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Resolutions and Representations of $GL(n)$

David A. Buchsbaum*

In his 1890 paper, [7], Hilbert "finished off" invariant theory by proving his Basis Theorem and his Syzygy Theorem. Recall that the syzygy theorem, i.e. the statement that every homogeneous ideal in the polynomial ring $k[X_1, \cdots, X_n]$ has a free resolution of length not greater than $n-1$, was proved to facilitate the calculation of the number, $H(d)$, of independent forms of degree d in the ring of invariants of a linear group action. (It also shows quite trivially that the function $H(d)$ is a polynomial function.) It is therefore very satisfying to see how, almost one hundred years later, we can turn things around and use representation theory to facilitate the calculation of explicit free resolutions of large classes of ideals (and modules). The aim of this paper is to illustrate how this is done in some special cases. In no case will we do a complete computation, but merely do enough to show how to employ the technique. It will become clear that a good computer program involving the Littlewood-Richardson rule as well as the illustrated counting process would be of enormous help.

§ 1. The generic $m\times n$ matrix

The generic $m\times n$ matrix $X=(X_{ij})$ with $1\leq i\leq m$, $1\leq j\leq n$, $m\geq n$, is regarded as a map from a free module of rank m to one of rank n. A number of important ideals and modules are associated with this matrix: The ideals I_p generated by the minors of X of order p; the various exterior powers of the cokernel of this linear map.

In order to consider this situation in a more intrinsic way, let us suppose we are working over a field, R, of characteristic zero, and that F and G are vector spaces of dimensions m and n respectively. We may then construct the symmetric algebra $S(F\otimes G)=S$ over R, and let $\bar{F}=S\otimes_R F$, $\bar{G}=S\otimes_R G$. To define a homogeneous map ϕ: $\bar{F}\rightarrow\bar{G}^*$ of degree 1, it suffices to define a map ϕ_0: $F\rightarrow S_1\otimes_R G^*$, where S_1 stands for the component of degree 1 in the graded ring $S=S(F\otimes G)$, i.e. $S_1=F\otimes G$.

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Under the natural isomorphism $\mathrm{Hom}_R(G, G) \approx G \otimes G^*$, the identity map I_G goes to an element C_G in $G \otimes G^*$. We define the map ϕ_0 by setting

$$\phi_0(x) = x \otimes C_G \in F \otimes G \otimes G^* = S_1 \otimes G^*.$$

If we choose a basis $\{x_1, \cdots, x_m\}$ for F and $\{y_1, \cdots, y_n\}$ for G, then $\{x_i \otimes y_j\}$ is a basis for $F \otimes G$ and S is the polynomial ring in the variables $X_{ij} = x_i \otimes y_j$. It is easy to see that the map $\phi\colon \bar{F} \to \bar{G}^*$ has $X = (X_{ij})$ as matrix with respect to the bases $\{1 \otimes x_i\}$ of $\bar{F}$ and $\{1 \otimes y_j^*\}$ of $\bar{G}^*$. Thus we see that our map ϕ is "the generic map" over R, but it is defined without recourse to a choice of basis. From our map ϕ we obtain, for each $p \leq n$, the map $\Lambda^p\phi\colon \Lambda^p\bar{F} \to \Lambda^p\bar{G}^*$, and hence the pairing $w_p\colon \Lambda^p\bar{G} \otimes \Lambda^p F \to S$. The image of w_p is the ideal I_p in S generated by the $p \times p$ minors of X. Hence our ideal I_p is also defined in a basis-free way, so that we may expect that any resolution of I_p over $S(F \otimes G)$ should be independent of choice of basis, i.e. the modules in such a resolution should be $GL(F) \times GL(G)$-modules. Since we are working over a field of characteristic zero, these modules are completely reducible and will therefore be direct sums of tensor products of irreducible representations of $GL(F)$ and $GL(G)$. We illustrate how this fact can be exploited to compute the first step of the resolution of I_p over S (or of S/I_p over S). That is, we want to determine the kernel of the map

$$\Lambda^p\bar{G} \otimes \Lambda^p\bar{F} \xrightarrow{w_p} S.$$

Using the fact that $\Lambda^p\bar{G} \otimes \Lambda^p\bar{F} = S \otimes_R \Lambda^p G \otimes \Lambda^p F$ and that the map w_p is homogeneous of degree p, we analyze w_p degree by degree. In degree 0, we have

$$S_0 \otimes \Lambda^p G \otimes \Lambda^p F \longrightarrow S_p(F \otimes G)$$

which is just the inclusion of $\Lambda^p F \otimes \Lambda^p G$ in $S_p(F \otimes G)$. It should be pointed out here that we use very heavily the Cauchy decomposition of $S(F \otimes G)$, i.e. $S(F \otimes G) = \sum_\lambda L_\lambda F \otimes L_\lambda G$ where the sum is over all partitions λ, and $L_\lambda F (L_\lambda G)$ denotes the Schur module (i.e. the irreducible representation of $GL(F)$ $(GL(G))$ associated to the partition λ. Thus $S_p(F \otimes G) = \Lambda^p F \otimes \Lambda^p G \oplus \cdots$, so that $\Lambda^p F \otimes \Lambda^p G$ is included in $S_p(F \otimes G)$.

In degree 1, we have

$$S_1 \otimes \Lambda^p G \otimes \Lambda^p F \longrightarrow S_{p+1} \quad \text{or}$$
$$(F \otimes G) \otimes \Lambda^p G \otimes \Lambda^p F \longrightarrow \Lambda^{p+1} F \otimes \Lambda^{p+1} G \oplus L_{(p,1)} F \otimes L_{(p,1)} G \oplus \cdots.$$

Using the Pieri formula, we have

$$F\otimes\Lambda^p F=\Lambda^{p+1}F\oplus L_{(p,1)}F, \qquad G\otimes\Lambda^p G=\Lambda^{p+1}G\oplus L_{(p,1)}G.$$

Hence

$$(F\otimes G)\otimes\Lambda^p G\otimes\Lambda^p F$$
$$=\Lambda^{p+1}F\otimes\Lambda^{p+1}G\oplus\Lambda^{p+1}F\otimes L_{(p,1)}G\oplus L_{(p,1)}F\otimes\Lambda^{p+1}G\oplus L_{(p,1)}F\otimes L_{(p,1)}G.$$

As the representations $\Lambda^{p+1}F\otimes L_{(p,1)}G$ and $L_{(p,1)}F\otimes\Lambda^{p+1}G$ do not appear in $S_{p+1}(F\otimes G)$, these irreducible representations of $GL(F)\times GL(G)$ must be in the kernel of w_p. Working on the assumption that the summands $\Lambda^{p+1}F\otimes\Lambda^{p+1}G$ and $L_{(p,1)}F\otimes L_{(p,1)}G$ in $(F\otimes G)\otimes\Lambda^p G\otimes\Lambda^p F$ are mapped isomorphically onto the corresponding summands in $S_{p+1}(F\otimes G)$, we conclude that the kernel of w_p in degree 1 is precisely $\Lambda^{p+1}F\otimes L_{(p,1)}G\oplus L_{(p,1)}F\otimes\Lambda^{p+1}G$. (That this assumption is true can be verified by looking at what happens to the canonical tableaux, but one need not do this at this point.) The next step is to look at degree k:

$$S_k(F\otimes G)\otimes\Lambda^p G\otimes\Lambda^p F\longrightarrow S_{p+k}(F\otimes G).$$

Here again one uses the Cauchy decomposition of S_k and S_{p+k}, the Pieri formula, and matching up of corresponding irreducible components and discovers that those components of $S_k(F\otimes G)\otimes\Lambda^p G\otimes\Lambda^p F$ which are not included in $S_{p+k}(F\otimes G)$ are contained in

$$S_{k-1}(F\otimes G)\otimes(\Lambda^{p+1}F\otimes L_{(p,1)}G\oplus L_{(p,1)}F\otimes\Lambda^{p+1}G).$$

Thus, the next step in our resolution of S/I_p is

$$S\otimes(\Lambda^{p+1}F\otimes L_{(p,1)}G\oplus L_{(p,1)}F\otimes\Lambda^{p+1}G),$$

and one now proceeds to find the rest of the resolution in the same way.

The next illustration of this technique is that of determining the resolution of $\Lambda^p M$ where M is the cokernel of the map $\phi\colon \bar{F}\to\bar{G}^*$. We know that

$$\bar{F}\otimes\Lambda^{p-1}\bar{G}^*\longrightarrow\Lambda^p\bar{G}^*\longrightarrow\Lambda^p M\longrightarrow 0$$

is exact, and we want to find the kernel of

$$\bar{F}\otimes\Lambda^{p-1}\bar{G}^*\longrightarrow\Lambda^p\bar{G}^*.$$

Since we are working over $S(F\otimes G)$, it is more convenient to work with $\bar{G}$ rather than $\bar{G}^*$ so, using the identification of $\Lambda^k\bar{G}^*$ with $\Lambda^{n-k}\bar{G}$, we have to find the kernel of

$$\beta\colon \bar{F}\otimes\Lambda^{n-p+1}\bar{G}\longrightarrow\Lambda^{n-p}\bar{G} \quad\text{or}\quad S\otimes F\otimes\Lambda^{n-p+1}G\xrightarrow{\ \delta\ }S\otimes\Lambda^{n-p}G.$$

We see that this map is defined as the following composition:

$$S_k \otimes F \otimes \Lambda^{n-p+1} G \xrightarrow{\delta} S_k \otimes F \otimes G \otimes \Lambda^{n-p} G \xrightarrow{\mu} S_{k+1}(F \otimes G) \otimes \Lambda^{n-p} G$$

where δ is obtained by diagonalizing $\Lambda^{n-p+1}G$ and μ is the multiplication in $S(F \otimes G)$ tensored with the identity on $\Lambda^{n-p}G$. The map β is homogeneous of degree 1, so again we compute its kernel degree by degree.

In degree 0, we have the injection

$$F \otimes \Lambda^{n-p+1} G \longrightarrow F \otimes G \otimes \Lambda^{n-p} G.$$

In degree 1, we have

$$(F \otimes G) \otimes F \otimes \Lambda^{n-p+1} G \longrightarrow S_2(F \otimes G) \otimes \Lambda^{n-p} G.$$

Using Cauchy decomposition and Pieri, we have on the left:

$$\Lambda^2 F \otimes \Lambda^{n-p+2} G \oplus \Lambda^2 F \otimes L_{(n-p+1,1)} G \oplus S_2 F \otimes \Lambda^{n-p+2} G \oplus S_2 F \otimes L_{(n-p+1,1)} G;$$

and on the right:

$$\Lambda^2 F \otimes \Lambda^{n-p+2} G \oplus \Lambda^2 F \otimes L_{(n-p+1,1)} G \oplus \Lambda^2 F \otimes L_{(n-p,2)} G$$
$$\oplus S_2 F \otimes L_{(n-p,1,1)} G \oplus S_2 F \otimes L_{(n-p+1,1)} G.$$

Thus $S_2 F \otimes \Lambda^{n-p+2} G$ must be in the kernel. It is also fairly straightforward to show that

$$(1) \quad S_k(F \otimes G) \otimes S_2 F \otimes \Lambda^{n-p+2} G \longrightarrow S_{k+1}(F \otimes G) \otimes F \otimes \Lambda^{n-p+1} G$$
$$\longrightarrow S_{k+2}(F \otimes G) \otimes \Lambda^{n-p} G$$

is "exact by counting" for $k < n-p$. But suppose $k = n-p$. Since $\Lambda^{n-p+1} F \otimes \Lambda^{n-p+1} G \otimes F \otimes \Lambda^{n-p+1} G$ is in $S_{n-p+1}(F \otimes G) \otimes F \otimes \Lambda^{n-p+1} G$, we see that $\Lambda^{n-p+2} F \otimes L_{(n-p+1,n-p+1)} G$ is in the middle term of (1). This term cannot be accounted for in $S_{n-p}(F \otimes G) \otimes S_2 F \otimes \Lambda^{n-p+2} G$, nor can it appear in the right-most term of (1). Therefore, in addition to $S \otimes S_2 F \otimes \Lambda^{n-p+2} G$, we must also adjoin the term $S \otimes \Lambda^{n-p+2} F \otimes L_{(n-p+1,n-p+1)} G$ in the next stage of our resolution. (Note that the map of $S \otimes \Lambda^{n-p+2} F \otimes L_{(n-p+1,n-p+1)} G$ into $S \otimes F \otimes \Lambda^{n-p+1} G$ is of degree $n-p+1$.) It is fairly easy to verify that no other terms must be added at this point so that

$$\Lambda^{n-p+2} \bar{F} \otimes L_{(n-p+1,n-p+1)} \bar{G} \oplus S_2 \bar{F} \otimes \Lambda^{n-p+2} \bar{G} \longrightarrow \bar{F} \otimes \Lambda^{n-p+1} \bar{G} \longrightarrow \Lambda^{n-p} \bar{G}$$

is "exact by counting".

By continuing in this way, one obtains a candidate (at least in characteristic 0) for a resolution of $\Lambda^p M$.

By now it is apparent why a computer would expedite this process. Even without the computer though, M. Artale and A. Miller in [5] and [7] have used this technique to produce resolutions of the module M_λ, where M_λ is the cokernel of the map $L_\lambda F \to L_\lambda G^*$, for certain partitions λ.

Before proceeding with our next section, some more should be said about the work of Artale and Miller cited above, because it is connected with a rather interesting conjecture. In [6], Eisenbud and I had shown that when λ is a hook, the module M_λ is perfect. In [3], Akin, Weyman and I showed that the ideal $(I_n)^k$ is an imperfect ideal for $k \geq 2$. In fact, $hd_S\, S/(I_n)^k = k(m-n)+1$ for $k=1, \cdots, n$. It seems natural to conjecture, then, that M_λ should be perfect if and only if λ is a hook, i.e. that the Durfee square of λ is equal to 1. As there is as yet no known relationship between Durfee square size and perfection, it seemed that the first approach to this conjecture would be the construction of explicit minimal resolutions of M_λ. Artale did this for $\lambda=(2, 2)$ and Miller for $\lambda=(3, 2)$ in characteristic 0 using, as was said, the counting technique illustrated here. In fact, although they did not write down maps and prove acyclicity for arbitrary λ, their work has been complete enough to strengthen the conjecture to the following:

$$hd_s M_\lambda = k(m-n)+1$$

where k is the Durfee square of the partition λ. (Recall that the partition $\lambda=(\lambda_1, \cdots, \lambda_t)$ has Durfee square k if $\lambda_i \geq i$ for $i=1, \cdots, k$ and $\lambda_{k+1} \leq k$.)

The construction of a minimal resolution of M_λ of length $k(m-n)+1$ would be interesting on two counts. For one, it would establish a connection between the two disparate notions of Durfee square and perfection. For another, I believe that as λ ranges over all partitions, these minimal resolutions of M_λ would probably reflect all possible situations that can arise in graded resolutions with respect to degrees of maps, Betti numbers, etc.

§ 2. The generic Pfaffian

In this section, we consider the generic skew-symmetric matrix $Y=(Y_{ij})$ where $1 \leq i, j \leq n$, $Y_{ii}=0$, and $Y_{ij}=-Y_{ji}$. The problem is to find resolutions of the ideals J_p generated by the pfaffians of Y of order p. The matrix Y is thought of as a skew-symmetric map from F to F^* where F is a vector space of dimension n over a field, R, of characteristic 0. Our first step is to formulate this in a basis-free way and we do this by considering the ring $S=S(\Lambda^2 F)$, the symmetric algebra of $\Lambda^2 F$ over R. We let $\bar{F}=S \otimes F$ and we define the map $\alpha_0\colon \bar{F} \to \bar{F}^*$ as the composition

$$F \xrightarrow{1\otimes C_F} F\otimes F\otimes F^* \longrightarrow \Lambda^2 F\otimes F^* = S_1\otimes F^*.$$

The map $\alpha\colon \bar{F}\to\bar{F}^*$ is the homogeneous map of degree 1 obtained from the map α_0. If we choose a basis $\{x_1, \cdots, x_n\}$ for F, and $\{x_1^*, \cdots, x_n^*\}$ the dual basis for F^*, the matrix of α with respect to these bases is $(x_i\wedge x_j)$ and thus skew-symmetric. The map $\alpha\colon \bar{F}\to\bar{F}^*$ gives us the corresponding element $\sigma\in\bar{F}^*\otimes\bar{F}^*$ which is in the kernel of $\bar{F}^*\otimes\bar{F}^*\to S_2(\bar{F}^*)$ due to skew-symmetry. Thus $\sigma\in\Lambda^2\bar{F}^*$ and the divided power $\sigma^{(p)}$ is in $\Lambda^{2p}\bar{F}^*$. The natural pairing $\beta\colon \Lambda^{2p}\bar{F}^*\otimes\Lambda^{2p}\bar{F}\to S$ defines a map $w_p\colon \Lambda^{2p}\bar{F}\to S$ by $w_p(x) = \beta(\sigma^{(p)}\otimes x)$. The image of w_p is the ideal J_p generated by the pfaffians of α of order p. A closer analysis of the map w_p shows that it is homogeneous of degree p; in degree 0 it defines the usual "pfaffian map"

$$\Lambda^{2p}F\longrightarrow S_p(\Lambda^2 F). \tag{2}$$

What we will do now is calculate the kernel of w_p to get the first term of our resolution of S/J_p. In degree 0, the map is (2) above, which is known to be an injection. Hence we move on to degree 1 and look at

$$S_1\otimes\Lambda^{2p}F\longrightarrow S_{p+1}(\Lambda^2 F). \tag{3}$$

Since $S_1=\Lambda^2 F$, the Pieri formula gives us

$$S_1\otimes\Lambda^{2p}F=\Lambda^{2p+2}F\oplus L_{(2p+1,1)}F\oplus L_{(2p,2)}F.$$

It is well-known that $S(\Lambda^2F)=\sum_\lambda L_{2\lambda}F$ where λ runs over all partitions $\lambda=(\lambda_1, \cdots, \lambda_k)$ and $2\lambda=(2\lambda_1, \cdots, 2\lambda_k)$. Thus $S_{p+1}(\Lambda^2F)=\Lambda^{2p+2}F\oplus L_{(2p,2)}F \oplus\cdots$ and we see immediately that $L_{(2p+1,1)}F$ must be in the kernel of the map (3). To show that

$$S_k\otimes L_{(2p+1,1)}F\longrightarrow S_{k+1}\otimes\Lambda^{2p}F\longrightarrow S_{p+k+1}(\Lambda^2F)$$

is "exact by counting", one must use the Littlewood-Richardson rule to decompose the products $L_{2\lambda}F\otimes L_{(2p+1,1)}F$ that occur in $S_k\otimes L_{(2p+1,1)}F$, but this does turn out to be the case so that

$$L_{(2p+1,1)}\bar{F}\longrightarrow\Lambda^{2p}\bar{F}\longrightarrow S(\Lambda^2F)$$

is exact by counting.

§ 3. Summary

The method described in the preceding sections has some evident disadvantages: it is lengthy; it does not provide the maps, only the terms of a resolution; it does not prove exactness, but gives only "exactness by counting"; it doesn't set an a priori bound on the degrees of the maps in the resolution; it only seems to work in characteristic 0. What, then, is the advantage of this procedure? The major advantage that has appeared is that it does give some idea of terms that must appear in certain resolutions, especially those terms not in the "linear strand" (i.e. boundary map of degree 1) of the complex. In practice, bounds on the degrees of the maps suggest themselves after a few calculations. For example, K. Akin and I were led to conclude that the Poincaré resolution of the coordinate ring of the Grassmannian is linear, a fact then confirmed by R. Buchweitz.

As for the definition of the maps, this is often a difficult undertaking. Of course one could keep track of the various decompositions and identifications involved in the counting process, and in that way the map would be evident. In practice again, one makes an intelligent guess and then tries to establish exactness, usually by repeated application of the acyclicity lemma and by known decompositions of the terms in the complex.

The last disadvantage mentioned is perhaps the most provocative, i.e. the fact that all the representation theory applied holds only in characteristic 0. Since all the examples considered in this article are universally defined, that is one does not have to assume that R is a field to state the problems, it seems a severe limitation to restrict oneself to the case when R is a field of characteristic 0 or, at most, contains such a field. The challenge, then, to find characteristic-free forms of the resolutions partially described in sections 1 and 2, is what prompted K. Akin, J. Weyman and myself to study integral representations of $GL(n)$. In [3], we succeeded in finding such a resolution for the ideal I_{n-1} of submaximal minors over any commutative ring R. This involved the definition of Schur and Weyl modules and Schur complexes as described in [4]. Deeper analysis of these problems then led to the study of Z-forms of rational representations of $GL(n)$ by Akin and myself. The results obtained thus far can be found in [1] and [2].

What seems to emerge from the evidence so far gathered is that, despite the disadvantages listed, the counting procedure, complemented by effective use of the computer, can provide a new practical tool for determining a large class of free resolutions. Furthermore, as is usual in the study of new procedures, the disadvantages themselves provide impetus for new and interesting fields of investigation.

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Department of Mathematics
Brandeis University
Waltham, Massachusetts 02154
USA