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# On the Definition of a Euclid Ring

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There are several definitions of the notion of a Euclid ring, and we start with historical survey of these definitions. In this article, we mean by a ring a commutative ring with identity, and we propose the following definition:

A ring $R$ is a Euclid ring if there is a pair of an ordered set $W$ with minimum condition and a mapping $\varphi$ of $R$ into $W$ satisfying the condition that for $a, b \in R$, there are $q, r \in R$ such that

$$b=qa+r \text{ with either } r=a \text{ or } \varphi r<\varphi a.$$

This is a modified version of the one which was given by Nagata [3]. The definition given by Samuel [4] is more general than the classical definition and is more restrictive than ours. As was shown by Nagata [2], there is an integral domain which is a Euclid ring in the sense of Samuel, but not in the classical sense. Thus, Samuel's definition is essentially more general than the classical one. But, our new definition does not increase the family of Euclid rings than Samuel's definition, though the choice of algorithm surely enlarges by our generalization.)

We would like to discuss advantage of our new definition, including our proof of the following fact:

The direct sum of a finite number of Euclid rings is again a Euclid ring.

## § 1. Historioal survey

The classical definition of a Euclid ring can be stated as follows (see, for instance, van der Waerden [5]):

An integral domain $R$ is a Euclid ring if there is a mapping $\varphi$ of $R-\{0\}$ into the set $N$ of natural numbers which satisfies two conditions

(1) if $a, b$ are non-zero elements of $R$ then $\varphi(ab)\geq\varphi a$, and

(2) if $a, b \in R$ and $a\neq 0$, then there are $q, r \in R$ with

$$b=qa+r \text{ and either } r=0 \text{ or } \varphi r<\varphi a.$$

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There is a modified definition by Zariski-Samuel [6]; they called "Euclidean domain" instead of "Euclid ring," and the modification is:

(i) the mapping $\varphi$ is $R \to N$, namely $\varphi 0$ is defined, and

(ii) in (2) "either $r=0$ or" is taken away.

This modification implies that $\varphi 0$ is smaller than any other $\varphi a$, and therefore this modification is nothing but to modify $\varphi$ so that $\varphi 0=1$ and $\varphi a = 1 + (\text{original } \varphi a)$.

By the way, in connection with the theory of integers, in some literature, the word Euclidean is used in order to mean that the ring of integers of the algebraic number field concerned is a Euclid ring with $\varphi$ being the norm map. This is rather a very special case, and we are not going to discuss the case.

There was an important remark by Motzkin [1] that (i) as for the value domain of $\varphi$, we can take any set of ordinal numbers and (ii) the condition (1) in the classical definition is not necessary, in proving the most important results on Euclid rings, namely, Euclid algorithm works and the ring is a principal ideal ring.

In view of the remark by Motzkin, Samuel [4] generalized the definition as follows:

A ring $R$ is called a Euclid ring if there is a pair of a well-ordered set $W$ and a mapping $\varphi$ of $R$ into $W$ such that for given $a, b \in R$ with $a \neq 0$, there exist $q, r \in R$ with

$$b = qa + r \quad \text{and} \quad \varphi r < \varphi a.$$

He proved there, among other things, that the direct sum of a finite number of Euclid rings is again a Euclid ring and that if $R$ is a Euclid ring with certain $\varphi$, $R$ has another mapping $\varphi'$ to $W$ such that $\varphi' a \leq \varphi a$ for any $a \in R$ and such that $R$ is a Euclid ring satisfying the condition (1) with respect to $\varphi'$. He also introduced the notion of the smallest algorithm, which automatically satisfies the condition (1).

Under the definition, rings which are not integral domains can be Euclid rings, and therefore the family of Euclid rings was enlarged. But Samuel [4] asked if the family of Euclid domains was enlarged by the definition. Nagata [2] answered the question by showing an example of a Euclid domain $R$ such that under the smallest algorithm, the value domain is $N \times N$ (with lexicographical order), hence cannot be a Euclid ring in the classical sense.

Then Nagata [3] noticed that the value domain $W$ need not be a well-ordered set; it is enough to be an ordered set with minimum condition, and he proposed the following definition:

A ring $R$ is called a Euclid ring if there is a pair of an ordered set $W$

with minimum condition and a mapping $\varphi$ of $R-\{0\}$ to $W$ such that for $a, b \in R$, $a \neq 0$, there are $q, r \in R$ with

$$b = qa + r \text{ and either } r = a \text{ or } \varphi r < \varphi a.$$

It was shown also that under this definition, we have an easy proof of the fact that the direct sum of a finite number of Euclid rings is again a Euclid ring, and that the family of Euclid rings was not enlarged by this generalization.

## § 2. New definition

We now modify the definition of Nagata [3] as follows:

A ring $R$ is called a *Euclid ring* if there is a pair of an ordered set $W$ with minimum condition and a mapping $\varphi$ of $R$ into $W$ satisfying the condition:

For $a, b \in R$, there are $q, r \in R$ such that

$$b = qa + r \text{ and either } r = a \text{ or } \varphi r < \varphi a.$$

Under the circumstances, we say that $(R, W, \varphi)$ is a Euclid ring and that $r$ is a *right residue* at the division of $b$ by $a$. $\varphi$ is called an *algorithm* on $R$.

Note that if $a=0$, then $r$ must be $b$, hence this definition implies that $\varphi 0 > \varphi b$ if $b \neq 0$. Thus, on one hand, a Euclid ring in this sense is a Euclid ring in the sense of Nagata [3], and on the other hand, if $(R, W, \varphi)$ is a Euclid ring in the sense of Nagata [3], then we add one new element, say $s$, to $W$ defining $s$ to be bigger than every element of $W$, and then define $\varphi 0 = s$, and we see that $R$ becomes a Euclid ring under the new definition.

Thus, this new definition is practically the same as in Nagata [3], but there are some conveniences in handling Euclid rings. In order to show this fact, we give proofs of some facts on Euclid rings.

**Theorem 1.** *A Euclid ring is a principal ideal ring.*

For this, the usual proof works quite well.

**Theorem 2.** *If $(R_i, W_i, \varphi_i)$ $(i=1, \cdots, n)$ are Euclid rings, then the direct sum $R = R_1 + \cdots + R_n$ is a Euclid ring with the ordered set $W = W_1 \times \cdots \times W_n$, in which $(a_1, \cdots, a_n) \geq (b_1, \cdots, b_n)$ if and only if $a_i \geq b_i$ for all $i$, and the mapping $\varphi = (\varphi_1, \cdots, \varphi_n)$.*

*Proof.* Consider two elements $(c_1, \cdots, c_n)$, $(d_1, \cdots, d_m)$ of $R$. Then $d_i = q_i c_i + r_i$ with a right residue $r_i$ at the division of $d_i$ by $c_i$. Then

$(d_1, \cdots, d_n)=(q_1, \cdots, q_n)(c_1, \cdots, c_n)+(r_1, \cdots, r_n)$ and $(r_1, \cdots, r_n)$ is a right residue. Q.E.D.

We now prove the following fact already proved in [3]:

**Theorem 3.** *If $(R, W, \varphi)$ is a Euclid ring, then there is a pair of a well-orderd set $W'$ and a mapping $\varphi'$ of $R$ into $W'$ so that $(R, W', \varphi')$ is a Euclid ring.*

For the proof, it suffices to prove the following

**Theorem 4.** *If $W$ is an ordered set with minimum condition, then there is a mapping $\theta$ of $W$ into a suitable well-ordered set $W'$ so that if $a>b$ in $W$ then $\theta a>\theta b$ in $W'$.*

*Proof.* We take a well-ordered set $W'$ of bigger cardinality than $W$. We define $\theta$ inductively. Namely, we are to define $\theta^{-1}(w)$, considering an element $w$ of $W'$ such that for all $y$ in $W'$ with $y<w$, $\theta^{-1}(y)$ are already defined. Let $M_w$ be the complement of $T_w=\bigcup_{y<w}\theta^{-1}(y)$ with respect to $W$. If $M_w$ is empty, then $\theta$ is already defined, and we assume that $M_w$ is not empty. Then we define $\theta^{-1}(w)$ to be the set of minimal elements in $M_w$. Thus we define $\theta$ on the union $U$ of all $\theta^{-1}(w)$ $(w \in W')$. $\theta(U)\neq W'$ because the cardinality of $w'$ is bigger than that of $W$. Because of the minimum condition on $W$, if the union $U$ is not $W$, then we can go on. Thus $\theta$ is a mapping of $W$ into $W'$. If $a>b$ in $W$, and if $\theta a=w$, then $b<a \in M_w$. Since $a$ is minimal in $M_w$, $b \notin M_w$. Therefore $b \in T_w$, i.e., $\theta b<w=\theta a$. Q.E.D.

As for the following assertion ([2], Proposition 4.4) we do not have easy proof without using Theorem 3 above.

**Theorem 5.** *If $(R, W, \varphi)$ is a Euclid ring and if $\psi$ is a ring homomorphism of $R$ onto $R'$, then $R'$ is a Euclid ring. If $W$ is well-ordered, then an algorithm $\varphi'$ on $R'$ is obtained by*

$$\varphi'(a')=\min\{\varphi a \mid \psi a=a'\}.$$

*Proof.* By Theorem 3, we may assume that $W$ is well-ordered. Now, if $a', b' \in R'$, then there are $a, b \in R$ such that $\psi a=a'$. $\varphi a=\varphi'(a')$, $\psi b=b'$. Then $b=qa+r$ with a right residue $r$, namely, either $r=a$ or $r<a$. In the former case, we have $b'=q'a'+a'$ $(q'=\psi q)$. In the latter case, $b'=q'a'+r'$ and $\varphi'(r')\leq\varphi r<\varphi a=\varphi'(a')$. Q.E.D.

Also for the proof of the following theorem ([2], Proposition 4.5), we need Theorem 3 above:

**Theorem 6.** *If $(R, W, \varphi)$ is a Euclid ring and if $S$ is a multiplicatively closed subset of $R$ not containing 0, then the ring $R_S$ is a Euclid ring. If $W$ is well-ordered, then an algorithm $\varphi'$ of $R_S$ is obtained by*

$$\varphi'(a') = \min \{\varphi a \mid a \in R,\ a' = a/s \text{ with some } s \in S\}.$$

*Proof.* By Theorem 3, we may assume that $W$ is well-ordered. Now, if $a', b' \in R_S$, then there are $a, b \in R$ such that $a' = a/s$, $s \in S$, $\varphi a = \varphi'(a')$, $b' = b/s_1$, $s_1 \in S$. Then $b = qa + r$ $(q, r \in R)$ with either $a = r$ or $\varphi r < \varphi a$. If $a = r$, then $b = (q+1)a$ and therefore $b' = ((q+1)s/s_1 - 1)a' + a'$. If $r < a$, then $b' = (qs/s_1)a' + (r/s_1)$, and $\varphi'(r/s_1) \leq \varphi r < \varphi a = \varphi'(a')$. Q.E.D.

As is well known, any local principal ideal ring is either an integral domain or an Artin local ring. Therefore each principal ideal ring is the direct sum of a finite number of principal ideal rings such that each of them is an integral domain or an Artin local ring. Therefore Theorems 5 and 6 imply:

**Theorem 7** ([2], Theorem 4.1). *A ring $R$ is a Euclid ring if and only if it is the direct sum of a finite number of Euclid rings $R_1, \cdots, R_n$ such that each $R_i$ is either an integral domain or an Artin local ring.*

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