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An Algorithm to Compute the Dimensions of Algebras A and A-Modules from their Generators and Relations

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Introduction

Let k be a field and suppose an associative k-algebra A (not necessarily commutative) is given in terms of a finite set of generators and a set of finitely many defining relations among them. Moreover, suppose a left A-module M, which is finite-dimensional over k, is also given in terms of its finite generators and relations. In this note, we present an algorithm which gives the exact k-dimension of M, starting with these presentations of A and M. Not only dimensionality, our algorithm also produces explicit matrix representations of the generators of A acting on M.

If the algebra A itself is finite-dimensional over k, our algorithm can be applied to the regular A-module A to find the k-dimension of A itself and we can display the regular representation in the form of matrices.

On the other hand, so long as M is finite-dimensional, A may as well be an infinite-dimensional algebra. So, for example, the algorighm is also applicable to representations of a Lie algebra, because they can be regarded as representations of an associative algebra called the universal enveloping algebra of the Lie algebra.

One of the features of our algorithm is that it manipulates only finite-dimensional objects during all its process. Therefore it is executable on computers. In fact, we have an experience of executing this algorithm on a computer to find the dimension of an algebra defined by A. Gyoja ($H(\mathfrak{S}_4)$ in his notation, see [G]), which turned out to be 204.

The key idea in this note is that the so-called Todd-Coxeter coset enumeration method (see [T-C], [L]) in the group theory can be modified so that it could be applied to the present problem.

Notation and basic definitions

Throughout this note, let k be a field.

Let S be a set. We denote by $k\langle S\rangle$ the non-commutative polynomial

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k-algebra with the elements of S as variables. Let $R \subset k\langle S\rangle$. We denote by $k\langle S|R\rangle$ the quotient k-algebra of $k\langle S\rangle$ by its two-sided ideal generated by the elements of R. The algebra $k\langle S|R\rangle$ will be called the *k-algebra generated by S and having defining relations R.*

Put $A = k\langle S|R\rangle$. All A-modules considered in this note shall be left A-modules. Let T be a set. We denote by AT the free A-module with the elements of T as A-free basis. Let $U \subset AT$. We denote by AT/AU the quotient A-module of AT by its A-submodule generated by the elements of U. This A-module shall be called the *A-module generated by T and having U as defining relations.*

However, we should recall that the algebra A is also given in terms of its generators and relations: $A = k\langle S|R\rangle$. Therefore, in order to specify an element of $U \subset AT$, actually we are obliged to write elements of A in terms of the generators of A. In other words, we specify a subset $\tilde{U} \subset k\langle S\rangle T$ and take U to be the canonical image of $\tilde{U}$ into AT via the natural projection: $k\langle S\rangle T \to AT$ induced by the canonical homomorphism: $k\langle S\rangle \to A = k\langle S|R\rangle$.

§ 1. Partial modules

In the usual theoretical construction of $A = k\langle S|R\rangle$ or $M = AT/AU$, we go through infinite-dimensional objects such as $k\langle S\rangle$ and its two-sided ideal generated by R. On the contrary, in our method we insist on constructing A or M through only finite-dimensional objects. In retribution, intermediate products we have to deal with are no more A-modules, although what we obtain in the end is a fully developed A-module which we exactly wish for.

In this section, we shall define some notions (especially "partial modules") describing those incomplete intermediate objects. Roughly speaking, a partial module is a vector space with the action of each generator of A defined only on its subspace.

Definition 1. Let V be a k-vector space. A *partial transformation* of V will mean a pair (D, L) of a k-subspace D of V and a k-linear map $L: D \to V$.

Remark. D is called the *domain* of (D, L) or L.

Definition 2. Let S and T be sets. A *partial module for* (S, T) will mean a triple $\mathcal{M} = (V, \mathcal{L}, \iota)$ such that

1° V is a k-vector space,

2° $\mathcal{L}$ is a family of partial transformations (D_s, L_s) of V indexed by the elements s of S, and

3° ι is a map of T into V.

Remark. V is called the *underlying vector space* of $\mathcal{M}$. We will say that $\mathcal{M}$ is *finite-dimensional* if its underlying vector space is a finite-dimensional vector space over k. For each $s \in S$, (D_s, L_s) (or sometimes simply L_s itself) is called the (*partial*) *action* of s on $\mathcal{M}$. We sometimes omit the words "for (S, T)" if the situation is clear enough.

Definition 3. Let $\mathcal{M}=(V, \mathcal{L}, \iota)$ $(\mathcal{L}=\{(D_s, L_s)\}_{s\in S})$ and $\mathcal{M}'=(V', \mathcal{L}', \iota')$ $(\mathcal{L}'=\{(D_s', L_s')\}_{s\in S})$ be partial modules for (S, T). A *morphism* ϕ (of partial modules) from $\mathcal{M}$ to $\mathcal{M}'$ will mean a k-linear map ϕ: $V\to V'$ such that

1° for each $s \in S$, $\phi(D_s)$ is contained in D_s' and the following diagram is commutative:

$$\begin{array}{ccc} D_s & \xrightarrow{\phi} & D_s' \\ {\scriptstyle L_s}\downarrow & & \downarrow{\scriptstyle L_s'} \\ V & \xrightarrow{\phi} & V' \end{array}$$

and

2° for each $t \in T$, $\phi(\iota(t))=\iota'(t)$.

Remark. For fixed S and T, the partial modules for (S, T) with the morphisms just defined form a category. In this category, $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic if there is a morphism ϕ: $\mathcal{M}\to\mathcal{M}'$ such that ϕ: $V\to V'$ is a linear isomorphism and $\phi(D_s)=D_s'$ for all $s \in S$.

Definition 4. Let $\mathcal{M}=(V, \mathcal{L}, \iota)$ $(\mathcal{L}=\{(D_s, L_s)\}_{s\in S})$ be a partial module for (S, T). By a *monomial* in S we mean a product $w=s_1s_2\cdots s_p$ $(s_i \in S, i=1, 2, \cdots, p)$ of elements of S in $k\langle S\rangle$. For such a monomial w in S, we denote by (D_w, L_w) the partial transformation of V defined by

$$D_w=\left\{v \in V \;\middle|\; \begin{array}{l} v \in D_{s_p}, \\ L_{s_p}(v) \in D_{s_{p-1}}, \\ \quad\cdots\cdots \\ L_{s_2}(L_{s_3}(\cdots(L_{s_p}(v))\cdots)) \in D_{s_1} \end{array}\right\},$$

$$L_w(v)=L_{s_1}(L_{s_2}(\cdots(L_{s_p}(v))\cdots)) \qquad \text{for } v \in D_w.$$

Also, for any element $x=\sum_w c_w w \in k\langle S\rangle$ (where $c_w \in k$, w runs through monomials in S, a finite sum), we define the partial transformation (D_x, L_x) by

$$D_x=\bigcap_{c_w\neq 0} D_w,$$

$$L_x(v)=\sum_w c_w L_w(v) \qquad \text{for } v \in D_x.$$

An element $y=\sum_{w,t} c_{w,t}wt \in k\langle S\rangle T$ (where w are monomials in S and $t \in T$, a finite sum) is called *termwise defined* in $\mathscr{M}$ if $\iota(t) \in D_w$ holds whenever $c_{w,t}\neq 0$. If this is the case, the element $\sum_{w,t} c_{w,t}L_w(\iota(t))$ of V is called the *image* of y in $\mathscr{M}$ and denoted by $y_{\mathscr{M}}$.

§ 2. The statements of the main theorem and the two lemmas

In this section, we shall describe the algorithm as an iteration of a certain unit process, and give a logical characterization of the output of that unit process, using the notion of a kind of universality relative to its input. Our main result is stated as follows.

Theorem. *Let S be a set, $R\subset k\langle S\rangle$, and put $A=k\langle S|R\rangle$. Also let $\tilde{U}\subset k\langle S\rangle T$, U be the canonical image of $\tilde{U}$ in AT, and put $M=AT/AU$. Moreover suppose S, R, T, U are all finite. Construct a sequence of partial modules for (S, T) and morphisms*

$$\mathscr{M}^{(0)}\xrightarrow{\phi^{(1)}}\mathscr{M}^{(1)}\xrightarrow{\phi^{(2)}}\mathscr{M}^{(2)}\xrightarrow{\phi^{(3)}}\cdots$$

as follows:

a) *$\mathscr{M}^{(0)}$ is determined by the Baby Lemma described below, and*

b) *each $\phi^{(i)}$ and $\mathscr{M}^{(i)}$ $(i\geqq 1)$ are determined inductively from $\mathscr{M}^{(i-1)}$ by the Growth Lemma described below.*

Then it holds that

1° *if $\phi^{(i)}$ is surjective as a linear map, then $\mathscr{M}^{(i)}$ naturally has a structure of an A-module and is isomorphic to M, and*

2° *there exists an i such that $\phi^{(i)}$ is surjective if and only if* $\dim_k M$ *is finite.*

It is to be noted that, as is stated in the statement of the theorem, termination of the process is detected by a concrete condition that $\phi^{(i)}$ is surjective. Two lemmas mentioned in the theorem are stated just below. However, their statements give only logical characterization of the objects; explicit constructions are made in the proofs given in succeeding sections.

Lemma 1 (the Baby Lemma). *Let S, T, $\tilde{U}$ be as in the theorem. Then there exists a unique (up to isomorphism) partial module $\mathscr{M}$ for (S, T) satisfying the following three conditions. Moreover, this $\mathscr{M}$ is finite-dimensional.*

a1) *any element $\tilde{u} \in \tilde{U}$ is termwise defined in $\mathscr{M}$,*

a2) *for all $\tilde{u} \in \tilde{U}$, its image $\tilde{u}_{\mathscr{M}}$ in $\mathscr{M}$ is equal to zero, and*

b) *$\mathscr{M}$ is universal with respect to the conditions* a1) *and* a2).

Lemma 2 (the Growth Lemma). *Let S, R, T be as in the theorem* *and $\mathcal{M}$ be a partial module for (S, T). Then there exists a unique (up to isomorphism) partial module $\mathcal{M}'$ and a morphism $\phi\colon \mathcal{M}\to\mathcal{M}'$ satisfying the following four conditions. Moreover, if $\mathcal{M}$ is finite-dimensional, so is $\mathcal{M}'$.*

a1) *for any $s \in S$, we have $\phi(V)\subset D'_s$, where $\mathcal{M}'=(V', \mathcal{L}', \iota')$ and $\mathcal{L}'=\{(D'_s, L'_s)\}_{s\in S}$, namely D'_s is the domain of the partial action of s on $\mathcal{M}'$,*

a2) *for any $r \in R$, we have $\phi(V)\subset D'_r$, where D'_r is derived from $\mathcal{L}'$ by the rule described in Definition* 4,

a3) *for any $r \in R$, we have $L'_r(\phi(V))=0$, and*

b) *$(\mathcal{M}', \phi)$ is universal with respect to the conditions* a1)–a3).

§ 3. Fundamental operations

In order to prove the two lemmas above by explicit construction, it is needed to construct a partial module satisfying a universality condition with respect to several items simultaneously. We shall decompose this condition into a series of small parts and attain them one by one. We provide two fundamental procedures or operations, namely free extension and quotient, as the ultimate constituents for such decomposition.

Definition 5. Let $\mathcal{M}=(V, \mathcal{L}, \iota)$ be a partial module for (S, T), W a k-subspace of V, and s an element of S. A *free extension of $\mathcal{M}$ with respect to* (s, W) is a pair $(\mathcal{M}', \phi)$ of a partial module $\mathcal{M}'=(V', \mathcal{L}', \iota')$ and a morphism $\phi\colon \mathcal{M}\to\mathcal{M}'$ satisfying

a) $\phi(W)\subset D'_s$ where $\mathcal{L}'=\{(D'_\sigma, L'_\sigma)\}_{\sigma\in S}$, and

b) $(\mathcal{M}', \phi)$ is universal with respect to the condition a).

Lemma 3. *There exists a unique (up to isomorphism) free extension $\mathcal{M}'$ of $\mathcal{M}$ with respect to (s, W). Moreover, if $\mathcal{M}$ is finite-dimensional then so is $\mathcal{M}'$.*

Proof. We put $\mathcal{M}'$ and ϕ as follows. Fix a subspace W_1 of W such that $W=(W\cap D_s)+W_1$ (vector space direct sum). Bring a new vector space W'_1 ("new" means that V and W'_1 have only 0 in common) of the same dimension as W_1 and fix a linear isomorphism $i\colon W_1\xrightarrow{\simeq} W'_1$. Put $V'=V\oplus W'_1$. For each $\sigma \in S$, we define (D'_σ, L'_σ) as follows:

$$\text{for } \sigma=s, \begin{cases} D'_s=D_s+W_1 \text{ (direct sum)}, \\ L'_s|_{D_s}=L_s, \\ L'_s|_{W_1}=i\colon W_1\xrightarrow{\simeq} W'_1, \end{cases}$$

for $\sigma\neq s$, put $D'_\sigma=D_\sigma$ and $L'_\sigma=L_\sigma$.

Moreover we define ι' by $\iota'(t)=\iota(t) \in V \subset V'$ for all $t \in T$. Let ϕ be the natural inclusion: $V \hookrightarrow V'$.

In other words, $\mathcal{M}'$ is obtained from $\mathcal{M}$ by

1) leaving unchanged the actions of the elements of S different from s, and

2) adding the new "image" W_1' of W_1 under s, so that the domain of s should include W.

From this construction, it is easily seen that our $\mathcal{M}'$ satisfies the condition a). To prove that it also satisfies b), suppose we have another partial module $\mathcal{M}^\#$ and a morphism $\phi^\#$: $\mathcal{M} \to \mathcal{M}^\#$ satisfying a) and we shall show that we can define a morphism ψ: $\mathcal{M}' \to \mathcal{M}^\#$ such that $\phi^\# = \psi \circ \phi$, and that such ψ is unique. First we define ψ as follows:

i) $\psi|_V = \phi^\#$,

ii) $\psi|_{W_1'} = L_s^\# \circ \phi^\# \circ i^{-1}$. (Note that this composition is legal because $\phi^\# \circ i^{-1}(W_1') = \phi^\#(W_1) \subset D_s^\#$.)

From i) we see that $\phi^\# = \psi \circ \phi$ as a linear map. Let us show that ψ is a morphism of partial modules. The condition 1° in Definition 3 for $\sigma \in S-\{s\}$ and the condition 2° there follow from i). The condition 1° for s is satisfied because $D_s' = D_s + W_1$ and on D_s we have i) and $\phi^\#$ commutes with the action of s; on W_1, ii) can be paraphrased as "for $v \in W_1$, $\psi(L_s'(v)) = L_s^\#(\phi^\#(v))$" using i).

Now, a morphism ψ: $\mathcal{M}' \to \mathcal{M}^\#$ satisfying $\phi^\# = \psi \circ \phi$ is unique because, on V, ψ must coincide with $\phi^\#$ and for any element $L_s'(v) \in W_1'$ ($v \in W_1$) we must have $\psi(L_s'(v)) = L_s^\#(\phi^\#(v))$. This proves the part b).

The uniqueness of $(\mathcal{M}', \phi)$ follows from the standard argument concerning universality.

If V is finite-dimensional, so is $V' = V + W_1'$ because $\dim W_1' = \dim W_1 \leqq \dim V$. □

It is convenient to define a procedure which expands the domain of the action of a monomial, using the one just defined.

Definition 6. Let $\mathcal{M}=(V, \mathcal{L}, \iota)$ be a partial module for (S, T), W a k-subspace of V, and $w=s_1 s_2 \cdots s_p$ be a monomial in S. A *free extension of* $\mathcal{M}$ *with respect to* (w, W) is a pair $(\mathcal{M}', \phi)$ formed by a partial module $\mathcal{M}'=(V', \mathcal{L}', \iota')$ and a morphism ϕ: $\mathcal{M} \to \mathcal{M}'$ satisfying

a) $\phi(W) \subset D_w'$, where (D_w', L_w') is derived from $\mathcal{L}'$ by the rules in Definition 4, and

b) the pair $(\mathcal{M}', \phi)$ is universal with respect to the condition a).

This is attained by a successive free extensions with respect to generators (i.e. elements of S) as follows:

$$
\begin{array}{l}
\mathcal{M} \\
\phi^1 \downarrow \text{ free extension with respect to } (s_p, W) \\
\mathcal{M}^1 \\
\phi^2 \downarrow \text{ free extension with respect to } (s_{p-1}, L^1_{s_p}(W)) \\
\mathcal{M}^2 \\
\phi^3 \downarrow \text{ free extension with respect to } (s_{p-2}, L^2_{s_{p-1}}(L^1_{s_p}(W))) \\
\mathcal{M}^3 \\
\downarrow \\
\vdots \\
\phi^p \downarrow \\
\mathcal{M}^p = \mathcal{M}'
\end{array}
$$

Here, for each $i\ (1 \leqq i \leqq p)$, $L^i_s\ (s \in S)$ is the partial action of s on $\mathcal{M}^i$. Moreover, since each morphism ϕ^i is injective, we have identified W with its image under ϕ^1, $L^1_{s_p}(W)$ with its image under ϕ^2, and so on. If $\mathcal{M}$ is finite-dimensional, then so is $\mathcal{M}'$ since each of the steps preserves finiteness of dimension.

Next we introduce an operation called quotient, which introduces new relations into a partial module.

Definition 7. Let $\mathcal{M} = (V, \mathcal{L}, \iota)$ be a partial module for (S, T), and let K_0 be a k-subspace of V. A *quotient of* $\mathcal{M}$ *with respect to* K_0 is a pair $(\mathcal{M}', \phi)$ of a partial module $\mathcal{M}'$ and a morphism $\phi: \mathcal{M} \to \mathcal{M}'$ satisfying the following conditions:

a) $\phi(K_0) = 0$, i.e. $K_0 \subset \operatorname{Ker} \phi$,

b) the pair $(\mathcal{M}', \phi)$ is universal with respect to the condition a).

Remark. Often $\mathcal{M}'$ itself is called the quotient.

Lemma 4. *In the same situation as in Definition 7, a quotient* $\mathcal{M}'$ *of* $\mathcal{M}$ *with respect to* K_0 *exists and is unique up to isomorphism. If* $\mathcal{M}$ *is finite-dimensional, so is* $\mathcal{M}'$.

Proof. First we shall construct such $\mathcal{M}'$. Let K be the smallest $\mathcal{L}$-invariant subspace containing K_0. Here, a subspace W of V is called $\mathcal{L}$-invariant if, for every $s \in S$, $L_s(W \cap D_s) \subset W$ holds (in other words, as far as the action of s is defined on it, any element of W is sent into W by the action of s.) Put $V' = V/K$ and let $\phi: V \to V'$ be the natural projection. For each $s \in S$, put

$$D'_s = \phi(D_s) \simeq D_s/K \cap D_s.$$

Since $L_s(K \cap D_s) \subset K$, L_s induces a linear map L'_s: $D'_s \to V'$ with the following commutative diagram:

$$\begin{array}{ccc} D_s & \xrightarrow{L_s} & V \\ \phi\downarrow & & \downarrow\phi \\ D'_s \simeq D_s/K \cap D_s & \xrightarrow{L'_s} & V/K \end{array}$$

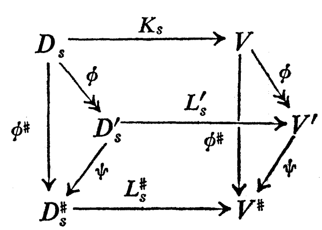
Finally put $\iota'(t) = \phi(\iota(t))$ $(t \in T)$. Then ϕ: $\mathscr{M} \to \mathscr{M}'$ is a morphism and since $K_0 \subset K$, $\mathscr{M}'$ satisfies the condition a).

In order to prove that $\mathscr{M}'$ also satisfies the condition b), suppose we have another pair $(\mathscr{M}^\#, \phi^\#)$ satisfying a) and we shall show the existence and the uniqueness of the morphism ψ: $\mathscr{M}' \to \mathscr{M}^\#$ satisfying $\phi^\# = \psi \circ \phi$. First of all, since ϕ is a surjective linear map, ψ is determined uniquely as a linear map as long as it exists. Next, in order to prove the existence of a linear map ψ with $\phi^\# = \psi \circ \phi$, we have only to show $K \subset \operatorname{Ker} \phi^\#$. Since $K_0 \subset \operatorname{Ker} \phi^\#$ and K is the smallest $\mathscr{L}$-invariant subspace containing K_0, it is sufficient to show the $\mathscr{L}$-invariance of $\operatorname{Ker} \phi^\#$. Since $\phi^\#$ is a morphism, the statement 1° of Definition 3 implies that for any $s \in S$,

$$\phi^\#(L_s(\operatorname{Ker} \phi^\# \cap D_s)) = L_s^\#(\phi^\#(\operatorname{Ker} \phi^\# \cap D_s)) = L_s^\#(0) = 0,$$

that is, $L_s(\operatorname{Ker} \phi^\# \cap D_s) \subset \operatorname{Ker} \phi^\#$, proving the $\mathscr{L}$-invariance of $\operatorname{Ker} \phi^\#$.

To prove that ψ is a morphism, we must check the conditions 1° and 2° of Definition 3. The commutativity of ψ with the partial action of $s \in S$ (the lower parallelogram of the diagram below) follows from the commutativity of the upper parallelogram (with respect to ϕ) and the large rectangle (with respect to $\phi^\#$) and the surjectivity of $D_s \to D'_s$. The condition concerning the images of elements of T follows similarly.



The uniqueness of $(\mathscr{M}', \phi)$ follows from the standard argument concerning universality. If V is finite-dimensional, so is V' since it is a quotient vector space of V. □

Remark. The smallest $\mathscr{L}$-invariant subspace K containing K_0 is obtained as follows: for $i=1, 2, \cdots$, define an increasing sequence $\{K_i\}$ of subspaces of V inductively by $K_i=K_{i-1}+\sum_{s\in S} L_s(K_{i-1}\cap D_s)$ and let K be its limit $\bigcup_i K_i$. If V is finite-dimensional, its limit is always attained by some K_i.

§ 4. Proof of the substantial two lemmas

In this section, we prove the two lemmas we stated in Section 2 by describing explicitly how to construct the partial modules claimed in the lemmas, using the fundamental operations several times.

Proof of the Baby Lemma. First of all, we bring the following partial module $\mathscr{M}^0=(V^0, \mathscr{L}^0, \iota^0)$ for (S, T):

$$\begin{cases} V^0=kT \text{ (k-vector space with the elements of T as k-basis),} \\ \text{if we put } \mathscr{L}^0=\{(D^0_s, L^0_s)\}_{s\in S}, \text{ then } D^0_s=0 \text{ for all } s\in S, \text{ and } \iota(t)=t \text{ (t on} \\ \qquad \text{the right-hand side represents the basis element of } V^0=kT). \end{cases}$$

Then $\mathscr{M}^0$ satisfies the condition:

for any partial module $\mathscr{M}$ for (S, T), there exists a unique morphism $\mathscr{M}^0\to\mathscr{M}$.

In other words, $\mathscr{M}^0$ is the most universal partial module. We begin our construction with this $\mathscr{M}^0$. The first target is the one universal with respect to the condition a1) in the statement of this lemma. The condition a1) can be restated in the following way. Let $\{w_1t_1, w_2t_2, \cdots, w_qt_q\}$ (w_i is a monomial in S, $t_i\in T$ $(1\leqq i\leqq q)$) be the set of all "terms" appearing in the expression of the elements $\tilde{u}\in\tilde{U}$ as k-linear combinations of elements of the form wt, where w is a monomial in S and $t\in T$. Now a1) means

$$(*)\qquad t_j\in D_{w_j}\quad (1\leqq j\leqq q).$$

This is attained by the following succession of free extensions (with respect to monomials):

$$\begin{array}{l} \mathscr{M}^0 \\ \big\downarrow \text{ free extension with respect to } (w_1, kt_1) \\ \mathscr{M}^1 \\ \big\downarrow \text{ free extension with respect to } (w_2, kt_2) \\ \mathscr{M}^2 \\ \big\downarrow \\ \vdots \\ \big\downarrow \text{ free extension with respect to } (w_q, kt_q) \\ \mathscr{M}^q \end{array}$$

Here, since all the morphisms $\mathscr{M}^i \to \mathscr{M}^{i+1}$ are injective, we regard $t_j \in kT = V^0$ as an element of the underlying space of any $\mathscr{M}^i (0 \leqq i \leqq q)$. kt_j is the one-dimensional subspace spanned by t_j. The universality of $\mathscr{M}^q$ with respect to the q conditions $(*)$ follows by combining the universality of each $\mathscr{M}^i$ concerning each of these q conditions.

The partial module $\mathscr{M}$ having the universality property with respect to both a1) and a2) can be constructed from $\mathscr{M}^q$ by the quotient procedure with respect to the subspace K_0 of the underlying space of $\mathscr{M}^q$ spanned by the images $\tilde{u}_{\mathscr{M}^q}$ of all the "module relations" $\tilde{u} \in \tilde{U}$ in $\mathscr{M}^q$. The required properties of $\mathscr{M}$ can be deduced from the universality property of $\mathscr{M}^q$ and the property of quotient operation.

$\mathscr{M}$ is finite-dimensional because so is $\mathscr{M}^0$ and both free extension and quotient preserve this property. □

Proof of the Growth Lemma. Put $S = \{s_1, s_2, \cdots, s_l\}$ and let $\{w_1, w_2, \cdots, w_m\}$ be the set of all monomials in S appearing in the expressions of the "algebra relations" $r \in R$ as k-linear combinations of monomials in S. The partial module $\mathscr{M}^*$ having the universality property with respect to a1) and a2) can be constructed by the following chain of free extensions:

$\mathscr{M}$

$\downarrow$ free extension with respect to (s_1, V)

$\mathscr{M}^1$

$\downarrow$ free extension with respect to (s_2, V)

$\mathscr{M}^2$

$\downarrow$

$\vdots$

$\downarrow$ free extension with respect to (s_l, V)

$\mathscr{M}^l$

$\downarrow$ free extension with respect to (w_1, V)

$\mathscr{M}^{l+1}$

$\downarrow$ free extension with respect to (w_2, V)

$\mathscr{M}^{l+1}$

$\downarrow$

$\vdots$

$\downarrow$ free extension with respect to (w_m, V)

$\mathscr{M}^{l+m} = \mathscr{M}^*$

Since all morphisms $\mathcal{M}^i \to \mathcal{M}^{i+1}$ are injective, we have written the image of V (the underlying space of $\mathcal{M}$) in each $\mathcal{M}^i$ simply as V.

The required $\mathcal{M}'$ can be constructed from $\mathcal{M}^*$ by the quotient operation with respect to K_0 spanned by the set $\{L_r^*(v) \mid v \in V, r \in R\}$. Here, we denote $\mathcal{M}^* = (V^*, \mathcal{L}^*, \iota^*)$, $\mathcal{L}^* = \{(D_s^*, L_s^*)\}_{s \in S}$ and L_r^* is derived from $\mathcal{L}^*$ by the rules in Definition 4. The property of $\mathcal{M}^*$ assures that any $v \in V$ lies in D_r^* for any $r \in R$. The required universality property of $\mathcal{M}'$ with respect to a1)–a3) follows from that of each step. Again the uniqueness follows from the universality.

This whole process preserves the finiteness of dimension because each of the steps does. □

Remarks. 1° $\mathcal{M}'$ in the Growth Lemma satisfies the following conditions relative to $\mathcal{M}$.

a) If $y \in k\langle S \rangle T$ is termwise defined in $\mathcal{M}$, then for any $s \in S$, sy is termwise defined in $\mathcal{M}'$. Moreover, if $y_{\mathcal{M}} = 0$ then $(sy)_{\mathcal{M}'} = 0$.

b) If $y \in k\langle S \rangle T$ is termwise defined in $\mathcal{M}$, then for any algebra relation $r \in R$, ry is termwise defined in $\mathcal{M}'$ and its image $(ry)_{\mathcal{M}'}$ is equal to 0.

These follow easily by noting that the image of each term of y in $\mathcal{M}'$ lies in D_s' for all $s \in S$ and also in D_r' for all $r \in R$.

2° In the proof of the Growth Lemma, the quotient operation was performed only at the end of the process in order to simplify the description. In actual calculation, it is possible and often more efficient to split the quotient operation into several parts and perform free extension and quotient alternately, keeping the "logical order" that, for $v \in V$ and $r \in R$, $\phi(v) \in D_r'$ must be attained before taking quotient with respect to $L_r'(\phi(v))$.

§ 5. Proof of the theorem

Proof of the theorem. 1° Suppose $\phi^{(i)}: \mathcal{M}^{(i-1)} \to \mathcal{M}^{(i)}$ is a surjective linear map and put $\mathcal{M}^{(i)} = (V^{(i)}, \mathcal{L}^{(i)}, \iota^{(i)})$, $\mathcal{L}^{(i)} = \{(D_s^{(i)}, L_s^{(i)})\}_{s \in S}$. Since $\phi^{(i)}(V^{(i-1)}) = V^{(i)}$ by hypothesis, the condition a1) of the Growth Lemma tells that the domain $D_s^{(i)}$ of $L_s^{(i)}$ covers the whole $V^{(i)}$. Hence V is provided with a left $k\langle S \rangle$-module structure if we define the action of each $s \in S$ to be $L_s^{(i)}$. By the condition a3) in the Growth Lemma, any $r \in R$ acts trivially (as 0) on $V^{(i)}$. Hence $V^{(i)}$ can be regarded as a left $A = k\langle S \mid R \rangle$-module and there exists an A-module homomorphism $\tilde{\alpha}$ from AT to $V^{(i)}$ sending each $t \in AT$ to $\iota^{(i)}(t) \in V^{(i)}$. Since we have a partial module morphism from $\mathcal{M}^{(0)}$ to $\mathcal{M}^{(i)}$, the image $\tilde{u}_{\mathcal{M}^{(i)}}$ of any module relation $\tilde{u} \in \tilde{U}$ in $\mathcal{M}^{(i)}$ is equal to 0. If $u \in U$ is the canonical image of $\tilde{u} \in \tilde{U}$ in AT, then $\tilde{\alpha}$ maps u onto $\tilde{u}_{\mathcal{M}^{(i)}} = 0$, so that $\tilde{\alpha}$ induces an A-module homomorphism α: $M=AT/AU\to V^{(i)}$.

On the other hand, the left A-module M can be naturally regarded as a partial module for (S, T) by defining L_s to be the action of $\bar{s}\in A$ (image of $s\in k\langle S\rangle$ in $A=k\langle S|R\rangle$) and $\iota(t)$ to be $\bar{t}\in M$ (image of $t\in T\subset AT$ in $M=AT/AU$). Since the image of any $u\in U$ in M is 0, the universality of the Baby partial module $\mathcal{M}^{(0)}$ assures that there exists a partial module morphism $\mathcal{M}^{(0)}\to M$. The action of any $s\in S$ is defined on all over M and any $r\in R$ acts as 0, so that the conditions a1)–a3) for $\mathcal{M}'$ in the Growth Lemma is always satisfied by M. Hence, the universality of $\mathcal{M}'$ in the Growth Lemma assures that, for any $\mathcal{M}$ such that there exists a partial module morphism $\mathcal{M}\to M$, there also exists a morphism $\mathcal{M}'\to M$. Applying this argument inductively starting with $\mathcal{M}=\mathcal{M}^{(0)}$, we have a partial module morphism $\mathcal{M}^{(j)}\to M$ for any $j=1, 2, \cdots$, especially β: $\mathcal{M}^{(i)}\to M$ for our i. Moreover, if we regard $\mathcal{M}^{(i)}$ as an A-module as above, β can also be regarded as an A-module homomorphism.

Now $\beta\circ\alpha$ is an A-module endomorphism of M mapping every $\bar{t}\in M$ onto itself. Since M is generated by $\{\bar{t}\}_{t\in T}$ as an A-module, we have $\beta\circ\alpha=\mathrm{id}$. On the other hand, $\alpha\circ\beta$ is also an A-module endomorphism of $\mathcal{M}^{(i)}$ mapping every $\iota^{(i)}(t)$ onto itself. Since $\mathcal{M}^{(i)}$ is obtained from $\mathcal{M}^0$ (in the proof of the Baby Lemma) by a repetition of free extension and quotient, any element of $V^{(i)}$ is obtained from $\{\iota^{(i)}(t)\}_{t\in T}$ by applying $L_s^{(i)}$ $(s\in S)$ and forming linear combinations several times.*) This means that $\mathcal{M}^{(i)}$, regarded as an A-module, is generated by $\{\iota^{(i)}(t)\}_{t\in T}$. Hence $\alpha\circ\beta=\mathrm{id}$.

Therefore we conclude that $\mathcal{M}^{(i)}\simeq M$.

2° $\Rightarrow$) $M\simeq\mathcal{M}^{(i)}$ is finite-dimensional because so is $\mathcal{M}^{(0)}$ and this property is preserved under the process described in the Growth Lemma.

$\Leftarrow$) Let $\{e_\alpha\}$ be a k-basis of M (note that the number is finite) and put

$$\bar{t}=\sum c_{t\alpha}e_\alpha \quad (c_{t\alpha}\in k),$$
$$\bar{s}e_\alpha=\sum d_{s\alpha\beta}e_\beta \quad (d_{s\alpha\beta}\in k).$$

Moreover, for each e_α, fix an element $y_\alpha\in k\langle S\rangle T$ whose image in M under the natural projection is e_α. Then the proof will be complete if we show the following two lemmas:

Lemma A. *For a sufficiently large i, the following* a) *and* b) *hold in* $\mathcal{M}^{(i-1)}$:

a) *for all* α *and* $s\in S$, sy_α *is termwise defined, and*

b) *if we put* $Y=\{t-\sum c_{t\alpha}y_\alpha, sy_\alpha-\sum d_{s\alpha\beta}y_\beta\}_{t\in T, s\in S, \alpha, \beta}$, *then any element y of Y is termwise defined and its image* $y_{\mathcal{M}^{(i-1)}}$ *is equal to* 0.

Lemma B. *For an i satisfying* a) *and* b) *above,* $\phi^{(i)}$ *is a surjective linear map.*

In order to show Lemma A, we first show the following

Lemma A′. *In* $\mathcal{M}^{(j)}$ $(j=0, 1, 2, \cdots)$,

a) *any element of* $k\langle S\rangle_j T$ *is termwise defined, and*

b) *any element* y *of* $k\langle S\rangle_j \tilde{U}+\sum_{l+m<j} k\langle S\rangle_l Rk\langle S\rangle_m T$ *is termwise defined and its image* $y_{\mathcal{M}^{(j)}}$ *is equal to* 0.

Here, $k\langle S\rangle_j$ denotes the subspace of $k\langle S\rangle$ spanned by the monomials in S of length j at the longest.

Lemma A′ clearly holds for $j=0$. It is shown inductively on j by Remark 1° to the Growth Lemma and the following facts:

$$k\langle S\rangle_j T=Sk\langle S\rangle_{j-1}T, \quad \text{and}$$
$$k\langle S\rangle_j \tilde{U}+\textstyle\sum_{l+m<j} k\langle S\rangle_l Rk\langle S\rangle_m T$$
$$=S(k\langle S\rangle_{j-1}\tilde{U}+\textstyle\sum_{l+m<j-1} k\langle S\rangle_l Rk\langle S\rangle_m T)+Rk\langle S\rangle_{j-1}T.$$

Since $\{e_\alpha\}$, S and T are all finite sets, we have only finitely many sy_α and elements of Y. So they are all contained in $k\langle S\rangle_j T$ for a sufficiently large j. Besides, Y is contained in the kernel of the natural projection $k\langle S\rangle T\to M$, which is $k\langle S\rangle\tilde{U}+k\langle S\rangle Rk\langle S\rangle T$. Being finite, Y is contained in $k\langle S\rangle_j\tilde{U}+\sum_{l+m<j} k\langle S\rangle_l Rk\langle S\rangle_m T$ for a sufficiently large j. This proves Lemma A.

To prove Lemma B, we note that a) and b) in Lemma A implies that, if we put $W=\sum_\alpha k(y_\alpha)_{\mathcal{M}^{(i-1)}}$, W contains all $\iota^{(i-1)}(t)$ and W is closed under all $L_s^{(i-1)}$. From the construction of $\mathcal{M}^{(i-1)}$, such a subspace W must coincide with the whole space $V^{(i-1)}$.*) Therefore the process to make $\mathcal{M}^{(i)}$ from $\mathcal{M}^{(i-1)}$ contains no substantial free extensions, so that $\phi^{(i)}$ is a surjective linear map. □

Remark. To be precise, the property of $\mathcal{M}^{(i)}$ used in the *) part is formulated as follows. A partial module $\mathcal{M}=(V, \mathcal{L}, \iota)$ for (S, T) is said to be *generated by* (S, T) if V is the only $\mathcal{L}$-invariant subspace of V containing all $\iota(t)$. It is equivalent to saying that, if we put $V_0=\sum_{t\in T} k\iota(t)$ and $V_i=V_{i-1}+\sum_{s\in S} L_s(V_{i-1}\cap D_s)$ $(i=1, 2, \cdots)$, then $V=\bigcup_{i=1}^\infty V_i$. $\mathcal{M}^{(0)}$ is obviously generated by (S, T) and this property is also preserved by free extension and quotient, so that every $\mathcal{M}^{(i)}$ is generated by (S, T).

In fact, $\mathcal{M}^{(i)}$ satisfies a stronger condition that the set of images of all elements of $k\langle S\rangle T$ which are termwise defined in $\mathcal{M}^{(i)}$ coincides with the whole space $V^{(i)}$. When this is the case, let us say that $\mathcal{M}^{(i)}$ is *termwise generated by* (S, T). This is also inductively shown using the following two arguments.

1° Suppose $\mathscr{M}$ is termwise generated by (S, T). Let W be a subspace of V and suppose that W is spanned by the images of elements of $k\langle S\rangle T$ which are termwise defined in $\mathscr{M}$ and whose images lie in W. Then the free extension $\mathscr{M}'$ of $\mathscr{M}$ with respect to (w, W) (w is any monomial in S) is also termwise generated by (S, T). This is because the underlying space of $\mathscr{M}'$ is spanned by the images of the elements of $k\langle S\rangle T$ of the form $s_{q+1}s_{q+2}\cdots s_p w't$, where $(w't)_{\mathscr{M}} \in W$ and $w=s_1s_2\cdots s_p$, $0\leqq q\leqq p$.

2° If $\mathscr{M}$ is termwise generated by (S, T), then so is a quotient of $\mathscr{M}$ because it is a surjective image of $\mathscr{M}$.

In the process of the Growth Lemma, the free extension is always performed with respect to the image of V, and it always satisfies the condition for W in 1° above. As a result, $\mathscr{M}^{(i)}$ is also termwise generated by (S, T) for any i.

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