Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 285–301

Theory and Applications of Universal Linkage

Bernd Ulrich*

Introduction

This paper is based on a talk given at the Kyoto conference. It is in some sense a report on joint work with C. Huneke ([9]), but also contains several new results. Proofs are only included as far as they differ from the ones given in ([9]), or if they are proofs of new results.

Two proper ideals I and J in a local Gorenstein ring R are said to be *linked* (write $I \sim J$) if there is a R-regular sequence $\underline{\alpha}=\alpha_1, \cdots, \alpha_g$ in $I \cap J$ such that $J=(\underline{\alpha}): I$ and $I=(\underline{\alpha}): J$. This definition was introduced by Peskine and Szpiro who rediscovered and formalized the notion of linkage in their paper [16]. To turn linkage into an equivalence relation one considers the linkage class of an ideal I, which is the set of all R-ideals obtained from I by a finite sequence of links. We say that I is *licci* if I is in the *li*nkage *c*lass of a *c*omplete *i*ntersection ideal. It is one of the main themes in linkage theory to find necessary and sufficient conditions for two ideals to be in the same linkage class, or at least to give a characterization of licci ideals.

So far a complete solution to this problem exists only for ideals of low codimension: Let I be an ideal of grade at most two, then Apéry and Gaeta have shown that I is licci if and only if I is perfect ([1], [4]), and Hartshorne and Rao generalized this result to the non-perfect case ([17]). Moreover J. Watanabe has shown that a perfect ideal of grade 3 is licci, if R/I is Gorenstein ([22]).

While these results are the only known general sufficient conditions for two ideals to be in the same linkage class, various authors were more successful in finding necessary conditions for ideals to belong to the same linkage class, i.e., in finding properties which are invariant under linkage. First note that perfectness is preserved by linkage ([16]). As further examples for invariant properties we only mention conditions on the depth of conormal modules and Koszul homology modules ([2], [3], [5], [6], [7]). These results can be effectively used to show that certain ideals do not belong to the same linkage class ([6], [7], [14], [20]). In particular

Received December 17, 1985.
* Supported in part by the National Science Foundation.

one can construct infinitely many linkage classes of arithmetically Gorenstein curves in $\boldsymbol{P}^5$ (arithmetically Cohen-Macaulay curves in $\boldsymbol{P}^4$), contrasting the above mentioned fact that there is only one linkage class of arithmetrically Gorenstein curves in $\boldsymbol{P}^4$ (arithmetically Cohen-Macaulay curves in $\boldsymbol{P}^3$).

The problem of finding properties invariant under linkage, can be further narrowed to the following question: What is the structure of licci ideals? As one of the reasons for studying licci ideals one should mention that this notion not only allows a unified treatment of known examples ([1], [4], [22]), but also creates new examples of ideals: Let k be a field, let X be generic alternating $2n\times 2n$ matrix, let Y be a generic $2n\times 1$ matrix (for $n\geq 2$), let $R=k[X, Y]_{(X,Y)}$, and let I be the R-ideal generated by the Pfaffian of X and the entries of the product matrix XY. Then I is licci and moreover I is a Gorenstein ideal of deviation 2 and grade $2n-1$. These ideals were introduced in [8] and extensively studied in [11], and so far (up to trivial operations) they are the only known Gorenstein ideals of deviation 2. (For further interesting examples of licci ideals we refer to [12]).

In this paper we will mainly—although not exclusively—study the structure of licci ideals. In particular we will investigate the singular locus and the divisor class group of algebras defined by such ideals. Our main tool is the concept of universal linkage which was introduced in [9] and which allows us to replace any sequence of links by a sequence of links with very specific properties.

§ 1. Universal linkage

Throughout the paper we will use the following notations: Let (R, m) be a Noetherian local ring, let I be an R-ideal, and let M be a finitely generated R-module. Then $\nu(M)$ will be the minimal number of generators of M, $d(I)=\nu(I)-\text{grade}(I)$ will be the deviation of I, $r(R)$ the type of R (in case R is Cohen-Macaulay), K_R the canonical module of R (in case it exists), and for a finite set of indeterminates X we write $R(X)=R[X]_{mR[X]}$. The ideal I is said to be Cohen-Macaulay, Gorenstein, or regular if R/I has any of these properties. We say that R satisfies (G_k) if R_p is Gorenstein for all $p\in \text{Spec}(R)$ with $\dim R_p\leq k$, and I satisfies (CI_k) if I_p is a complete intersection for all $p\in V(I)$ with $\dim (R/I)_p\leq k$. In Spec(R) we consider the subsets $\text{Sing}(R)=\{p\mid R_p$ is not regular$\}$, $NG(R)=\{p\mid R_p$ is not Gorenstein, $NCI(I)=\{p\mid p\in V(I)$, I_p is not a complete intersection$\}$. For a matrix C with entries in R we denote by $I_t(C)$ the R-ideal generated by all $t\times t$ minors of C.

We need one more definition concerning linkage:

Definition 1.1. Let R be a local Gorenstein ring and let I and J be two R-ideals. We say that J is *minimally linked* to I (write $I \to J$) if I and J are linked with respect to a regular sequence $\alpha_1, \cdots, \alpha_g$ which forms part of a minimal set of generators for I.

Remark 1.2 ([16]). Let R be a local Gorenstein ring, let I be an unmixed ideal of grade g in R, let $\underline{\alpha}=\alpha_1, \cdots, \alpha_g \subset I$ be a regular sequence with $(\underline{\alpha}) \neq I$, and set $J=(\underline{\alpha}): I$. Then

a) I and J are linked.

b) I is Cohen-Macaulay if and only if J is Cohen-Macaulay.

c) If I is Cohen-Macaulay, then $K_{R/J} \cong I/(\underline{\alpha})$, $K_{R/I} \cong J/(\underline{\alpha})$.

From Remark 1.2 one obtains immediately:

Remark 1.3.

a) $r(R/J) \geq d(I)$

b) $r(R/J)=d(I)$ if and only if $I \to J$ is minimal.

Definition 1.4. Let (R_i, I_i) be pairs, where R_i are Noetherian rings, and $I_i \subset R_i$ are ideals or $I_1=R_1$ or $I_2=R_2$.

a) We write $(R_1, I_1) \cong (R_2, I_2)$, if there is an isomorphism $\varphi: R_1 \to R_2$ with $\varphi(I_1)=I_2$.

b) We say that (R_1, I_1) and (R_2, I_2) are *equivalent* and write $(R_1, I_1) \equiv (R_2, I_2)$, if there are finite sets of indeterminates X and Z such that $(R_1[X], I_1R_1[X]) \cong (R_2[Z], I_2R_2[Z])$.

c) Let R_1 and R_2 be local. We say that (R_1, I_1) and (R_2, I_2) are *generically equivalent* and write $(R_1, I_1) \approx (R_2, I_2)$, if there are finite sets of indeterminates X and Z such that $(R_1(X), I_1R_1(X)) \cong (R_2(Z), I_2R_2(Z))$.

We can now define generic and universal links as was done in [9].

Let R be a (not necessarily local) Gorenstein ring, let I be an unmixed R-ideal of height $g>0$, let $\underline{f}=f_1, \cdots, f_n$ be a generating sequence of I, let Y be a generic $g \times n$ matrix, write $T=R[Y]$, set

$$\alpha_i=\sum_{j=1}^{n} Y_{ij}f_j, \qquad 1 \leq i \leq g,$$

and define a T-ideal $L_1(\underline{f})=(\underline{\alpha})T: IT$. By 1.2.a, $L_1(\underline{f})$ and IT are (locally) linked with respect to the quasiregular sequence $\underline{\alpha}$. One can show that up to equivalence of pairs in the sense of 1.4.b, the definition of $(T, L_1(\underline{f}))$ only depends on I, but not on the choice of $\underline{f}$ ([9]). We may therefore write $L_1(I)$ instead of $L_1(\underline{f})$, and iterate the above process to define $L_i(I)=L_1(L_{i-1}(I))$ for $i>1$. We call $L_i(I)$ a *i-th generic link of I.*

In addition to the above assumptions let (R, m) be local, and set $S=T_{mT}=R(Y)$ and $L^1(\underline{f})=L_1(\underline{f})S$. Now up to generic equivalence in the sense of 1.4.c, $(S, L^1(\underline{f}))$ only depends on I, and we may write $L^1(I)=L^1(\underline{f})$

([9]). For $i>1$ define inductively $L^i(I)=L^1(L^{i-1}(I))$ if $L^{i-1}(I)$ is not equal to the unit ideal (1), and $L^i(I)=(1)$ otherwise. We call $L^i(I)$ a *i-th universal link of I.*

Before we can list the main properties of generic and universal links we first need more definitions:

Definition 1.5. Let (R, I) and (S, J) be pairs, where R and S are Noetherian local rings, and $I\subset R$, $J\subset S$ are ideals or $I=R$ or $J=S$.

a) We say that (S, J) is a *deformation* of (R, I) if there is a sequence $\underline{a}$ in S which is regular on S and S/J such that $(S/(\underline{a}), (J+\underline{a})/(\underline{a}))=(R, I)$.

b) We say that (R, I) is *smoothable in codimension k* if (R, I) has a deformation (S, J) with $\operatorname{char}(S)=\operatorname{char}(R)$ such that S/J satisfies Serre's condition (R_k).

c) We say that (S, J) is *essentially a deformation* of (R, I), if there is a sequence of pairs (S_i, J_i) of Noetherian local rings S_i and $J_i\subset S_i$, $1\leq i\leq n$, such that $(S_1, J_1)=(R, I)$, $(S_n, J_n)=(S, J)$, and for any $1\leq i\leq n-1$ one of the following three conditions hold:

i) (S_{i+1}, J_{i+1}) is a deformation of (S_i, J_i),

ii) $(S_{i+1}, J_{i+1})=((S_i)_p, (J_i)_p)$ for some $p\in \operatorname{Spec}(S_i)$,

iii) $(S_{i+1}, J_{i+1})\approxeq(S_i, J_i)$.

Now we are able to state one of the main results from [9]:

Theorem 1.6. *Let (R, m) be a local Gorenstein ring, let I be a Cohen-Macaulay R-ideal with grade $I>0$, and let $I\sim I_1\sim\cdots\sim I_n$ be any sequence of links in R. In some $T=R[Z]$ consider a sequence of generic links $IT\sim L_1(I)T\sim\cdots\sim L_n(I)T$, and in some $S=R(X)$ consider a sequence of universal links $IS\sim L^1(I)S\sim\cdots\sim L^n(I)S$.*

a) *There exists $q\in\operatorname{Spec}(T)$ with $m\subset q$ such that $(T_q, L_i(I)T_q)$ is a deformation of (R, I_i) for all i.*

b) *$(S, L^i(I))$ is essentially a deformation of (R, I_i) for all i.*

c) *$(S, L^{2i}(I))$ is essentially a deformation of (R, I) for all even $2i$.*

Since most ring theoretic properties are preserved under essentially a deformation it is now clear from Theorem 1.6.b, c that one can replace an arbitrary sequence of links by a new sequence of (universal) links such that in each even step, $L^{2i}(I)$ shares most "good" properties of I_{2i} as well as I. For most purposes we can even avoid the ring extension S of R, which was necessary to define the universal links, and restrict ourselves to a sequence of links inside R:

Theorem 1.7 ([10]). *Let R be a local Gorenstein ring with infinite residue class field, let I be a Cohen-Macaulay R-ideal with grade $I>0$, and* *let $I \sim I_1 \sim \cdots \sim I_n$ be a sequence of links in R. Then there exists a sequence of links in R, $I \sim J_1 \sim \cdots \sim J_n$, such that for all i,*

a) $\nu(J_{2i}) \leq \min\{\nu(I_{2i}), \nu(I)\}$, $\nu(J_i) \leq \nu(I_i)$,

b) $r(R/J_{2i}) \leq \min\{r(R/I_{2i}), r(R/I)\}$, $r(R/J_i) \leq r(R/I_i)$.

Moreover for some t, $0 \leq t \leq n$, the links $I \to J_1 \to \cdots J_t$ are minimal and J_i are complete intersections for all $i > t$.

Corollary 1.8. *Let R be a regular local ring with infinite residue class field, and let I be a licci Gorenstein R-ideal. Then there exists a sequence of links in R, $I \sim J_1 \sim \cdots \sim J_{2s}$ such that J_{2s} is regular and J_{2i} are Gorenstein for all even $2i$.*

Proof. Let $I \sim I_1 \sim \cdots \sim I_n$ be a sequence of links with I_n a complete intersection. By Theorem 1.7 there exists a sequence of links $I \sim J_1 \sim \cdots \sim J_n$ such that $r(R/J_{2i}) \leq r(R/I) = 1$ for $0 \leq 2i \leq n$, and $\nu(J_n) \leq \nu(I_n) = \text{grade}(I)$. Thus, J_{2i} are Gorenstein for $0 \leq 2i \leq n$, and J_n is a complete intersection in the regular local ring R. But then we can find a sequence of links $J_n \sim J_{n+1} \sim \cdots \sim J_{2s}$ with J_i complete intersections for $n \leq i \leq 2s$, and J_{2s} regular. □

The following result, which will be generalized in Theorem 2.4, follows immediately from Corollary 1.8 and a theorem by Kustin and Miller ([13], 3.11).

Corollary 1.9. *Let k be an infinite field, $R = k[[x_1, \cdots, x_n]]$, and let I be a licci Gorenstein R-ideal. Then (R, I) is smoothable in codimension 6.*

§ 2. Licci ideals

Corollary 1.8 illustrates that Theorems 1.6 and 1.7 can be used for arguments using induction on the number of steps needed to link a licci ideal to a regular ideal. To start the induction we have to investigate ideals which are doubly linked to regular ideals, and in the light of Theorem 1.6.a it suffices to study localizations of the second generic link of regular ideals.

To do this we first introduce some notation.

Let R be a regular local ring, and let I be a regular R-ideal of grade $g \geq 3$ generated by $x_1, \cdots, x_g$. Let Y be a generic $g \times g$ matrix, $\Delta = \det(Y)$, $T = R[Y]$, $\alpha_i = \sum_{j=1}^{g} Y_{ij} x_j$ for $1 \leq i \leq g$. Then $K = (\alpha_1, \cdots, \alpha_g)T : IT$ is a first generic link of I, and it is well known that in this case K is a prime ideal generated by $\alpha_1, \cdots, \alpha_g, \Delta$.

Let Z be a $g \times (g+1)$ matrix of indeterminates over T, set $S = T[Z]$, $\beta_i = \sum_{j=1}^{g} Z_{ij} \alpha_j + Z_{i\,g+1} \Delta$ for $1 \leq i \leq g$. Then $J = (\beta_1, \cdots, \beta_g)S : KS$ is a second generic link of I.

Before we describe the structure of J, which will be done in Theorem 2.1, we need more definitions. Consider the $g\times(g+1)$ matrix

$$V=(v_{ij})=\begin{pmatrix}V_1\\ \vdots\\ V_g\end{pmatrix}=\left(\operatorname{adj}(Y)\,\middle|\,\begin{matrix}-x_1\\ \vdots\\ -x_g\end{matrix}\right),$$

the $g\times g$ matrix W which consists of the first g columns of Z, the $2g\times(g+1)$ matrix $U=\begin{pmatrix}Z\\ V\end{pmatrix}$, and the determinants $\Delta_i=\det\begin{pmatrix}Z\\ V_i\end{pmatrix}$ for $1\leq i\leq g$.

Finally define the S-ideals

$$q_1=(x_1,\cdots,x_g,I_{g-1}(Y),I_{g-2}(WY),I_g(W))S,$$

$$q_2=\left(K,\left\{I_g\begin{pmatrix}Z\\ V_i\end{pmatrix}\,\middle|\,1\leq i\leq g\right\}\right)S.$$

Theorem 2.1. *Let J be the second generic link of a regular ideal as described above. Then*

a) $J=(\beta_1,\cdots,\beta_g,\Delta_1,\cdots,\Delta_g)S=(\beta_1,\cdots,\beta_g,I_{g+1}(U))S$

b) $NG(S/J)=V(q_2)$, $NCI(J)=V(q_1)\cup V(q_2)$

c) $p_1=\sqrt{q_1}$ *is a prime ideal in S, $q_1S_{p_1}$ is prime,* $\dim(S/J)_{p_1}=7$, *there is a regular local ring $\bar{S}$ and an $\bar{S}$-ideal $\bar{J}$ such that $(S/J)_{p_1}\cong\bar{S}/\bar{J}$ and $\bar{J}$ is a Gorenstein ideal of grade* 3 *and deviation* 2.

d) $p_2=\sqrt{q_2}$ *is a prime ideal in S, $q_2S_{p_2}$ is prime,* $\dim(S/J)_{p_2}=4$, *there is a regular local ring $\bar{S}$ and an $\bar{S}$-ideal $\bar{J}$ such that $(S/J)_{p_2}\cong\bar{S}/\bar{J}$ and $\bar{J}$ is a perfect ideal of grade* 2 *and deviation* 1.

The complete proof of Theorem 2.1 can be found in [9]. With the following proposition we will only show part of Theorem 2.1.d. We include the proof, since it is different from the one given in [9], and also yields a somewhat better result.

Proposition 2.2. *Let $p_2=\sqrt{q_2}$. Then $p_2=\sqrt{KS+I_g(U)}$. Moreover, p_2 is prime, $q_2S_{p_2}$ is prime, and* $\dim(S/J)_{p_2}=4$.

Proof. The proof proceeds through several partial claims. We will use the same notations as in Theorem 2.1. For $p\in\operatorname{Spec}(S)$ set $k(p)=S_p/pS_p$, and let " $-$ " denote residue classes in $k(p)$.

Claim 1. Let $p\in\operatorname{Spec}(S)$ with $p\supset K$. Then $\operatorname{rank}_{k(p)}\bar{V}\leq 1$.

It suffices to show that $\operatorname{rank}_{k(KS)}(\bar{V})\leq 1$. But obviously $\bar{Y}\bar{V}=\bar{0}$ over $k(KS)$, and $I_{g-1}(Y)\not\subset KS$, hence $\operatorname{rank}_{k(KS)}(\bar{Y})=g-1$. Therefore $\operatorname{rank}_{k(KS)}(\bar{V})\leq g-(g-1)=1$.

Claim 2. Let $p \in \mathrm{Spec}(S)$ such that for some $1 \leq i \leq g$, $p \supset KS + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$, but $p \not\supset I_1(V_i)$. Then $p \supset KS + I_g(U)$.

It suffices to show that $\mathrm{rank}_{k(p)}(\bar{U}) \leq g-1$. By assumption,

$$\mathrm{rank}_{k(p)}\left(\begin{pmatrix} \bar{Z} \\ \bar{V}_i \end{pmatrix}\right) \leq g-1, \quad \mathrm{rank}_{k(p)}(\bar{V}_i) = 1,$$

and by Claim1, $\mathrm{rank}_{k(p)}(\bar{V}) \leq 1$. Hence

$$\mathrm{rank}_{k(p)}(\bar{U}) = \mathrm{rank}_{k(p)}\left(\begin{pmatrix} \bar{Z} \\ \bar{V} \end{pmatrix}\right) = \mathrm{rank}_{k(p)}\left(\begin{pmatrix} \bar{Z} \\ \bar{V}_i \end{pmatrix}\right) \leq g-1.$$

Claim 3. Let $p \in \mathrm{Spec}(S)$ be a minimal prime containing q_2. Then $p \not\supset I_1(V)$.

This is proved in [9], Lemma 3.18.

Claim 4. Let $p \in \mathrm{Spec}(S)$ be a minimal prime containing q_2. Then $p \supset KS + I_g(U)$, and $\mathrm{ht}(p) \leq g+4$.

By claim 3, $p \not\supset I_1(V_i)$ for some $1 \leq i \leq g$, and by assumption, $p \supset q_2 \supset KS + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$. Therefore $p \supset KS + I_g(U)$ by Claim 2. We now show that p is a minimal prime ideal containing $KS + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$. Let q be a prime ideal with $p \supset q \supset KS + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$, then $q \not\supset I_1(V_i)$ since $p \not\supset I_1(V_i)$, and thus by Claim 2, $q \supset KS + I_g(U) \supset q_2$. But $p \supset q$ and p was a minimal prime ideal of q_2, hence $p = q$. Therefore p is a minimal prime over $KS + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$, and hence $\mathrm{ht}(p) \leq g+4$, since $\mathrm{ht}(KS) = g$, and $I_g\begin{pmatrix} Z \\ V_i \end{pmatrix}$ is generated by the $g \times g$ minors of a $(g+1) \times (g+1)$ matrix.

Claim 5. Let v_{ij} be an entry of V. First note that $v_{ij} \in T = R[Y]$, hence $S' = S[v_{ij}^{-1}] = R[Y, v_{ij}^{-1}][Z] = T'[Z]$, where $T' = R[Y, v_{ij}^{-1}]$. Now define a $g \times (g+1)$ matrix $Z' = (Z'_{lk})$ by setting $Z'_{lj} = Z_{lj}$ for $1 \leq l \leq g$, $Z'_{lk} = \det\begin{pmatrix} Z_{lk} & Z_{lj} \\ v_{ik} & v_{ij} \end{pmatrix}$ for $1 \leq l \leq g$, $1 \leq k \leq g+1$, $k \neq j$. Since v_{ij} in invertible in T', it is clear that $S' = T'[Z] = T'[Z']$. In particular, Z'', the $g \times g$ matrix obtained from Z' by deleting the j-th column is also a matrix of indeterminates over T'. We claim that $q_2 S' = KS' + I_{g-1}(Z'')S'$, and that $q_2 S'$ is a prime ideal of height $g+4$.

Again, since v_{ij} is a unit in S', it is clear that $I_{g-1}(Z'')S' = I_g\begin{pmatrix} Z \\ V_i \end{pmatrix} S'$. Therefore $KS' + I_{g-1}(Z'')S' = KS' + I_g\begin{pmatrix} Z \\ V_i \end{pmatrix} S' \subset q_2 S'$. On the other hand,

$K \subset T'$ and Z'' is a $g \times g$ matrix of indeterminates over T', hence $KS' + I_{g-1}(Z'')S'$ is a prime ideal of height $g+4$. Therefore $KS' + I_{g-1}(Z'')S' = q_2S'$, since $\mathrm{ht}(q_2S') \leq g+4$ by Claim 4.

Claim 6. Let $p \in \mathrm{Spec}(S)$ be a minimal prime of q_2. Then $v_{11} \notin p$.

By Claim 3, there is an entry v_{ij} of V with $v_{ij} \notin p$. Let $S' = S[v_{ij}^{-1}]$. Then q_2S' is a prime ideal by Claim 5, and hence $pS' = q_2S'$. Suppose that $v_{11} \in p$, then $v_{11} \in pS' = q_2S'$, where $q_2S' = KS' + I_{g-1}(Z'')S'$ by Claim 5. Since v_{11} and K are contained in T' and Z'' is a matrix of indeterminates over T', the inclusion $v_{11} \in KS' + I_{g-1}(Z'')S'$ would imply $v_{11} \in KT'$, and hence $v_{11} \in K$, since K is a prime ideal in $T = R[Y]$. But v_{11} is an entry of $\mathrm{adj}(Y)$, and $K = (\alpha_1, \cdots, \alpha_g, \det(Y))R[Y]$, hence $v_{11} \notin K$. Thus we have shown $v_{11} \notin p$.

Now we can complete the proof of Proposition 2.2. In Claim 4 we had already seen that $\sqrt{q_2} = \sqrt{KS + I_g(U)}$. By Claim 6, v_{11} is a regular element on $S/\sqrt{q_2}$, hence $\sqrt{q_2}$ is a prime ideal of height $g+4$ once we have shown that $\sqrt{q_2}S[v_{11}^{-1}]$ is a prime ideal of height $g+4$. But this is clear by Claim 5. Since $q_2S[v_{11}^{-1}]$ is prime and $v_{11} \notin p_2$, it also follows that $q_2S_{p_2}$ is prime. □

By combining Theorems 1.6 and 2.1 the following result was shown in [9].

Corollary 2.3. *Let R be a regular local ring, and let I be a licci R-ideal which is not a complete intersection. Then (R, I) has essentially a deformation (S, J) with $S/J \cong (P[X]/I')_{(m_P, X)}$, where P is a regular local ring, and I' is a $P[X]$-ideal, where either*

a) *X is a generic alternating 5×5 matrix, and $I' = \mathrm{Pf}_4(X)$, the ideal generated by the 4×4 Pfaffians of X,*

or

b) *X is a generic 2×3 matrix, and $I' = I_2(X)$.*

By means of Corollary 2.3 many questions concerning licci non-complete-intersection ideals can be reduced to studying the most simple ideals described in Corollary 2.3. a,b. Mainly using this observation we are able to prove one of the main results in [9] (c.f. Theorem 2.4 below), which gives an upper bound for the codimension of the non-complete-intersection locus of licci ideals, and which shows in particular that Hartshorne's conjecture holds true for licci ideals. A local version of Hartshorne's conjecture can be stated in the following way: Let R be a regular local ring, and let I be an R-ideal of grade g which is not a complete intersection, then (R, I) is not smoothable in codimension $2g+1$.

Theorem 2.4. *Let R be a regular local ring with infinite residue class field, and let I be a licci R-ideal which is not a complete intersection.*

a) *If I is Gorenstein, then (R, I) is smoothable in codimension* 6, *but not in codimension* 7.

b) *If I is not Gorenstein, then (R, I) is smoothable in codimension* 3, *but (R, I) has no deformation (S, J) with S/J satisfying (G_4).*

Proof. We only prove that (R, I) is smoothable in codimension 6 if I is Gorenstein. The other parts have been shown in [9]. First notice that grade $I \geq 3$, since I is Gorenstein, but not a complete intersection.

Since I is licci and Gorenstein we may use Corollary 1.8 to conclude that there is a sequence of links in R, $I \sim I_1 \sim \cdots \sim I_{2n}$ such that I_{2n} is regular and I_{2i} is Gorenstein for all $0 \leq 2i \leq 2n$. We will prove the claim by induction on n.

Let $n \geq 1$. Since I_2 is Gorenstein it follows from the induction hypothesis that (R, I_2) is smoothable in codimension 6, i.e., (R, I_2) has a deformation $(\tilde{R}, \tilde{I}_2)$ with $\tilde{R}/\tilde{I}_2$ being (R_6). By [9], 2.16, there is a sequence of links in $\tilde{R}$, $\tilde{I} \sim \tilde{I}_1 \sim \tilde{I}_2$, such that $(\tilde{R}, \tilde{I})$ is a deformation of (R, I). Replacing $I \sim I_1 \sim I_2$ by $\tilde{I} \sim \tilde{I}_1 \sim \tilde{I}_2$, we may therefore assume that R/I_2 satisfies (R_6). In some $T = R[Z]$ consider a second generic link $L_2(I_2) \subset T$. By Theorem 1.6.a there exists $q \in \operatorname{Spec}(T)$ such that $(T_q, L_2(I_2)T_q)$ is a deformation of (R, I).

We now show that $L_2(I_2)T_q$ satisfies (CI_6). Let $p \in \operatorname{Spec}(T)$ with $q \supset p \supset L_2(I_2)$ and $\dim(T/L_2(I_2))_p \leq 6$. Write $p' = p \cap R$, then either $p' \supset I_2$ and $\dim(R/I_2)_{p'} \leq 6$ or $I_2R_{p'} = R_{p'}$. Hence $I_2R_{p'}$ is either regular or the unit ideal. Now $L_2(I_2)R_{p'}[Z]$ can be considered as a second generic link of the regular ideal or unit ideal $I_2R_{p'}$, $(L_2(I_2)R_{p'}[Z])_p = L_2(I_2)T_p$ is Gorenstein, and $\dim(T/L_2(I_2))_p \leq 6$. Since $(R_{p'}[Z], L_2(I_2)R_{p'}[Z])$ is equivalent to the second generic link of a regular ideal considered in Theorem 2.1, we may now use Theorem 2.1 to conclude that $L_2(I_2)T_p$ is a complete intersection (the case where $I_2R_{p'} = R_{p'}$ is trivial).

Thus we have seen that (R, I) has a deformation $(T_q, L_2(I_2)T_q)$ with $L_2(I_2)T_q$ satisfying (CI_6). In [10] we show that if a pair (R, I) of a regular local ring R and a perfect R-ideal I has a deformation (S, J) with J satisfying (CI_6), then (R, I) is smoothable in codimension 6.

Therefore (R, I) is smoothable in codimension 6, if I is a licci Gorenstein ideal. □

Corollary 2.5. *Let $R = k[[x_1, \cdots, x_n]]$, let I be a licci R-ideal, and assume that $A = R/I$ rigid (i.e. has only trivial infinitesimal deformations over k), but not regular. Then* $\operatorname{codim}(\operatorname{Sing}(A)) = 7$ *if A is Gorenstein, and* $\operatorname{codim}(\operatorname{Sing}(A)) = 4$ *otherwise.*

Combining Corollary 2.3 and Theorem 2.4 we are now able to show an improved version of Corollary 2.3.

Theorem 2.6. *Let R be a regular local ring, and let I be a licci R-ideal which is not a complete intersection.*

a) *If I is Gorenstein then (R, I) has essentially a deformation (S_1, J_1) with $S_1/J_1 \cong (P[X]/\mathrm{Pf}_4(X))_{(m_P, X)}$ where P is a regular local ring, and X is a generic alternating 5×5 matrix.*

b) *If I is not Gorenstein then (R, I) has essentially a deformation (S_2, J_2) with $S_2/J_2 \cong (P[X]/I_2(X))_{(m_P, X)}$ where P is a regular local ring, and X is a generic 2×3 matrix.*

Proof. By Corollary 2.3 we already know that a licci R-ideal I which is not a complete intersection has essentially a deformation of the form (S_1, J_1) or (S_2, J_2). If I is Gorenstein, then so is any essential deformation of I, and hence (S_2, J_2) cannot be essentially a deformation of (R, I). This proves part a).

To prove b) we now assume that I is not Gorenstein. We may also assume that grade $I\geq 3$, because the result is well known for ideals of grade two. Since I is licci, there is a sequence of links $I\sim I_1\sim\cdots\sim I_{2n}$ where I_{2n} is a complete intersection. In some $S=R(Z)$ we consider a sequence of universal links, $IS\sim L^1(I)S\sim\cdots\sim L^{2n}(I)S$. By Theorem 1.6.b, $(S, L^{2n}(I)S)$ is essentially a deformation of (R, I_{2n}), and hence $L^{2n}(I)S$ is a complete intersection (or the unit ideal). Since on the other hand I is not Gorenstein, there is an integer l, $1\leq l\leq n$, such that $L^{2l}(I)S$ is Gorenstein (or the unit ideal), but $L^{2l-2}(I)S$ is not Gorenstein. Because by Theorem 1.6.c $(S, L^{2l-2}(I)S)$ is essentially a deformation of (R, I), we may replace (R, I) by $(S, L^{2l-2}(I)S)$ to assume that there is a sequence of links $I\sim I_1\sim I_2$ with I_2 Gorenstein and licci and that the residue class field of R is infinite.

By Theorem 2.4.a there is a deformation $(\tilde{R}, \tilde{I}_2)$ of (R, I_2) such that $\tilde{R}/\tilde{I}_2$ satisfies (R_6), and by [9], Lemma 2.16, we can lift the links $I\sim I_1\sim I_2$ to a sequence of links in $\tilde{R}$, $\tilde{I}\sim\tilde{I}_1\sim\tilde{I}_2$, such that $(\tilde{R}, \tilde{I})$ is a deformation of (R, I). Hence replacing (R, I) by $(\tilde{R}, \tilde{I})$ we may assume that I is licci, but not Gorenstein, and that there is a sequence of links $I\sim I_1\sim I_2$ with R/I_2 satisfying (R_6).

By Theorem 2.4.b, there exists an element $p\in NG(R/I)$ with $\dim(R/I)_p\leq 6$. Note that $IR_p\sim I_1R_p\sim I_2R_p$ where I_2R_p is regular. Obviously (R_p, IR_p) is essentially a deformation of (R, I). Now we may replace (R, I) by (R_p, IR_p) to assume that I is not Gorenstein, and that there is a sequence of links $I\sim I_1\sim I_2$ with I_2 being regular.

Now choose the particular second generic link J of I_2 in S, as described in Theorem 2.1, and let $q \in \text{Spec}(S)$ such that (S_q, J_q) is a deformation of (R, I) (c.f. Theorem 1.6.a). Then $q \in NG(S/J) = V(p_2)$, where $p_2 = \sqrt{q_2}$ is the prime ideal described in Theorem 2.1. Now a suitable deformation of (S_{p_2}, J_{p_2}) is essentially a deformation of (R, I), and has the desired properties by Theorem 2.1. □

We now apply Theorem 2.6 to show some facts about the divisor class group $\text{Cl}(A)$ of a normal domain $A = R/I$, where I is a licci ideal in a regular local ring R. First we need several lemmas.

Lemma 2.7. *Let R be a regular local ring, and let I be a perfect R-ideal such that $A = R/I$ is (G_2). For some $r \geq 1$ let $\otimes^r K_A$ denote the r-th tensor power of the canonical module of A. Let (S, J) be essentially a deformation of (R, I), and write $B = S/J$.*

If $\text{Hom}_A(\otimes^r K_A, A) \cong A$, *then* $\text{Hom}_B(\otimes^r K_B, B) \cong B$.

Proof. (Compare to [15], proof of Theorem 1). By the definition of essentially a deformation it obviously suffices to prove the claim for the case that $A \cong B/xB$, where x is a B-regular element. To this end we only have to show that $\text{Hom}_B(\otimes^r K_B, B) \otimes_B A \cong \text{Hom}_A(\otimes^r K_A, A)$.

Let T be the B-torsion of $\otimes^r K_B$, and write $N = (\otimes^r K_B)/T$, $M = N_B \otimes_B A$. Then $M_p = (\otimes^r K_A)_p$ for all $p \in \text{Spec}(A)$ with $\dim A_p \leq 2$. We have to prove that

$$\text{Hom}_B(N, B) \underset{B}{\otimes} A \cong \text{Hom}_A(M, A). \tag{2.8}$$

The exact sequence

$$0 \longrightarrow B \xrightarrow{\mu_x} B \longrightarrow A \longrightarrow 0$$

induces an exact sequence

$$\begin{aligned} &0 \longrightarrow \text{Hom}_B(N, B) \xrightarrow{\mu_x} \text{Hom}_B(N, B) \longrightarrow \text{Hom}_A(M, A) \\ &\longrightarrow \text{Ext}^1_B(N, B) \xrightarrow{\mu_x} \text{Ext}^1_B(N, B) \longrightarrow \text{Ext}^1_B(N, A). \end{aligned} \tag{2.9}$$

We will show that

$$\text{Ext}^1_A(M, A) = 0. \tag{2.10}$$

Since x is a regular element on N, it is easy to see that $\text{Ext}^1_B(N, A) \cong \text{Ext}^1_A(M, A)$, and hence (2.10) and (2.9) imply by Nakayama's Lemma that $\text{Ext}^1_B(N, B) = 0$. But then (2.8) follows from (2.9).

Also (2.10) is clear since M is (R_2) and $\text{Hom}_A(M, A)$ is (S_3). To see

(2.10) suppose that $\mathrm{Ext}^1_A(M, A) \neq 0$ and choose a minimal element q in $\mathrm{Supp}(\mathrm{Ext}^1_A(M, A)) \neq \phi$. For all $p \in \mathrm{Spec}(A)$ with $\dim A_p \leq 2$ we assumed that A_p is Gorenstein, and hence M_p is free. Therefore $\dim A_q \geq 3$. Now consider an exact sequence

$$0 \longrightarrow U \longrightarrow F \longrightarrow M_q \longrightarrow 0$$

where F is a free A_q-module. Applying $(-)^* = \mathrm{Hom}_{A_q}(-, A_q)$ we obtain an exact sequence

$$(2.11) \qquad 0 \longrightarrow M_q^* \longrightarrow F^* \longrightarrow U^* \longrightarrow (\mathrm{Ext}^1_A(M, A))_q \longrightarrow 0.$$

Since $M_q^* \cong \mathrm{Hom}_A(M, A)_q \cong A_q$, and depth $A_q \geq 3$, and since depth $U^* \geq 2$, we now conclude from (2.11) that $\mathrm{depth}(\mathrm{Ext}^1_A(M, A))_q \geq 1$. But this is impossible, since by the choice of q, $\mathrm{depth}(\mathrm{Ext}^1_A(M, A))_q = 0$. Thus we have proved (2.10). □

Lemma 2.12. *Let R be a regular local ring, let I be a licci R-ideal, let $A = R/I$, and assume that A is (G_2).*

Then A is Gorenstein if and only if for some $r \geq 1$, $\mathrm{Hom}_A(\otimes^r K_A, A) \cong A$.

Proof. We assume that $\mathrm{Hom}_A(\otimes^r K_A, A) \cong A$ and show that A is Gorenstein. So suppose that A is not Gorenstein. Then it follows from Theorem 2.6.b that (R, I) has essentially a deformation (S, J) with $S/J \cong B \cong (P[X]/I_2(X))_{(m_P, X)}$, where P is a regular local ring and X is a generic 2×3 matrix. Here B is a normal domain whose divisor class group $\mathrm{Cl}(B)$ is isomorphic to $\boldsymbol{Z}$ and is generated by the class of the canonical module of B, $[K_B]$. In particular, $[K_B]$ has infinite order in $\mathrm{Cl}(B)$. On the other hand, the assumption $\mathrm{Hom}_A(\otimes^r K_A, A) \cong A$ and Lemma 2.7 imply that $\mathrm{Hom}_B(\otimes^r K_B, B) \cong B$. But then $-[K_B]$ would have finite order in $\mathrm{Cl}(B)$, which is impossible. □

Corollary 2.13. *Let R be a regular local ring, let I be a licci R-ideal such that $A = R/I$ is normal, and (G_2), but not Gorenstein.*

Then $\mathrm{Cl}(A)$ is not torsion.

Example 2.14. Let k be a field, let G be a finite group with $\mathrm{char}\, k \nmid |G|$ acting linearly on $k[x_1, \cdots, x_n]$ and assume that G contains no pseudoreflections except the identity and that $G \not\subset \mathrm{SL}(n, k)$. Then the ring of invariants $A = (k[x_1, \cdots, x_n]_{(x_1, \cdots, x_n)})^G$ is not Gorenstein ([21]). Assume that A is (R_2) and write $A = R/I$ where R is a regular local ring. Then $\mathrm{Cl}(A)$ is finite ([18]) and hence Corollary 2.13 implies that I is not licci.

§ 3. Homogeneous linkage and class groups

In this section we will study linkage of homogeneous (not necessarily licci) ideals. The main results are concerned with linkage of powers of the maximal ideal, linkage of determinantal ideals, and the divisor class group of algebras defined by certain homogeneous licci ideals.

When applying the method of universal linkage to the graded case we have to face the problem that universal links of homogeneous ideals are not necessarily homogeneous. However any universal link can be obtained as a localization of a generic link which in fact can be given many different gradings. This leads to the definition of homogeneous generic links with a given weight, which we are going to describe now. Let k be a field, let $R'=k[x_1, \cdots, x_e]$ be a graded polynomial ring with $\deg x_i \in \boldsymbol{Z}$, and let I' be an equicodimensional ideal of height $g>0$ having a homogeneous resolution of the form

$$(3.1) \qquad F_{\bullet}: 0 \longrightarrow \bigoplus_i R'(-n_{gi}) \longrightarrow \cdots \longrightarrow \bigoplus_i R'(-n_{1i}) \longrightarrow R' \longrightarrow R'/I' \longrightarrow 0.$$

Of course the existence of (3.1), makes I' locally perfect. Let $f_1, \cdots, f_n$ be a homogeneous generating sequence of I' with $\deg f_i = n_{1i}$, let $Y=(Y_{ij})$ be a generic $g \times n$ matrix, set $T'=R'[Y]$, and $\alpha_i = \sum_{j=1}^n Y_{ij} f_j$, $1 \leq i \leq g$. Fix an integer d and extend the grading of R' to *a* $\boldsymbol{Z}$-grading of T' by setting $\deg Y_{ij} = d - \deg f_j$, $1 \leq i \leq g$, $1 \leq j \leq n$. Then $\alpha_1, \cdots, \alpha_g$ form a quasiregular T'-sequence consisting of homogeneous elements of degree d. Therefore the first generic link $L_1(I') = (\alpha_1, \cdots, \alpha_g)T' : I'T'$ is also a homogeneous ideal, which we will denote by $L_1(I'; d)$. By induction on i we define $L_i(I'; d) = L_1(L_{i-1}(I'; d); d)$, and call this homogeneous ideal a *i-th homogeneous generic link of I' with weight d.*

A homogeneous T'-resolution of $T'/L_1(I'; d)$ can be obtained in the following way: Consider the homogeneous Koszul complex of $\alpha_1, \cdots, \alpha_g$:

$$(3.2) \qquad K_{\bullet}: 0 \longrightarrow T'(-gd) \longrightarrow \cdots \longrightarrow T'^g(-d) \longrightarrow T' \longrightarrow T'/(\alpha_1, \cdots, \alpha_g) \longrightarrow 0,$$

and let $u_{\bullet}: K_{\bullet} \to F_{\bullet} \otimes_{R'} T'$ be a homogeneous morphism of complexes:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \bigoplus_i T'(-n_{gi}) & \longrightarrow & \cdots & \longrightarrow & \bigoplus_i T'(-n_{1i}) & \longrightarrow & T' \\ & \uparrow u_g & & \uparrow u_{g-1} & & \uparrow u_1 & & \| \\ 0 \longrightarrow & T'(-gd) & \longrightarrow & T'^g(-gd+d) & \longrightarrow & \cdots & \longrightarrow & T'. \end{array}$$

Then up to shifts the T'-dual of the mapping cone of $u_{\bullet}$ yields a homogeneous resolution of the link $T'/L_1(I';d)$ (c.f. [16]):

$$0\longrightarrow\bigoplus_i T'(-gd+n_{1i})\longrightarrow\cdots\longrightarrow\bigoplus_i T'(-gd+n_{gi})\oplus T'^g(-d)\longrightarrow T'\longrightarrow T'/L_1(I';d)\longrightarrow 0. \tag{3.3}$$

Now let $R'=k[x_1,\cdots,x_e]$ be a graded polynomial ring with $\deg x_i>0$, and let I' be a homogeneous perfect R'-ideal of grade $g>0$. Then R'/I' has a homogeneous minimal R'-resolution of the form

$$0\longrightarrow\bigoplus_{i=1}^{b_g} R'(-n_{gi})\longrightarrow\cdots\longrightarrow\bigoplus_{i=1}^{b_1} R'(-n_{1i})\longrightarrow R'\longrightarrow R'/I'\longrightarrow 0. \tag{3.4}$$

In our later discussion the following condition will play a central role:

$$(*)\qquad \max_i\{n_{gi}\}\leq(g-1)\min_i\{n_{1i}\}.$$

We first need a lemma:

Lemma 3.5. *Let $R'=k[x_1,\cdots,x_e]$ be a graded polynomial ring with $\deg x_i>0$ for $1\leq i\leq e$, let I' be a homogeneous perfect R'-ideal such that condition $(*)$ is satisfied. Write $d=\min_i\{n_{1i}\}$. Then there exists a n-th homogeneous generic link of I' with weight d, $L_n(I';d)\subset T'=R'[Y]$ such that*

a) *$L_n(I';d)$ is generated by homogeneous elements of degrees at least d.*

b) *The variables Y_{ij} have non-positive degrees.*

Proof. By induction using (3.1) and (3.3) we obtain homogeneous resolutions of any homogeneous generic links $L_{2l}(I';d)\subset T'_{2l}$, and $L_{2l+1}(I';d)\subset T'_{2l+1}$:

$$0\longrightarrow\bigoplus_{i=1}^{b_g} T'_{2l}(-n_{gi})\oplus T'^{lg}_{2l}(-gd+d)\longrightarrow\cdots\longrightarrow\bigoplus_{i=1}^{b_1} T'_{2l}(-n_{1i})\oplus T'^{lg}_{2l}(-d)\longrightarrow T'_{2l}\longrightarrow T'_{2l}/L_{2l}(I';d)\longrightarrow 0, \tag{3.6}$$

$$0\longrightarrow\bigoplus_{i=1}^{b_1} T'_{2l+1}(-gd+n_{1i})\oplus T'^{lg}_{2l+1}(-gd+d)\longrightarrow\cdots\longrightarrow\bigoplus_{i=1}^{b_g} T'_{2l+1}(-gd+n_{gi})\oplus T'^{lg+g}_{2l+1}(-d)\longrightarrow T'_{2l+1}\longrightarrow T'_{2l+1}/L_{2l+1}(I';d)\longrightarrow 0. \tag{3.7}$$

Now a) readily follows from (3.6) and (3.7) since $n_{1i}\geq d$ for all i, and $gd-n_{gi}\geq d$ for all i, because condition $(*)$ is satisfied.

Part b) can be shown by induction on $n \geq 0$ (set $L_0(I'; d) = I' \subset R'$). By part a) there exists a homogeneous generating sequence $f_1, \cdots, f_s$ of $L_n(I'; d) \subset T'$ such that $\deg f_i \geq d$, $1 \leq j \leq s$. If we use this particular generating sequence to define $L_{n+1}(I'; d)$ we have to adjoin variables of degrees $d - \deg f_j$, $1 \leq j \leq s$. However $d - \deg f_j \leq 0$, $1 \leq j \leq s$. □

Corollary 3.8. *Let* $R' = k[x_1, \cdots, x_e]$ *be a graded polynomial ring with* deg $x_i = 1$ *for* $1 \leq i \leq e$, *let* $I' = \bigoplus_{i=d}^{\infty} I'_i$ *be a homogeneous perfect* R'*-ideal with* $I'_d \neq 0$, *and assume that* R'/I' *satisfies* $(*)$. *Write* $R = k[x_1, \cdots, x_e]_{(x_1, \cdots, x_e)}$, $m = (x_1, \cdots, x_e)R$, $I = I'R$, *and let* J *be an* R*-ideal in the linkage class of* I.

Then $J \subset m^d$,

Proof. Consider a sequence of links $I \sim I_1 \sim \cdots \sim I_n = J$, and a n-th homogeneous generic link as in Lemma 3.5, $L_n(I'; d) \subset R'[Y] = k[x_1, \cdots, x_e, Y_{ij}]$. Then it follows from Lemma 3.5 that $L_n(I'; d)$ is generated by homogeneous polynomials in $x_1, \cdots, x_e, Y_{ij}$ of degrees at least d and that all the variables Y_{ij} have non-positive degrees. Therefore $L_n(I'; d) \subset (x_1, \cdots, x_e)^d R'[Y]$. On the other hand, it is clear from the definition that $L_n(I'; d)R \subset R[Y]$ is also an n-th generic link of I. Hence by Theorem 1.6.a, there is a $q \in \operatorname{Spec}(R[Y])$ with $(x_1, \cdots, x_e)R = m \subset q$ such that $(R[Y]_q, L_n(I'; d)R[Y]_q)$ is a deformation of (R, J). Since $L_n(I'; d)R[Y]_q \subset m^d R[Y]_q$ it is then clear that $J \subset m^d$. □

Example 3.9. Let $e \geq 3$, $R = k[x_1, \cdots, x_e]_{(x_1, \cdots, x_e)}$, $m = (x_1, \cdots, x_e)R$. Let d be a positive integer and let J be any R-ideal in the linkage class of m^d, then $J \subset m^d$. This fact follows immediately from Corollary 3.8 since from the Eagon-Northcott complex one sees that $(x_1, \cdots, x_e)^d$ satisfies $(*)$ for $e \geq 3$, and $d \geq 2$.

The following corollary is an improved version of [20], Theorem 2.

Corollary 3.10. *Let* $R' = k[x_1, \cdots, x_e]$ *be a graded polynomial ring with* deg $x_i = 1$ *for* $1 \leq i \leq e$, $R = k[x_1, \cdots, x_e]_{(x_1, \cdots, x_e)}$, *for* $j = 1, 2$ *let* $2 \leq t_j \leq r_j \leq s_j$, *let* C_j *be* $r_j \times s_j$ *matrices with linear entries in* R', *let* $I_j = I_{t_j}(C_j)$ *with* grade $(I_j) = (r_j - t_j + 1)(s_j - t_j + 1) > 2$.

If I_1 *and* I_2 *belong to the same linkage class, then* $t_1 = t_2$.

Proof. By Corollary 3.8 we only have to show that R'/I'_1 and R'/I'_2 satisfy condition $(*)$. But this can be done directly or follows from [9], 6.14 and 6.3. □

Corollary 3.11. *Let* $R' = k[x_1, \cdots, x_e]$ *be a graded polynomial ring with* deg $x_i > 0$ *for* $1 \leq i \leq e$, *let* I' *be a homogeneous perfect* R'*-ideal, and assume that* R'/I' *satisfies* $(*)$. *Write* $R = k[x_1, \cdots, x_e]_{(x_1, \cdots, x_e)}$, $I = I'R$.

Then I is not licci in R.

Proof. We may assume that $\deg x_1 = \min\{\deg x_i\}$. Let $I' = \bigoplus_{i=d}^{\infty} I'_i$ with $I'_d \neq 0$. Obviously $d \geq \deg x_1$. If $d = \deg x_1$, then $\max\{n_{gi}\} \geq g \deg x_1 = gd$ which contradicts (*). Therefore $d > \deg x_1$.

Now suppose that I is licci, then there is a sequence of links $I \sim I_1 \sim \cdots \sim I_n = (x_1, \cdots, x_g)$. As in the proof of Corollary 3.8 we consider $L_n(I'; d) \subset R'[Y]$, and since $L_n(I'; d)$ is generated by forms of degrees at least $d > \deg x_1$ in $x_1, \cdots, x_e, Y_{ij}$, and all the variables Y_{ij} have non-positive degrees, it now follows that $L_n(I'; d) \subset (x_1^2, x_2, \cdots, x_e)R'[Y]$. Also by Theorem 1.6.a there exists $q \in \operatorname{Spec}(R[Y])$ with $(x_1, \cdots, x_e) \subset q$ such that $(R[Y]_q, L_n(I'; d)R[Y]_q)$ is a deformation of (R, I_n). Hence the inclusion $L_n(I'; d)R[Y]_q \subset (x_1^2, x_2, \cdots, x_e)R[Y]_q$ implies that $(x_1, \cdots, x_g) = I_n \subset (x_1^2, x_2, \cdots, x_e)$, which is impossible. □

The following result is an improved version of [9], Corollary 5.19.

Theorem 3.12. *Let $R' = k[x_1, \cdots, x_e]$ be a graded polynomial ring with* $\deg x_i > 0$ *for* $1 \leq i \leq e$, *let I' be a homogeneous R'-ideal such that $I = I'R'_{(x_1, \cdots, x_e)}$ is licci. Assume that in the resolution* (3.4), $n_{1i} = n_{1j}$ *for* $1 \leq i, j \leq b_1$ (*i.e. the minimal homogeneous generators of I' have the same degrees*), *and $n_{gi} = n_{gj}$ for $1 \leq i, j \leq b_g$ (i.e. the minimal homogeneous generators of the canonical module $K_{R'/I'}$ have the same degrees*), *and that $A = (R'/I')^\wedge$ is a rigid k-algebra.*

Then $\operatorname{Cl}(A) = \mathbf{Z}[K_A]$. *In particular, A is factorial if A is Gorenstein, and* $\operatorname{Cl}(A) \cong \mathbf{Z}$ *if A is not Gorenstein.*

Proof. Since I is licci it follows from Corollary 3.11 that (*) is not satisfied, and therefore $\max_i\{n_{gi}\} > (g-1)\min_i\{n_{1i}\}$. Moreover by assumption, $\max_i\{n_{gi}\} = \min_i\{n_{gi}\}$ and $\max_i\{n_{1i}\} = \min_i\{n_{1i}\}$, and therefore

$$(**) \qquad \min_i\{n_{gi}\} > (g-1)\max_i\{n_{1i}\}.$$

Since I is licci, and A is rigid, it is known that A satisfies (R_3) (e.g. Corollary 2.5 or [19], 2.3). Finally in [8], Theorem 4.2, it was shown that if (**) is satisfied, and A is rigid and (R_2), then $\operatorname{Cl}(A) = \mathbf{Z}[K_A]$. The additional information that $\operatorname{Cl}(A) \cong \mathbf{Z}$ in case A is not Gorenstein, follows from Corollary 2.13. □

References

[1] R. Apéry, Sur les courbes de première espèce de l'espace à trois dimensions, C. R. Acad. Sci., **220** (1945), 271–272.

[2] R.-O. Buchweitz, Contributions à la théorie des singularités, Thesis, l'Université Paris VII, 1981.

[3] R.-O. Buchweitz and B. Ulrich, Homological properties which are invariant under linkage, preprint.
[4] F. Gaeta, Quelques progrès récents dans la classification des variétés algébriques d'un espace projective. Deuxième Colloque de Géométrie Algébrique, Liège 1952.
[5] J. Herzog and R. Waldi, Differentials of linked curve singularities, Arch. Math., **42** (1984), 335–343.
[6] C. Huneke, Linkage and the Koszul homology of ideals, Amer. J. Math., **104** (1982), 1043–1062.
[7] ——, Numerical invariants of liaison classes, Invent. Math., **75** (1984), 301–325.
[8] C. Huneke and B. Ulrich, Divisor class groups and deformations, Amer. J. Math., **107** (1985), 1265–1303.
[9] ——, The structure of linkage, to appear in Ann. of Math.
[10] ——, Algebraic linkage, to appear in Duke Math. J.
[11] A. Kustin, The minimal free resolutions of the Huneke-Ulrich deviation two Gorenstein ideals, J. Algebra, **100** (1986), 265–304.
[12] A. Kustin and M. Miller, Deformation and linkage of Gorenstein algebras, Trans. Amer. Math. Soc., **284** (1984), 501–533.
[13] ——, Tight double linkage of Gorenstein algebras, J. Algebra, **95** (1985), 384–397.
[14] A. Kustin, M. Miller and B. Ulrich, Linkage theory for algebras with pure resolutions, J. Algebra, **102** (1986), 199–228.
[15] J. Lipman, Rings with discrete divisor class group: Theorem of Danilov-Samuel, Amer. J. Math., **101** (1979), 203–211.
[16] C. Peskine and L. Szpiro, Liaison des variétés algébriques, I, Invent. Math., **26** (1974), 271–302.
[17] P. Rao, Liaison among curves in P^3, Invent. Math., **50** (1979), 205–217.
[18] B. Singh, Invariants of finite groups acting on a local unique factorization domain, J. Indian Math. Soc., **34** (1970), 31–38.
[19] B. Ulrich, Liaison and deformation, J. Pure Appl. Algebra, **39** (1986), 165–175.
[20] ——, Rings of invariants and linkage of determinantal ideals, Math. Ann., **274** (1986), 1–17.
[21] K. Watanabe, Certain invariant subrings are Gorenstein II, Osaka J. Math., **11** (1974), 379–388.
[22] J. Watanabe, A note on Gorenstein rings of embedding codimension three, Nagoya Math. J., **50** (1973), 227–232.

Department of Mathematics
Michigan State University
East Lansing, Michigan 48824
USA