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The Dilworth Number of Artinian Rings and Finite Posets with Rank Function

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Introduction

In my paper [13] I proved that for any ideal $\mathfrak{a}$ of an Artinian local ring A and for any non-unit element y in A there is an inequality $\mu(\mathfrak{a}) \leqq l(A/yA)$. Thus we are naturally led to consider the numbers $d(A) := \mathrm{Max}\,\{\mu(\mathfrak{a})\}$ and $r(A) := \mathrm{Min}\,\{l(A/yA)\}$. The present paper has two purposes: (1) To give a combinatorial interpretation of the number $d(A)$ and (2) to study one case where the equality $d(A)=r(A)$ holds.

Although the problems concerning the number of generators of ideals have drawn considerable attention (for examples Sally [9]), the number $d(A)$ of an Artinian ring A does not seem to have ever been considered explicitly. But as soon as one tries to compute the number, taking an example of Artinian ring of "monomial type", one realizes that this is quite a combinatorial question, and fortunately some theorems and certain ideas in combinatorics are available for the purpose. To mention some of these, Dilworth's theorem, Sperner property and symmetric chain decomposition of posets. I called the number $d(A)$ the Dilworth number of the Artinian ring A because with an Artinian ring A of monomial type a poset is naturally associated and $d(A)$ coincides with what the combinatorists call the Dilworth number of the poset.

As to the number $r(A)$, I called it the Rees number because Rees [8] defined the notion of general elements of local rings in a general setting. The definition of a general element in the Artinian case adopted in [13] and in the present paper is slightly different from his: namely we say that y is a general element of A if $l(A/yA)=r(A)$ provided that A has an infinite residue field. The significance of this number is that it bounds the number of generators of ideals of the ring. I.e., $d(A) \leqq r(A)$. Now a natural question arises: when does the equality hold? To answer this question seems very difficult, and because a general theory cannot be expected at this time, what we do here is to consider a certain class of

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Artinian rings which satisfy some extreme condition, which is equivalent to $\mathrm{Max}_i \{\mu(\mathfrak{m}^i)\}=r(A)$, where $\mathfrak{m}$ is the maximal ideal. It should be mentioned that this is a ring-theoretic version of the Sperner property of posets (or at least a derivative of it). As in the theory of posets this condition is handy to deal with. Even stronger and in some sense easier condition is what we called the strong Stanley property (Definition 3.1), which might better be described by saying that it is a "graded Artinian ring on which the hard Lefschetz theorem holds". I called this the (strong) Stanley property because in his papers [11] and [12] Stanley used the hard Lefschetz theorem to obtain some combinatorial results, which was quite inspiring to my problem.

It will be shown that the strong Stanley property is closed under taking tensor product and quotient by an ideal of the form $0:f$ provided that f is sufficiently general. Thus we obtain a large class of Artinian rings having the Stanley property. They give us, in particular, the equality $d(A)=r(A)$, which was our original purpose.

When a graded Artinian ring satisfies the Stanley property, it has automatically a unimodal Hilbert function. But the equality $d(A)=r(A)$ is satisfied by many Artinian rings with non-unimodal Hilbert function. For example let B be a polynomial ring over a field modulo a power of the homogeneous maximal ideal with canonical module B^*, and let $A=B\oplus B^*$ be the ring obtained by the principle of idealization. Then it can be proved that $d(A)=r(A)$. But as was pointed out by Stanley [10], A does not in general have a unimodal Hilbert function. This suggests possibility of some other methods to deal with the question about the equality. Nevertheless the rings with the Stanley property are of independent interest by themselves.

The reader is assumed to be a commutaive ring theorist without specialized knowledge of combinatorics. Therefore all the necessary definitions and theorems in combinatorics that we use are collected in Section 1. But this is not meant to be anything like a systematic introduction to the subject. This is just to familiarize the reader with the terminology as quickly as possible. For details we refer to [1] and [2].

In Section 2 we consider, except in the last theorem, only Artinian rings of monomial type. We show that in such a ring the number $d(A)$ can be interpreted as the Dilworth number of the poset formed by the monomials of the ring (Lemma 2.4). Moreover we give a combinatorial proof for the inequality $d(A)\leq r(A)$. It is hoped that the proof elucidates the nature of the number and the inequality. Proposition 2.5 and Theorem 2.6 should show how the combinatorial theory is helpful to the study of Artinian rings.

In Section 3 we define weak and strong Stanley property for graded

Artinian rings. The weak Stanley property has a numerical characterization (Proposition 3.2) while the strong Stanley property is characterized in terms of the Lie algebra sl_2 (Proposition 3.4). For basic facts of the representation theory of sl_2 we refer to [7]. Theorem 3.8 is our main result; however, more interesting than the theorem itself is Example 3.9, which suggests that most Gorenstein rings have the strong Stanley property.

As in [13], we use the following notation; l for length, μ for the minimal number of generators and τ for the type of an ideal, hence $\mu(\mathfrak{a})=l(\mathfrak{a}/\mathfrak{a}\mathfrak{m})$ and $\tau(\mathfrak{a})=l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a})$.

§ 1. Dilworth's theorem

A (finite) digraph $D=(V, A)$ consists of a set of vertices $V=\{p_1, p_2, \cdots, p_n\}$ and a set of ordered pairs $A\subset V\times V$ of vertices. A is called the set of arcs. Note that a digraph can have loops. The adjacency matrix $M=(m_{ij})$ of D is the $n\times n$ matrix such that $m_{ij}=1$ if $(p_i, p_j)\in A$ and $m_{ij}=0$ otherwise. $(p_i, p_j)\in A$ may be written $p_i\to p_j$. Since a digraph is determined by its adjacency matrix, any 0-1 matrix may be thought of as defining a digraph. The reachability matrix $R=(r_{ij})$ of D is defined by $r_{ij}=1$ if there is a directed path, i.e,. a sequence $p\to p'\to p''\to\cdots$, starting with p_i and ending with p_j, and $r_{ij}=0$ otherwise. If M is the adjacency matrix and if $M^k=(m_{ij}^{(k)})$, then $m_{ij}^{(k)}$ is the number of directed paths of length k starting at p_i and ending at p_j. Hence M and R are related as follows: Let $\tilde{M}=(\tilde{m}_{ij})=M+M^2+M^3+\cdots$ ($\tilde{M}$ may contain ∞). Then $r_{ij}=1$ if $\tilde{m}_{ij}>0$ and $r_{ij}=0$ if $\tilde{m}_{ij}=0$. (R may be called the "0-1-fication" of $\tilde{M}$.)

If $D=(V, A)$ is a digraph, any subset A' of A defines a digraph (V, A') called a spanning subdigraph of D. When $D=(V, A)$ is a digraph we usally say that V, the set of vertices, is a digraph, in which case A should be clear from the context. For example a poset (partially ordered set) is a digraph.

Let P be a poset. A subset of P is a chain if any two elements in it are comparable. A subset of P is an independent set (or antichain) if any two elements in it are incomparable. Now let us state Dilworth's theorem.

Theorem 1.1 (Dilworth, 1950). *For a (finite) poset P, the maximum cardinality of independent sets is equal to the minimum number of disjoint chains into which P is decomposed.*

For proof see, for example, [1] 8.14, also [4] p. 61.

The common number in the theorem is called the Dilworth number of P, and is denoted by $d(P)$.

Let $Y=(y_{ij})$ be any matrix. A matching of Y is a subset M of $\{(i,j)\mid y_{ij}\neq 0\}$ having the following property:

$$(i,j),(i',j')\in M,\ (i,j)\neq(i',j') \quad\Rightarrow\quad i\neq i' \quad\text{and}\quad j\neq j'.$$

The matching number of Y is the maximum cardinality of matchings of Y. This will be denoted by $\beta(Y)$.

From a proof of Dilworth's theorem (for example [4] p. 61), one knows the following

Theorem 1.2. *Let $P=\{p_1, p_2, \cdots\}$ be a poset, and let $Z=(z_{ij})$ be the relation matrix: $z_{ij}=1$ if $p_i<p_j$ and $z_{ij}=0$ otherwise. Then $d(P)=|P|-\beta(Z)$.*

A poset P is said to be graded if it has a rank function $\mathrm{rk}\colon P\to\{0, 1, 2, \cdots\}$ satisfying (1) $\mathrm{rk}(a)=0$ if a is a minimal element, and (2) $\mathrm{rk}(b)=\mathrm{rk}(a)+1$ if $a<b$ and there are no elements properly between a and b. For such a poset we write $P_k=\{a\in P\mid \mathrm{rk}(a)=k\}$. P is said to have the Sperner property if $d(P)=\mathrm{Max}_k|P_k|$.

Let $e_1, e_2, \cdots, e_n$ be positive integers and let $M(e_1, e_2, \cdots, e_n)$ be the set of integral vectors $\{(a_1, a_2, \cdots, a_n)\mid 0\leqq a_i\leqq e_i$ for all $i\}$. Define the relation $(a_i)\leqq(a_i')$ to mean $a_i\leqq a_i'$ for all i. Then $M(e_1, \cdots, e_n)$ is a graded poset with $\mathrm{rk}((a_i))=\sum a_i$. This is a called a lattice of multisets of a set. Note that this is (isomorphic to) a divisor lattice. This is also described as a chain product: $M(e_1, \cdots, e_n)=M(e_1)\times\cdots\times M(e_n)$.

The following theorem is interesting to us because it gives us the Dilworth number of a monomial complete intersection. (See Proposition 2.5.) Conversely any ring theoretic proof of Proposition 2.5 should prove this theorem.

Theorem 1.3 (deBruijn, Tengbergen, Kruyswijk, 1952). *$M(e_1, \cdots, e_n)$ have the Sperner property. More precisely, $d(P)=\#\{(a_i)\mid\sum a_i=m\}$, where $m=[\frac{1}{2}\sum e_i]$.*

Various proofs are known. For example see [1] Chapter 8 and [5].

§ 2. The Dilworth number of Artinian rings

Definition 2.1. Let $(A, \mathfrak{m})$ be an Artinian ring. Define:

$$d(A)=\mathrm{Max}\,\{\mu(\mathfrak{a})\mid \mathfrak{a}\text{ ideal in }A\},$$
$$r(A)=\mathrm{Min}\,\{l(A/yA)\mid y\in\mathfrak{m}\}.$$

$d(A)$ will be called the Dilworth number and $r(A)$ the Rees number of A.

Remark 2.2. An element $y \in \mathfrak{m}$ is called a general element of A if $l(A/yA)=r(A')$, where $A'=A(X)$ is the polynomial ring $A[X]$ in one variable localized at $\mathfrak{m}A[X]$. Provided that $A/\mathfrak{m}$ is infinite, a general element exists, and it is an element of $\mathfrak{m}\setminus\mathfrak{m}^2$. In partiuclar, if A is homogeneously graded over an infinite field, a homogeneous general element exists, and it is a linear form. For details of general elements we refer to [13], Appendix.

The following inequality was proved in [13].

Theorem 2.3. $d(A)\leqq r(A)$.

Proof. In Theorem 1, [13], put $R=A$, $\mathfrak{a}=0$.

Now we consider Artinian rings of the form $k[X_1, \cdots, X_n]$/(monomials), where k is a field, i.e., a polynomial ring over k modulo an ideal generated by monomials. Let A be such a ring, and let P be the set of all the monomials in A. Then P is made into a poset by defining $p\leqq p'$ if and only if p divides p'. We will denote this poset by $P(A)$. For example, if $A=k[X_1, \cdots, X_n]$/(powers of $X_1, \cdots, X_n$), then $P(A)$ is a divisor lattice. If $A=k[X_1, \cdots, X_n]/(X_1^2, \cdots, X_n^2)$, then $P(A)$ is a Boolean lattice. More generally, suppose $B=k[X_1, \cdots, X_n]/I(\Delta)$ is a Stanley-Reisner ring, with Δ, a simplicial complex, and let $A=B/(X_1^2, \cdots, X_n^2)$. Then $P(A)$ is the lattice formed by the faces of Δ. Notice that a set of monomials is an independent set in $P(A)$ if and only if it is a minimal set of generators of an ideal in A. Hence the following lemma shows that $d(A)=d(P(A))$, i.e., the Dilworth number of the Artinian ring A, as defined above, is the same as that of $P(A)$ in the original combinatorial sense.

Lemma 2.4. *For A as above, the following three numbers are equal:*
(1) $d=\mathrm{Max}\,\{\mu(\mathfrak{a})\,|\,\mathfrak{a}$ *ideal*$\}$
(2) $d'=\mathrm{Max}\,\{\mu(\mathfrak{a})\,|\,\mathfrak{a}$ *homogeneous ideal*$\}$
(3) $d''=\mathrm{Max}\,\{\mu(\mathfrak{a})\,|\,\mathfrak{a}$ *monomial ideal*$\}$

Proof. First we prove $d=d'$. Obviously $d\geqq d'$. Assume $d>d'$. Let $I=(a_1, a_2, \cdots, a_d)$ with $\mu(I)=d$. For $a\in A$, let us denote by a° the initial form of a. Then the assumption $d>d'$ implies that $\mu((a_1^\circ, \cdots, a_d^\circ))<d$. Hence one of a_i°'s, say a_1°, is a linear combination of the others with homogeneous coefficients: $a_1^\circ=\sum_{i=2}^d b_i a_i^\circ$. Consider the set $\{a_1-\sum_{i=2}^d b_i a_i, a_2, \cdots, a_d\}$. This obviously generates I, but the degree of the first generator has been increased at least by 1. This process, if repeated, will lead us to a contradiction because the totality of degrees of generators should be bounded. We can prove $d'=d''$ in the same way. I.e., among the monomials of the same degree, we put a lexicographic order, and for a

homogeneous element a, we call the exponent of the least monomial that occurs in a the "fine degree" of a. Then the assumption $d'>d''$ would increase the fine degrees of generators infinitely. Q.E.D.

Now we can apply Theorem 1.3 to obtain the following

Proposition 2.5. *Let A be a monomial complete intersection, i.e., $A=k[X_1, \cdots, X_n]/(X_1^{e_1+1}, \cdots, X_n^{e_n+1})$. Then $d(A)=\mu(\mathfrak{m}^m)$, where $m=[\frac{1}{2}\sum e_i]$.*

Let A be as before, and let $y=X_1+\cdots+X_n$. Note that y is a general element of A. (To prove this we may assume that k is infinite. Then a general element will be of the form $y'=t_1X_1+\cdots+t_nX_n$ with $t_i \in k^*$. Since A is defined by monomials, the substitution $X_i \to t_i^{-1}X_i$ is an automorphism of A, sending y' to y.) Consider the multiplication $y: A \to A$ as a linear map over k. This will be represented by a 0-1 matrix on the set of monomials as a basis. Thus it defines a digraph $D(A)$ on the set $P(A)$ of monomials of A. It is easy to see that the arcs of $D(A)$ generate the relations of $P(A)$. In other words, the "0-1-fication" of $1+y+y^2+y^3+\cdots$ (considered as a matrix) is the relation matrix of $P(A)$. Let $Y=y+y^2+\cdots$. Obviously $\mathrm{rank}(y)=\mathrm{rank}(Y)$, because y is nilpotent. Morevoer $\beta(Y)\geqq\mathrm{rank}(Y)$. These and Theorem 1.2 imply $d(P(A))=|P(A)|-\beta(Y) \leqq |P(A)|-\mathrm{rank}(y)=l(A/yA)=r(A)$. This is an alternative proof for Theorem 2.3 in the monomial case. It has turned out that the inequality in the theorem is essentially the general inequality $\mathrm{rank}(Y)\leqq\beta(Y)$ which holds for any matrix Y. What is interesting is that it has a generalization to arbitrary Artinian rings (i.e., Theorem 2.3) and moreover in many cases the inequality is actually an equality (cf. Theorem 3.8).

We will show another example of analogy that stands between posets and rings. Let P be a poset and P^* its dual poset (i.e., the poset obtained by reversing all the orders of P). Then obviously $d(P)=d(P^*)$. When this is translated in terms of rings it gives us a non-trivial result as follows:

Theorem 2.6. *Let $(A, \mathfrak{m})$ be an arbitrary ring. Then the number $\mathrm{Max}\,\{\tau(\mathfrak{a}) \mid \mathfrak{a}\subset A\}$ is equal to $d(A)$.*

(This is the translation of $d(P)=d(P^*)$ because P corresponds to the canonical module K of A and μ in K corresponds to τ in A.)

Proof. Notice that if $\mathfrak{a}$ has the property that $\mu(\mathfrak{a})\geqq\mu(\mathfrak{b})$ for all $\mathfrak{b}$ such that $\mathfrak{b}\supset\mathfrak{a}$, then $\mathfrak{a}\mathfrak{m}:\mathfrak{m}=\mathfrak{a}$. In fact put $\mathfrak{b}=\mathfrak{a}\mathfrak{m}:\mathfrak{m}$. Then $\mathfrak{m}\mathfrak{b}=\mathfrak{m}\mathfrak{a}$. Hence $\mathfrak{b}\supsetneqq\mathfrak{a}$ implies that $\mu(\mathfrak{b})>\mu(\mathfrak{a})$. Dually, assume that $\mathfrak{a}$ has the property: $\tau(\mathfrak{a})\geqq\tau(\mathfrak{b})$ for all $\mathfrak{b}\subset\mathfrak{a}$. Then it follows that $\mathfrak{m}(\mathfrak{a}:\mathfrak{m})=\mathfrak{a}$. (In fact put $\mathfrak{b}=\mathfrak{m}(\mathfrak{a}:\mathfrak{m})$. Then $\mathfrak{b}:\mathfrak{m}=\mathfrak{a}:\mathfrak{m}$. If $\mathfrak{b}\subsetneqq\mathfrak{a}$, then $\tau(\mathfrak{b})=l(\mathfrak{b}:\mathfrak{m}/\mathfrak{b})>l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a})=\tau(\mathfrak{a})$.) Let $T=\mathrm{Max}\,\tau(\mathfrak{a})$, and let $\mathfrak{a}$ be such that $\tau(\mathfrak{a})=T$. Then

$\mu(\mathfrak{a}:\mathfrak{m})=l(\mathfrak{a}:\mathfrak{m}/\mathfrak{m}(\mathfrak{a}:\mathfrak{m}))=l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a})=\tau(\mathfrak{a})$. Hence $d(A)\geqq T$. The dual argument shows that $d(A)\leqq T$ as well. Q.E.D.

§ 3. The Stanley property of Artinian rings

In this section we consider only Artinian rings homogeneously graded over a field k of characteristic 0. Hence whenever we write $A=\bigoplus_{i=0}^{n} A_i$, it will be assumed that $A_0=k$, char $k=0$, $A=A_0[A_1]$, and moreover $A_n\neq 0$. If $f\in A_d$, then the notation like $f\colon A_i\to A_{i+d}$ will mean the homogeneous part of the multiplication $f\colon A\to A$, considered as a k-vector space homomorphism. We also use the notations $0:f$ and Ker $[f\colon A\to A]$, and $(0:f)_i$ and Ker $[f\colon A_i\to A_{i+d}]$ interchangably.

Definition 3.1. For $A=\bigoplus_{i=0}^{n} A_i$ as above, we say that A has the strong Stanley property (SSP) if there exists $g\in A_1$ such that $g^{n-2i}\colon A_i\to A_{n-i}$ is bijective for $i=0, 1, 2, \cdots, [n/2]$. We say that A has the weak Stanley property (WSP) if (1) the Hilbert function $i\to\dim A_i$ is unimodal and (2) there exists $g\in A_1$ such that $g\colon A_i\to A_{i+1}$ is either injective or surjective for every i. In these cases we will say that the pair (A, g) has SSP or WSP in the obvious sense.

It is easy to see that SSP implies WSP. Let us make one more definition: We call $\mathrm{Max}_i\{\dim A_i\}$ the Sperner number of A, and denote it by $s(A)$. Using this we may characterize WSP as follows.

Proposition 3.2. *$A=\bigoplus_{i=0}^{n} A_i$ has WSP if and only if $s(A)=r(A)$. In this case $d(A)=r(A)$.*

Proof. Suppose (A, g) has WSP. Then

$$r(A)\leqq l(A/gA)=\sum_{i=0}^{n}\dim_k(A_i/gA_{i-1})=s(A).$$

But $s(A)\leqq r(A)$ holds generally. Thus we have $s(A)=r(A)$. This argument shows the converse, too. For the second statement notice that, in general, $s(A)\leqq d(A)\leqq r(A)$, the second inequality being Theorem 2.3.

Remark 3.3. Suppose (A, g) has WSP. Then the above proof shows that g is a general element of A. In this case (A, g') has WSP for any (homogeneous) general element g' of A.

Proposition 3.4. *Let $A=\bigoplus_{i=0}^{n} A_i$ be a homogeneously graded Artinian ring. Then (A, g) has SSP if and only if there exists a representation $\rho\colon sl_2\to\mathrm{End}_k(A)$, of the special linear Lie algebra sl_2 with $\rho\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\cdot g$*

(=*multiplication by g*) *such that the weight space decomposition coincides with the natural grading decomposition.*

Proof. This is essentially the algebraic part of the hard Lefschetz theorem in algebraic geometry, for which we refer to [6]. Here we just outline the proof. First assume that we have a representation of sl_2 having the property stated. The key point is that a finite dimensional representation of sl_2 is completely reducible, and for each positive integer s, there is a unique $V(s)$ of $(s+1)$-dimensional irreducible sl_2-module. The module structure of $V(s)$ is given as follows: With a basis $\{v_0, v_1, \cdots, v_s\}$ of $V(s)$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} v_i = v_{i+1}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v_i = i(s+1-i)v_{i-1}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v_i = (s-2i)v_i$, where we set $v_{-1} = v_{s+1} = 0$. From this the assertion follows easily. Conversely assume SSP of A. In order to find a representation ρ with the required property, we want to choose a basis of A so that $\cdot g$ is represented by a Jordan canonical form. This can be done as follows: First of all, $1, g, g^2, \cdots, g^n$ will be a part of the basis, giving us a block of the Jordan canonical form of $\cdot g$. Now let $\bar{a} \in \operatorname{Ker}[g: A_{n-1} \to A_n]$. By SSP of A, there is $a \in A_1$ such that $ag^{n-2} = \bar{a}$. Then the elements $a, ag, ag^2, \cdots, ag^{n-2}$, none of these being dependent of the previously chosen elements, will be another part of the basis. If $\operatorname{Ker}[g: A_{n-1} \to A_n]$ has $\bar{a}'$, independent of $\bar{a}$, then we choose $a' \in A_1$ such that $g^{n-2}a' = \bar{a}'$, and let $a', a'g, \cdots, a'g^{n-2}$ be a third part of the basis. If this process is carried out to the end, this obviously gives us a basis of A on which $\cdot g$ is written as a Jordan canonical form. Now we may construct ρ block for blcok, so that $\rho\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = g$ and $\rho\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is a "degree -1 map" and if v is an element of degree i in the basis then $\rho\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v = (n-2i)v$. This completes the proof.

Corollary 3.5. *Suppose that the Artinian rings* (A, g) *and* (B, h) *with* $A_0 = B_0 = k$ *have SSP. Then so does* $(A \otimes B, g \otimes 1 + 1 \otimes h)$. *In particular,* $k[X_1, \cdots, X_n]/(X_1^{e_1}, \cdots, X_n^{e_n})$ *has SSP with* $X_1 + \cdots + X_n$ *as a general element.*

Proof. This follows immediately from the tensor representation of sl_2.

Definition 3.6. Let $A = \bigoplus_{i=0}^n A_i$ be a graded Artinian ring. An element $f \in A_d$ is said to be general if, equivalently:

(1) $l(A/fA) = \operatorname{Min}\{l(A/f'A) \mid f' \in A_d\}$;

(2) $l(A/0:f) = \operatorname{Max}\{l(A/0:f') \mid f' \in A_d\}$.

Remark 3.7. (1) What we have been saying "a general element of A" coincides with a general linear form in the above definition, provided that it is homogeneous. For details of general elements see [13] Appendix.

(2) The set $\{f \in A_d \mid \text{general}\}$ is a Zariski open set of A_d considered as an affine space.

Theorem 3.8. *Suppose that a homogeneously graded Artinian ring* $A = \bigoplus_{i=0}^{n} A_i$ *has SSP with a general element* $g \in A_1$ *having the property in the definition of SSP. Let* $f \in A_d$. *Then:*

(0) g^i *is a general element of degree* i *for every* i.

(1) *If* f *is general, then* $A/0:f$ *has a unimodal Hilbert function.*

(2) *If* f *is general for both* A *and* $A/0:g$, *then* $(A/0:f, \bar{g})$ *has WSP.*

(3) *If* f *is general for all* $A/0:g^i$, $i=0, 1, \cdots, n$, *then* $(A/0:f, \bar{g})$ *has SSP.*

Proof. (0) This follows immediately from the fact that for every j the matrix g^i: $A_j \to A_{j+i}$ has full rank.

(1) f and g^d have the same rank because they are both general of degree d, from which it follows that f: $A_i \to A_{i+d}$ is either injective or surjective according as $\dim A_i \leqq \dim A_{i+d}$ or $\dim A_i \geqq \dim A_{i+d}$. Note that the i-th graded piece of $A/0:f$ is $A_i/\mathrm{Ker}\,[A_i \xrightarrow{\cdot f} A_{i+d}]$. Hence we may identify $(A/0:f)_i = A_i$ or A_{i+d} according as f: $A_i \to A_{i+d}$ is injective or surjective. Let $m = \mathrm{Min}\,\{i \mid \dim A_i > \dim A_{i+d}\}$. Then with the identification made above $A/0:f$ decomposes:

$$(*) \qquad A/0:f = A_0 \oplus A_1 \oplus \cdots \oplus A_{m-1} \oplus A_{m+d} \oplus A_{m+d+1} \oplus \cdots \oplus A_n.$$

Thus the unimodality of the Hilbert function of $A/0:f$ follows easily from that of A.

(2) With the identification $(*)$ above, the action of g on $A/0:f$ may be described as

$$A_0 \xrightarrow{g} A_1 \xrightarrow{g} \cdots \xrightarrow{g} A_{m-1} \xrightarrow{gf} A_{m+d} \xrightarrow{g} A_{m+d+1} \xrightarrow{g} \cdots \xrightarrow{g} A_n.$$

In fact the only part which might not be clear is $A_{m-1} \to A_{m+d}$. But recall the identification $A_{m+d} = A_m/\mathrm{Ker}\,[A_m \xrightarrow{f} A_{m+d}] = fA_m$. This shows that the map $A_{m-1} \to A_{m+d}$ is indeed given by $\cdot gf$. To prove WSP of $A/0:f$, it suffices to show that the matrix gf: $A_{m-1} \to A_{m+d}$ has full rank. Since f is general of degree d for $A/0:g$, we have $l(A/0:g+fA) = \mathrm{Min}\,\{l(A/0:g+f'A) \mid f' \in A_d\}$ or equivalently, $l(A/(0:g):f) = \mathrm{Max}\,\{l(A/(0:g):f') \mid f' \in A_d\}$, i.e., $l(A/0:gf) = \mathrm{Max}\,\{l(A/0:gf') \mid f' \in A_d\}$. Note that $g^d \in A_d$ gives us this maximum value because g^{d+1} is a general element. This shows that the matrix gf: $A_{m-1} \to A_{m+d}$ has the same rank as g^{d+1}: $A_{m-1} \to A_{m+d}$.

Thus it has full rank. This proves (2).

(3) As in the proof of (2), SSP is proved if we can show that $g^i f$: $A_j \to A_{j+d+i}$ has full rank for every i and every j. This follows, exactly in the same manner as in the proof of (2), from the hypothesis that f is general for all $A/0:g^i$.

Example 3.9. Recall that every Artinian Gorenstein ring, graded over a field k, can be written as $k[X_1, \cdots, X_n]/(X_1^{e_1}, \cdots, X_n^{e_n}):f$ for some homogeneous form f of some degree, with some exponents $e_1, \cdots, e_n$. (See [3], Proposition 1.3, also [14], Lemma 4.) We may apply Theorem 3.8 (3) to $A=k[X_1, \cdots, X_n]/(X_1^{e_1}, \cdots, X_n^{e_n})$ with $g=X_1+\cdots+X_n$. Hence if f is sufficiently general, this ring has SSP. In view of Remark 3.7, (2), it turns out that "most" Gorenstein rings have SSP. In these rings, in particular, the Dilworth number and the Rees number coincide.

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