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# Maximal Analytic Spread in Birational Extensions of Regular Local Rings

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The purpose of this note is to give a different viewpoint and a different proof for the result in [Sy] which states that if $(R, m) \subsetneq (S, n)$ are $d$-dimensional regular local rings with the same quotient field, then the analytic spread, $l(mS)$, is at most $d-1$. The different viewpoint comes from dropping the assumption that $(S, n)$ is regular and seeing what it means for $l(mS)$ to be equal to $d$. This turns out to be a very useful tool for studying birational extensions of regular local rings. The different proof evolved after communications from Craig Huneke and David Rees. Huneke showed me that certain hypotheses in [Sy] implied that the natural map from $R/m$ to $S/n$ is an isomorphism.

**Theorem.** *Let $(R, m)$ be a $d$-dimensional regular local ring. Let $(S, n)$ be a $d$-dimensional local ring which birationally dominates $R$. If $l(mS)=d$, then*

(i) *$S$ is dominated by the $m$-adic prime divisor of $R$*

(ii) *$R/m=S/n$*

(iii) *the natural map of the associated graded ring of $R$ to the associated graded ring of $S$ is injective.*

Note that (ii) and (iii) imply that a minimal basis for $m$ in $R$ is a subset of a minimal basis of any ideal $I$ in $S$ which contains $mS$.

**Corollary.** *Let $(R, m)$ be a $d$-dimensional regular local ring. Let $(S, n)$ be a $d$-dimensional regular local ring which birationally dominates $R$. If $S \neq R$, then $l(mS)<d$.*

*Proof of the Corollary.* Suppose $S \neq R$ and $l(mS)=d$. Say $x_1, \cdots, x_d$ is a minimal basis for $m$. Zariski's Main Theorem implies that $\mathrm{ht}(mS) < d$, so since $S$ is regular, there is a relation

$$u_1x_1+\cdots+u_dx_d \in n^2,$$

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with $u_i$ in $S$, not all in $n$. But, by (ii), $u_i \equiv v_i \bmod n$ with $v_i$ in $R$, not all in $m$. Thus, there is a relation

$$v_1x_1+\cdots+v_dx_d \in n^2.$$

However, by choice of the $x_i$, $v_1x_1+\cdots+v_dx_d \in m\backslash m^2$. This contradicts (iii).

*Proof of the Theorem.* We may assume that $R/m$ is an infinite field and take $x_1, \cdots, x_d$ to be a minimal basis for $m$ in $R$ and an analytically independent set in $S$. The analytical independence of the $x_i$ in $S$ implies that $nS[m/x_1]$ is a prime ideal so we can form the rings $V=R[m/x_1]_{mR[m/x_1]}$ and $W=S[m/x_1]_{nS[m/x_1]}$. $V$ is the $m$-adic prime divisor of $R$. $W\neq K$, by analytic independence again, so $V=W$. This proves (i). Thus, if $\nu$ is the $m$-adic valuation, $\nu(z)>0$ for all $z$ in $n$. $\nu$ has minimum value 1, so $\nu(z)\geq r$ for all $z$ in $n^r$. This means that $n^r\cap R=m^r$ for all $r$. This proves (iii). The dimension inequality

$$d \geq \dim S+\text{tr. d. } S/n\colon R/m$$

shows that $S/n$ is algebraic over $R/m$ and, since the residue field $V/m(V)$ is pure transcendental over $R/m$, it follows that $S/n=R/m$.

We conclude by giving one illustration of how the theorem can be used to find properties of normal local domains which birationally dominate 2-dimensional regular local rings. (A very rich source for information about such normal local domains is, of course, Lipman's paper [L]. Huneke and also Sally have some more recent results, too.)

**Proposition.** *Let $(R, m)$ be a 2-dimensional regular local ring. Let $(S, n)$ be a local domain which birationally dominates $R$. If $S$ is a UFD, then $S$ is regular.*

*Proof.* We will assume that $S$ has dimension 2. Suppose that $S$ is not regular. Then we may assume that $R$ is "maximally regular" in $S$, i.e., if $R\subseteq R_1\subset S$ with $R_1$ regular local, then $R=R_1$. For if there is no maximally regular local ring containing $R$ and contained in $S$, we could construct a valuation domain between $R$ and $S$ using quadratic transformations as in Zariski's factorization theorem, cf. [A]. Now $R$ maximally regular in $S$ means that $mS$ is not principal. Thus, $l(mS)=2$. Consequently, the number of generators of any prime ideal containing $mS$ is at least 2, by the Theorem. However, $mS$ has height 1 so $S$ cannot be a UFD.

## References

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