Maximal Buchsbaum Modules over Regular Local Rings and a Structure Theorem for Generalized Cohen-Macaulay Modules

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§ 1.   Introduction

The purpose of this paper is to give, applying a decomposition theorem of maximal Buchsbaum modules over regular local rings, a structure theorem for generalized Cohen-Macaulay modules relative to so-called standard systems of parameters.   Before stating the results more precisely, let us recall the definition of Buchsbaum modules and generalized Cohen-Macaulay modules respectively (see (2.8) and (4.1) for a further detail).

Throughout let A denote a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module of $\dim_A M = s$.   Then M is said to be *Buchsbaum* (resp. *generalized Cohen-Macaulay*), if there is given a numerical invariant $I_A(M)$ of M so that the equality

$$I_A(M) = \ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$$

holds for any parameter ideal $\mathfrak{q}$ for M (resp. the supremum $\sup_{\mathfrak{q}}[\ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)]$ is finite, where $\mathfrak{q}$ runs over parameter ideals for M, and the equality

$$I_A(M) = \sup_{\mathfrak{q}} [\ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)]$$

holds) (here $\ell_A(M/\mathfrak{q}M)$ and $e_{\mathfrak{q}}(M)$ respectively denote the length of $M/\mathfrak{q}M$ and the multiplicity of M relative to $\mathfrak{q}$).   This condition is equivalent to saying that any system $x_1, x_2, \cdots, x_s$ of parameters for M forms a d-sequence on M (resp. there is an integer $N \geq 1$ such that any system $x_1, x_2, \cdots, x_s$ of parameters for M contained in $\mathfrak{m}^N$ forms a d-sequence on M), that is

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$$(x_1, \cdots, x_{i-1})M : x_i x_j = (x_1, \cdots, x_{i-1})M : x_j$$

for any $1 \leq i \leq j \leq s$ ([12] and [16]). Recall that M is a generalized Cohen-Macaulay A-module if and only if

$$\ell_A(H_{\mathfrak{m}}^i(M)) < \infty \quad (i \neq s),$$

where $H_{\mathfrak{m}}^i(M)$ denotes the ith local cohomology module of M relative to $\mathfrak{m}$ ([21]). Every Buchsbaum module is, by definition, generalized Cohen-Macaulay and an A-module M is Cohen-Macaulay if and only if M is Buchsbaum and $I_A(M) = 0$. We say that A is a Buchsbaum (resp. generalized Cohen-Macaulay) ring, if A is a Buchsbaum (resp. generalized Cohen-Macaulay) module over itself. A Buchsbaum module M is called *maximal*, if $\dim_A M = \dim A$.

The research on Buchsbaum rings was started from an answer of W. Vogel [28] to a problem of D.A. Buchsbaum [3] and the readers may consult [8], [9], [23], and [24] for general references on Buchsbaum rings and modules (see [17] for a connection with singularities). The development of the theory of Buchsbaum rings has inspired a further generalization of the Cohen-Macaulay property, that is to say, *generalized Cohen-Macaulay module* which is explored (somewhat independently) by several authors [2], [12], [14], [20], [21], [26], and [27]. Among these papers we would like to recommend the article [14] which unifies the other researches, adopting the abstract terminology of unconditioned strong d-sequence (see (2.1) for definition).

Now assume that A contains (as a subring) a regular local ring R with maximal ideal $\mathfrak{n}$ such that (1) A is a module-finite extension of R, (2) $R/\mathfrak{n} \cong A/\mathfrak{m}$ and (3) $\dim_A M = \dim R = s \geq 1$. Let E_i $(0 \leq i \leq s)$ denote the ith syzygy module of the residue field $R/\mathfrak{n}$ of R. For a given R-module E and an integer $h \geq 0$, let E^h denote the direct sum of h copies of E. With the above notation, the main result of this paper is stated as follows.

Theorem (1.1). *Let* $\underline{x} = x_1, x_2, \cdots, x_s$ *be a regular system of parameters for* R *and put* $\mathfrak{q} = (x_1, x_2, \cdots, x_s)A$. *Then the following conditions are equivalent.*

(1) *M is a generalized Cohen-Macaulay A-module and* $I_A(M) = \ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$.

(2) *M is a Buchsbaum R-module.*

(3) *The idealization $R \propto M$ is a Buchsbaum ring.*

(4) *$\underline{x}$ is an unconditioned strong d-sequence on M.*

(5) *$M \cong \bigoplus_{i=0}^{s} E_i^{h_i}$ as R-modules for some integers $h_i \geq 0$.*

When this is the case, the integers h_i's in condition (5) are given by $h_i = \ell_A(H_{\mathfrak{m}}^i(M))$ $(0 \leq i < s)$ and

$$h_s = e_q(M) - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \cdot h_i$$

and hence uniquely determined by M.

For the special case $A = R$ (that is A is regular), Theorem (1.1) provides a decomposition theorem of maximal Buchsbaum A-modules; as is well-known, every maximal Cohen-Macaulay module over a regular local ring A is free and our theorem (1.1) claims that any maximal Buchsbaum A-module is a direct sum of syzygy modules of the residue field $A/\mathfrak{m}$.

Provided M is a generalized Cohen-Macaulay A-module, a system $x_1, x_2, \cdots, x_s$ of parameters for M is said to be standard if the difference $\ell_A(M/\mathfrak{q}M) - e_q(M)$ (here $\mathfrak{q} = (x_1, x_2, \cdots, x_s)A$) attains the supremum

$$I_A(M) = \sup_q [\ell_A(M) - e_q(M)].$$

If one chooses an integer $N \geq 1$ so that every system of parameters for M contained in $\mathfrak{m}^N$ forms a d-sequence on M, then any system of parameters for M contained in $\mathfrak{m}^N$ is standard (c.f. (4.3)). Standard systems of parameters are the main target in the research on generalized Cohen-Macaulay modules: by [20] and [26], various results of the articles [8], [9], [10], [22], [23], and [24] on systems of parameters of Buchsbaum rings are analogously extended to the assertions on standard systems of parameters for generalized Cohen-Macaulay modules. However, in equi-characteristic case, passing to the $\mathfrak{m}$-adic completion, the problems are reduced to those in the situation to Theorem (1.1) (see (4.5)) and we can immediately get, from the case of Buchsbaum rings, the assertions of [20] and [26] on standard systems of parameters for generalized Cohen-Macaulay modules. Furhtermore once Theorem (1.1) is demonstrated, in order to analyse generalized Cohen-Macaulay modules in equi-characteristic case we no longer need even the theory of general Buchsbaum modules but a precise exploration of the very special Buchsbaum modules E_i's and the action of x_i's on them over the regular local ring R. Such a reduction might be practical also in the study of Buchsbaum rings, to compute various invariants relative to parameter ideals (cf. (4.6)).

Let us explain how to organize this paper. In what follows we assume, to predicate the validity of Theorem (1.1) in the research of generalized Cohen-Macaulay modules, the readers to have the knowledge on Buchsbaum modules (resp. generalized Cohen-Macaulay modules) within [24] (resp. [21]). To supply a self-contained proof of Theorem (1.1), we need a few preliminaries which we summarize in section 2, mainly in terms of unconditioned strong d-sequences. Theorem (1.1) will be proved in section 4. Section 3 is devoted to a characterization of

maximal Buchsbaum modules over a regular local ring; a series of equi-valent conditions to the Buchsbaum property shall be given, which clarifies the interest of (1.1) as a decomposition theorem of maximal Buchsbaum modules over regular local rings.

Througrout this paper let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let M always stand for a finitely generated A-module. For each $i \in Z$, let $H_{\mathfrak{m}}^i(\cdot)$ denote the ith local cohomology functor relative to $\mathfrak{m}$.

§ 2. Preliminaries

Throughout this section let $\underline{a} = a_1, a_2, \cdots, a_s \ (s \geq 1)$ denote a sequence of elements in A. We put $\mathfrak{q} = (a_1, a_2, \cdots, a_s)A$.

First of all we recall the following

Definition (2.1) ([16]). *The sequence $\underline{a}$ is said to be a d-sequence on M, if $(a_1, \cdots, a_{i-1})M : a_i a_j = (a_1, \cdots, a_{i-1})M : a_j$ for any $1 \leq i \leq j \leq s$.*

When $a_1^{n_1}, a_2^{n_2}, \cdots, a_s^{n_s}$ form a d-sequence on M for any integers $n_1, n_2, \cdots, n_s \geq 1$, $\underline{a}$ is called a strong d-sequence on M. A strong d-sequence on M is termed unconditioned, when it forms in any order a strong d-sequence on M.

Let us summarize below some basic facts on d-sequences.

Lemma (2.2). *Assume that $\underline{a}$ is a d-sequence on M. Then*
(1) *For any $1 \leq t \leq s$, $a_1, a_2, \cdots, a_t$ is a d-sequence on M.*
(2) *$a_2, a_3, \cdots, a_s$ is a d-sequence on $M/a_1 M$, provided $s \geq 2$.*
(3) *$a_1, a_2, \cdots, a_s$ is a d-sequence on $M/[(0) : a_1]_M$.*
(4) *$(a_1, \cdots, a_{i-1})M : a_i = (a_1, \cdots, a_{i-1})M : a_i^n = (a_1, \cdots, a_{i-1})M : \mathfrak{q}$
for any $1 \leq i \leq s$ and $n \geq 1$.*
(5) *$[(a_1, \cdots, a_{i-1})M : a_i] \cap \mathfrak{q}^n M = [(a_1, \cdots, a_{i-1})\mathfrak{q}^{n-1})]M$ for any $1 \leq i \leq s$ and $n \geq 1$.*

Proof. See, e.g., Proof of [13, (4.2)] for assertion (5). The proof of the other assertions is standard.

Remark (2.3). Let N be an A-submodule of M. Then we say that $\underline{a}$ is a relative M-regular sequence with respect to N, if

$$[(a_1, \cdots, a_{i-1})N : a_i]_N \subset (a_1, \cdots, a_{i-1})M$$

for any $1 \leq i \leq s$ ([5]). Assertion (5) of (2.2) claims that once $\underline{a}$ is a d-sequence on M, it forms a relative M-regular sequence with respect to $\mathfrak{q}M$.

For each subset I of $\{1, 2, \cdots, s\}$ we put $\mathfrak{q}_I = (a_i \mid i \in I)A$. We begin with the following

Lemma (2.4). *Let $b \in A$ and assume that $\underline{a}$ forms an unconditioned strong d-sequence on M/bM. Then*

$$(a_1^{n_1}, \cdots, a_s^{n_s})M : b = \sum_{I \subset \{1,2,\cdots,s\}} \prod_{i \in I} a_i^{n_i-1} \cdot [\mathfrak{q}_I M : b]$$

for any integers $n_1, n_2, \cdots, n_s \geq 1$.

Proof. It suffices to show that

$$(a_1^{n_1}, \cdots, a_s^{n_s})M : b \subset \sum_{I \subset \{1,2,\cdots,s\}} \prod_{i \in I} a_i^{n_i-1} \cdot [\mathfrak{q}_I M : b]$$

which we prove by induction on s. Suppose $s = 1$ and let $x \in a_1^{n_1}M : b$. We write $bx = a_1^{n_1}y$ with $y \in M$. Then as $bM : a_1^{n_1} = bM : a_1$, we have $a_1 y = bz$ for some $z \in M$; hence $b(x - a_1^{n_1-1}z) = 0$ and so

$$x \in [(0) : b]_M + a_1^{n_1-1} \cdot [a_1 M : b].$$

Now assume that $s \geq 2$ and that our inclusion holds for $s - 1$. Then since $a_2, a_3, \cdots, a_s$ is an unconditioned strong d-sequence on $M/(a_1^{n_1}M + bM)$ (c.f. (2.2) (2)), we find

$$(a_1^{n_1}, \cdots, a_s^{n_s})M : b \subset \sum_{I \subset \{2,\cdots,s\}} \prod_{i \in I} a_i^{n_i-1} \cdot [(a_1^{n_1}M + \mathfrak{q}_I)M : b].$$

Because

$$(a_1^{n_1}M + \mathfrak{q}_I)M : b \subset [\mathfrak{q}_I M : b] + a_1^{n_1-1} \cdot [(a_1 M + \mathfrak{q}_I M) : b]$$

by the case where $s = 1$ (pass to $M/\mathfrak{q}_I M$), we get

$$(a_1^{n_1}, \cdots, a_s^{n_s})M : b \subset \sum_{I \subset \{2,\cdots,s\}} \prod_{i \in I} a_i^{n_i-1} \cdot [\mathfrak{q}_I M : b]$$

$$+ \sum_{I \subset \{2,\cdots,s\}} \prod_{i \in I \cup \{1\}} a_i^{n_i-1} \cdot [\mathfrak{q}_{I \cup \{1\}} M : b]$$

$$= \sum_{I \subset \{1,2,\cdots,s\}} \prod_{i \in I} a_i^{n_i-1} \cdot [\mathfrak{q}_I M : b]$$

as required.

Proposition (2.5). *Assume that $\underline{a}$ is an unconditioned strong d-sequence on M. Then*

$$(a_1^{n_1}, \cdots, a_{s-1}^{n_{s-1}})M : a_s^{n_s} = (a_1^{n_1}, \cdots, a_{s-1}^{n_{s-1}})M : \mathfrak{q}$$

for any integers $n_1, n_2, \cdots, n_s \geq 1$.

Proof. Since $a_1, a_2, \cdots, a_{s-1}$ forms an unconditioned strong d-sequence on $M/a_s^{n_s}M$, we get by (2.4) that

$$(a_1^{n_1}, \cdots, a_{s-1}^{n_{s-1}})M : a_s^{n_s} = \sum_{I \subset \{1,2,\cdots,s-1\}} \sum_{i \in I} a_i^{n_i-1} \cdot [\mathfrak{q}_I M : a_s^{n_s}]$$

which directly implies

$$(a_1^{n_1}, \cdots, a_{s-1}^{n_{s-1}})M : a_s^{n_s} \subset (a_1^{n_1}, \cdots, a_{s-1}^{n_{s-1}})M : \mathfrak{q}$$

as $\mathfrak{q}_I M : a_s^{n_s} = \mathfrak{q}_I M : \mathfrak{q}$ (c.f. (2.2) (4)).   The opposite inclusion is obvious.
    For each $p \in \mathbf{Z}$, let

$$H_p(a_1, a_2, \cdots, a_s ; M)$$

denote the pth homology module of the Koszul complex $K_p(a_1, a_2, \cdots, a_s ; M)$.   Recalling that $K_{\cdot}(a_1, a_2, \cdots, a_s ; A)$ is self-dual, we define

$$H^p(a_1, a_2, \cdots, a_s ; M) = H_{s-p}(a_1, a_2, \cdots, a_s ; M).$$

Then $\{H^p(a_1^n, a_2^n, \cdots, a_s^n ; M)_{n \geq 1}$ naturally forms an inductive system of A-modules, whose limit we shall denote by $H_{\underline{a}}^p(M)$:

$$H_{\underline{a}}^p(M) = \varinjlim_n H^p(a_1^n, a_2^n, \cdots, a_s^n ; M)$$

c.f. [[6], see also [15]).
    The next result plays a dominant rôle throughout this paper.

Theorem (2.6).   *Assume that $\underline{a}$ is an unconditioned strong d-sequence on M.   Then*
    (1) $\mathfrak{q} \cdot H_{\underline{a}}^p(M/(a_1^{n_1}, a_2^{n_2}, \cdots, a_j^{n_j})M) = (0)$ *for* $0 \leq j < s, 0 \leq p < s-j$, *and* $n_1, n_2, \cdots, n_j \geq 1$.
    (2)   *The canonical homomorphisms*

$$\phi_M^p : H^p(a_1, a_2, \cdots, a_s ; M) \longrightarrow H_{\underline{a}}^p(M)$$

are surjective for all $p \neq s$.
    (3)   *In case* $\ell_A(M/\mathfrak{q}M) < \infty$, *the length* $\ell_A(H_{\underline{a}}^p(M))$ *is finite for any* $p \neq s$ *and the equality*

$$\ell_A(H_{\underline{a}}^p(M/(a_1, a_2, \cdots, a_j)M)) = \sum_{i=0}^{j} \binom{j}{i} \cdot \ell_A(H_{\underline{a}}^{i+p}(M))$$

$$\left(resp. \ \ell_A(H_p(a_1, a_2, \cdots, a_j ; M)) = \sum_{i=0}^{j-p} \binom{j}{p+i} \cdot \ell_A(H_{\underline{a}}^i(M)) \right)$$

holds for $0 \leq j < s$ *and* $0 \leq p < s-j$ *(resp.* $1 \leq j \leq s$ *and* $1 \leq p \leq j$).

Proof.   (1)   For $j = s-1$, our assertion is obvious as

$$(a_1^{n_1}, a_2^{n_2}, \cdots, a_{s-1}^{n_{s-1}})M : a_s^2 = (a_1^{n_1}, a_2^{n_2}, \cdots, a_{s-1}^{n_{s-1}})M : \mathfrak{q}$$

by (2.5).   Assume that $0 \leq j < s-1$ and that assertion (1) is true for $j+1$.
We put $b = a_{j+1}^2$ and consider the exact sequence

$$0 \longrightarrow K \longrightarrow E \overset{b}{\longrightarrow} E \longrightarrow F \longrightarrow 0$$

where $E = M/(a_1^{n_1}, \cdots, a_j^{n_j})M$ and $F = M/(a_1^{n_1}, \cdots, a_j^{n_j}, b)M$.   Then as
$\mathfrak{q}K = (0)$ by (2.5), we have

$$H_{\mathfrak{q}}^0(K) = K \quad \text{and} \quad H_{\mathfrak{q}}^p(K) = (0) \quad \text{for all } p > 0.$$

Therefore similarly as in Proof of [12, Lemma 1.2], we get a long exact
sequence

$$0 \longrightarrow K \longrightarrow H_{\mathfrak{q}}^0(E) \overset{b}{\longrightarrow} H_{\mathfrak{q}}^0(E) \longrightarrow H_{\mathfrak{q}}^0(F) \longrightarrow$$

(♯)
$$H_{\mathfrak{q}}^1(E) \overset{b}{\longrightarrow} H_{\mathfrak{q}}^1(E) \longrightarrow H_{\mathfrak{q}}^1(F) \longrightarrow \cdots$$

$$H_{\mathfrak{q}}^p(E) \overset{b}{\longrightarrow} H_{\mathfrak{q}}^p(E) \longrightarrow H_{\mathfrak{q}}^p(F) \longrightarrow \cdots$$

of A-modules.   Let $0 \leq p < s-j$ and $x \in H_{\mathfrak{q}}^p(E)$.   Then $\mathfrak{q}^t x = (0)$ for some
$t \geq 1$ —— choose such t as small as possible and we claim that $t = 1$.   In
fact if $t \geq 2$, we have $b(\mathfrak{q}^{t-2}x) = (0)$ as $b \in \mathfrak{q}^2$.   Hence

$$\mathfrak{q}^{t-2}x \subset [(0) : b]_{H_{\mathfrak{q}}^p(E)}$$

and consequently

$$\mathfrak{q} \cdot (\mathfrak{q}^{t-2}x) = (0),$$

because $[(0) : b]_{H_{\mathfrak{q}}^p(E)}$ is a homomorphic image of $H_{\mathfrak{q}}^{p-1}(F)$ in the exact
sequence (♯) and because $\mathfrak{q}H_{\mathfrak{q}}^{p-1}(F) = (0)$ by the hypothesis of induction
on j.   Of course this contradicts the choice of t and we conclude that
$\mathfrak{q} \cdot H_{\mathfrak{q}}^p(E) = (0)$ as required.

(2)   We may assume that $s \geq 2$ and that assertion (2) is true for $s-1$.   Let $U = [(0) : \mathfrak{q}]_M$.   Then as

$$U \cap \mathfrak{q}M = (0)$$

(c.f. (2.2) (5)), we find the canonical homomorphisms

$$H^p(a_1, a_2, \cdots, a_s; U) \longrightarrow H^p(a_1, a_2, \cdots, a_s; M)$$

are injective for all $p \in Z$.   Therefore as

$$H_{\mathfrak{a}}^p(U) = (0) \quad \text{for } p \neq 0 \text{ and } U = H_{\mathfrak{a}}^0(M),$$

passing to M/U we may assume that a_1 is a nonzerodivisor for M.   Then as $a_1 \cdot H_{\mathfrak{a}}^p(M) = (0)$ for $p \neq s$ by (1), by virtue of the exact sequence

$$0 \longrightarrow M \overset{a_1}{\longrightarrow} M \longrightarrow M/a_1 M \longrightarrow 0$$

we get for each $p < s$ a commutative diagram

$$
\begin{array}{ccc}
H^{p-1}(a_1, a_2, \cdots, a_s; M/a_1 M) & \longrightarrow & H^p(a_1, a_2, \cdots, a_s; M) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi_M^p} \\
H_{\mathfrak{a}}^{p-1}(M/a_1 M) & \longrightarrow H_{\mathfrak{a}}^p(M) \longrightarrow 0
\end{array}
$$

with exact bottom row.   So it suffices to show that ψ is onto.

Let $\underline{b} = a_2, a_3, \cdots, a_s$ and note that the canonical epimorphisms

$$H^{p-1}(a_1^n, a_2^n, \cdots, a_s^n; M/a_1 M) \longrightarrow H^{p-1}(a_2^n, \cdots, a_s^n; M/a_1 M) \quad (n \geq 1)$$

induces in isomorphism

$$H_{\mathfrak{a}}^{p-1}(M/a_1 M) \overset{\sim}{\longrightarrow} H_{\underline{b}}^{p-1}(M/a_1 M).$$

Then we immediately find, by the hypothesis of induction on s and by the following commutative diagram

$$
\begin{array}{ccc}
H^{p-1}(a_1, a_2, \cdots, a_s; M/a_1 M) & \longrightarrow H^{p-1}(a_2, \cdots, a_s; M/a_1 M) \longrightarrow 0 \\
\downarrow{\scriptstyle \psi} & \downarrow{\scriptstyle \phi_{M/a_1 M}^{p-1}} \\
H_{\mathfrak{a}}^{p-1}(M/a_1 M) & \overset{\sim}{\longrightarrow} H_{\underline{b}}^{p-1}(M/a_1 M)
\end{array}
$$

with exact top row, that ψ is onto as required.

(3)   First of all we check the following

Claim. $\ell_A(H_p(a_1, a_2, \cdots, a_j; M)) < \infty$ for $p \geq 1$ and $1 \leq j \leq s$.

Proof of Claim.   Since $\underline{a}$ is a relative M-regular sequence with respect to $\mathfrak{q}M$ (c.f. (2.3)), we find by [5] that

$$a_j \cdot H_p(a_1, a_2, \cdots, a_{j-1}; M) = (0)$$

for any $p \geq 1$ and $2 \leq j \leq s$.   Consequently we get short exact sequences

$$0 \longrightarrow H_p(a_1, \cdots, a_{j-1}; M)$$
$$\longrightarrow H_p(a_1, \cdots, a_j; M) \longrightarrow H_{p-1}(a_1, \cdots, a_{j-1}; M) \longrightarrow 0 \quad (p \geq 2)$$

and

$$0 \longrightarrow H_1(a_1, \cdots, a_{j-1}; M) \longrightarrow H_1(a_1, \cdots, a_j; M)$$
$$\longrightarrow (a_1, \cdots, a_{j-1})M : a_j/(a_1, \cdots, a_{j-1})M \longrightarrow 0.$$

As

$$[(a_1, \cdots, a_{j-1})M : a_j] \cap \mathfrak{q}M = (a_1, \cdots, a_{j-1})M$$

by (2.2) (5), $(a_1, \cdots, a_{j-1})M : a_j/(a_1, \cdots, a_{j-1})M$ is naturally contained in $M/\mathfrak{q}M$ and hence the (ascending) induction on j proves that

$$\ell_A(H_p(a_1, a_2, \cdots, a_j; M)) < \infty$$

for all $p \geq 1$ and $1 \leq j \leq s$.

We put $h^p(M) = \ell_A(H_\mathfrak{q}^p(M))$. Notice that $h^p(M)$ is finite for $0 \leq p < s$, which follows by the above claim from assertion (2). Let us prove the first equality in assertion (3). Let $U = [(0) : a_1]_M$. Then as

$$H_\mathfrak{q}^0(U) = U \quad \text{and} \quad H_\mathfrak{q}^p(U) = (0) \quad \text{for } p \neq s,$$

we get a long exact sequence

$$0 \longrightarrow U \longrightarrow H_\mathfrak{q}^0(M) \xrightarrow{a_1} H_\mathfrak{q}^0(M) \longrightarrow H_\mathfrak{q}^0(M/a_1 M) \longrightarrow \cdots$$
$$H_\mathfrak{q}^p(M) \xrightarrow{a_1} H_\mathfrak{q}^p(M) \longrightarrow H_\mathfrak{q}^p(M/a_1 M) \longrightarrow \cdots$$

of A-modules (c.f. Proof of [12, Lemma 1.2]) which canonically splits into short exact sequences

$$0 \longrightarrow H_\mathfrak{q}^p(M) \longrightarrow H_\mathfrak{q}^p(M/a_1 M) \longrightarrow H_\mathfrak{q}^{p+1}(M) \longrightarrow 0$$

for $0 \leq p < s-1$, because in our case $a_1 \cdot H_\mathfrak{q}^p(M) = (0)$ for any $p \neq s$ (c.f. (1)). Repeating this argument and appealing to the isomorphism

$$H_\mathfrak{q}^p(M/a_1 M) \cong H_\mathfrak{b}^p(M/a_1 M)$$

where $\underline{b} = a_2, a_3, \cdots, a_s$, we get short exact sequences

$$0 \longrightarrow H_\mathfrak{q}^p(M/(a_1, \cdots, a_j)M)) \longrightarrow H_\mathfrak{q}^p(M/(a_1, \cdots, a_{j+1})M)$$
$$\longrightarrow H_\mathfrak{q}^{p+1}(M/(a_1, \cdots, a_j)M)) \longrightarrow 0$$

for $0 \leq j < s$ and $0 \leq p < s-j-1$. Hence the induction on j yields the required equality

$$\ell_A(H_\mathfrak{q}^p(M/(a_1, a_2, \cdots, a_j)M)) = \sum_{i=0}^{j} \binom{j}{i} \cdot h^{p+i}(M)$$

for $0 \leq j < s$ and $0 \leq p < s - j$. Let us prove the second equality by induction on j. For $j = 1$, this is obvious since

$$H_1(a_1; M) = [(0): a_1]_M = H_{\mathfrak{a}}^0(M).$$

Assume that $s \geq j \geq 2$ and that our assertion is true for $j - 1$. Then by the proof of the above claim we get

$$\ell_A(H_p(a_1, \cdots, a_j; M)) = \ell_A(H_{p-1}(a_1, \cdots, a_{j-1}; M)$$
$$+ \ell_A(H_p(a_1, \cdots, a_{j-1}; M))$$

for $p \geq 2$ and

$$\ell_A(H_1(a_1, \cdots, a_j; M)) = \ell_A(H_1(a_1, \cdots, a_{j-1}; M))$$
$$+ \ell_A((a_1, \cdots, a_{j-1})M: a_j)/(a_1, \cdots, a_{j-1})M).$$

Therefore we have, by the hypothesis of induction on j together with the first equality in assertion (3), the length of

$$H_p(a_1, \cdots, a_{j-1}; M) \quad \text{and}$$
$$(a_1, \cdots, a_{j-1})M: a_j/(a_1, \cdots, a_{j-1})M = H_{\mathfrak{a}}^0(M/(a_1, \cdots, a_{j-1})M)$$

in terms of $h^i(M)$ ($0 \leq i < s$) and so the required equality follows straightforward by standard arguments on binomial coefficients, which finishes the proof of (2.6).

Remark (2.7). Theorem (2.6) is true without the Noetherian assumption on A and M. This point of view is one of the main theme of the joint work [14], in which further properties of unconditioned strong d-sequences are discussed with some consequences.

Corollary (2.8) ([16] and [19])[*]. *Let* $\dim_A M = s$. *Then the following conditions are equivalent.*

(1) *M is a Buchsbaum A-module.*

(2) *Any system of parameters for M forms a d-sequence on M.*

(3) *Any system of parameters for M forms an unconditioned strong d-sequence on M.*

When this is the case, $\mathfrak{m} \cdot H_{\mathfrak{m}}^i(M) = (0)$ *for* $i \neq s$ *and*

$$I_A(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H_{\mathfrak{m}}^i(M)).$$

Proof. (1)$\Rightarrow$(2) Let $a_1, a_2, \cdots, a_s$ be a system of parameters for M. Then by [23, Satz 10] we know that $\underline{a}$ is a weak M-sequence, that is

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[*] The equivalence [(1) $\Leftrightarrow$ (2)] (resp. the last assertion) was given by [16] (resp. [19]).

$$(a_1, \cdots, a_{i-1})M: a_i = (a_1, \cdots, a_{i-1})M: \mathfrak{m}$$

for any $1 \leq i \leq s$. Consequently for $1 \leq i \leq j \leq s$ we get

$$(a_1, \cdots, a_{i-1})M: a_i a_j = (a_1, \cdots, a_{i-1})M: \mathfrak{m}$$
$$= (a_1, \cdots, a_{i-1})M: a_j,$$

because both the systems $a_1, \cdots, a_{i-1}, a_i a_j$ and $a_1, \cdots, a_{i-1}, a_j$ form subsystems of parameters for M. Thus $\underline{a}$ is a d-sequence on M.

(2)$\Rightarrow$(3) This is obvious.

(3)$\Rightarrow$(1) and the last assertions. Let $\underline{a} = a_1, a_2, \cdots, a_s$ be a system of parameters for M. Then as $\underline{a}$ is an unconditioned strong d-sequence on M, we get by (2.6) (3) that

$$\ell_A((a_1, \cdots, a_{s-1})M: a_s/(a_1, \cdots, a_{s-1})M) = \ell_A(H^0_{\underline{a}}(M/(a_1, \cdots, a_{s-1})M)$$
$$= \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H^i_{\mathfrak{m}}(M)).$$

Since

$$\mathfrak{q} \cdot [(a_1, \cdots, a_{i-1})M: a_i] \subset (a_1, \cdots, a_{i-1})M$$

for $1 \leq i \leq s$ (c.f. (2.2) (4)), our system $\underline{a}$ of parameters for M is reducing[*] and so

$$\ell_A(M/\mathfrak{q}M) - e_\mathfrak{q}(M) = \ell_A((a_1, \cdots, a_{s-1})M: a_s/(a_1, \cdots, a_{s-1})M)$$

(c.f. [1]). Accordingly as

$$H^i_{\underline{a}}(M) = H^i_{\mathfrak{m}}(M)$$

for any $i \in \mathbf{Z}$, we have

$$\ell_A(M/\mathfrak{q}M) - e_\mathfrak{q}(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H^i_{\mathfrak{m}}(M))$$

which guarantees that M is a Buchsbaum A-module with

$$I_A(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H^i_{\mathfrak{m}}(M)).$$

Because $\mathfrak{q} \cdot H^i_{\mathfrak{m}}(M) = (0)$ $(i \neq s)$ by (2.6) (1), we also get $\mathfrak{m} \cdot H^i_{\mathfrak{m}}(M) = (0)$ $(i \neq s)$ as required.

Corollary (2.9). *Let $0 \to N \to F \to M \to 0$ be a short exact sequence of finitely generated A-modules and assume that $\dim_A N = \dim_A F = \dim_A M =$*

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[*] See [1, p. 643].

s.   Then N is Buchsbaum, if F is Cohen-Macaulay and M is Buchsbaum.

Proof.   Let $\underline{a} = a_1 a_2, \cdots, a_s$ be a system of parameters for N.   Then we can choose by [18, Theorem 124] a system $\underline{b} = b_1, b_2, \cdots, b_s$ of parameters for F so that $(a_i - b_i)N = (0)$ for all i; hence replacing $\underline{a}$ by $\underline{b}$, we may assume without loss of generality that $\underline{a}$ is a system of parameters for F too.   Then since $\underline{a}$ is an F-regular sequence, we get an exact sequence

$$0 \longrightarrow H_1(a_1, a_2, \cdots, a_s; M) \longrightarrow N/\mathfrak{q}N \longrightarrow F/\mathfrak{q}F \longrightarrow M/\mathfrak{q}M \longrightarrow 0$$

of A-modules, which implies by (2.6) (3) that

$$\ell_A(N/\mathfrak{q}N) - e_\mathfrak{q}(N) = \ell_A(H_1(a_1, a_2, \cdots, a_s; M)) - I_A(M)$$

$$= \sum_{i=0}^{s-1} \binom{s}{i+1} \cdot \ell_A(H_\mathfrak{m}^i(M)) - I_A(M).$$

Thus the difference $\ell_A(N/\mathfrak{q}N) - e_\mathfrak{q}(N)$ does not depend on $\mathfrak{q}$ and so N is Buchsbaum.

Let $B = A \propto M$ denote the idealization of M over A and let $\psi : B \to A$ be the A-algebra map defined by $\psi(a, x) = a$ for $(a, x) \in B$.   (Thus the additive group of B coincides with the direct sum $A \oplus M$ and the multiplication in B is defined by $(a, x) \cdot (b, y) = (ab, ay + bx)$.)   Then B is a Noetherian local ring with maximal ideal $\mathfrak{n} = \mathfrak{m} \times M$ and $\dim B = \dim A$.   Furthermore we have the following

Proposition (2.10).   *Suppose that A is a Cohen-Macaulay ring and $\dim_A M = \dim A$.   Then the following conditions are equivalent.*
   (1)   *B is a Buchsbaum ring.*
   (2)   *M is a Buchsbaum A-module.*
When this is the case, $I(B) = I_A(M)$.

Proof.   Let $\mathfrak{Q} = (f_1, f_2, \cdots, f_d)B$ $(d = \dim A)$ be a parameter ideal of B and write $f_i = (a_1, x_i)$ with $a_i \in A$, $x_i \in M$.   We put $\mathfrak{q} = (a_1, a_2, \cdots, a_d)A$.   Then $\dim A/\mathfrak{q} = 0$, as $\mathfrak{q} = \mathfrak{Q}A$ and so $f_1, f_2, \cdots, f_d$ act on A, via ψ, as a regular sequence.   Hence putting $N = \text{Ker } \psi$, we have $\mathfrak{Q} \cap N = \mathfrak{Q}N$ which yields by the exact sequence $0 \to N \to B \overset{\psi}{\to} A \to 0$ that

$$\ell_B(B/\mathfrak{Q}) = \ell_A(A/\mathfrak{q}) + \ell_B(N/\mathfrak{Q}N)$$

$$= \ell_A(A/\mathfrak{q}) + \ell_A(M/\mathfrak{q}M).$$

Accordingly as $e_\mathfrak{q}(B) = e_\mathfrak{Q}(A) + e_\mathfrak{q}(M)$, we get

(♯) $$\ell_B(B/\mathfrak{Q}) - e_\mathfrak{Q}(B) = \ell_A(M/\mathfrak{q}M) - e_\mathfrak{q}(M).$$

Hence the implication [(2)$\Rightarrow$(1)].

Conversely suppose that B is a Buchsbaum ring and let $\mathfrak{q} = (a_1, a_2, \cdots, a_d)A$ be a parameter ideal for M. Then by [18, Theorem 124] we can choose a system $\underline{b} = b_1, b_2, \cdots, b_d$ of parameters for A so that

$$(b_i - a_i \mid 1 \leq i \leq d) \cdot M = (0).$$

Hence replacing $\underline{a}$ by $\underline{b}$, we may assume that $\underline{a}$ is a system of parameters for A too. Then letting $f_i = (a_i, 0)$ $(1 \leq i \leq d)$, we get by the equality ($\sharp$) that

$$I(B) = \ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$$

(here $\mathfrak{q} = (a_1, a_2, \cdots, a_d)A$) whence M is Buchsbaum and $I(B) = I_A(M)$.

In [9] the author gave some basic results on systems of parameters for Buchsbaum rings. The principle of idealization (2.10) is useful to extend them to the assertions on systems of parameters for Buchsbaum modules.

We close this section with a simple remark (2.11) on boundary modules of the Koszul complex $K_{\cdot}(a_1, a_2, \cdots, a_s; A)$; it is fairly obvious but we shall give a proof for confirmation, because (2.11) plays a key rôle in the proof of (1.1) (see Proof of (3.5) (1)).

Lemma (2.11). *Let B_p denote the pth boundary module of $K_{\cdot}(a_1, a_2, \cdots, a_s; A)$. Then*

$$\left(\prod_{i=1}^{s} a_i \right) \cdot B_p \subset (a_i^2 \mid 1 \leq i \leq s) \cdot B_p,$$

if $0 \leq p \leq s - 2$.

Proof. Let $X_1, X_2, \cdots, X_s$ be a basis of $K_1(a_1, a_2, \cdots, a_s; A)$. We put for each subset I of $\{1, 2, \cdots, s\}$ with $\sharp I = p + 1$

$$X_I = X_{i_1} \wedge X_{i_2} \wedge \cdots \wedge X_{i_{p+1}}$$

where $I = \{i_1, i_2, \cdots, i_{p+1}\}$ with $i_1 < i_2 < \cdots < i_{p+1}$. Let ∂ denote the differentiation of the complex $K_{\cdot}(a_1, a_2, \cdots, a_s; A)$. Then for any $1 \leq j \leq s$ we have

$$a_j \partial(X_I) = \sum_{\alpha=1}^{p+1} (-1)^{\alpha+1} a_{i_\alpha} \partial(X_j \wedge X_{I \setminus \{i_\alpha\}}),$$

because

$$\partial(X_j \wedge X_I) = a_j X_I - \sum_{\alpha=1}^{p+1} (-1)^{\alpha+1} a_{i_\alpha} X_j \wedge X_{I \setminus \{i_\alpha\}}.$$

So we may write with $c_\alpha \in (a_i^2 \mid i \in I)A$

$$(ba_j)\partial(X_I) = \sum_{\alpha=1}^{p+1} c_\alpha \partial(f_\alpha),$$

where $b = \prod_{i \in I} a_i$ and $f_\alpha = X_j \wedge X_{I \setminus \{i_\alpha\}}$. Therefore choosing j so that $j \notin I$, we get

$$\left(\prod_{i=1}^{s} a_i\right) \cdot \partial(X_I) \in (a_i^2 \mid i \in I)B_p$$

whence

$$\left(\prod_{i=1}^{s} a_i\right) \cdot B_p \subset (a_i^2 \mid 1 \leq i \leq s) \cdot B_p$$

as required.

§ 3.   Maximal Buchsbaum modules over regular local rings

In this section we assume that A is a regular local ring of dim $A = d$ ≥ 1.   Let

$$F: \quad 0 \longrightarrow F_d \xrightarrow{\partial_d} F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 = A \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

be a minimal free resolution of $A/\mathfrak{m}$ and put for each $0 \leq i \leq d$

$$E_i = \operatorname{Im}(F_i \xrightarrow{\partial_i} F_{i-1}) \qquad (i \neq 0),$$
$$= A/\mathfrak{m} \qquad (i = 0).$$

Our purpose is to prove the following

Theorem (3.1).   *Let M be a finitely generated A-module of* dim$_A$ $M = d$ *and let $\underline{x} = x_1, x_2, \cdots, x_d$ denote a regular system of parameters for A. Then the following conditions are equivalent.*

   (1)   *M is a Buchsbaum A-module.*
   (2)   *The idealization $A \propto M$ is a Buchsbaum ring.*
   (3)   *$\underline{x}$ is an unconditioned strong d-sequence on M.*
   (4)   *The canonical homomorphisms*

$$\phi_M^i : \operatorname{Ext}_A^i(A/\mathfrak{m}, M) \longrightarrow H_{\mathfrak{m}}^i(M) = \varinjlim_t \operatorname{Ext}_A^i(A/\mathfrak{m}^t, M)$$

are surjective for all $i \neq d$.
   (5)   *$M \cong \bigoplus_{i=0}^{d} E_i^{h_i}$ for some integers $h_i \geq 0$.*
When this is the case, the integers h_i's in assertion (5) are given by $h_i = \ell_A(H_{\mathfrak{m}}^i(M))$ $(0 \leq i < d)$ and

$$h_d = \operatorname{rank}_A M - \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot h_i$$

and hence uniquely determined by M.

For this purpose we prepare a few preliminary steps.

Let $E^* = \mathrm{Hom}_A(E, A)$ for each A-module E. We begin with the following

Lemma (3.2). $E_i^* \cong E_{d+1-i}$ *for* $1 \leq i < d$ *and* $E_d \cong A$.

This follows from the well-known fact that the minimal free resolution F of $A/\mathfrak{m}$ is given by the Koszul complex

$$K_.(x_1, x_2, \cdots, x_d; A)$$

generated by a regular system $x_1, x_2, \cdots, x_d$ of parameters for A and is self-dual.

The next two results (3.3) and (3.4) are due to [7] (see (3.1) and (3.2)). As the proof of them is quite short, we shall give it for completeness.

Lemma (3.3). *Let* $1 \leq i \leq d$. *Then*
(1) E_i *is a maximal Buchsbaum A-module.*
(2) $H_{\mathfrak{m}}^i(E_i) = A/\mathfrak{m}$ *if* $i < d$ *and* $H_{\mathfrak{m}}^p(E_i) = (0)$ $(p \neq i, d)$.
(3) $\mathrm{rank}_A E_i = \binom{d-1}{i-1}$ *and* $\ell_A(E_i/\mathfrak{m}E_i) = \binom{d}{i}$.

Proof. The assertion $\ell_A(E_i/\mathfrak{m}E_i) = \binom{d}{i}$ comes from the fact $\mathrm{rank}_A F_i$
$= \binom{d}{i}$. That $\mathrm{rank}_A E_i = \binom{d-1}{i-1}$ and assertions (1) and (2) follow by induction on i from the exact sequence

$$0 \longrightarrow E_i \longrightarrow F_{i-1} \longrightarrow E_{i-1} \longrightarrow 0$$

(c.f. (2.9) and the proof).

Proposition (3.4). *Let* $h_0, h_1, \cdots, h_d \geq 0$ *be integers such that* $h_i \neq 0$ *for some* $1 \leq i \leq d$. *Let* $M = \bigoplus_{i=0}^{d} E_i^{h_i}$. *Then*
(1) M *is a maximal Buchsbaum A-module.*
(2) $\ell_A(H_{\mathfrak{m}}^i(M)) = h_i$ *for* $0 \leq i < d$.
(3) $\mathrm{rank}_A M = \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot h_i$.

Proof. Assertions (2) and (3) follow from assertions (2) and (3) of (3.3). Clearly $\dim_A M = d$ by the choice of h_i's. To see that M is Buchsbaum, let q be a parameter ideal for M. Then as $\dim_A M = d$, q is a parameter ideal for all E_i's $(1 \leq i \leq d)$ too; hence

$$\ell_A(M/\mathfrak{q}M) - e_\mathfrak{q}(M) = h_0 + \sum_{i=1}^{d-1} h_i \cdot [\ell_A(E_i/\mathfrak{q}E_i) - e_\mathfrak{q}(E_i)]$$

$$= h_0 + \sum_{i=1}^{d-1} h_i \cdot I_A(E_i).$$

Thus M is Buchsbaum.

Let

$$0 \longrightarrow N \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$$

be a short exact sequence of A-modules with F finitely generated and free. Assume that $\dim_A M = d$ and that the canonical homomorphisms

$$\phi_M^i : \mathrm{Ext}_A^i(A/\mathfrak{m}, M) \longrightarrow H_\mathfrak{m}^i(M)$$

are surjective for all $i \neq d$. Assume further that N is isomorphic to a direct sum of some copies of E_i's $(1 \leq i \leq d)$, say $N = \bigoplus_{i=1}^d E_i^{r_i}$ with integers $r_i \geq 0$. Consider the dual sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} F^* \xrightarrow{f^*} N^* \xrightarrow{\partial} \mathrm{Ext}_A^1(M, A) \longrightarrow 0$$

where ∂ denotes the connecting homomorphism. Then putting

$$N' = \bigoplus_{i=1}^{d-1} E_i^{r_i},$$

we have the following

Lemma (3.5). (1) $\partial(N'^*) = (0)$.

(2) $\mathrm{Ker}\, \partial \cong \bigoplus_{i=1}^d E_i^{s_i}$ and $M^* \cong \bigoplus_{i=1}^d E_i^{t_i}$ *for some non-negative integers* s_i, t_i.

Proof. (1) Let

$$p : N \longrightarrow N'$$

denote the projection. We have to show the composite of

$$N'^* \xrightarrow{p^*} N^* \xrightarrow{\partial} \mathrm{Ext}_A^1(M, A)$$

is zero. Let $\hat{A}$ (resp. I) denote the $\mathfrak{m}$-adic completion of A (resp. the injective envelope $E_A(A/\mathfrak{m})$ of $A/\mathfrak{m}$). Then the local duality theorem [6, Theorem 6.3] claims that for any finitely generated A-module E and $i \in \mathbf{Z}$, there is a natural isomorphism

$$\hat{A} \bigotimes_A \mathrm{Ext}_A^i(E, A) \cong \mathrm{Hom}_A(H_\mathfrak{m}^{d-i}(E), I)$$

of $\hat{A}$-modules. Applying it to our situation, we see that our assertion $\partial \circ p^* = 0$ is equivalent to saying that the composite of

$$H_{\mathfrak{m}}^{d-1}(M) \xrightarrow{\tau} H_{\mathfrak{m}}^d(N) \xrightarrow{p} H_{\mathfrak{m}}^d(N')$$

is zero where $\tau: H_{\mathfrak{m}}^{d-1}(M) \to H_{\mathfrak{m}}^d(N)$ denotes the connecting map of local cohomology modules.

Now we take a regular system $\underline{x} = x_1, x_2, \cdots, x_d$ of parameters for A and recall how to calculate the local cohomology modules $H_{\mathfrak{m}}^{d-1}(M)$ and $H_{\mathfrak{m}}^d(N)$. Firstly consider the commutative diagram

$$
(\sharp) \quad
\begin{array}{ccccc}
H^{d-1}(x_1, \cdots, x_d; M) & \xrightarrow{\rho} & H^d(x_1, \cdots, x_d; N) & \xrightarrow{p} & H^d(x_1, \cdots, x_d; N') \\
\downarrow{\psi_M} & & \downarrow{\phi_N} & & \downarrow{\phi_{N'}} \\
H_{\underline{x}}^{d-1}(M) & \xrightarrow{\tau} & H_{\underline{x}}^d(N) & \xrightarrow{p} & H_{\underline{x}}^d(N')
\end{array}
$$

of A-modules, where the vertical maps are canonical homomorphisms into inductive limits and ρ denotes the connecting homomorphism of Koszul cohomology modules. Then as

$$H^{d-1}(x_1, x_2, \cdots, x_d; M) = \mathrm{Ext}_A^{d-1}(A/\mathfrak{m}, M),$$

by our standard assumption on M the map ψ_M is onto. Consequently assertion (1) is equivalent to saying that the composite of

$$H^{d-1}(x_1, \cdots, x_d; M) \xrightarrow{\rho} H^d(x_1, \cdots, x_d; N) \xrightarrow{\phi_N} H_{\underline{x}}^d(N) \xrightarrow{p} H_{\underline{x}}^d(N')$$

is zero, which straightforward follows from the second square in the diagram $(\sharp)$ because the map $\phi_{N'}: N'/\mathfrak{m}N' = H^d(x_1, \cdots, x_d; N') \to H_{\underline{x}}^d(N')$ is zero by (2.11) (notice that the structure maps $f_{n,m}$ $(1 \leqq n \leqq m)$ of the inductive system

$$\{N'/(x_1^n, x_2^n, \cdots, x_d^n)N' = H^d(x_1^n, x_2^n, \cdots, x_d^n; N')\}_{n \geqq 1}$$

considered are defined by

$$
f_{n,m}(z \bmod (x_1^n, x_2^n, \cdots, x_d^n)N')
$$
$$
= \left(\prod_{i=1}^d x_i \right)^{m-n} \cdot z \bmod (x_1^m, x_2^m, \cdots, x_d^m)N' \quad \text{for } z \in N'.
$$

Thus $\partial(N'^*) = (0)$ as required.

(2) Recall that

$$\mathfrak{m} \cdot \mathrm{Ext}_A^1(M, A) = (0)$$

by local duality, since $\mathfrak{m} \cdot H_{\mathfrak{m}}^{d-1}(M) = (0)$. Let $F' = E_d^{r_d}$. Then $N* = N'* \oplus F'*$ clearly and so, as $\partial(N'*) = (0)$ by (1), in the exact sequence

$$0 \longrightarrow M* \xrightarrow{\;g*\;} F* \xrightarrow{\;f*\;} N* \xrightarrow{\;\partial\;} \mathrm{Ext}_A^1(M, A) \longrightarrow 0$$

we get $\partial(F'*) = \mathrm{Ext}_A^1(M, A)$.   Consequently we have an isomorphism

$$(\sharp\sharp) \quad \mathrm{Ker}\,\partial \cong N'* \oplus E_1^s \oplus A^t$$

for some non-negative integers s and t, because $F'*$ is free and $\mathrm{Ext}_A^1(M, A)$ is a vector space over $A/\mathfrak{m}$.   As $N'*$ is isomorphic to a direct sum of some copies of E_i's $(1 \leq i \leq d)$ (see (3.2)), we get by $(\sharp\sharp)$ the former part of our assertion (2).   The latter part is now standard.

We are now ready to prove Theorem (3.1).

Proof of Theorem (3.1).
(1)$\Leftrightarrow$(2) See (2.10).
(1)$\Rightarrow$(3) See (2.8).
(3)$\Rightarrow$(4) See (2.6) (2).
(4)$\Rightarrow$(1) See [24, Theorem 1].
(4)$\Rightarrow$(5) Let $V = H_{\mathfrak{m}}^0(M)$.   Then as $V \cap \mathfrak{m}M = (0)$ (c.f., e.g., [23, Hilfssatz 11]), V is a direct summand of M.   Therefore passing to M/V, we may assume $t = \mathrm{depth}_A M \geq 1$.   If $t = d$, then M is free and there is nothing to prove.   Assume that $t < d$ and that our assertion (5) is true for any finitely generated A-module N of $\dim_A N = d$ and $\mathrm{depth}_A N = t+1$ which satisfies also condition (4).

Firstly we choose a presentation

$$(\sharp) \quad 0 \longrightarrow N \xrightarrow{\;f\;} F \xrightarrow{\;g\;} M \longrightarrow 0$$

of M so that F is finitely generated and free.   Then $\mathrm{depth}_A N = t+1$ clearly. Furthermore N satisfies condition (4).   In fact, applying the functors $\mathrm{Ext}_A^i(A/\mathfrak{m}, \cdot)$ and $H_{\mathfrak{m}}^i(\cdot)$ to the exact sequence $(\sharp)$, as F is Cohen-Macaulay we get a commutative square

$$\begin{array}{ccc} \mathrm{Ext}_A^{i-1}(A/\mathfrak{m}, M) & \xrightarrow{\;\sim\;} & \mathrm{Ext}_A^i(A/\mathfrak{m}, N) \\ \downarrow{\scriptstyle \phi_M^{i-1}} & & \downarrow{\scriptstyle \phi_N^i} \\ H_{\mathfrak{m}}^{i-1}(M) & \xrightarrow{\;\sim\;} & H_{\mathfrak{m}}^i(N) \end{array}$$

for each $i < d$.   Because the vertical map ϕ_M^{i-1} is onto by assumption (4), so is ϕ_N^i as required.   Thus the hypothesis on t yields that N is isomorphic to a direct sum of some copies of E_i's $(1 \leq i \leq d)$ and consequently by (3.5) (2), $M*$ is.

Secondly consider the dual sequence

$$0 \longrightarrow M^* \xrightarrow{\ g^*\ } F^* \xrightarrow{\ f^*\ } N^* \xrightarrow{\ \partial\ } \mathrm{Ext}^1_A(M, A) \longrightarrow 0$$

of ($\sharp$). We put $W = \mathrm{Ker}\, \partial$. Then by (3.5) (2), W is isomorphic to a direct sum of some copies of E_i's, whence W is a maximal Buchsbaum A-module (see (3.4)). Therefore the implication [(1) $\Rightarrow$ (4)] guarantees that W satisfies condition (4). Furthermore taking the A-dual of

$$(\sharp\sharp) \quad 0 \longrightarrow M^* \xrightarrow{\ g^*\ } F^* \xrightarrow{\ h\ } W \longrightarrow 0$$

again, we get a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & W^* & \xrightarrow{\ h^*\ } & F^{**} & \xrightarrow{\ g^{**}\ } & M^{**} & \xrightarrow{\ \tau\ } & \mathrm{Ext}^1_A(W, A) & \longrightarrow & 0 \\
& & & & \| & & \uparrow h_M & & & & \\
0 & \longrightarrow & N & \xrightarrow{\ f\ } & F & \xrightarrow{\ g\ } & M & \longrightarrow & 0 & &
\end{array}
$$

with exact rows, where $h_M : M \to M^{**}$ is the canonical map. As M is torsionfree (c.f., e.g., (4.1)), chasing the above diagram we have the sequence

$$0 \longrightarrow M \xrightarrow{\ h_M\ } M^{**} \xrightarrow{\ \tau\ } \mathrm{Ext}^1_A(W, A) \longrightarrow 0$$

to be exact too, hence $M = \mathrm{Ker}\, \tau$.

Lastly, because W satisfies condition (4) and M^* is a direct sum of some copies of E_i's, by (3.5) (2) applied to the sequence ($\sharp\sharp$) we get $M = \mathrm{Ker}\, \tau$ is a direct sum of some copies of E_i's. Since both the last assertions and the implication [(5) $\Rightarrow$ (1)] of (3.1) are proved in (3.4), this completes the proof of (3.1).

As immediate consequences of (3.1) we have

Corollary (3.6). *Suppose that M is a maximal Buchsbaum A-module. Then so is $M^* = \mathrm{Hom}_A(M, A)$.*

Corollary (3.7). *Let M be a finitely generated A-modules. Then the following conditions are equivalent.*
(1) *M is an indecomposable maximal Buchsbaum A-module.*
(2) *$M \cong E_i$ for some $1 \leq i \leq d$.*
When this is the case, the integer i is given by $i = \mathrm{depth}_A M$.

Remark (3.8). (1) Every maximal Buchsbaum A-module M of $\mathrm{depth}_A M \geq 2$ enjoys the property that M is reflexive and the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free for any $\mathfrak{p} \in \mathrm{Spec}\, A \setminus \{\mathfrak{m}\}$ (c.f. (4.1)). Modules having this

property are called vector bundles and our theorem (3.1) may have some interest as an assertion which concerns the structure of certain vector bundles over regular local rings.

(2)   The implication [(4)⇒(1)] (resp. [(1)⇒(4)]) in (3.1) is due to [24, Theorem 1] (resp. [22, Satz 2]).   In graded case, D. Eisenbud and the author [4] gave a restricted form of the equivalence [(1)⇔(5)] in (3.1) (c.f. [4, Theorem 3.1 (b)]).   The implication [(4)⇒(5)] in (3.1) is closely related to [4, Theorem 3.2] too.   The assumption in [4, Theorem 3.2] is the dual form of (4) in (3.1) (c.f. [22, Satz 3]).

§ 4.   Proof of Theorem (1.1)

Let M be a finitely generated A-module of $\dim_A M = s \geq 1$.   First of all we note

Proposition and Definition (4.1).   *The following conditions are equivalent.*

(1)   *M is a generalized Cohen-Macaulay A-module.*

(2)   *$\ell_A(H_{\mathfrak{m}}^i(M)) < \infty$ for $i \neq s$.*

(3)   *There is an integer $N \geq 1$ such that any system of parameters for M contained in $\mathfrak{m}^N$ forms a d-sequence (and hence an unconditioned strong d-sequence) on M.*

When this is the case,

$$I_A(M) = \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H_{\mathfrak{m}}^i(M))$$

and furthermore for each $\mathfrak{p} \in \mathrm{Supp}_A M \setminus \{\mathfrak{m}\}$, $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module of $\dim_{A_{\mathfrak{p}}} M_{\mathfrak{p}} = s - \dim A/\mathfrak{p}$.

Proof.   (1)⇔(2) and the last assertion.   See [21].
(2)⇔(3) See [12].

We begin with the following

Lemma (4.2).   *Assume that M is a generalized Cohen-Macaulay A-module and let $x \in A$ such that $\dim_A M/xM = s - 1$.   Then*

(1)   *x is $M/H_{\mathfrak{m}}^0(M)$-regular.*

(2)   *In case $s \geq 2$, M/xM is again a generalized Cohen-Macaulay A-module and $I_A(M/xM) \leq I_A(M)$.   The equality $I_A(M/xM) = I_A(M)$ holds if and only if $x \cdot H_{\mathfrak{m}}^i(M) = (0)$ for all $i \neq s$.*

Proof.   (1)   See, e.g., [21, (3.3) Satz].
(2)   As $\ell_A([0] : (x]_M) < \infty$ by (1), we have similarly as in Proof of [12, Lemma 1.2] a long exact sequence

$$0 \longrightarrow [(0): x]_M \longrightarrow H^0_{\mathfrak{m}}(M) \xrightarrow{\ x\ } H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(M/xM) \longrightarrow \cdots$$

$$H^i_{\mathfrak{m}}(M) \xrightarrow{\ x\ } H^i_{\mathfrak{m}}(M) \longrightarrow H^i_{\mathfrak{m}}(M/xM) \longrightarrow \cdots$$

of local cohomology; hence for any $i < s - 1$,

$$\ell_A(H^i_{\mathfrak{m}}(M/xM)) = \ell_A(H^i_{\mathfrak{m}}(M)/xH^i_{\mathfrak{m}}(M)) + \ell_A([(0): x]_{H^{i+1}_{\mathfrak{m}}(M)})$$
$$= \ell_A(H^i_{\mathfrak{m}}(M)) + \ell_A(H^{i+1}_{\mathfrak{m}}(M)) < \infty.$$

Consequently M/xM is generalized Cohen-Macaulay and

$$I_A(M/xM) = \sum_{i=0}^{s-2} \binom{s-2}{i} \cdot \ell_A(H^i_{\mathfrak{m}}(M/xM))$$
$$\leqq \sum_{i=0}^{s-1} \binom{s-2}{i} \cdot [\ell_A(H^i_{\mathfrak{m}}(M)) + \ell_A(H^{i+1}_{\mathfrak{m}}(M))]$$
$$= \sum_{i=0}^{s-1} \binom{s-1}{i} \cdot \ell_A(H^i_{\mathfrak{m}}(M)) = I_A(M)$$

$\left(\text{recall } \binom{s-1}{i} = \binom{s-2}{i} + \binom{s-2}{i-1} \right)$; thus $I_A(M/xM) \leqq I_A(M)$. It is now obvious that $I_A(M/xM) = I_A(M)$ if and only if

$$\ell_A(H^i_{\mathfrak{m}}(M/xM)) = \ell_A(H^i_{\mathfrak{m}}(M)) + \ell_A(H^{i+1}_{\mathfrak{m}}(M))$$

for any $i < s - 1$, that is $x \cdot H^i_{\mathfrak{m}}(M) = (0)$ for all $i \neq s$.

Proposition (4.3) (c.f. [14]). *Let* $\mathfrak{q} = (x_1, x_2, \cdots, x_s)A$ *be a parameter ideal for* M. *Then the following conditions are equivalent.*
(1) *M is a generalized Cohen-Macaulay A-module and* $\underline{x} = x_1, x_2, \cdots, x_s$ *is a standard system of parameters for* M, *that is* $I_A(M) = \ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M)$.
(2) *$\underline{x}$ is an unconditioned strong d-sequence on* M.

Proof. (2)$\Rightarrow$(1) See Proof of the implication [(3)$\Rightarrow$(1)] in (2.8).
(1)$\Rightarrow$(2) For a given system $\underline{y} = y_1, y_2, \cdots, y_s$ of parameters for M, we define

$$I(y_1, y_2, \cdots, y_s; M) = \ell_A((y_1, \cdots, y_{s-1})M : y_s/(y_1, \cdots, y_{s-1})M).$$

Then as $\underline{y}$ is a reducing system of parameters for M (c.f. [21, (2.11) Satz]), we have by [1, Corollary 4.8] that

$$I(y_1, y_2, \cdots, y_s; M) = \ell_A(M/\mathfrak{Q}M) - e_{\mathfrak{Q}}(M)$$

(here $\mathfrak{Q} = (y_1, y_2, \cdots, y_s)A$); so the invariant $I(y_1, \cdots, y_s; M)$ does not

depend on the order of elements $y_1, y_2, \cdots, y_s$ in the system $\underline{y}$. Notice that

$$(\sharp) \quad I(y_1, \cdots, y_{s-1}, y_s^{n+1}; M) \geqq I(y_1, \cdots, y_{s-1}, y_s^n; M)$$

for any integers $n \geqq 1$, which directly follows from the definition.

First of all we note the following

Claim. $I(x_1^{n_1}, x_x^{n_2}, \cdots, x_s^{n_s}; M) = I_A(M)$ for any integers $n_1, n_2, \cdots,$ $n_s \geqq 1$.

Proof of Claim. Because

$$I(x_1, x_2, \cdots, x_s; M) = I_A(M) = \sup_{\mathfrak{q}} \left[\ell_A(M/\mathfrak{q}M) - e_\mathfrak{q}(M) \right],$$

it is enough to check that

$$I(x_1^{n_1}, x_2^{n_2}, \cdots, x_s^{n_s}; M) \geqq I(x_1, x_2, \cdots, x_s; M)$$

which can be reduced (by suitable repeated permutations) to the inequality $(\sharp)$ above. Hence the result.

Let us prove by induction on s that the claim implies assertion (2). If $s = 1$, we have

$$[(0) : x_1]_M = [(0) : x_1^n]_M$$

for all $n \geqq 1$ and there is nothing to prove. Assume $s \geqq 2$ and that our implication is true for $s - 1$. Let σ be any permutation of $1, 2, \cdots, s$ and $n_1, n_2, \cdots, n_s \geqq 1$ be integers. We put $\overline{M} = M/x_{\sigma(1)}^{n_1} M$. Then the above claim implies

$$
\begin{aligned}
I_A(\overline{M}) &\geqq I(x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s)}^{n_s}; \overline{M}) \\
&= \ell_A((x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s-1)}^{n_s-1})\overline{M} : x_{\sigma(2)}^{n_s}/(x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s-1)}^{n_s-1})\overline{M}) \\
&= \ell_A((x_{\sigma(1)}^{n_1}, x_{\sigma(i)}^{n_2}, \cdots, x_{\sigma(s-1)}^{n_s-1})M : x_{\sigma(s)}^{n_s}/(x_{\sigma(1)}^{n_1}, x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s-1)}^{n_s-1})M) \\
&= I_A(x_{\sigma(1)}^{n_1}, x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s)}^{n_s}; M) = I_A(M),
\end{aligned}
$$

whence by (4.2) we find that

$$I_A(\overline{M}) = I(x_{\sigma(2)}^{n_2}, \cdots, x_{\sigma(s)}^{n_s}; \overline{M})$$

and $x_{\sigma(1)}^{n_1} \cdot H_\mathfrak{m}^i(M) = (0)$ for any $i \neq s$; so

$$x_k \cdot H_\mathfrak{m}^i(M) = (0)$$

for any $i \neq s$ and $1 \leqq k \leqq s$. Furthermore the hypothesis of induction on

s yields that $x_{\sigma(2)}, x_{\sigma(3)}, \cdots, x_{\sigma(s)}$ is an unconditioned strong d-sequence on $\overline{M}$.

Now let $1 \leq i \leq j \leq s$ and let $f \in M$ such that

$$(x_{\sigma(i)}^{n_i} \cdot x_{\sigma(j)}^{n_j}) f \in L = (x_{\sigma(1)}^{n_1}, \cdots, x_{\sigma(i-1)}^{n_{i-1}}) M.$$

Then if $i \geq 2$, we have $x_{\sigma(j)}^{n_j} \cdot f \in L$ at once (pass to $\overline{M} = M/x_{\sigma(1)}^{n_1} M$). If $i = 1$, firstly we find that $f \in H_{\mathfrak{m}}^0(M)$ since

$$\ell_A([(0) : x_{\sigma(1)}^{n_1} \cdot x_{\sigma(j)}^{n_j}]_M) < \infty$$

(c.f. (4.2) (1)). Consequently $x_{\sigma(j)} f = 0$, because $x_k \cdot H_{\mathfrak{m}}^0(M) = (0)$ for any $1 \leq k \leq s$ as we have proved above. Thus $x_{\sigma(2)}^{n_1}, x_{\sigma(3)}^{n_2}, \cdots, x_{\sigma(s)}^{n_s}$ form a d-sequence on M, which completes the proof of (4.3).

Remark (4.4). We say that M is quasi-Buchsbaum, if $\mathfrak{m} \cdot H_{\mathfrak{m}}^i(M) = (0)$ for any $i \neq s$. (As is noted in (2.8), any Buchsbaum A-modules is quasi-Buchsbaum but the converse is not true in general, cf. [11]). It is not difficult to show that any system of parameters for a quasi-Buchsbaum module which is contained in $\mathfrak{m}^2$ is standard, see [10, (2.10)].

We are now ready to prove Theorem (1.1).

Proof of Theorem (1.1).

As $e_{\mathfrak{q}}(M) = \mathrm{rank}_R M$, the last assertion follows from (3.1). The equivalence of assertions (2), (3), (4), and (5) is due to (3.1).

(1)$\Rightarrow$(4)   See (4.3).

(2)$\Rightarrow$(1)   As $H_{\mathfrak{m}}^i(M) = H_{\mathfrak{n}}^i(M)$ for any $i \in \mathbf{Z}$, M is a generalized Cohen-Macaulay A-module. Furthermore as $R/\mathfrak{n} \cong A/\mathfrak{m}$, the equalities $I_A(M) = I_R(M)$ and $\ell_A(M/\mathfrak{q}M) - e_{\mathfrak{q}}(M) = \ell_R(M/\mathfrak{n}M) - e_{\mathfrak{n}}(M)$ follow and so $\underline{x}$ must be a standard system of parameters for the A-module M, because $I_R(M) = \ell_R(M/\mathfrak{n}M) - e_{\mathfrak{n}}(M)$ by assumption (2). This finishes the proof of Theorem (1.1).

For the rest of this section we assume that A contains a field and let $\hat{A}$ (resp. $\hat{M}$) denote the $\mathfrak{m}$-adic completion of A (resp. M). We choose a coefficient field k of $\hat{A}$ and put

$$R = k[[x_1, x_2, \cdots, x_s]] \quad (\subset \hat{A}),$$

where $x_1, x_2, \cdots, x_s$ denotes a system of parameters for M. Recall that R is a regular local ring and that $\hat{M}$ is a finitely generated R-module of $\dim_R \hat{M} = \dim R = s$. Let E_i $(0 \leq i \leq s)$ denote the ith syzygy module of the residue field k of R. Then as an immediate consequence of Theorem (1.1) we have

Corollary (4.5). *The following conditions are equivalent.*

(1)   *M is a generalized Cohen-Macaulay A-module and* $\underline{x} = x_1, x_2,$
$\cdots, x_s$ *is a standard system of parameters for M.*

(2) $\hat{M}$ *is a Buchsbaum R-module.*

(3)   *The idealization* $R \propto \hat{M}$ *is a Buchsbaum ring.*

(4) $\underline{x}$ *is an unconditioned strong d-sequence on M.*

(5) $\hat{M} \cong \bigoplus_{i=0}^{s} E_i^{h_i}$ *as R-modules for some integers* $h_i \geq 0.$
When this is the case, the integers h_i'*s in condition (5) are given by*

$$h_i = \ell_A(H_{\mathfrak{m}}^i(M)) \quad (0 \leq i < s) \quad \text{and} \quad h_s = e_{\mathfrak{q}}(M) - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \cdot h_i$$

(here $\mathfrak{q} = (x_1, x_2, \cdots, x_s)A$*).*

Proof.   Passing to $A/(0) : M$, we may assume that $\underline{x}$ is a system of parameters for A too.   Hence (1) $\hat{A}$ is a module-finite extension of R, (2) $k = R/\mathfrak{n} = \hat{A}/\mathfrak{m}\hat{A}$ where $\mathfrak{n} = (x_1, x_2, \cdots, x_s)R$, and (3) $\dim_{\hat{A}} M = \dim R = s \geq 1$.   Thus our assertion (3.5) follows from (1.1).

Passing to the $\mathfrak{m}$-adic completion, by virtue of (4.5) we can extend in equi-characteristic case the results of [8] and [9] on systems of parameters for Buchsbaum rings to those on standard systems of parameters for generalized Cohen-Macaulay modules.   In (4.6) we shall summarize a few of them, whose proof is now routine and we leave it to readers.

With the same notation as in (4.5), suppose that M is a generalized Cohen-Macaulay A-module and $x_1, x_2, \cdots, x_s$ is a standard system of parameters for M.   Let $\mathfrak{q} = (x_1, x_2, \cdots, x_s)A$.   We denote by G (resp. $G(M)$) the associated graded ring of A (resp. the associated graded module of M) relative to $\mathfrak{q}$.   Then $G(M)$ is a graded G-module and hence the local cohomology modules $H_{\mathfrak{P}}^i(G(M))$ of $G(M)$ relative to the maximal ideal $\mathfrak{P} = \mathfrak{m}G + G_+$ of G are naturally graded, whose homogeneous components with degree n ($n \in \mathbf{Z}$) shall be denoted by $[H_{\mathfrak{P}}^i(G(M))]_n$.

Corollary (4.6).   (1)   *Let* $f_i = x_i \bmod \mathfrak{q}^2$.   *Then* $\underline{f} = f_1, f_2, \cdots, f_s$ *is an unconditioned strong d-sequence on* $G(M)$.

(2) $[H_{\mathfrak{P}}^i(G(M))]_n = H_{\mathfrak{m}}^i(M) \quad (n = -i),$
$$= (0) \qquad (n \neq -i)$$
for $0 \leq i < s$ *and* $[H_{\mathfrak{P}}^s(G(M))]_n = (0)$ ($n > -s$).

(3) $(x_1^{n_1}, x_2^{n_2}, \cdots, x_k^{n_k})M \cap \mathfrak{q}^n M = \sum_{i=1}^{k} (x_i^{n_i} \mathfrak{q}^{n-n_i}) \cdot M$ *for any* $1 \leq k \leq s,$
$n \in \mathbf{Z}$, *and* $n_1, n_2, \cdots, n_s \geq 1$.

(4) $(x_1^n, \cdots, x_k^n)M : \left(\prod_{i=1}^{k} x_i \right)^{n-1} = (x_1, \cdots, x_k)M$
$$+ \sum_{i=1}^{k} (x_1, \cdots, \hat{x}_i, \cdots, x_k)M : x_i$$

and

$$\ell_A\left(\left[(x_1^n, \cdots, x_k^n)M : \left(\prod_{i=1}^{k} x_i\right)^{n-1}\right] \Big/ (x_1, \cdots, x_k)M\right) = \sum_{i=0}^{k-1} \binom{k}{i} \cdot \ell_A(H_{\mathfrak{m}}^i(M))$$

for $1 \leqq k \leqq s$ *and* $n \geqq 2$.

(5) $e_{\mathfrak{q}}(M) = \sum_{i=1}^{s-1} \binom{s-1}{i-1} \cdot \ell_A(H_{\mathfrak{m}}^i(M)) + h_s$ *(here* h_s *is the integer given by* (4.5) (5)). *In particular*

$$e_{\mathfrak{q}}(A) \geqq 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot \ell_A(H_{\mathfrak{m}}^i(A))$$

for the case $M = A$.[*)]

Let $I = E_A(A/\mathfrak{m})$ denote the injective envelope of $A/\mathfrak{m}$.   Then an A-module K is called a canonical module of M, if

$$\hat{A} \bigotimes_A K \cong \operatorname{Hom}_A(H_{\mathfrak{m}}^s(M), I)$$

as $\hat{A}$-modules.   If M possesses a canonical module K, then it is uniquely determined (up to isomorphisms) by M and shall be denoted by K_M (c.f. [15]).

Corollary (4.7) ([25]).   *Assume that M possesses a canonical module K_M.   Then K_M is a generalized Cohen-Macaulay A-module and $x_1, x_2, \cdots, x_s$ is a standard system of parameters for K_M too.   In particular if M is a Buchsbaum A-module, then so is K_M.*

Proof.   We may assume A to be complete.   Then by [15, 5.13] we have an isomorphism

$$K_M \cong \operatorname{Hom}_R(M, R)$$

of A-modules, while $\operatorname{Hom}_R(M, R)$ is a Buchsbaum R-module by (3.6). Hence K_M is a Buchsbuam R-module and so our assertions follow from (4.5) at once.

[*)] For the case $M = A$ in (4.5), the integer h_d is positive because R is a direct summand of $\hat{A}$ (cf. M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J., 51 (1973), 25–43).

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