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Separated Ultraproducts and Big Cohen-Macaulay Modules

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This short note is meant to be a brief introduction to my theory which is developed in my paper, "Toward the construction of big Cohen-Macaulay modules." (Nagoya Math. J. 103 (1986), 96–125.) The reader should consult with my original paper for the details.

Section 1

Let $(T, \mathfrak{m})$ be an Artin local ring (commutative with unity) and let R be a T-algebra. We consider the following

Question. When does there exist a (non-trivial) T-injective R-module?

Roughly speaking my answer to this question is that *many T-algebras have such modules, while only a few do not.* The aim of this section is to explain this fact. For the convenience I make the following

Definition. A T-algebra R is a *rich* T-algebra if R has a T-injective R-module. Non-rich T-algebras are called *poor.*

The following lemma is an immediate consequence.

Lemma. *Flat* $\Longrightarrow$ *Rich* $\Longrightarrow$ *Pure.*

Proof. If R is T-flat and $E=E_T(T/\mathfrak{m})$, then $\mathrm{Hom}_T(R, E)$ is a T-injective R-module, hence R is a rich T-algebra. If M is a T-injective R-module, then M is a direct sum of copies of E as a T-module, so there is an injection of E into M. Taking the dual of this injection, one has the mapping f: $\mathrm{Hom}_T(M, E) \to \mathrm{Hom}_T(E, E)=T$. Thus there is an element x in $\mathrm{Hom}_T(M, E)$ satisfying $f(x)=1$. Define an R-module map g by $g(1)=x$. ($\mathrm{Hom}_T(M, E)$ is an R-module.) Then the composition $f \cdot g$ is a T-homomorphism of R onto T, hence T is a pure subring of R.

The converses of the implications in Lemma do not hold in general.

Example. (1) Let $T=k[\![x, w]\!]/(x^2, w^4)$ and let

$$R=k[\![x, y, z, w]\!]/(x^2, w^4, xw-yz, x^2z-y^3, yw^2-z^3, xz^2-y^2w)$$

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where k is a field. Then one can show that R is rich but not flat over T.

(2) Let V be a discrete valuation ring with a prime element t, and let $T=V/t^2V$, $R=T[X]/(tX, X^2+t)$. Then it can be seen that T is a direct summand of R, hence pure, and poor.

We would like to show a theorem giving some conditions for 'poorness'. Before that, we should note the fact that the family of all poor T-algebras is closed under specialization.

Theorem 1. *There is a family of poor T-algebras* $\{u_l(T)\}_{l=1,2,3,\cdots}$ *such that a T-algebra R is poor if and only if R contains a specialization of some* $u_l(T)$. *Each* $u_l(T)$ *is uniquely determined by a minimal free resolution of* $T/\mathfrak{m}$ *as a T-module.*

The concrete description of $u_l(T)$ is not difficult but rather complicated. So we do not give its concrete form here, but we would like to give an example to explain it.

Example. Let V be a discrete valuation ring with a prime element t and let $T=V/t^2V$. Let $Y_n=(y_1^{(n)}, y_2^{(n)}, \cdots, y_n^{(n)}, 0, 0, \cdots)$ be a $1\times\infty$ matrix of indeterminates and 0 for each n, and let z_{ij}, w_n be indeterminates and $E=(1, 0, 0, \cdots)$. Then

$$u_2(T)=T[Y_1, Y_2, z_{11}, w_1, w_2]/(tY_1, tY_2-z_{11}(Y_1, t), E-w_1(Y_1, t)-w_2(Y_2, t))$$

and

$$u_3(T)=\frac{T[Y_1, Y_2, Y_3, z_{11}, z_{21}, z_{22}, w_1, w_2, w_3]}{\begin{pmatrix} tY_1, tY_2-z_{11}(Y_1, t), tY_3-z_{21}(Y_1, t)-z_{22}(Y_2, t), \\ E-w_1(Y_1, t)-w_2(Y_2, t)-w_3(Y_3, t) \end{pmatrix}}.$$

One can observe that $R=T[X]/(tX, X^2+t)$ is obtained from $u_3(T)$ by the following specialization;

$$\begin{aligned} &Y_1\longrightarrow(-x, 0, 0, \cdots),\ Y_2\longrightarrow(1, 0, 0, \cdots),\ Y_3\longrightarrow(0, 0, x, 0, \cdots) \\ &z_{11}\longrightarrow x,\ z_{21}\longrightarrow 0,\ z_{22}\longrightarrow 0, \\ &w_1\longrightarrow 0,\ w_2\longrightarrow 1,\ w_3\longrightarrow x. \end{aligned}$$

Hence R is a poor T-algebra.

In the sence of Theorem 1 one can say that only a few T-algebras are poor because all poor algebras are dominated by generic ones. Now in the next section we shall consider some sufficient conditions for a special kind of T-algebras to be rich.

Section 2

In this section $(T, \mathfrak{m}, k)$ is always a regular local ring and R is a finite local T-algebra such that $d=\dim(T)=\dim(R)$. First we recall the definition of big Cohen-Macaulay modules.

Let $x=\{x_1, x_2, \cdots, x_d\}$ be a system of parameters of R. An R-module M (not necessarily finitely generated) is called a big Cohen-Macaulay module with respect to x if x is a regular sequence on M, i.e., $xM \neq M$ and x_i is a non-zero divisor on $M/(x_1, \cdots, x_{i-1})M$ for all i. M. Hochster proved that, for any system of parameters x of an *equicharacteristic* local ring R, there exists a big Cohen-Macaulay module with respect to x. However this existence problem is still open in general.

Before stating the second theorem of this paper, we need one more definition.

Definition. R is a *very rich T-algebra* if R/I is a rich T/I-algebra for any $\mathfrak{m}$-primary ideal I, or equivalently $R/\mathfrak{m}^n R$ is a rich $T/\mathfrak{m}^n$-algebra for each n.

Theorem 2. *The following two conditions are equivalent.*

(1) *R is a very rich T-algebra.*

(2) *R has a big Cohen-Macaulay module with respect to some* (*any*) *system of parameters of R.*

One shows that this theorem reduces the existence problem of big Cohen-Macaulay modules to a problem for Artin rings. The proof of the implication (2)$\Rightarrow$(1) is done by using the usual big CM argument and is not difficult. The most surprising part of Theorem 2 is in showing the implication (1)$\Rightarrow$(2). We postpone the proof of this part until the last of this paper and we next introduce the notion of separated ultraproducts of modules which will be necessary in this proof.

Section 3

Let $\{(R_i, \mathfrak{n}_i)\}_{i\in I}$ be a family of local rings and let $\{M_i\}_{i\in I}$ be a family of R_i-modules indexed by I.

$\mathscr{F} \subset 2^I$ is an ω-incomplete ultrafilter on I if (1) $\phi \notin \mathscr{F}$, (2) if $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$, (3) if $A \in \mathscr{F}$ and $A \subset B \subset I$ then $B \in \mathscr{F}$, (4) if $A \notin \mathscr{F}$ then $I-A \in \mathscr{F}$ and (5) there is a countable family $\{A_i\}_{i=1,2,\dots}$ of elements in $\mathscr{F}$ satisfying $\bigcap_{i=1}^{\infty} A_i \notin \mathscr{F}$.

By Zorn's lemma if $\sharp(I)=\infty$ then there is an ω-incomplete ultrafilter $\mathscr{F}$ on I. In the following of this paper we always fix such a filter $\mathscr{F}$ on I.

Let $\{P_i\}_{i\in I}$ be a family of propositions. We say that P_i holds for almost all i (abbr. P_i for a.a. i) if $\{i \in I \mid P_i \text{ is true}\} \in \mathscr{F}$.

Definition. $\widetilde{\prod}_{i\in I} M_i$ is called the separated ultraproduct of M_i and is defined as $\prod_{i\in I} M_i/\sim$ where $\sim$ is the equivalence relation given by the following; $(a_i)_{i\in I}\sim(b_i)_{i\in I}$ if and only if, for any integer n, $a_i-b_i\in\mathfrak{n}_i^n M_i$ for a.a. i. We denote the class of $(a_i)_{i\in I}$ in $\widetilde{\prod}_{i\in I} M_i$ by $(a_i)^{\sim}_{i\in I}$.

In the following for the simplicity we assume that there is an integer n such that the embedding dimension of R_i is not bigger than n for a.a. i. Under this assumption the following facts are easily observed.

Facts. (1) $\widetilde{\prod}_{i\in I} R_i$ is a Noetherian local ring with maximal ideal $\tilde{\mathfrak{n}}=\{(a_i)^{\sim}\mid a_i\in\mathfrak{n}_i \text{ for a.a. } i\}$ and $\widetilde{\prod}_{i\in I} M_i$ is a $\widetilde{\prod}_{i\in I} R_i$-module.

(2) $\widetilde{\prod}_{i\in I} M_i$ is always complete and separated in $\tilde{\mathfrak{n}}$-adic topology. In particular $\widetilde{\prod}_{i\in I} R_i$ is a complete local ring.

Let $(R, \mathfrak{n})$ be a local ring and let $N\subset M$ be R-modules. Then the Artin-Rees number of $N\subset M$ is defined as follows:

$$a_R(N, M)=\inf\{r\mid \mathfrak{n}^n M\cap N=\mathfrak{n}^{n-r}(\mathfrak{n}^r M\cap N) \text{ for all } n>r\}.$$

In general the computation of separated ultraproducts is quite difficult or impossible. However in some cases it is possible to compute it by the following theorem which I call the exactness theorem.

Theorem 3. *Let* $0\to M_i'\to M_i\to M_i''\to 0$ *be an exact sequence of* R_i-*modules for* $i\in I$. *Assume that there is an integer* r *satisfying* $a_{R_i}(M_i', M_i)<r$ *for a.a.* i. *Then the following sequence of* $\widetilde{\prod}_{i\in I} R_i$-*modules is exact*:

$$0\longrightarrow \widetilde{\prod_{i\in I}} M_i'\longrightarrow \widetilde{\prod_{i\in I}} M_i\longrightarrow \widetilde{\prod_{i\in I}} M_i''\longrightarrow 0.$$

As an application of this theorem one can compute separated ultraproducts in some cases.

Example. (1) If $R_i=R$ for all $i\in I$ and k is a coefficient field of $\hat{R}$, then the separated ultrapower $\tilde{R}=\widetilde{\prod} R$ is isomorphic to $\hat{R}\hat{\otimes}_k k^*$ where k^* is the ultrapower of the field k in the usual sence. In general it can be proved that $\tilde{R}$ is always faithfully flat over R.

(2) If $I=N$ and $R_i=k[\![x, y]\!]/(x^2+y^i)$ then $\widetilde{\prod}_{i\in I} R_i=k^*[\![x, y]\!]/(x^2)$.

Section 4

We now show the brief sketch of the proof of the implication from (1) to (2) in Theorem 2.

Recall that T is a regular local ring and R is a finite local T-algebra such that $d=\dim(T)=\dim(R)$. Let $x=\{x_1, \cdots, x_d\}$ be a regular system of parameters for T and let $T_n=T/(x_1^n, \cdots, x_d^n)$ and $R_n=R/(x_1^n, \cdots, x_d^n)R$ for all n. By the assumption there is a T_n-injective R_n-module M_n for each n, where one should remark that every M_n is T_n-free. Consider $\tilde{M}=\tilde{\prod}_{i\in N} M_i$. We claim that $\tilde{M}$ is a big Cohen-Macaulay module with respect to x. Equivalently we show the following

Claim. $x_1(s_i)^\sim+\cdots+x_k(t_i)^\sim+x_{k+1}(u_i)^\sim=0$ $(0\leq k<d)$ implies $(u_i)^\sim\in(x_1, \cdots, x_k)\tilde{M}$.

Under the assumption of this claim, one can easily prove that, for any integer n, u_i belongs to $(x_1, \cdots, x_k, x_{k+1}^{n-1}, x_{k+2}^n, \cdots, x_d^n)M_i$ for a.a. i. If we denote $\bar{M}_i=M_i/(x_1, \cdots, x_k)M_i$ and denote the class of u_i in $\bar{M}_i$ by $\bar{u}_i$, then this shows that, for any n, $\bar{u}_i$ belongs to $(x_{k+1}^{n-1}, x_{k+2}^n, \cdots, x_d^n)\bar{M}_i$ for a.a. i. This implies $(\bar{u}_i)^\sim=0$ in $\tilde{\prod}\bar{M}_i$.

On the other hand by Theorem 3 the following sequence is exact:

$$0\longrightarrow\tilde{\prod}(x_1, \cdots, x_k)M_i\xrightarrow[f]{}\tilde{\prod}M_i\xrightarrow[g]{}\tilde{\prod}\bar{M}_i\longrightarrow0$$

(One can check the validity of the assumption of Theorem 3 in this case.) As we have seen above, we have $g((u_i)^\sim)=(\bar{u}_i)^\sim=0$. Hence there is an element $(x_1a_i+\cdots+x_kb_k)^\sim$ in $\tilde{\prod}_{i\in I}(x_1, \cdots, x_k)M_i$ which satisfies $f((x_1a_i+\cdots+x_kb_k)^\sim)=(u_i)^\sim$, equivalently $(u_i)^\sim=x_1(a_i)^\sim+\cdots+x_k(b_i)^\sim\in(x_1, \cdots, x_k)\tilde{M}$. This completes the proof of the claim, hence Theorem 2.

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