automorphisms, inner automorphisms and outer automorphisms of X, respectively. The canonical homomorphism $X \to \mathrm{Int}\,X$ is defined by $x \to \mathrm{Int}\,x$, where $(\mathrm{Int}\,x)y = xyx^{-1}$ $(x, y \in X)$, and Res: $\mathrm{Int}\,\tilde{\mathfrak{F}} \to \mathrm{Aut}\,\mathfrak{F}$ is the restriction homomorphism.

(1–2) This definition, starting from $\boldsymbol{P}^1 \backslash \{0, 1, \infty\}$ over $\boldsymbol{Q}$, can of course be generalized to the case of an arbitrary scheme over any field. But here, we want to look closely at this special case which is *rigid* with respect to deformations and which gives a *canonical* representation of G_Q determined only by l. While the ordinary linear representation of the Galois group is a representation in the automorphism group of a *vector space*, our representation is in the (outer) automorphism group of the Galois group $\mathfrak{F} = \mathrm{Gal}\,(M/K)$ which is isomorphic to the *free pro-l group of rank* 2. (In the general case, what corresponds to Gal (M/K) is the geometric part of the pro-l fundamental group of the given scheme, which, except for the case of curves, is usually either difficult to determine or too small.)

We may also replace "pro-l" by either "almost pro-l", or "profinite". Namely:

(i) *The almost pro-l case.* Choose any *finite Galois* extension K'/K unramified outside $t=0, 1, \infty$, and define M to be the maximum pro-l extension of K' unramified outside $t=0, 1, \infty$. Then Gal (M/K) is a *free almost pro-l group* of rank 2 in the sense of [10], i.e., the completion of the abstract free group F of rank 2 with respect to the pro-l topology of some normal subgroup $F' \subset F$ of finite index. If an intermediate field $\boldsymbol{Q}^*$ $(\boldsymbol{Q} \subset \boldsymbol{Q}^* \subset \bar{\boldsymbol{Q}})$ is such that $K'/\boldsymbol{Q}^*(t)$ is a Galois extension, we obtain a representation of G_{Q^*} in Out (Gal (M/K)).

(ii) *The profinite case.* Take M to be the maximum Galois extension of K unramified outside 0, 1, ∞. Then Gal (M/K) is the free profinite group of rank 2, and we obtain a canonical representation of G_Q in Out (Gal (M/K)).

These two cases are equally important as the pro-l case, but we shall mainly restrict our attention to the pro-l case and occasionally give remarks related to other cases.

As a final remark here, we note that one may also replace the base field $\boldsymbol{Q}$ by any other perfect field k with characteristic $\neq l$ (without changing the structure of Gal(M/K)). But the only basic cases are $k=\boldsymbol{Q}$ and $k=\boldsymbol{F}_p=\boldsymbol{Z}/p$ $(p\neq l)$. The representation φ_k for other cases can be obtained from $\varphi_{\boldsymbol{Q}}$ or $\varphi_{\boldsymbol{F}_p}$ by restriction. (Even then, the study of $\varphi_{\boldsymbol{Q}_l}$ ($\boldsymbol{Q}_l$: the l-adic number field) is of an independent interest.) We shall mainly consider the case over $\boldsymbol{Q}$, and abbreviate as $\varphi=\varphi_{\boldsymbol{Q}}$.

(1–3) Two basic problems are:

(P1) What is the kernel of φ?

(P2) What is the image of φ?

(1–4) *About* (P1). (i) Let k_φ denote the Galois extension over $\boldsymbol{Q}$ corresponding to the kernel of φ. Then k_φ has the following interpretation. For any intermediate field k of $\bar{\boldsymbol{Q}}/\boldsymbol{Q}$, a *k-model* of M will mean any intermediate field M_k of $M/k(t)$ such that $M_k\cdot\bar{\boldsymbol{Q}}=M$ and $M_k\cap\bar{\boldsymbol{Q}}=k$. It will be called a *Galois* k-model if moreover $M_k/k(t)$ is a Galois extension. Now, in each of the pro-l, almost pro-l and the profinite case, the group Gal(M/K) has trivial center. (In fact, a free pro-l (resp. almost pro-l, profinite) group of finite rank>1 has trivial center.) From this follows immediately that *k_φ is the smallest algebraic extension of $\boldsymbol{Q}$ for which M has a Galois k_φ-model.*

Incidentally, as for non-Galois models of M, there is a convenient $\boldsymbol{Q}$-model $M_{\boldsymbol{Q}}$ used by Beylĭ [3] (also by Deligne [6] and the author [9]) cf. [9] I § 4.

(ii) In the profinite case, Belyĭ [3] proved that φ is *injective*. He proved this by showing that every algebraic curve defined over an algebraic number field can be realized as a finite covering of $\boldsymbol{P}^1$ unramified outside 0, 1, ∞. In particular, an elliptic curve with any given absolute invariant $j\in\bar{\boldsymbol{Q}}$ is so, and this leads to that $k_\varphi=\bar{\boldsymbol{Q}}$.

(iii) In the pro-l case, k_φ cannot be as large as $\bar{\boldsymbol{Q}}$, because *k_φ must be a pro-l extension of the cyclotomic field $\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$ unramified outside l* ([5], [9] § I). Thus, one may ask:

(P1′) Is k_φ in the pro-l case the maximum pro-l extension of $\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$ unramified outside l?

Our present knowledge is so narrow, and we cannot even put it as a conjecture.

A closely related geometric question is this:

(P3) Which curve over $\bar{\boldsymbol{Q}}$ (or $\bar{\boldsymbol{Q}}_l$) can be realized as an l-covering of $\boldsymbol{P}^1$ unramified outside 0, 1, ∞?

Here, an *l-covering* means a finite covering such that the degree of *its Galois closure* is a *power of l.*

We know that such curves have good reduction outside l [9] § I. As for the special fiber above l of the *integral closure of $\boldsymbol{P}^1_{\boldsymbol{Z}_l}$* in such a covering of $\boldsymbol{P}^1_{\bar{\boldsymbol{Q}}_l}$, what we know at present is the following elementary

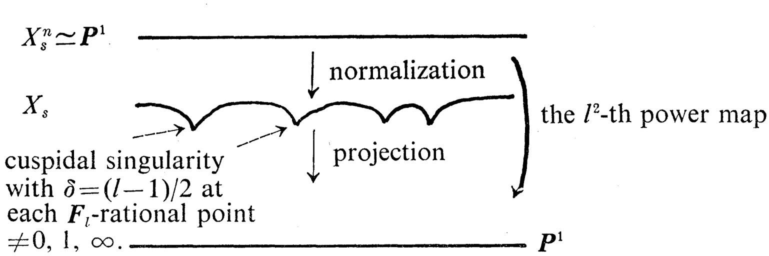
Theorem 1. *Let* $\boldsymbol{P}^1_{\boldsymbol{Z}_l}=\operatorname{Spec}\boldsymbol{Z}_l[t]\cup\operatorname{Spec}\boldsymbol{Z}_l[t^{-1}]$ *be the projective t-line over* $\boldsymbol{Z}_l$ *with the geometric general fiber* $\boldsymbol{P}^1_{\bar{\boldsymbol{Q}}_l}$. *Let* $X_{\bar{\eta}}/\boldsymbol{P}^1_{\bar{\boldsymbol{Q}}_l}$ *be an l-covering of degree* l^n $(n\geqslant 1)$ *unramified outside* $t=0, 1, \infty$, *and X be the integral closure of* $\boldsymbol{P}^1_{\boldsymbol{Z}_l}$ *in* $X_{\bar{\eta}}$. *Then the special fiber* X_s *of X is an integral scheme, and its normalization* X_s^n, *considered as a* $\boldsymbol{P}^1_{\bar{\boldsymbol{F}}_l}$*-scheme via the projection* $X_s\to\boldsymbol{P}^1_{\bar{\boldsymbol{F}}_l}$, *is isomorphic to*

$$\operatorname{Spec}\bar{\boldsymbol{F}}_l[t^{1/l^n}]\cup\operatorname{Spec}\bar{\boldsymbol{F}}_l[t^{-1/l^n}],$$

i.e., the unique purely inseparable covering $\boldsymbol{P}^1_{\bar{\boldsymbol{F}}_l}\to\boldsymbol{P}^1_{\bar{\boldsymbol{F}}_l}$ *of degree* l^n.

For example, if $X_{\bar{\eta}}$ is the Fermat covering of level l corresponding to the function field $\bar{\boldsymbol{Q}}_l(t^{1/l},(1-t)^{1/l})$, then X_s is the projective t_1-line $(t_1=t^{1/l^2})$ *with cuspidal singularities* at each of the $(l-2)$ distinct $\boldsymbol{F}_l$-rational points $t_1=a$ of $\boldsymbol{P}^1_{\bar{\boldsymbol{F}}_l}\backslash\{0, 1, \infty\}$, and the completion of the local ring of X_s at $t_1=a$ is given by

$$\bar{\boldsymbol{F}}_l[[T^2, T^l]],\qquad T=t_1-a.$$



The proof of Theorem 1 is reduced to this Fermat case by passage to the Frattini subcovering of (a suitable enlargement of) $X_{\bar{\eta}}$.

(1–5) *About* (P2). The Galois group $\mathfrak{F}=\operatorname{Gal}(M/K)$ is equipped with the conjugacy classes of three special subgroups, the inertia groups above $t=0, 1, \infty$. The outerly action of $G_{\boldsymbol{Q}}$ on $\mathfrak{F}$ respects this structure. To be more precise, call any place of M over $\bar{\boldsymbol{Q}}$ lying above $t=0, 1, \infty$, a *cuspidal place* of M. Then the inertia group in M/K of a cuspidal place is (topologically) generated by a single element, and hence is a quotient of $\hat{\boldsymbol{Z}}$. It is isomorphic to $\boldsymbol{Z}_l$ (resp. $(\boldsymbol{Z}/m)\times\boldsymbol{Z}_l$ with some $m\not\equiv 0 \pmod{l}$, resp. $\hat{\boldsymbol{Z}}$), according to whether the case is pro-l (resp. almost pro-l, resp. profinite). A *primitive parabolic element* is a generator of one of such an inertia group, and an $\mathfrak{F}$-conjugacy class of such an element is a *primitive parabolic conjugacy class*. Call $\tilde{\Phi}$ the group of all $\sigma\in\operatorname{Aut}\mathfrak{F}$ that raises each primitive parabolic conjugacy class c to some power c^α ($\alpha \in \hat{\boldsymbol{Z}}^\times$). Here, α depends on σ but not on c. Define $\Phi=\tilde{\Phi}/\mathrm{Int}\,\mathfrak{F}\subset\mathrm{Out}\,\mathfrak{F}$. Then $\varphi(G_{\boldsymbol{Q}})$ is contained in Φ. Moreover, it is contained in the $\mathfrak{S}_3$-symmetric part $S\Phi$ defined as follows. Let the symmetric group $\mathfrak{S}_3$ act on $K=\bar{\boldsymbol{Q}}(t)$ as the group of linear fractional transformations stabilizing and acting on $\{0, 1, \infty\}$ as the group of permutations (the "λ-group"). The fixed field is $\bar{\boldsymbol{Q}}(j)$, with

$$j=2^8(t^2-t+1)^3/t^2(1-t)^2.$$

The exact sequence of Galois groups

$$1\longrightarrow\mathfrak{F}\longrightarrow\mathrm{Gal}\,(M/\bar{\boldsymbol{Q}}(j))\longrightarrow\mathfrak{S}_3\longrightarrow 1$$

defines an *injective* homomorphism $\mathfrak{S}_3\hookrightarrow\mathrm{Out}\,\mathfrak{F}$. We define $S\Phi$ to be the centralizer of $\mathfrak{S}_3$ in Φ. One checks easily that $\varphi(G_{\boldsymbol{Q}})\subset S\Phi$. These are rather obvious restrictions of the image. Deligne thinks that the image is much smaller than $S\Phi$, and our study of the ψ-representation ([9] II, IV; cf. § 3) seems to support this. See also [11] "the $\mathfrak{S}_4$-symmetricity of $F_\rho * F_\rho$", and [1].

The situation is completely parallel in the profinite case. Namely, the groups Φ and $S\Phi$ are defined analogously, and $\varphi(G_{\boldsymbol{Q}})\subset S\Phi$. As φ is injective in this case, φ induces an *isomorphism* between $G_{\boldsymbol{Q}}$ and its image.

Now for the "coordinate system". Recall that the topological fundamental group $\Gamma=\pi_1(\boldsymbol{P}^1_{\boldsymbol{C}}\backslash\{0, 1, \infty\})$ is a free group of rank 2 generated by such loops x_C, y_C, z_C around 0, 1, ∞, respectively that $x_C y_C z_C=1$. Therefore, its completion $\mathfrak{F}$ is free pro-l (resp. almost pro-l, or profinite, depending on the case) with rank 2, generated by such x and y that x, y and $z=(xy)^{-1}$ each generates some inertia group above 0, 1 and ∞, respectively. The group $\tilde{\Phi}$ can be expressed as

$$\tilde{\Phi}=\left\{\sigma \in \mathrm{Aut}\,\mathfrak{F};\; \begin{matrix}\sigma x \sim x^\alpha \\ \sigma y \sim y^\alpha \\ \sigma z \sim z^\alpha\end{matrix}\ (\exists\alpha \in \hat{\boldsymbol{Z}}^\times)\right\}.$$

As for the exponent α, if the case is pro-l (resp. profinite), it is uniquely determined by σ as an element of $\boldsymbol{Z}_l^\times$ (resp. $\hat{\boldsymbol{Z}}^\times$), called the norm $N(\sigma)$ of σ. For $\rho \in G_{\boldsymbol{Q}}$, $\chi(\rho)=N(\varphi(\rho))$ is the l-cyclotomic (resp. the cyclotomic) character describing the action of ρ on the group of l-powerth (resp. all) roots of unity.

As for the freedom in the choice of (x, y): If we only impose that they generate $\mathfrak{F}$ and each of $x, y, z=(xy)^{-1}$ is primitive parabolic above 0, 1, ∞, respectively, then the choice of (x, y) is up to $\tilde{\Phi}$-transforms. But if we further impose that they come from x_C, y_C via an embedding $\bar{\boldsymbol{Q}}\subset\boldsymbol{C}$, then this will define a narrower class. The triple (x_C, y_C, z_C) described above is, roughly speaking, unique up to simultaneous Γ-conjugation, and it is precisely so if we impose x_C any y_C to be positively oriented. The class of (x, y) thus defined is unique up to $\widetilde{\varphi(G_Q)}$-transformations, where $\widetilde{\varphi(G_Q)}$ is the preimage of $\varphi(G_Q)$ in Aut $\mathfrak{F}$. This means that, for a free pro-l (or profinite) group $\mathfrak{F}$ of rank 2 given together with a set of free generators (x, y), there is a uniquely determined subgroup of Out $\mathfrak{F}$ that corresponds with $\varphi(G_Q)$. What is THIS?

(1–6) *Similar problems for* φ_{Q_l}. The representation $\varphi_{Q_l}: G_{Q_l} \to \Phi$ can be identified with the restriction of φ_Q to the decomposition group of an (arbitrary) extension $\bar{l}/l$ to $\bar{Q}$. Therefore, Image $\varphi_{Q_l} \subset$ Image φ_Q, and as for the kernel, the Galois extension k_{φ_l}/Q_l corresponding to Ker φ_{Q_l} can be identified with the $\bar{l}$-adic completion of k_φ.

At present, we have nothing to add about the image. As for k_{φ_l}/Q_l, it is obvious that k_{φ_l} is *some* pro-l extension over $Q_l(\mu_{l^\infty})$. But at least when l is a *regular prime*, $k_{\varphi_l}/Q_l(\mu_{l^\infty})$ cannot be the maximum pro-l extension. In fact, denote by Σ_l the maximum pro-l extension over $Q_l(\mu_{l^\infty})$ (or equivalently, over $Q_l(\mu_l)$), and Σ' the maximum pro-l extension over $Q(\mu_{l^\infty})$ (or equivalently, over $Q(\mu_l)$) unramified outside l. Call Σ'_l the $\bar{l}$-adic completion of Σ', which can be regarded as a Galois subextension of $\Sigma_l/Q_l(\mu_{l^\infty})$. The minimum number of generators of the pro-l groups Gal $(\Sigma_l/Q_l(\mu_l))$ and Gal $(\Sigma'/Q(\mu_l))$ are $l+1$ and $\frac{1}{2}(l+1)$ respectively. (Here and in the following, when $l=2$, $\frac{1}{2}(l+1)$ should be replaced by 2.) Now, if l is regular, then it follows (by Frattinization and the Hilbert classfield theory) that Gal $(\Sigma'/Q(\mu_l)) \simeq$ Gal $(\Sigma'_l/Q_l(\mu_l))$ canonically, and from [13] Satz 11.5 follows directly that this group is a free pro-l group of rank $\frac{1}{2}(l+1)$. Since $k_\varphi \subset \Sigma'$, we conclude that if l is regular, the minimum number of generators of Gal $(k_{\varphi_l}/Q_l(\mu_l))$ is *at most* $\frac{1}{2}(l+1)$, and if furthermore (P1′) is valid then Gal $(k_{\varphi_l}/Q_l(\mu_l))$ must be a free pro-l group of rank $\frac{1}{2}(l+1)$.

(P4) What is k_{φ_l}? What is the structure of the pro-l group Gal $(k_{\varphi_l}/Q_l(\mu_l))$? (Is it free with rank $\frac{1}{2}(l+1)$?)

Unfortunately, the choice of standard generators of the Demůskin group Gal $(\Sigma_l/Q_l(\mu_l))$ seems "too arbitrary" to study Gal $(k_{\varphi_l}/Q_l(\mu_l))$ as its quotient.

(1–7) φ_{F_p} *for* $p \neq l$. The representation φ_{F_p}: Gal $(\bar{F}_p/F_p) \to \Phi$ is determined by the image of the Frobenius element. Its conjugacy class is the same as the one determined by the φ_Q-image of a Frobenius above p. This conjugacy class is contained in the subset $\{\sigma \in \Phi; N(\sigma)=p\}$, but as shown in [9] I, this set consists of more than one Φ-conjugacy class (and in fact, infinitely many Φ-conjugacy classes; cf. Kanako [12]).

(P5) Can one give a good parametrization of Φ-conjugacy classes and pinpoint the Frobenius conjugacy class?

§2. Approximation of φ; the canonical filtration of G_Q

(2–1) In Sections 2 and 3, we shall consider two different types of "approximations" of φ. First, in Section 2, we restrict ourselves to the pro-l case and consider the first type. This arises from the filtration $\{\mathfrak{F}(m)\}_{m\geqslant 1}$ of the free pro-l group $\mathfrak{F}=\mathrm{Gal}(M/K)$ by the descending central series; $\mathfrak{F}(1)=\mathfrak{F}$, $\mathfrak{F}(m+1)=[\mathfrak{F}, \mathfrak{F}(m)]$ $(m\geqslant 1)$. Here, [,] is the commutator operation (closure of the algebraic commutator). Take any positive integer m. Then each outer automorphism of $\mathfrak{F}$ induces an outer automorphism of $\mathfrak{F}/\mathfrak{F}(m+1)$ and hence an automorphism of its center $\mathfrak{F}(m)/\mathfrak{F}(m+1)$. Therefore, Φ and hence also G_Q act outerly on $\mathfrak{F}/\mathfrak{F}(m+1)$, and in particular on $\mathfrak{F}(m)/\mathfrak{F}(m+1)$. We have three things here to look at.

(i) First, $\mathfrak{F}(m)/\mathfrak{F}(m+1)$ is a free $\boldsymbol{Z}_l$-module of finite rank

$$\rho_m=\frac{1}{m}\sum_{d\mid m}\mu\left(\frac{m}{d}\right)2^d; \qquad \text{(Witt)};$$

hence $\mathrm{Aut}(\mathfrak{F}(m)/\mathfrak{F}(m+1))\simeq GL_{\rho_m}(\boldsymbol{Z}_l)$. But for any $\sigma\in\mathrm{Aut}\,\mathfrak{F}$, the action of σ on $\mathfrak{F}(m)/\mathfrak{F}(m+1)$ is determined by its action on $\mathfrak{F}/\mathfrak{F}(2)\simeq\boldsymbol{Z}_l^{\oplus 2}$. In particular, for $\sigma\in\Phi$, it acts on $\mathfrak{F}(m)/\mathfrak{F}(m+1)$ via scalar multiplication by $N(\sigma)^m$. Therefore, this representation of G_Q in $\mathfrak{F}(m)/\mathfrak{F}(m+1)$ is simply the scalar representation given by $\rho\to\chi(\rho)^m$ $(\rho\in G_Q)$.

(ii) Each quotient $\mathfrak{F}/\mathfrak{F}(m+1)$ is a finitely generated pro-l group, and is nilpotent with finite level. So, as Deligne did in [6], one may look at its Malcev's Lie algebra $\mathfrak{g}_m$ over $\boldsymbol{Q}$, and try to determine the Galois image in $\mathrm{Der}(\mathfrak{g}_m)/\mathrm{Int}(\mathfrak{g}_m)$; the algebra of outer derivations of $\mathfrak{g}_m$. By using the Belyǐ lifting of φ, one may replace Out $\mathfrak{F}$ by Aut $\mathfrak{F}$, and $\mathrm{Der}(\mathfrak{g}_m)/\mathrm{Int}(\mathfrak{g}_m)$ by $\mathrm{Der}(\mathfrak{g}_m)$. In [6], Deligne gives a description of the Galois image in $\mathrm{Der}(\mathfrak{g}_m)$ modulo some ideal. It corresponds to some essential part of the study [9] of the Galois representation in $\mathrm{Out}(\mathfrak{F}/[\mathfrak{F}(2), \mathfrak{F}(2)])$ (cf. [9] IV §7).

(iii) Let $\Phi(m)$ $(m\geqslant 1)$ denote the kernel of the homomorphism $p_m^1:\Phi\to\mathrm{Out}(\mathfrak{F}/\mathfrak{F}(m+1))$. Then $\Phi(1)$ is the kernel of the norm N: $\Phi\to\boldsymbol{Z}_l^\times$, and $\{\Phi(m)\}_{m\geqslant 1}$ gives a descending filtration of Φ. For $m\geqslant 2$, $\Phi(m)$ is the same as the group $\Phi_1(m)$ of [9]. In particular, $\Phi(1)=\Phi(2)=\Phi(3)$, and $[\Phi(m), \Phi(n)]\subset\Phi(m+n)$ $(m, n\geqslant 1)$. For each $m\geqslant 2$, the quotient $\mathrm{gr}^m\,\Phi=\Phi(m)/\Phi(m+1)$ is a free $\boldsymbol{Z}_l$-module of rank $2\rho_m-\rho_{m+1}$. The group $\Phi/\Phi(1)\simeq\boldsymbol{Z}_l^\times$ acts on $\mathrm{gr}^m\,\Phi$ via conjugation $\mathrm{Int}\,\sigma$ $(\sigma\in\Phi)$, and this action is nothing but the α^m-multiplication $(\alpha\in\boldsymbol{Z}_l^\times)$. As for the symmetric part $S\Phi$ of Φ, we also put $S\Phi(m)=S\Phi\cap\Phi(m)$. Then $\mathrm{gr}^m\,S\Phi=S\Phi(m)/S\Phi(m+1)$ can be considered naturally as a submodule of $\mathrm{gr}^m\,\Phi$, and its rank is approximately 1/6 times rank $\mathrm{gr}^m \Phi$. More precisely, it is given by the following formula of Deligne*);

$$\mathrm{rank}\ \mathrm{gr}^m S\Phi = \alpha_m - \beta_{m+1} \qquad (m \geqslant 3,\ l \neq 2, 3),$$

with

$$\alpha_m = (r_m : \pi) = \frac{1}{3m} \sum_{\substack{d \mid m \\ m/d \not\equiv 0 \ (\mathrm{mod}\ 3)}} \left\{ \mu\left(\frac{m}{d}\right) 2^d - \varepsilon_m \right\},$$

$$\beta_m = (r_m : 1) = \frac{1}{6m} \left\{ \sum_{d \mid m} \delta\left(\frac{m}{d}\right) \mu\left(\frac{m}{d}\right) 2^d + 2\varepsilon_m \right\},$$

where

$$\varepsilon_m = \begin{cases} -1 \cdots m = 3^\alpha \\ 2 \cdots m = 2 \cdot 3^\alpha \\ 0 \cdots \text{otherwise} \end{cases}, \qquad \delta(m) = \begin{cases} 1 \cdots & m \equiv \pm 1 \\ 3 \cdots & 3 \\ 4 \cdots & \pm 2 \\ 6 \cdots & 0 \end{cases} \quad (\mathrm{mod}\ 6).$$

Here, r_m is the character of the $\mathfrak{S}_3$-action on $\mathfrak{F}(m)/\mathfrak{F}(m+1)$, and π is the irreducible character of $\mathfrak{S}_3$ with degree 2. For small m we have

m	2	3	4	5	6	7	8	9	10	11	12
rank $\mathrm{gr}^m \Phi$	0	1	0	3	0	6	4	13	12	37	40
rank $\mathrm{gr}^m S\Phi$	0	1	0	1	0	2	1	4	2	9	7

Now call $G_{Q(m)}$ the kernel of $p_m \circ \varphi: G_Q \to \mathrm{Out}(\mathfrak{F}/\mathfrak{F}(m+1))$. In other words, $G_{Q(m)} = \varphi^{-1}(\Phi(m))$. Then $\{G_{Q(m)}\}_{m \geqslant 1}$ gives a descending filtration of G_Q such that $\bigcap_m G_{Q(m)} = G_{k_\infty}$. We have

$$Q(\boldsymbol{\mu}_{l^\infty}) = Q(1) = Q(2) = Q(3) \subset Q(4) = Q(5) \subset Q(6) \cdots.$$

The basic properties of this tower are:

(a) $\{Q(m)\}_{m \geqslant 1}$ is an increasing sequence of Galois extensions of Q, with $Q(1) = Q(\boldsymbol{\mu}_{l^\infty})$.

(b) Each extension $Q(m)/Q(1)$ is pro-l, and is unramified outside l;

(c) $\mathrm{Gal}(Q(m+1)/Q(m))$ is central in $\mathrm{Gal}(Q(m+1)/Q(1))$.

(d) Consider $\mathrm{Gal}(Q(m+1)/Q(m))$ as a Z_l-module. Then the natural action of $Z_l^\times = \mathrm{Gal}(Q(1)/Q)$ on $\mathrm{Gal}(Q(m+1)/Q(m))$ is given by the α^m-multiplication ($\alpha \in Z_l^\times$).

*) This formula appears already in his letter [5]; the author also calculated it independently.

(2–2)

(P6) Determine the sequence $\{\boldsymbol{Q}(m)\}$ explicitly. Is each $\boldsymbol{Q}(m)$ maximal under the conditions (a) ~ (d)?

For each $m \geqslant 1$, the quotient $\mathrm{gr}^m G_{\boldsymbol{Q}} = \mathrm{Gal}(\boldsymbol{Q}(m+1)/\boldsymbol{Q}(m))$ can be identified with the submodule of $\mathrm{gr}^m S\Phi$ consisting of the image of $G_{\boldsymbol{Q}(m)}$. Hence $\mathrm{gr}^m G_{\boldsymbol{Q}}$ is a free $\boldsymbol{Z}_l$-module of finite rank $\leqslant$ rank $\mathrm{gr}^m S\Phi$.

(P6′) Determine rank $\mathrm{gr}^m G_{\boldsymbol{Q}}$ $(m \geqslant 1)$.

What we know at present about this rank is as follows.

Proposition 1. *We have* $c'_m \leqslant \mathrm{rank}\ \mathrm{gr}^m G_{\boldsymbol{Q}} \leqslant c_m$, *where*

$$c'_m = \begin{cases} 1 \cdots m\colon odd \geqslant 3 \\ 0 \cdots otherwise, \end{cases} \qquad c_m = \begin{cases} a_m - b_m + 1 \cdots m\colon odd \geqslant 3 \\ a_m \qquad\quad \cdots otherwise, \end{cases}$$

with $a_m = \mathrm{rank}\ \mathrm{gr}^m S\Phi$ *(given above), and* $b_m = [(m+3)/6]$ $(m\colon odd \geqslant 3)$. *Here,* $[*]$ $(* \in \boldsymbol{Q})$ *denotes the greatest integer* $\leqslant *$.

m	1	2	3	4	5	6	7	8	9	10	11
c_m	0	0	1	0	1	0	2	1	3	2	8
rank $\mathrm{gr}^m G_{\boldsymbol{Q}}$	0	0	1	0	1	0	1 or 2	0 or 1	$1 \leqslant\ \leqslant 3$	$0 \leqslant\ \leqslant 2$	$1 \leqslant\ \leqslant 8$

(P7) Can $\mathrm{gr}^m G_{\boldsymbol{Q}}$ be non-trivial for some even m?

When $l=2$, there exists some $m \geqslant 7$ such that rank $\mathrm{gr}^m G_{\boldsymbol{Q}} > c'_m$ (cf. § 3).

(2–3) *An interpretation of* $\boldsymbol{Q}(m)$ *in terms of the Belyĭ's* $\boldsymbol{Q}$*-models.* Let $M_{\boldsymbol{Q}}$ be the $\boldsymbol{Q}$-model of M corresponding to the Belyĭ's representative [9] I § 4, $K(m+1)$ $(m \geqslant 1)$ be the subextension of M/K corresponding to $\mathfrak{F}(m+1)$ and put $K(m+1)_{\boldsymbol{Q}} = K(m+1) \cap M_{\boldsymbol{Q}}$. Then $K(m+1)_{\boldsymbol{Q}}$ is a $\boldsymbol{Q}$-model of $K(m+1)$, but it is not Galois over $\boldsymbol{Q}(t)$.

Proposition 2. (i) $k = \boldsymbol{Q}(m)$ $(m \geqslant 1)$ *is the smallest Galois extension of* $\boldsymbol{Q}$ *which makes* $K(m+1)_{\boldsymbol{Q}} \cdot k/\boldsymbol{Q}(t) \cdot k$ *a Galois extension.*

(ii) $\boldsymbol{Q}(m+1)$ *coincides with the residue field of each cuspidal place of* $K(m+1)_{\boldsymbol{Q}} \cdot \boldsymbol{Q}(m)$.

The proof is an easy exercise using [9] I.

§ 3. Subrepresentations ϕ

(3–1) The approximation of φ of the second type is, roughly speaking, as follows. Suppose $\mathfrak{n} \subset \mathfrak{F}$ is a normal subgroup invariant by the action of the Galois group $G_{\boldsymbol{Q}^*}$, for some $\boldsymbol{Q} \subset \boldsymbol{Q}^* \subset \bar{\boldsymbol{Q}}$, and suppose we know already the (outerly) action of G_{Q^*} on the quotient $\mathfrak{g}=\mathfrak{F}/\mathfrak{n}$. Consider the "finer" quotient $\mathfrak{F}^*=\mathfrak{F}/[\mathfrak{n},\mathfrak{n}]$. The main subject of Section 3 is a certain group theoretic framework convenient to describe the G_{Q^*}-action on $\mathfrak{F}^*$. An advantage of considering this extension $\mathfrak{F}^*\to\mathfrak{g}$ is that $\mathfrak{F}^*$ has trivial center (under a mild assumption on $\mathfrak{n}$). The main "new part" to describe is the G_{Q^*}-action on $\mathfrak{n}^*=\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ which (again, under a mild assumption on $\mathfrak{n}$) can be identified with the projective limit $\varprojlim T_l(\mathrm{Jac}\, X_n^*)$ of the Tate module of the Jacobian of X_n^*, where $\{X_n^*/\boldsymbol{P}^1\}$ is the $\mathfrak{g}$-tower corresponding to $\mathfrak{n}$. The basic reference for Section 3 is [10].

(3–2) To be more precise, let $\mathfrak{F}$ be a free almost pro-l group of rank 2, $\{x, y\}$ be a generator of $\mathfrak{F}$, and put $z=(xy)^{-1}$. Let $\mathfrak{n}$ be a closed normal subgroup of $\mathfrak{F}$ such that $\mathfrak{n}$ is pro-l and

$$\mathfrak{n}\cap\langle x\rangle=\mathfrak{n}\cap\langle y\rangle=\mathfrak{n}\cap\langle z\rangle=1.$$

Put $\mathfrak{n}^*=\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$, $\mathfrak{F}^*=\mathfrak{F}/[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{g}=\mathfrak{F}/\mathfrak{n}$;

$$1\longrightarrow\mathfrak{n}^*\longrightarrow\mathfrak{F}^*\longrightarrow\mathfrak{g}\longrightarrow 1 \qquad \text{(exact).}$$

Denote by x^* (resp. y^*, z^*) the projection of x (resp. y, z) on $\mathfrak{F}^*$.

Theorem 2. (i) *The centralizer of x^* (resp. y^*, z^*) in $\mathfrak{F}^*$ is the cyclic group topologically generated by x^* (resp. y^*, z^*);*
(ii) *the center of $\mathfrak{F}^*$ is trivial.*

As for (i), one can prove a little more, that if $s^*x^*s^{*-1}$ ($s^*\in\mathfrak{F}^*$) is a power of x^* then s^* must be a power of x^* (and similarly for y^*, z^*). These proofs reduce easily to Lemma 5.3 of [10] by using free differentiations.

Now let us define the group "Φ for $\mathfrak{F}/[\mathfrak{n},\mathfrak{n}]$", called Ψ, as follows.

$$\Psi=\left\{\sigma\in\mathrm{Aut}\,\mathfrak{F}^*;\ \sigma\mathfrak{n}^*=\mathfrak{n}^*,\ \begin{matrix}\sigma x^*\sim x^{*\alpha}\\ \sigma y^*\sim y^{*\alpha}\\ \sigma z^*\sim z^{*\alpha}\end{matrix}(\exists\alpha\in\hat{\boldsymbol{Z}}^\times)\right\}/\mathrm{Int}(\mathfrak{F}^*,\mathfrak{n}^*),$$

where $\sim$ denotes conjugacy in $\mathfrak{F}^*$, and $\mathrm{Int}(\mathfrak{F}^*,\mathfrak{n}^*)$ denotes the group of all inner automorphisms of $\mathfrak{F}^*$ of the form $\mathrm{Int}\,n$ ($n\in\mathfrak{n}^*$). The exponent α is again determined uniquely by σ (call it also $\alpha=N(\sigma)$). This group contains a normal subgroup

$$\Psi_1=\left\{\sigma\in\mathrm{Aut}\,\mathfrak{F}^*;\ \begin{matrix}\sigma x^*\approx x^*\\ \sigma y^*\approx y^*\\ \sigma z^*\approx z^*\end{matrix}\right\}/\mathrm{Int}(\mathfrak{F}^*,\mathfrak{n}^*),$$

where $\approx$ denotes conjugacy by element of $\mathfrak{n}^*$.

Our first result related to these groups is the existence of a certain anti 1-cocycle $\varepsilon\colon \Psi\to\mathscr{A}^\times$, where $\mathscr{A}=\boldsymbol{Z}_l\,[[\mathfrak{g}]]$, the completed group algebra of $\mathfrak{g}$. Before stating this result, we need some preliminaries.

(3–3) Denote by $\mathscr{B}=\boldsymbol{Z}_l[[\mathfrak{F}]]$ (resp. $\mathscr{A}=\boldsymbol{Z}_l[[\mathfrak{g}]]$) the completed group algebra of $\mathfrak{F}$ (resp. $\mathfrak{g}$) over $\boldsymbol{Z}_l$, and $\pi\colon \mathscr{B}\to\mathscr{A}$ the projection. As $\mathfrak{F}$ is a free almost pro-l group of rank 2 generated by x and y, $\mathscr{B}$ is equipped with the free differentiations $\partial/\partial x$, $\partial/\partial y\colon \mathscr{B}\to\mathscr{B}$ defined as follows. Every element $\theta\in\mathscr{B}$ is expressed *uniquely* as

$$\theta=s(\theta)\cdot 1_{\mathfrak{F}}+\theta_1(x-1)+\theta_2(y-1) \qquad (\theta_1,\,\theta_2\in\mathscr{B})$$

where $s\colon \mathscr{B}\to\boldsymbol{Z}_l$ is the augmentation homomorphism. We define $\partial\theta/\partial x=\theta_1$, $\partial\theta/\partial y=\theta_2$ (cf. [10] Theorem 2.1).

Now the group Ψ acts naturally on the quotient $\mathfrak{g}$ of $\mathfrak{F}^*$, and hence also on $\mathscr{A}$ and $\mathscr{A}^\times$. Note that Ψ_1 is contained in the kernel of this action. A continuous map $\varepsilon\colon \Psi\to\mathscr{A}^\times$ will be called an anti 1-cocycle, if

$$\varepsilon(\sigma'\circ\sigma)=\sigma'(\varepsilon(\sigma))\cdot\varepsilon(\sigma'), \qquad \text{for all } \sigma,\,\sigma'\in\Psi.$$

(3–4) The following theorems are basic for the presentations of Ψ and Ψ_1.

Theorem 3. *There exists a unique continuous anti 1-cocycle*

$$\varepsilon\colon \Psi\to\mathscr{A}^\times$$

satisfying the following property. For any $\sigma\in\Psi$, any $\tilde\sigma\in\operatorname{Aut}\mathfrak{F}^*$ representing σ, and any $\alpha\in\hat{\boldsymbol{Z}}^\times$, s, $t\in\mathfrak{F}$ such that $sx^\alpha s^{-1}$ (resp. $ty^\alpha t^{-1}$) represents $\tilde\sigma x^*$ (resp. $\tilde\sigma y^*$) modulo $[\mathfrak{n},\mathfrak{n}]$, one has

$$\varepsilon(\sigma)=\pi\left(s-\frac{\partial(s-t)}{\partial x}(x-1)\right)=\pi\left(t-\frac{\partial(t-s)}{\partial y}(y-1)\right).$$

The proof is parallel to that of Theorem A in [10].

Theorem 4. (i) *For $\sigma\in\Psi$, $\varepsilon(\sigma)=1$ if and only if $\sigma=1$.*
(ii) *The restriction of ε to Ψ_1 gives an anti-isomorphism*

$$\varepsilon_1\colon \Psi_1\to[1+\mathscr{R}(\boldsymbol{x}-1)\cap\mathscr{R}(\boldsymbol{y}-1)]^\times$$

where $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) is the projection of x (resp. y) on $\mathfrak{g}\subset\mathscr{A}$, and $\mathscr{R}$ is the right ideal of $\mathscr{A}$ defined by

$$\begin{aligned}\mathscr{R}&=\{r\in\mathscr{A};\,(\boldsymbol{x}-1)r\in(\boldsymbol{x}\boldsymbol{y}-1)\mathscr{A}\}\\&=\{r\in\mathscr{A};\,(\boldsymbol{y}-1)r\in(\boldsymbol{y}\boldsymbol{x}-1)\mathscr{A}\}.\end{aligned}$$

Note that in the pro-l case every element of $1+\mathscr{R}(\boldsymbol{x}-1)\cap\mathscr{R}(\boldsymbol{y}-1)$ is invertible. The proof of Theorem 4 can be obtained by the combination of methods used in [9] II (proof of Theorem 3B) and [10] § 3.

Thus if $\varDelta$ denotes the image of the canonical homomorphism $\delta\colon\varPsi\to\operatorname{Aut}\mathfrak{g}$, then $\varPsi$ *can be embedded* into the semi-direct product $\varDelta\ltimes(\mathscr{A}^\times)^\circ$ via (δ,ε), where $(\mathscr{A}^\times)^\circ$ denotes the aniti-isomorphic dual of $\mathscr{A}^\times$.

(3–5) Consider now the restriction homomorphism $\mu\colon\varPsi\to\operatorname{Aut}\mathfrak{n}^*$. How can this be explicitly presented? The following two theorems answer this question.

Theorem 5 ([10] § 1). *Consider the pro-l abelian group $\mathfrak{n}^*$ as a left $\mathfrak{g}$-module by conjugation, and hence also as a left $\mathscr{A}$-module. Then as left $\mathscr{A}$-modules,*

$$\mathfrak{n}^*\xrightarrow{\sim}\mathscr{A}(\boldsymbol{x}-1)\cap\mathscr{A}(\boldsymbol{y}-1)\qquad(\textit{canonically}).$$

This is induced from the mapping

$$\mathfrak{n}\ni n\longrightarrow\pi(\partial n/\partial x)(\boldsymbol{x}-1)=-\pi(\partial n/\partial y)(\boldsymbol{y}-1)\in\mathscr{A}(\boldsymbol{x}-1)\cap\mathscr{A}(\boldsymbol{y}-1).$$

Theorem 6. *The action of $\sigma\in\varPsi$ on $\mathfrak{n}^*$, when translated to an action on $\mathscr{A}(\boldsymbol{x}-1)\cap\mathscr{A}(\boldsymbol{y}-1)$ via Theorem 5, is given as*

$$\alpha\longrightarrow\sigma(\alpha)\cdot\varepsilon(\sigma)\qquad(\alpha\in\mathscr{A}(\boldsymbol{x}-1)\cap\mathscr{A}(\boldsymbol{y}-1)).$$

The proof is completely parallel to that of Theorem C of [10].

(3–6) *The Galois representation in $\varPsi$.* A natural representation of the Galois group $G_{\boldsymbol{Q}^*}$ in $\varPsi$ arises when there is an infinite Galois extension L/K and its $\boldsymbol{Q}^*$-model $L^*/\boldsymbol{Q}^*(t)$ $(\boldsymbol{Q}\subset\boldsymbol{Q}^*\subset\bar{\boldsymbol{Q}})$, satisfying the following properties.

(i) L/K is unramified outside $t=0,1,\infty$;

(ii) L/K is an almost pro-l extension, i.e., $\mathfrak{g}=\operatorname{Gal}(L/K)$ contains an open normal pro-l subgroup;

(iii) the ramification index of each of $0,1,\infty$ in L is infinite;

(iv) $L^*\cdot\bar{\boldsymbol{Q}}=L$, $L^*\cap\bar{\boldsymbol{Q}}=\boldsymbol{Q}^*$ (but $L^*/\boldsymbol{Q}^*(t)$ need not be Galois).

There are many interesting examples of L, such as those obtained from the tower of Fermat (or Heisenberg) curves of level l^n $(n\to\infty)$, the tower of modular curves of level $2ml^n$ $(n\to\infty)$, etc. (cf. [10]). Since we shall later refer to the "Fermat case", we recall here what this means. With the notation of Section 2 (2–3), this is the pro-l case with $L=K(2)$, $\boldsymbol{Q}^*=\boldsymbol{Q}$, and $L^*=K(2)_{\boldsymbol{Q}}$ (hence $\mathfrak{g}=\mathfrak{F}/\mathfrak{F}(2)=\boldsymbol{Z}_l\times\boldsymbol{Z}_l$).

Now L and L^* being given, denote by M the maximum pro-l extension of L unramified outside 0, 1, ∞, and put

$$\mathfrak{n}=\mathrm{Gal}\,(M/L), \quad \mathfrak{F}=\mathrm{Gal}\,(M/K), \quad \mathfrak{g}=\mathrm{Gal}\,(L/K)$$
$$1\longrightarrow\mathfrak{n}\longrightarrow\mathfrak{F}\longrightarrow\mathfrak{g}\longrightarrow 1 \qquad \text{(exact)}.$$

Then $\mathfrak{F}$ is a free almost pro-l group of rank 2 generated by two such elements x, y that x, y and $z=(xy)^{-1}$ each generates some inertia group above 0, 1, ∞ respectively. Note that $\mathfrak{n}$ satisfies the assumptions of (3–2). Identify $G_{\boldsymbol{Q}^*}$ with $\mathrm{Gal}\,(L/L^*)=\mathrm{Gal}\,(M/L^*)/\mathfrak{n}$ in the canonical way, and for each $\rho\in G_{\boldsymbol{Q}^*}$ choose an element $\rho^*\in\mathrm{Gal}\,(M/L^*)$ which lifts ρ. Then the conjugation $\mathrm{Int}\,\rho^*$ induces an automorphism of $\mathfrak{F}$ which is well-defined by ρ *modulo* inner automorphisms by elements of $\mathfrak{n}$. Clearly, $\mathrm{Int}\,\rho^*$ stabilizes $\mathfrak{n}$ and hence also $[\mathfrak{n}, \mathfrak{n}]$. Thus, $\rho\rightarrow\mathrm{Int}\,\rho^*$ induces a homomorphism

$$\psi\colon G_{\boldsymbol{Q}^*}\longrightarrow\Psi.$$

Note that the natural action of $G_{\boldsymbol{Q}^*}$ on $\mathfrak{g}$, or on $\mathfrak{n}^*$ factors through ψ.

The composite $\varepsilon\circ\psi\colon G_{\boldsymbol{Q}^*}\rightarrow\mathscr{A}^\times$ is the anti 1-cocycle constructed and studied in [10]. (In [10], $\varepsilon\circ\psi$ is denoted as ψ. It is constructed without making explicit reference to the group Ψ; cf. also [9] § II for the case $\mathfrak{n}=[\mathfrak{F}, \mathfrak{F}]$.) The composite $\mu\circ\psi\colon G_{\boldsymbol{Q}^*}\rightarrow\mathrm{Aut}\,\mathfrak{n}^*$ is the natural action of $G_{\boldsymbol{Q}^*}$ on $\mathfrak{n}^*=\varprojlim T_l\,(\mathrm{Jac}\,X_n^*)$, and by Theorems 4, 5, this can be explicitly presented as the "twisted right multiplication" of

$$\varepsilon(\psi(\rho))\in\mathscr{A}^\times \quad \text{on} \quad \mathfrak{n}^*\simeq\mathscr{A}(x-1)\cap\mathscr{A}(y-1) \quad (\rho\in G_{\boldsymbol{Q}^*}).$$

(3–7)
(P8) What is the kernel $\mathrm{Ker}\,\psi=\mathrm{Ker}\,(\varepsilon\circ\psi)$?
(P9) What is the image of $\varepsilon\circ\psi$ in $\mathscr{A}^\times$?

In the Fermat case, both questions are closely related to the Vandiver conjecture, as shown by Coleman [4].

(3–8) Since Φ and Ψ are, roughly speaking, outer automorphism groups of $\mathfrak{F}$ and of $\mathfrak{F}^*=\mathfrak{F}/[\mathfrak{n}, \mathfrak{n}]$ respectively, one wants to connect them by a "canonical homomorphism" $\gamma\colon\Phi\rightarrow\Psi$ and study its image and the kernel. This would help obtain some information on the representation φ from that on ψ. But strictly speaking, there is no canonical homomorphism $\gamma\colon\Phi\rightarrow\Psi$ unless one replaces $\mathrm{Int}\,(\mathfrak{F}^*; \mathfrak{n}^*)$ by $\mathrm{Int}\,\mathfrak{F}^*$ and makes a further assumption on $\mathfrak{n}$ that $\mathfrak{n}$ is Φ-invariant. Here, the latter assumption on $\mathfrak{n}$ would not be so harmful, because we are mostly interested in the case where $\mathfrak{n}$ is a characteristic subgroup of $\mathfrak{F}$. But the former replacement would define a group Ψ' for which an analogue of Theorem 3 would be more complicated (at least in general). So, here, we simply restrict our attention to some suitable subgroups of Φ and Ψ, imposing the following assumption on $\mathfrak{n}$;

$$\langle x\rangle \cap \langle y\rangle \cap \langle z\rangle = (1) \qquad \text{on } \mathfrak{g}.$$

This is satisfied if $\mathfrak{g}$ has trivial center, or if $\mathfrak{F}$ is pro-l and $\mathfrak{n}\subset[\mathfrak{F},\mathfrak{F}]$. Let $\Psi_1\subset\Psi$ be as given in (3–2), and put

$$\Phi_1=\Phi_{1,\mathfrak{n}}=\left\{\sigma\in \mathrm{Aut}\,\mathfrak{F};\ \begin{matrix}\sigma x\approx x\\ \sigma y\approx y\\ \sigma z\approx z\end{matrix}\right\}/\mathrm{Int}\,(\mathfrak{F};\mathfrak{n}),$$

where $\approx$ denotes conjugacy by element of $\mathfrak{n}$, and $\mathrm{Int}\,(\mathfrak{F};\mathfrak{n})$ denotes the group of inner automorphisms of $\mathfrak{F}$ of the form $\mathrm{Int}\, n$ ($n\in\mathfrak{n}$). Under the above assumption on $\mathfrak{n}$, the canonical homomorphisms $\Phi_1\to\mathrm{Out}\,\mathfrak{F}$, $\Psi_1\to\mathrm{Out}\,\mathfrak{F}^*$ are injective; hence Φ_1 can also be considered as a subgroup of Φ. There is an obvious homomorphism

$$\gamma_1\colon \Phi_1\longrightarrow\Psi_1,$$

and we have $\gamma_1\circ\varphi(\rho)=\psi(\rho)$ for all $\rho\in G_{Q^*}$ such that $\varphi(\rho)\in\Phi_1$.

(P 10) What is the image of $\varepsilon\circ\gamma_1$ in $[1+\mathscr{R}(\boldsymbol{x}-1)\cap\mathscr{R}(\boldsymbol{y}-1)]^\times$?

In the Fermat case, this is answered in [9] III (Theorem 8). Namely, the image of $\varepsilon\circ\gamma_1$ in this case is precisely the "odd part" of $[1+\mathscr{R}(\boldsymbol{x}-1)\cap\mathscr{R}(\boldsymbol{y}-1)]=1+uvw\mathscr{A}$.

(3–9) Now put $\Theta=\mathrm{Ker}\,\gamma_1$ and $G_{k_\psi}=\mathrm{Ker}\,\psi$. It is easy to see that $\varphi(G_{k_\psi})\subset\Phi_1$, and then that $\varphi(G_{k_\psi})\subset\Theta$;

$$\begin{array}{ccccccccc} & & & & G_{k_\psi} & & & & \\ & & & \theta\swarrow & \downarrow\varphi & \searrow\text{trivial} & & & \\ 1 & \longrightarrow & \Theta & \longrightarrow & \Phi_1 & \xrightarrow[\gamma_1]{} & \Psi_1 & \longrightarrow & 1. \end{array}$$

This defines a representation $\theta\colon G_{k_\psi}\to\Theta$, which factors through a faithful representation $\mathrm{Gal}\,(k_\varphi k_\psi/k_\psi)\to\Theta$. The question whether φ is richer than ψ, for a given $\mathfrak{n}$, is equivalent to asking whether θ is non-trivial. Interesting concrete problems arise if one specifies $\mathfrak{n}$ and looks at the filtrations of these groups and morphisms compatible to the filtration of Φ defined in Section 2. But we shall restrict our attention to the Fermat case.

(3–10) Now, the Fermat case. In this case, $\Phi_1=\Phi(1)$, and Φ_1 and Ψ_1 coincide with the groups treated in [9] under the same symbols. Obviously, $\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})\subset k_\psi\subset k_\varphi$. Moreover, k_ψ is *abelian* over $\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$ [9]. Define a filtration $\{\Psi_1(m)\}_{m\geqslant 1}$ of Ψ_1 using the descending central series $\{\mathfrak{F}^*(m)\}_{m\geqslant 1}$ of $\mathfrak{F}^*=\mathfrak{F}/[\mathfrak{n},\mathfrak{n}]=\mathfrak{F}/[\mathfrak{F}(2),\mathfrak{F}(2)]$. Namely, $\Psi(m)$ is the kernel of the canonical homomorphism $\Psi_1\to \mathrm{Out}(\mathfrak{F}^*/\mathfrak{F}^*(m+1))$. Correspondingly, we define another filtration of $G_{\boldsymbol{Q}}$, by $G_{\boldsymbol{Q}[m]}=\psi^{-1}(\Psi_1(m))$. Then $\boldsymbol{Q}[1]=\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$, and $\{\boldsymbol{Q}[m]\}_{m\geqslant 1}$ satisfies all properties (a)$\sim$(d) for $\boldsymbol{Q}(m)$. (Besides this, $\boldsymbol{Q}[m]/\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$ is abelian, and $\cup\,\boldsymbol{Q}[m]=k_\psi$.) It is clear that $\boldsymbol{Q}[m]\subset\boldsymbol{Q}(m)$ for each $m\geqslant 1$.

Theorem 7. *The Galois group* $\mathrm{Gal}(\boldsymbol{Q}[m]/\boldsymbol{Q}[m+1])$ $(m\geqslant 1)$ *is a free* $\boldsymbol{Z}_l$*-module of rank* $c'_m=1$ $(m$*:odd*$\geqslant 3)$, $=0$ (*otherwise*).

This is a direct consequence of the combination of [11] Theorem B (first proved by Coleman [4]) and [8] Theorem B (by C. Soulé). From all these follow that $k_\varphi=k_\psi$ if and only if $\boldsymbol{Q}[m]=\boldsymbol{Q}(m)$ for all $m\geqslant 1$, or equivalently, rank $\mathrm{gr}^m G_{\boldsymbol{Q}}=c'_m$ for all $m\geqslant 1$.

Corollary. *θ in the Fermat case is non-trivial if and only if there exists some $m\geqslant 7$ with* rank $\mathrm{gr}^m G_{\boldsymbol{Q}}>c'_m$.

(Incidentally, Proposition 1 (§2) follows by using the above filtration of Ψ_1. In fact, $\mathrm{Gal}(\boldsymbol{Q}[m]/\boldsymbol{Q}[m+1])$ can be regarded as a submodule of $\mathrm{gr}^m\,\Psi_1$, and it lies in $\mathrm{gr}^m\,S\Psi_1^-$, where $-$ specifies the "odd part" [9] III, and S specifies the $\mathfrak{S}_3$-symmetric part (analogous to $S\Phi$). The number $b_m=[(m+3)/6]$ is the rank of $\mathrm{gr}^m\,S\Psi_1^-$, and this gives Proposition 1.)

At present, it is only for $l=2$ that we know the non-triviality of θ;

Proposition 3. *When $l=2$, θ is non-trivial.*

To prove the non-triviality of θ, it suffices to show that $k_\varphi/\boldsymbol{Q}(\boldsymbol{\mu}_{l^\infty})$ is *non*-abelian. When $l=2$, this last statement can be checked by using the following two special circumstances.

(i) When $l=2$, the modular curves of 2-power levels constitute a pro-2 tower of coverings of $\boldsymbol{P}^1$ unramified outside 0, 1, ∞ (because $\boldsymbol{P}^1$ can be regarded as the modular curve of level 2 with cusps at 0, 1, ∞).

(ii) There exists an elliptic curve over $\boldsymbol{Q}$ with conductor 2^7, which is a Weil curve and has no CM [15].*)

For $l>3$, one may try to use Heisenberg curves instead, but at present, the author does not know whether their Jacobians do not really have enough CM.

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