Advanced Studies in Pure Mathematics 10, 1987
Algebraic Geometry, Sendai, 1985
pp. 695–716

Mixed Hodge Structures on Cohomologies with Coefficients in a Polarized Variation of Hodge Structure

Yuji Shimizu

Introduction

Let U be a smooth quasi-projective variety over $\boldsymbol{C}$ and V a polarized variation of Hodge structure of weight m on U (cf. (0.3), [Gr1], [S]). A basic problem in the Hodge theory is to show the existence of a canonical mixed Hodge structure on the cohomology $H^i(U, V)$.

As is well known, the constant coefficient case $V=\boldsymbol{Q}_U$ is just Deligne's mixed Hodge theory [D2]. The case dim $U=1$ was treated by Zucker [Z1].

In this paper, we try to generalize Zucker's result to higher dimension in a special case. We assume U to be the complement of a smooth hypersurface Y in a projective smooth variety X (cf. (0.1)):

$$U \xhookrightarrow{j} X \xhookleftarrow{i} Y = X - U.$$

Under this assumption, we obtain a slight generalization of Zucker's theorems (cf. [Z1, (7.12), (13.11), (14.3)]).

Theorem 1 (cf. (2.5)). *There exists a natural pure Hodge structure of weight $m+i$ on $H^i(X, j_* V)$.*

Theorem 2 (cf. (3.1.7), (3.2.6)). *There exists a natural mixed Hodge structure of weight $\geqq m+i$ (resp. $\leqq m+i$) on $H^i(U, V)$ (resp. $H^i_c(U, V)$).*

Such a generalization of [Z1] was obtained independently by Zucker himself, as announced in [Z2, p. 182].

We shall indicate the proof of these theorems, which is reduced to the one-dimensional case [Z1] by our assumption on Y (cf. (0.5)).

For Theorem 1, we use the Hodge theory for V-valued L^2-forms in Section 2, in parallel with [Z1]. Once an L^2-complex is constructed, its properties are proved by means of local Künneth type formulas. Note that the norm estimate due to Schmid [S, (6.6)] is essential here.

Received September 14, 1985.
Revised March 27, 1986.

For Theorem 2, by the standard machinery of cohomological mixed Hodge complexes (cf. (0.4)), we have only to construct filtered complexes which enable us to compute $H^i(U, V)$ and $H^i_c(U, V)$.

The cohomology $H^i(U, V)$ is calculated by the logarithmic de Rham complex $K_C = \Omega^{\cdot}_X(\log Y) \otimes \bar{\mathscr{V}}$, where $\bar{\mathscr{V}}$ is Deligne's canonical prolongation of $\mathscr{V} = \mathcal{O}_U \otimes V$. In (3.1), we construct its filtration $W_{\bullet}$ so that the hypercohomology of each $\mathrm{Gr}^W_k K_C$ has a pure Hodge structure. It will turn out that $\mathrm{Gr}^W_0 K_C$ is quasi-isomorphic to the L^2-complex mentioned above and $\mathrm{Gr}^W_k K_C$ $(k>0)$ is the de Rham complex of a polarized variation of Hodge structure, obtained as a graded piece of the variation of the "limit" mixed Hodge structure of V, which is constructed with its prerequisite in Section 1.

The cohomology $H^i_c(U, V)$ is similarly treated in (3.2).

In (3.3), we collect the basic properties of the mixed Hodge structures constructed in (3.1), (3.2).

In Section 4, by Theorems 1, 2 or their proof, we verify that a part of Brylinski's conjectures on filtered $\mathscr{D}$-Modules [Br1] hold in our situation. To state our results, let us recall the language of $\mathscr{D}$-Modules (cf. (4.1)). Let $\mathscr{D}_X$ be the sheaf of rings of holomorphic linear differential operators, $D^b_{rh}(\mathscr{D}_X)$ the derived category of regular holonomic $\mathscr{D}_X$-Modules, $D^b_c(X, \boldsymbol{C})$ the derived category of $\boldsymbol{C}$-constructible sheaves. Kashiwara-Kawai [K1, 2], [KK1] and Mebkhout [M] proved that the following functor is an equivalence of categories (the Riemann-Hilbert correspondence)

$$DR_X = R\mathscr{H}om_{\mathscr{D}_X}(\mathcal{O}_X, \ ): D^b_{rh}(\mathscr{D}_X) \longrightarrow D^b_c(X, \boldsymbol{C}).$$

Then, in our situation, there exist regular holonomic $\mathscr{D}_X$-Modules, $\mathscr{L}$, $\mathscr{M}$ such that

$$DR_X(\mathscr{L}) = j_* V, \qquad DR_X(\mathscr{M}) = Rj_* V$$

and, if we identify V with its dual V^* by the polarization, then we have

$$\mathscr{L} = \mathscr{L}^*, \qquad DR_X(\mathscr{M}^*) = j_! V.$$

It is easy to see that in the present case $j_* V[2 \dim X]$ equals the intersection complex $IC^{\cdot}(X, V)$ of Deligne and Goresky-MacPherson [BBD], [GM]. Its hypercohomology is by definition the intersection cohomology $IH^i(X, V)$.

Then, our results are stated as follows.

Theorem 3 (cf. (4.2.1)). *$\mathscr{L}$ gives rise to a pure Hodge structure on $H^i(X, j_* V) = IH^i(X, V)$ constructed in Theorem* 1.

Theorem 4 (cf. (4.2.6)). *There exists a filtration $W_{\bullet}$ on $\mathcal{M}$ (resp. $\mathcal{M}^*$), which gives rise to the weight filtration on $H^i(U,V)$ (resp. $H^i_c(U, V)$) constructed in Theorem* 2.

Theorem 3 corresponds to the purity conjecture [Br1, II (2.1)] and restates Theorem 1. Theorem 4 corresponds to a conjecture on the mixed $\mathscr{D}$-Modules [Br1, II (2.3)] and we construct such a weight filtration on $\mathcal{M}$ and $\mathcal{M}^*$ in (4.2).

This paper is a refined version of the author's master's thesis (1983). After its redaction, several progresses have been made in this direction. First, Schmid's norm estimate was generalized to the several variable case by Kashiwara [K4] and by Cattani-Kaplan-Schmid [CKS1], by different methods. Secondly, Kashiwara-Kawai [KK2] and Cattani-Kaplan-Schmid [CKS2], independently, generalized the L^2-Poincaré lemma (cf. (2.4,3)) to the several variable case and showed that the intersection cohomology coincides with the L^2-cohomology and has a pure Hodge structure in the setting (0.1) except that Y is only assumed to be a divisor with normal crossings. Kashiwara-Kawai [KK3] also showed that the Hodge filtration is given by a good filtration of the minimal extension. Thirdly, Morihiko Saito proved the purity of the intersection cohomology with coefficients in a variation of "geometric origin" by his theory of Hodge modules [Sa]. Finally, Steenbrink-Zucker [SZ] generalized Zucker's results [Z1] allowing the coefficients to be a "good" variation of mixed Hodge structure.

I am grateful to Professor S. Zucker for valuable comments and to the referee for useful advice. I would like to express my gratitude to Prof. T. Shioda for encouragements. I would like to thank Prof. J-L. Verdier for giving me the chance to participate in the Colloque à Luminy "Systèmes différentiels et singularités" (1983), which has greatly inspired me. Thanks are due to Prof. H. Hironaka and the Committee on the Educational Project for Japanese Mathematical Scientists for financial support.

§ 0. Preliminaries

In this paragraph, we fix some notation to be constantly used in the sequel, and recall basic concepts in the mixed Hodge theory.

(0.1) We are always in the following situation:

X: a projective smooth algebraic variety over $\boldsymbol{C}$ (or a compact Kähler manifold)

Y: a smooth hypersurface of X

$U := X - Y \xrightarrow{j} X \xleftarrow{i} Y$: the inclusions.

We consider a polarized variation of Hodge structure V on U, the definition of which we recall in (0.3). $\mathcal{V} := \mathcal{O} \otimes V$ is the corresponding locally free $\mathcal{O}_U$-Module with an integrable connection.

We assume for simplicity the monodromy of V along Y to be unipotent. It is not necessary for the validity of the main results.

(0.2) A *mixed Hodge structure* (MHS) is a triple $H = (H_{\boldsymbol{Q}}, W, F)$ consisting of a finite dimensional $\boldsymbol{Q}$-vector space $H_{\boldsymbol{Q}}$, an increasing filtration W_k of $H_{\boldsymbol{Q}}$ and a decreasing filtration F^p of $H_{\boldsymbol{C}} := H_{\boldsymbol{Q}} \otimes \boldsymbol{C}$ such that $F^p \oplus \bar{F}^q = \mathrm{Gr}_k^W H_{\boldsymbol{C}}$ for $p+q=k+1$, where F^p is the induced filtration on $\mathrm{Gr}_k^W H_{\boldsymbol{C}}$ and $\bar{\ }$ means the complex conjugate with respect to $H_{\boldsymbol{R}} := H_{\boldsymbol{Q}} \otimes \boldsymbol{R}$.

The integer k such that $\mathrm{Gr}_k^W H_{\boldsymbol{Q}} \neq 0$ is called a weight of H. If H has only a single weight, we say that H is pure, or H is a (pure) Hodge structure (HS).

A polarization of a Hodge structure H of weight m is a bilinear pairing $S: H_{\boldsymbol{Q}} \otimes H_{\boldsymbol{Q}} \to \boldsymbol{Q}$ such that $S(F^p, F^q) = 0$ for $p+q=m+1$, $S(Cu, \bar{u}) > 0$ for $u \in H^{p,q} := F^p \cap \bar{F}^q$, $u \neq 0$, $p+q=m$, where C is the Weil operator on $H_{\boldsymbol{C}}$, defined by $C = i^{p-q}$ on $H^{p,q}$.

A graded polarization of a mixed Hodge structure H is a collection of polarizations S_k on each $\mathrm{Gr}_k^W H$.

For more details, see [D2, § 2].

(0.3) A *variation of mixed Hodge structure* (VMHS) (on a complex manifold X) is a triple $V = (V_{\boldsymbol{Q}}, \{W_k\}, \{\mathcal{F}^p\})$ consisting of a local system $V_{\boldsymbol{Q}}$, an increasing filtration $\{W_k\}$ by local subsystems of $V_{\boldsymbol{Q}}$, and a decreasing filtration $\{\mathcal{F}^p\}$ by locally free $\mathcal{O}$-subModules of $\mathcal{V} = \mathcal{O} \otimes V_{\boldsymbol{Q}}$, such that (i) $V_x = (V_{\boldsymbol{Q},x}, \{W_{k,x}\}, \{\mathcal{F}^p(x)\})$ defines a MHS at each point $x \in X$, and (ii) the natural connection ∇ on $\mathcal{V}$ satisfies $\nabla \mathcal{F}^p \subset \Omega_X^1 \otimes \mathcal{F}^{p-1}$ ("horizontality").

A graded polarization of a VMHS is a collection of morphisms of local systems $S_k: \mathrm{Gr}_k^W V_{\boldsymbol{Q}} \otimes \mathrm{Gr}_k^W V_{\boldsymbol{Q}} \to \boldsymbol{Q}$ which gives a graded polarization of MHS at each point of X.

When V has only a single weight, V is a variation of Hodge structure (VHS) introduced by Griffiths [Gr1, § 2] and a graded polarization is simply called a polarization.

For further details on VMHS, we refer to Steenbrink-Zucker [SZ] and Usui [U].

(0.4) The method to put a mixed Hodge structure on some (hyper-) cohomology is formulated in terms of cohomological mixed Hodge complexes [D2, III (8.1)].

A *cohomological mixed Hodge complex* (CMHC) (on a complex manifold X) is a triple $K=(K_{Z}, (K_{Q}, W), (K_{C}, W, F))$ consisting of a constructible complex K_{Z} of Z_X-Modules, a filtered complex (K_{Q}, W) of Q_X-Modules, and a bifiltered complex (K_{C}, W, F) of C_X-Modules, together with a quasi-isomorphism $K_{Z}\otimes Q\simeq K_{Q}$ and a filtered quasi-isomorphism $(K_{Q}, W)\otimes C\simeq (K_{C}, W)$, such that (i) the differential of $R\Gamma(X, \mathrm{Gr}_k^W K_C)$ is strictly compatible with F (cf. [D2, II (1.1)]) and (ii) $H^i(X, \mathrm{Gr}_k^W K_Q)$ together with F is a HS of weight $k+i$.

Deligne's fundamental theorem in [D2, III, (8.1.9)] asserts that $(H^i(X, K_Z), W$ on $H^i(X, K_Q), F$ on $H^i(X, K_C))$ is a MHS when K is a CMHC.

(0.5) We repeatedly reduce our proof to the one-dimensional case as follows: When the question is local, we can sometimes reduce ourselves to the following situation, localizing (0.1):

$$\begin{array}{ccccc} U=D^*\times D^{n-1} & \xrightarrow{j} & X=D^n & \xleftarrow{i} & Y=\{0\}\times D^{n-1} \\ \cup\!\!\!\shortmid & & & & \cup\!\!\!\shortmid \\ u & & & & y=(0, y') \end{array}$$

$z=(z_1, \cdots, z_n)$ denotes the standard coordinate: $Y=\{z_1=0\}$, $p: D^*\times D^{n-1}\to D^*$; $z\mapsto z_1$, $q: D^*\times D^{n-1}\to D^{n-1}$; $z\mapsto (z_2, \cdots, z_n)$ etc.

Thanks to the Nilpotent Orbit Theorem [S, (4.12)], we may sometimes assume that the original VHS V on U comes from a VHS V' on D^*: $V=p^{-1}V'$.

The local monodromy along Y is by definition the image in $\mathrm{GL}(V_u)$ of the generator of $\pi_1(U, u)\simeq Z$ for some point $u\in U$. Under the assumption of unipotence, we can consider its logarithm

$$N=\log T:=-\sum_{i\geqq 0}(I-T)^i/i$$

which is a finite sum. When the local monodromy is not unipotent, we can consider the logarithm of its unipotent part.

§ 1. Construction of the variation of limit mixed Hodge structure

In this paragraph, we construct a local system on Y and its filtrations after Zucker [Z1] using the limit mixed Hodge structure due to Schmid [S, § 6].

(1.1) Deligne's *canonical prolongation*: For any local system L on U, we put $\mathscr{L}=\mathcal{O}_U\otimes L$. Then, $\mathscr{L}$ has canonically an integrable connection $\nabla: \mathscr{L}\to\Omega_U^1\otimes\mathscr{L}$. According to [D1, II, § 5], there is a unique prolongation $\bar{\mathscr{L}}$ of $\mathscr{L}$ as a locally free $\mathcal{O}_X$-Module such that (i) the connection has at most logarithmic poles along Y with respect to any local frame of $\bar{\mathscr{L}}$, and (ii) the residue $\mathrm{Res}_Y \bar{\nabla}$ has eigenvalues with real parts in [0, 1).

By virtue of (i), we have a logarithmic connection $\bar{\nabla}: \bar{\mathscr{L}} \to \Omega_X^1(\log Y) \otimes \bar{\mathscr{L}}$. The assignment $L \mapsto \bar{\mathscr{L}}$ is functorial and preserves exact sequences. Moreover, if the local monodromy of L along Y is unipotent, then $\mathrm{Res}_Y \bar{\nabla}$ is nilpotent and the functor $L \mapsto \bar{\mathscr{L}}$ is compatible with $\mathscr{H}om$, $\otimes$, Λ.

(1.2) *Monodromy weight filtration* (cf. [S, (6.4)]): Given a finite-dimensional vector space E and a nilpotent endomorphism N of E, then there exists a unique increasing filtration $\{W_k\}$ on E such that (i) $NW_k \subset W_{k-2}$ and (ii) N^k induces an isomorphism $\mathrm{Gr}_k^W E \simeq \mathrm{Gr}_{-k}^W E$ for $k \geqq 0$. This filtration is called the monodromy weight filtration with respect to N (the N-filtration for short). The following relations hold:

$$W_{-k} = N^k W_k, \quad W_k = NW_{k+2} + \mathrm{Ker}\, N^{k+1} \quad (k \geqq 0).$$

Put $P_k := \mathrm{Ker}\,(N^{k+1}: \mathrm{Gr}_k^W E \to \mathrm{Gr}_{-k-2}^W E)$ (the primitive part). Then we have the decomposition (cf. [S, (6.4)]):

$$\mathrm{Gr}_k^W E \simeq \bigoplus_{j \geqq 0} N^j P_{k+2j}.$$

We also consider another filtration $\{Z_k\}$: $Z_k := \mathrm{Im}\, N + W_{k-1}$ $(k \geqq 0)$. Note that $Z_0 = \mathrm{Im}\, N$ and $\mathrm{Gr}_k^Z E \simeq P_{k-1}$ for $k \geqq 1$.

(1.3) Now we construct filtrations of the canonical prolongation $\bar{\mathscr{V}}$ of $\mathscr{V} = \mathcal{O}_U \otimes V$. Recall that we are in the setting (0.1).

We localize the situation (cf. (0.5)). Fix a point $u \in U$ and denote by V_u the stalk of V at u. By assumption, the local monodromy $T \in \mathrm{GL}\,(V_u)$ is unipotent and we can consider $N = \log T$, which is nilpotent.

By (1.2), we get filtrations $W_.$ and $Z_.$ on V_u. These are N-invariant subspaces of V_u and hence π_1-modules. Then these π_1-modules define local systems on U and we can consider their canonical prolongation (1.1). By uniqueness, these patch together and we get locally free $\mathcal{O}$-subModules $\bar{\mathscr{W}}_k$, $\bar{\mathscr{Z}}_k$ of $\bar{\mathscr{V}}$ defined in a tubular neighborhood of Y.

We now define local systems on Y:

Definition-Proposition (1.4). 1) *$\bar{\mathscr{V}}|_Y$ has a natural integrable connection ∇_Y and $\bar{\mathscr{V}}|_Y$ (resp. $\bar{\mathscr{W}}_k|_Y$, $\bar{\mathscr{Z}}_k|_Y$) is obtained from a unique local system V_Y (resp. $W_{Y,k}$, $Z_{Y,k}$) on Y.*

2) *$N \in \mathrm{End}\,(\bar{\mathscr{V}}|_Y)$ is induced by an element (also denoted by N) of $\mathrm{End}\,(V_Y)$.*

Proof. 1) We prove this only for $\bar{\mathscr{V}}|_Y$, since the same argument works for others. Recall the fact that the logarithmic connection $\bar{\nabla}$ induces an integrable connection on $\bar{\mathscr{V}}|_Y$, denoted by ∇_Y, as follows (cf. [D1, II (3.9)]): let $(z_1, \cdots, z_n)$ be a local coordinate of X such that $Y=\{z_1=0\}$ (cf. (0.5)). The covariant derivative $(\nabla_Y)_{\partial/\partial z_i}$ $(i=2, \cdots, n)$ is defined to be $(\nabla_{\partial/\partial z_i})|_Y$. These patch together to ∇_Y.

Once we have an integrable connection ∇_Y on $\bar{\mathscr{V}}|_Y$, the corresponding local system is given as the sheaf of horizontal sections of $\bar{\mathscr{V}}|_Y$, with respect to ∇_Y.

2) The residue $\mathrm{Res}_Y \nabla$ acts on $\bar{\mathscr{V}}|_Y$, which is horizontal with respect to ∇_Y (cf. [D1, II (3.10)]). We also have the relation $N=-2\pi i \cdot \mathrm{Res}_Y \nabla$ (cf. loc. cit. (3.11)). q.e.d.

The local systems $W_{Y,k}$, $Z_{Y,k}$ and the endomorphism N possess the same properties as in (1.2). As to the polarization, we can show:

Lemma (1.5). *The polarization S: $V\otimes V\to \boldsymbol{C}_U$ gives rise to a bilinear pairing S_Y: $V_Y\otimes V_Y\to \boldsymbol{C}_Y$ and non-degenerate bilinear forms $S_{Y,k}$ on* $\mathrm{Gr}_k^W V_Y$.

Proof. By the functoriality of the canonical prolongation, we obtain $\bar{S}$: $\bar{\mathscr{V}}\otimes\bar{\mathscr{V}}\to\mathscr{O}_X$ and its restriction S_Y: $\bar{\mathscr{V}}|_Y\otimes\bar{\mathscr{V}}|_Y\to\mathscr{O}_Y$. Regarding S as an element of $\mathscr{H}om_{\boldsymbol{C}}(V\otimes V, \boldsymbol{C}_U)$, it is horizontal with respect to the natural connection on $\mathscr{H}om(\mathscr{V}\otimes\mathscr{V}, \mathscr{O}_U)$. Then S_Y is horizontal with respect to the natural connection on $\mathscr{H}om(\bar{\mathscr{V}}|_Y\otimes\bar{\mathscr{V}}|_Y, \mathscr{O}_Y)$ induced from ∇_Y.

For $k\geqq 0$, letting u, v to be sections of $\mathrm{Gr}_k^W V_Y$, $\tilde{u}$, $\tilde{v}$ their representatives in V_Y, we put $S_{Y,k}(u, v):=S_Y(\tilde{u}, N^k\tilde{v})$ and, on $\mathrm{Gr}_{-k}^W V_Y$, we define $S_{Y,-k}$ so that N^k: $\mathrm{Gr}_k^W V_Y\simeq\mathrm{Gr}_{-k}^W V_Y$ is an isometry. Then the second assertion follows from [S, (6.4)]. q.e.d.

(1.6) Note that, in the local situation (0.5), $V_{Y,y}$ is identified with V_u by the map $V_u\to V_{Y,y}$ which sends v to the germ at $y=(0, y')$ of the section $\tilde{v}:=\exp(-(\log z)N/2\pi i)v$. Next we put:

$$\bar{\mathscr{F}}^p:=j_*\mathscr{F}^p\cap\bar{\mathscr{V}} \quad (\text{in } j_*\mathscr{V}), \quad \mathscr{F}_Y^p:=\bar{\mathscr{F}}^p|_Y.$$

By the Nilpotent Orbit Theorem [S, (4.12)], these are locally free.

From the preceding constructions, we obtain the data $(V_Y, W_{Y,\cdot}, \mathscr{F}_Y^\cdot)$ and $(V_Y, Z_{Y,\cdot}, \mathscr{F}_Y^\cdot)$. The following theorem may be regarded as a geometric aspect of Schmid's theory [S, (6.16)].

Theorem (1.7). *$(V_Y, \{W_{Y,k}\}, \{\mathscr{F}_Y^p\})$ is a gradedly polarized variation of mixed Hodge structure* (0.3).

Proof. First, we prove that the triple defines a MHS at each point $y\in Y$. The question being local, we may localize the situation (0.5). Thus we are reduced to the case $\dim X=1$. Then, under the identification V_u

$\simeq V_{Y,y}$ (1.6), we may identify $\mathscr{F}^p_Y(y)$ with the limit Hodge filtration $F^p_\infty := \lim_{z\to 0}\exp(-(\log z)N/2\pi i)F^p_z$. Then, Schmid's theorem [S, (6.16)] shows that $(V_{Y,y}, \{W_{Y,y}\}, \{\mathscr{F}^p_Y(y)\})$ is a MHS and the primitive part P_k of $\mathrm{Gr}^W_k V_{Y,y}$ is a HS of weight $m+k$, polarized by $(S_{Y,k})_y$ (cf. (1.5)). Therefore $\mathrm{Gr}^W_k V_{Y,y}$ itself is polarizable.

It remains to show the horizontality for V_Y, which follows from the relation $\bar{\nabla}\mathscr{F}^p \subset \Omega^1_X(\log Y)\otimes\mathscr{F}^{p-1}$. q.e.d.

Remark (1.8). 1) The identification of $\mathscr{F}^p_Y(y)$ with F^p_∞ depends on the choice of a local parameter, but it changes only by a factor $\exp(\alpha N)$ $(\alpha \in \boldsymbol{C})$, which does not affect the fact that $\mathscr{F}^p_Y(y)$ defines a MHS.

2) In the course of the proof of (1.7), it is also proved that P_k is a VHS of weight $m+k$, polarized by $S_{Y,y}$.

§ 2. Zucker's theorem —a slight generalization—

In this paragraph, we remark that Zucker's theorem [Z1, (7.12)] can be generalized to the setting (0.1) (Theorem 1 in Introduction). We briefly recall the steps of [Z1] and indicate the necessary modification.

Step 1. *The norm estimate*

The basic ingredient to compute the L^2-complex (cf. Step 2) is the following estimate of the Hodge norm. We assume the local setting (0.5).

Let us take an element $v \in V_u$ and denote by $\tilde{v}$ the corresponding section of $\mathscr{V}$ (1.6). The Hodge norm of $\tilde{v}$ at $z \in U$ is by definition $\|\tilde{v}\|^2_z := S(C\tilde{v}(z), \overline{\tilde{v}(z)})$, where S is the polarization of V and C is the Weil operator (0.2). Then the following proposition is due to Schmid [S, (6.6)] when $\dim X=1$ (cf. [Z1, (3.6), (3.7)) and a careful look at its proof shows the continuous dependence of the norm $\|\ \|_z$ on $(z_1, \cdots, z_n)$ so that we can reduce it to the case $\dim X=1$.

Proposition (2.2). *The following conditions are equivalent.*

(i) $v \in W_k - W_{k-1}$ *considered as a section of* V_Y.

(ii) $\|\tilde{v}\|^2_z \sim (-\log|z_1|)^k$ *as* $|z_1|\to 0$. *Here* $\sim$ *means that they have the same growth.*

We can also refer to the thorough generalization by Kashiwara [K4, (3.4.2)] and Cattani-Kaplan-Schmid [CKS1].

Step 2. *The* Poincaré *and* Dolbeault *lemmas for* L^2*-complexes*

There is a Kähler metric on U, which is quasi-isometric to the metric

$$\frac{i}{2}\left\{\frac{dz\wedge d\bar{z}}{|z_1|^2(\log|z_1|^2)}+\sum_{j=2}^{n} dz_j\wedge d\bar{z}_j\right\}$$

on $D^* \times D^{n-1}$ (cf. (0.5)) locally around Y. Then U is a complete manifold of finite volume with respect to this metric ([Z1, (3.2), (3.4)]).

Zucker introduced L^2-complexes with respect to the above metric to relate the cohomology of $j_* V$ to harmonic analysis (Step 3).

Definition (2.2). (i) Consider the following sheaves on a tubular neighborhood of Y $(i \geqq 1)$:

$$\Omega^i_{X,0} := \{\omega \in \Omega^i_X;\ \text{near } Y=\{z_1=0\},\ \omega \text{ does not contain the factor } dz_1\}$$
$$\Omega^i_{X,1} := \{\omega \in \Omega^i_X;\ \text{near } Y=\{z_1=0\},\ dz_1 \wedge \omega = 0\} = dz_1 \wedge \Omega^{i-1}_X$$
$$\Omega^i_{X,1}(\log Y) := dz_1/z_1 \wedge \Omega^{i-1}_X \text{ near } Y=\{z_1=0\}$$

(ii) Define a subcomplex M_C of $\Omega^{\cdot}_X(\log Y) \otimes \mathscr{V}$ $(i \geqq 1)$:

$$M^0_C := \mathscr{W}_0 + \mathscr{I}_Y \mathscr{V}$$
$$M^i_C := \Omega^i_{X,0} \otimes \{\mathscr{W}_0 + \mathscr{I}_Y \mathscr{V}\} + \Omega^i_{X,1}(\log Y) \otimes \{\mathscr{W}_{-2} + \mathscr{I}_Y \mathscr{V}\}$$

Here $\mathscr{W}_0$, $\mathscr{W}_{-2}$ are defined in (1.3) and $\mathscr{I}_Y$ denotes the ideal sheaf of Y.

(iii) Define $\Omega(V)^{\cdot}_{(2)}$ to be the subcomplex of $j_*(\Omega^{\cdot} \otimes V)$, which consists of forms, L^2 with respect to the Kähler metric of U and the Hodge norm of V. In the same way, we define $(\Omega^{\cdot} \otimes \mathscr{F}^p)_{(2)}$ (resp. $(\Omega^{\cdot} \otimes \mathscr{G}r^p)_{(2)}$) in $j_*(\Omega^{\cdot}_U \otimes \mathscr{F}^p)$ (resp. $j_*(\Omega^{\cdot}_U \otimes \mathscr{G}r^p)$), where $\mathscr{G}r^p := \mathscr{F}^p / \mathscr{F}^{p+1}$.

$F^p \Omega(V)^{\cdot}_{(2)} := (\Omega^i \otimes \mathscr{F}^{p-i})_{(2)}$ $(i \geqq 0)$, defines a filtration of $\Omega(V)^{\cdot}_{(2)}$.

(iv) Set

$\mathscr{L}(V)^i_{(2)} := \{V_C\text{-valued } L^2\text{-form } \omega \text{ of degree } i, \text{ for which } d\omega \text{ is also an } L^2\text{-form}\}$

$\mathscr{L}(\mathscr{G}r^p)^{r,s}_{(2)} := \{\mathscr{G}r^p\text{-valued } L^2\text{-form } \omega \text{ of type } (r,s), \text{ for which } \bar{\partial}\omega \text{ is also an } L^2\text{-form}\}$.

The differentials d, $\bar{\partial}$ are taken in the weak sense. $(\mathrm{Gr}^p_F \mathscr{L}(V))^{\cdot}_{(2)}$ is by definition the simple complex associated to $(\mathscr{L}(\mathscr{G}r^{p-r})^{r,s}_{(2)})$.

For these L^2-complexes, we can show the following (2.3). The question being local, we may reduce ourselves to the one-dimensional case (0.5) treated by Zucker by means of local Künneth type formulas such as

$$\begin{aligned} & M^{\cdot}_C(V) \simeq \underline{\underline{s}}[p^{-1} M^{\cdot}_C(V') \otimes q^{-1} M^{\cdot}_C(C_{D^{n-1}})] \\ (2.3) \qquad & \mathscr{L}(V)^{\cdot}_{(2)} \simeq \underline{\underline{s}}[p^{-1} \mathscr{L}(V')^{\cdot}_{(2)} \otimes q^{-1} \mathscr{L}(C_{D^{n-1}})^{\cdot}_{(2)}] \\ & (\mathrm{Gr}^p_F \mathscr{L}(V))^{\cdot}_{(2)} \simeq \bigoplus_r p^{-1} (\mathrm{Gr}^{p-r}_F \mathscr{L}(V'))^{\cdot}_{(2)} \hat{\otimes} q^{-1} \mathscr{L}(C_{D^{n-1}})^{r,\cdot}_{(2)} \end{aligned}$$

where $\hat{\otimes}$ denotes the completed tensor product. Similar formulas hold for $\Omega(V)^{\cdot}_{(2)}$, $\mathrm{Gr}^p_F \Omega(V)^{\cdot}_{(2)}$, etc. See [Z1, § 2].

Proposition (2.4). 1) *$\Omega(V)^{\cdot}_{(2)}$ coincides with $M^{\cdot}_{C}$ and gives a resolution of $j_* V_C$.* (cf. [Z1, (4.1), (4.4)])

2) *$(\Omega^i \otimes \mathscr{F}^p)_{(2)}$ and $(\Omega^i \otimes \mathscr{G}_i{}^p)_{(2)}$ are locally free $\mathscr{O}_X$-Modules.* (cf. [Z1, (5.2)])

3) *The following are quasi-isomorphisms*:

$$j_* V_C \longrightarrow \mathscr{L}(V)^{\cdot}_{(2)}, \qquad \mathrm{Gr}^p_F \Omega(V)^{\cdot}_{(2)} \longrightarrow (\mathrm{Gr}^p_F \mathscr{L}(V))^{\cdot}_{(2)}.$$

Note that the norm estimate (2.1) is indispensable for (2.4).

Step 3. *Harmonic analysis*

In order to get the Hodge decomposition by harmonic forms, we use harmonic analysis for L^2-complexes, on a non-compact manifold with complete metric. We do not state it here, since it is described in [Z1, § 7] and no change of the argument is necessary except for the reference to (2.3).

Denote by $\mathrm{Har}^i(V)$ (resp. $\mathrm{Har}^{r,s}(V)$) the space of harmonic V_C-valued i-forms (resp. (r, s)-forms). Then we have

$$\begin{aligned} &H^i(X, j_* V_C) \simeq H^i(X, M^{\cdot}_C) \\ &\qquad \simeq H^i(\Gamma(X, \mathscr{L}(V)^{\cdot}_{(2)})) \simeq \mathrm{Har}^i(V) \simeq \bigoplus_{r+s=m+i} \mathrm{Har}^{r,s}(V) \\ &\mathrm{Gr}^p_F H^i(X, M^{\cdot}_C) \simeq \mathrm{Har}^{p,q}(V) \qquad (p+q=m+i), \end{aligned}$$

so that the Hodge spectral sequence $E_1^{p,q} = H^{p+q}(X, \mathrm{Gr}^p_F M^{\cdot}_C) \Rightarrow H^{p+q}(X, M^{\cdot}_C)$ degenerates at E_1-terms. Note that in the second row the Kähler identities are necessary (cf. [Z1, (2.7), (7.5)]).

We have thus shown the following theorem which implies Theorem 1 in Introduction.

Theorem (2.5). *$(j_* V, (M^{\cdot}_C, F))$ is a cohomological Hodge complex of weight m* (0.4).

§ 3. Mixed Hodge structures on cohomologies

We prove Theorem 2 in Introduction in (3.1), (3.2) and supplement it by basic properties in (3.3). We always assume to be in the setting (0.5).

(3.1) Mixed Hodge structure on the ordinary cohomology

We will construct a cohomological mixed Hodge complex (0.4) which enable us to compute $H^i(U, V)$, as outlined in Introduction.

Note the relation $H^i(U, V) \simeq H^i(X, Rj_* V)$ and the following quasi-isomorphisms:

$$Rj_* V_C \longrightarrow j_*(\Omega^{\cdot}_U \otimes V_C) \longleftarrow \Omega^{\cdot}_X(\log Y) \otimes \bar{\mathscr{V}}. \tag{3.1.1}$$

The left morphism is quasi-isomorphic because j is a Stein morphism, and the quasi-isomorphy of the right one is due to Deligne [D1, II (3.13), (6.9)]. Then we put

$$K_{\boldsymbol{Q}} := Rj_* V_{\boldsymbol{Q}}, \qquad K_{\boldsymbol{C}} := \Omega^{\cdot}_X(\log Y) \otimes \bar{\mathscr{V}}. \tag{3.1.2}$$

On these complexes, we define filtrations W, F using the construction in Section 1.

Definition (3.1.3) (cf. [Z1, § 13]).

(i) $$W_k K^{\cdot}_{\boldsymbol{C}} := (Rj_* V_{\boldsymbol{Q}})^0 \longrightarrow (\operatorname{Im} d)^1 + (\operatorname{Ker} d)^1 \cap (Rj_* Z_{k,\boldsymbol{Q}})^1$$

for $k \geqq 0$ while $W_k K^{\cdot}_{\boldsymbol{Q}} := 0$ for $k < 0$. Here $Z_{k,\boldsymbol{Q}}$ is a local subsystem of $V_{\boldsymbol{Q}}$ in a tubular neighborhood of Y (cf. (1.3)). Note that $W_0 K^{\cdot}_{\boldsymbol{Q}} = \tau_{\leqq 0} Rj_* V_{\boldsymbol{Q}} = j_* V_{\boldsymbol{Q}}$, since $(Rj_* Z_{0,\boldsymbol{Q}})^1 \subset (\operatorname{Im} d)^1$ and that $W_k K^{\cdot}_{\boldsymbol{Q}}$ are subcomplexes of $\tau_{\leqq 1} Rj_* V_{\boldsymbol{Q}} = Rj_* V_{\boldsymbol{Q}}$.

(ii) $$W_k K^{\cdot}_{\boldsymbol{C}} := \Omega^{\cdot}_{X,0} \otimes \bar{\mathscr{V}} + \Omega^{\cdot}_{X,1}(\log Y) \otimes [\bar{\mathscr{Z}}_k + \mathscr{I}_Y \bar{\mathscr{V}}]$$

for $k \geqq 0$ while $W_k K^{\cdot}_{\boldsymbol{C}} := 0$ for $k < 0$. For notation, refer to (1.4) for $\bar{\mathscr{Z}}_k$, to (2.2) (i) for $\Omega^{\cdot}_{X,0}$, $\Omega^{\cdot}_{X,1}(\log Y)$. Here $\mathscr{I}_Y$ is the ideal sheaf of Y. The above expression makes sense only in a tubular neighborhood of Y. Since $W_k K_{\boldsymbol{C}} = \Omega^{\cdot}_U \otimes V_{\boldsymbol{C}}$ outside Y $(k \geqq 0)$, we understand that $W_k K_{\boldsymbol{C}}$ means $\Omega^{\cdot}_U \otimes V_{\boldsymbol{C}}$ (globally) on U.

(iii) $$F^p K^i_{\boldsymbol{C}} := \Omega^i_X(\log Y) \otimes \bar{\mathscr{F}}^{p-i}.$$

Here $\bar{\mathscr{F}}^q$ is as defined in (1.6). We also consider the induced filtration F on $W_k K^{\cdot}_{\boldsymbol{C}}$.

Gr^W_k of $K_{\boldsymbol{Q}}$ and $K_{\boldsymbol{C}}$ are calculated as follows:

Proposition (3.1.4).

$$\mathrm{Gr}^W_k K^{\cdot}_{\boldsymbol{Q}} \simeq \begin{cases} j_* V_{\boldsymbol{Q}} & \text{for } k=0 \\ P_{k-1,\boldsymbol{Q}}(-1)[-1] & \text{for } k>0. \end{cases}$$

Here P_{k-1} is the primitive part as in (1.2). $[-1]$ *is the usual degree shift while* (-1) *is the Tate twist.*

Proof. The equality $\mathrm{Gr}^W_0 K^{\cdot}_{\boldsymbol{Q}} = j_* V$ is trivial. For $k > 0$ note first that $\mathrm{Gr}^W_k K^{\cdot}_{\boldsymbol{Q}} \simeq R^1 j_*(Z_k/Z_{k-1})[-1]$. Under the isomorphism

$$R^1 j_* V_{\boldsymbol{Q}} \simeq V_Y/NV_Y(-1) \simeq V_Y/Z_{Y,0}(-1),$$

the subquotients $R^1 j_*(\mathrm{Gr}^Z_k V)$ and $\mathrm{Gr}^Z_k V_Y(-1)$ correspond. Since $\mathrm{Gr}^Z_k V_Y \simeq P_{k-1}$ by (1.2), we obtain the desired isomorphism. q.e.d.

Proposition (3.1.5). 1) *The inclusion $M_C^{\cdot}\to W_0K_C^{\cdot}$ is an F-filtered quasi-isomorphism.*

2) *The Poincaré residue gives an F-filtered quasi-isomorphism* $\mathrm{Gr}_k^W K_C^{\cdot} \simeq \Omega_Y^{\cdot}\otimes P_{k-1}(-1)[-1]$ *for* $k>0$.

Proof. 1) Since $M_C^{\cdot}=W_0K_C^{\cdot}$ $(=\Omega_U^{\cdot}\otimes V_C)$ on U, we may consider the assertion only locally around Y. So we may reduce it to the one-dimensional case [Z1, (9.1)] by the Künneth type formula. See (2.3) for $M_C^{\cdot}$. Similar formula holds for $W_0K_C^{\cdot}$.

2) By the Poincaré residue, we have

$$\begin{aligned}\mathrm{Gr}_k^W K_C^{\cdot} &\xrightarrow{\mathrm{Res}} \Omega_Y^{\cdot-1}\otimes(\mathscr{Z}_k/\mathscr{Z}_{k-1})|_Y(-1)\\ &\simeq \Omega_Y^{\cdot}\otimes \mathrm{Gr}_k^Z V_Y(-1)[-1]\simeq \Omega_Y^{\cdot}\otimes P_{k-1}(-1)[-1].\end{aligned}$$

We used (1.2) in the last equality. q.e.d.

To relate the filtration W on K_Q and K_C, we define a filtration W on $\tilde{K}_C^{\cdot}:=j_*(\Omega_U^{\cdot}\otimes V_C)$ as

$$W_k\tilde{K}_C^{\cdot}:=d[j_*(\Omega_{U,0}^{\cdot-1}\otimes V)]+j_*[\Omega_{U,0}^{\cdot}\otimes V+\Omega_{U,1}^{\cdot}\otimes Z_k] \qquad \text{for } k\geqq 0$$

while $W_k\tilde{K}_C^{\cdot}=0$ for $k<0$. Then we have the relation $\mathrm{Gr}_k^W\tilde{K}_C^{\cdot}=j_*V_C$ for $k=0$ and $\Omega_Y^{\cdot}\otimes P_{k-1}(-1)[-1]$ for $k>0$, similar to (3.1.5). Thus we get:

Lemma (3.1.6). *The morphisms in* (3.1.2) *are W-filtered quasi-isomorphisms.*

Consider the data $K=((K_Q, W), (K_C, W, F))$. Since P_{k-1} is a polarized VHS of weight $m+k-1$ by (1.7), $\mathrm{Gr}_k^W K$ is a cohomological Hodge complex of weight $m+k$ $(k\geqq 1)$ by a theorem of Deligne in [Z1, (2.9)]. Meanwhile, $\mathrm{Gr}_0^W K\simeq(j_*V_Q, (M_C^{\cdot}, F))$ in (3.1.4), (3.1.5) is a cohomological Hodge complex of weight m by (2.5). Therefore we conclude:

Theorem (3.1.7). *$((K_Q, W), (K_C, W, F))$ is a cohomological mixed Hodge complex of weight $\geqq m$.*

This implies the $H^i(U, V)$ part of Theorem 2 in Introduction.

(3.2) Mixed Hodge structure on the cohomology with compact supports

We will construct a cohomological mixed Hodge complex which enables us to compute $H_c^i(U, V)$.

Note the relation $H_c^i(U, V)\simeq H^i(X, j_!V)$ and consider the short exact sequence

$$(3.2.1) \qquad 0 \longrightarrow j_! V \longrightarrow j_* V \longrightarrow i_* i^* j_* V \longrightarrow 0.$$

$j_* V_C$ has a resolution $W_0 K_C^{\cdot}$ (3.1.5). $j_! V_C$ has the following subcomplex $K_{!,C}^{\cdot}$ of $W_0 K_C^{\cdot}$ as a resolution:

$$K_{!,C}^i := \mathscr{I}_Y \bar{\mathscr{V}} \text{ for } i=0,\ \Omega_{X,0}^i \otimes \mathscr{I}_Y \bar{\mathscr{V}} + \Omega_{X,1}^i \otimes \bar{\mathscr{V}} \text{ for } i>0.$$

Here $\Omega_{X,0}^i$, $\Omega_{X,1}^i$ are as introduced in (2.2) (i). Then, to $j_! V \to j_* V$ corresponds the inclusion $K_{!,C}^{\cdot} \to W_0 K_C^{\cdot}$ and to $i_* i^* j_* V$ the quotient complex $W_0 K_C^{\cdot}/K_{!,C}^{\cdot}$.

Then we put (cf. [Z1, § 14]):

$$(3.2.2) \qquad \begin{aligned} L_Q &:= [j_* V_Q \to i_* i^* j_* V] \\ L_C &:= \underline{\underline{s}}[W_0 K_C^{\cdot} \to W_0 K_C^{\cdot}/K_{!,C}^{\cdot}]. \end{aligned}$$

where $\underline{\underline{s}}$ denotes the associated simple complex. We will define filtrations W, F on L_Q, L_C instead of $K_{!,Q} := j_! V$, $K_{!,C}$. The reason for using L instead of $K_!$ will become clear when we consider the corresponding $\mathscr{D}_X$-Modules (4.2).

Note the following:

Lemma (3.2.3). 1) $W_0 K_C/K_{!,C} = \Omega_Y^{\cdot} \otimes \bar{\mathscr{V}}|_Y \oplus \Omega_Y^{\cdot -1} \otimes \bar{\mathscr{Z}}_0|_Y$.

2) *$W_0 K_C/K_{!,C}$ is quasi-isomorphic to the mapping cone of $N\colon \Omega_Y^{\cdot} \otimes \bar{\mathscr{V}}|_Y \to \Omega_Y^{\cdot} \otimes \bar{\mathscr{Z}}_0|_Y$ induced from $N\colon V_Y \to Z_{Y,0}$.*

Proof. 1) is straightforward by means of the Poincaré residue. As for 2), considering 1), we have only to note that the differential of the complex $\Omega_Y^{\cdot} \otimes \bar{\mathscr{V}}|_Y \oplus \Omega_Y^{\cdot -1} \otimes \bar{\mathscr{Z}}_0|_Y$ is given by

$$\begin{pmatrix} d^i & (-1)^i 1 \otimes N \\ 0 & d^{i-1} \end{pmatrix} \qquad \text{q.e.d.}$$

Now we can define filtrations W, F on L using (3.2.3).

Definition (3.2.4) (cf. [Z1, § 14]).

$$\text{(i)} \qquad W_k L_Q^{\cdot} := \begin{cases} \operatorname{Ker}(N|\, W_{Y,k+1})[-1] & \text{for } k<0 \\ L_Q & \text{for } k=0 \end{cases}$$

$$\text{(ii)} \qquad W_k L_C^{\cdot} := \begin{cases} \underline{\underline{s}}[\Omega_Y^{\cdot} \otimes \bar{\mathscr{W}}_{k+1}|_Y \xrightarrow{N} \Omega_Y^{\cdot} \otimes \bar{\mathscr{W}}_{k-1}|_Y][-1] & \text{for } k<0 \\ L_C & \text{for } k=0 \end{cases}$$

(iii) We consider the filtration F on the complexes $K_{!,C}$, $W_0 K_C$, L_C and $W_k L_C$ induced from F on K_C (3.1.3) (iii).

We calculate Gr_k^W of L as follows:

Proposition (3.2.5). 1) $\mathrm{Gr}_k^W L_C^{\cdot}=P_{k+1,Q}[-1]$ *for* $k<0$, $j_* V_Q$ *for* $k=0$.
2) $\mathrm{Gr}_k^W L_C^{\cdot}=\Omega_Y^{\cdot}\otimes P_{k+1,C}[-1]$ *for* $k<0$, $W_0 K_C$ *for* $k=0$
3) $\mathrm{Gr}_k^W L_Q \otimes_Q C \simeq \mathrm{Gr}_k^W L_C$.

Proof. 3) follows from 1), 2) and (3.1.5). For $k<0$,

$$\begin{aligned}\mathrm{Gr}_k^W L_Q &\simeq \mathrm{Gr}_{k+1}^W(\mathrm{Ker}\,(N\,|\,V_{Y,Q}))[-1]\\ &=\mathrm{Ker}\,(N\,|\,\mathrm{Gr}_{k+1}^W V_{Y,Q})[-1]=P_{k+1,Q}[-1]\\ \mathrm{Gr}_k^W L_C &\simeq \underline{s}[\Omega_Y^{\cdot}\otimes \mathrm{Gr}_{k+1}^W \mathscr{V}|_Y \xrightarrow{N} \Omega_Y^{\cdot}\otimes \mathrm{Gr}_{k-1}^W \mathscr{V}|_Y][-1]\\ &=\underline{s}[\Omega_Y^{\cdot}\otimes \mathrm{Gr}_{k+1}^W V_Y \xrightarrow{N} \Omega_Y^{\cdot}\otimes \mathrm{Gr}_{k-1}^W V_Y][-1]\\ &=\Omega_Y^{\cdot}\otimes \mathrm{Ker}\,(N\,|\,\mathrm{Gr}_{k+1}^W V_Y)[-1]=\Omega_Y^{\cdot}\otimes P_{k+1,C}[-1].\end{aligned}$$

For $k=0$, we have $\mathrm{Gr}_0^W L_Q=j_* V_Q$, since $\mathrm{Ker}\,(N\,|\,(V_Y/W_{Y,0}))=0$. Similarly for $\mathrm{Gr}_0^W L_C$. q.e.d.

Thus $\mathrm{Gr}_k^W L$ is a cohomological Hodge complex of weight $m+k$ by (2.5) for $k=0$ and by [Z1, (2.9)] for $k<0$. Therefore we conclude:

Theorem (3.2.6). *$((L_Q, W), (L_C, W, F))$ is a cohomological mixed Hodge complex of weight $\leqq m$.*

This implies the $H_c^i(U, V)$ part of Theorem 2 in Introduction.

(3.3) Functorial properties

We can prove several functorial properties for the mixed Hodge structures constructed in (3.1), (3.2). We state them except the duality without proofs, because they were proved in [Z1].

(3.3.1) Functoriality: The MHS's on $H^i(X, j_* V)$, $H^i(U, V)$, and $H_c^i(U, V)$ are functorial in (U, V) in an obvious way. (cf. [Z1, § 8])

(3.3.2) Long exact sequence: We have the following exact sequences of MHS (cf. [Z1, (14.3), (14.5)]):

(i) $\cdots \longrightarrow H^i(X, j_* V) \longrightarrow H^i(U, V) \longrightarrow H^{i-1}(Y, \mathrm{Coker}\, N) \xrightarrow{\partial} \cdots$

(ii) $\cdots \longrightarrow H^{i-1}(Y, \mathrm{Ker}\, N) \xrightarrow{\partial} H_c^i(U, V) \longrightarrow H^i(X, j_* V) \longrightarrow \cdots$

Here N is the endomorphism of V_Y (cf. (1.4), 2)). The sequence (i) comes from the Leray spectral sequence for $j: U \longrightarrow X$, and (ii) comes from the exact sequence (3.2.1) and we used the relations $R^1 j_* V \simeq \mathrm{Coker}\, N$ and $i^* j_* V \simeq \mathrm{Ker}\, N$.

(3.3.3) Leray spectral sequence: Let $\bar{f}: \bar{X} \longrightarrow \bar{S}$ be a projective morphism between compact Kähler manifolds $\bar{X}$ and $\bar{S}$, Σ a smooth hypersurface of $\bar{S}$. We put $Y=\bar{f}^{-1}(\Sigma)$, $S=\bar{S}-\Sigma$, and $X=\bar{X}-Y$. We assume that $f=\bar{f}|_X$ is smooth and Y is a divisor with normal crossings on $\bar{X}$. Then the Leray spectral sequence for f:

$$_LE_2^{p,q}=H^p(S, R^qf_*\boldsymbol{Q})\Longrightarrow H^{p+q}(X, \boldsymbol{Q})$$

is a spectral sequence in the category of MHS (cf. [Z1, § 15]).

(3.3.4) Duality: The duality between $H^{\cdot}(U, V)$ and $H_c^{\cdot}(U, V)$, which is induced via cup-product from the polarization S, is a duality of MHS. More precisely, the cup-product pairing

$$S\colon H_c^i(U, V_{\boldsymbol{Q}})\otimes H^{2n-i}(U, V_{\boldsymbol{Q}})\longrightarrow H_c^{2n}(U, \boldsymbol{Q}(-m))\simeq \boldsymbol{Q}(-n-m)$$

($n=\dim U$, $m=$the weight of V), satisfies the following conditions:

1) $S(W_kH_c^i, W_\ell H^{2n-i})=0$ for $k+\ell<2n+2m$, and
2) the induced pairings are the duality pairings of HS:

$$S_k\colon \mathrm{Gr}_k^W H_c^i\otimes \mathrm{Gr}_\ell^W H^{2n-i}\longrightarrow \boldsymbol{Q}(-n-m) \qquad (k+\ell=2n+2m),$$

i.e., $S_k(F^p\mathrm{Gr}_k^W H_c^i, F^q\mathrm{Gr}_\ell^W H^{2n-i})=0$ for $p+q>n+m$, $k+\ell=2n+2m$, and S_k gives a perfect pairing between $F^p\mathrm{Gr}_k^W H_c^i$ and $F^q\mathrm{Gr}_\ell^W H^{2n-i}$ for $p+q=n+m$, $k+\ell=2n+2m$.

There is a proof of the duality in Fujiki [F] when $V=\boldsymbol{Q}_U$ and also in Steenbrink-Zucker [SZ, (4.30)] when $\dim U=1$ for V a "good" VMHS.

The above pairing S is given by the pairings of complexes

$$K_{!,\boldsymbol{Q}}\otimes K_{\boldsymbol{Q}}=j_!V_{\boldsymbol{Q}}\otimes Rj_*V\longrightarrow j_!\boldsymbol{Q}_U(-m)$$
$$K_{!,\boldsymbol{C}}\otimes K_{\boldsymbol{C}}=K_{!,\boldsymbol{C}}\otimes \Omega_X^{\cdot}(\log Y)\otimes\bar{\mathcal{V}}\longrightarrow \Omega_X^n(\log Y)[-n].$$

Note that $H_c^{2n}(U, \boldsymbol{Q})\simeq H^{2n}(X, \boldsymbol{Q})\simeq \boldsymbol{Q}(-n)$ and $H^n(X, \Omega_X^n)\simeq \boldsymbol{C}$.

Replace $K_!$ by L and let us see first that $W_{-1}L_{\boldsymbol{Q}}\otimes W_0K_{\boldsymbol{Q}}$ goes to zero. By (3.1.3, i) and (3.2.4, i), it is equal to $\mathrm{Ker}\,(N|\,W_{Y,0})[-1]\otimes j_*V_{\boldsymbol{Q}}$. This goes to zero, because, over U, $\mathrm{Ker}\,N$ is zero, and, over Y, the sections of $j_*V_{\boldsymbol{Q}}$ are N-invariant while $N(\mathrm{Ker}\,N)=0$.

Therefore we may divide the above pairing into the following two parts:

3) $\qquad \mathrm{Gr}_0^W L_{\boldsymbol{Q}}\otimes W_0K_{\boldsymbol{Q}}\longrightarrow \boldsymbol{Q}_X(-m)$

4) $\qquad W_{-1}L_{\boldsymbol{Q}}\otimes K_{\boldsymbol{Q}}/W_0K_{\boldsymbol{Q}}\longrightarrow \boldsymbol{Q}_Y(-m-1)[-2].$

Replacing $K_{\boldsymbol{C}}$ by the mapping cone $[K_{\boldsymbol{C}}/W_0K_{\boldsymbol{C}}\to W_0K_{\boldsymbol{C}}[1]]$ (cf. (4.2.2)) and using a similar argument, we get

3′) $\mathrm{Gr}_0^W L_C W_0 K_C \longrightarrow \Omega_X^n[-n]$

4′) $W_{-1}L_C \otimes K_C/W_0K_C \longrightarrow \Omega_Y^{n-1}[-n].$

By (3.1.4), (3.2.5), $\mathrm{Gr}_0^W L_Q \simeq j_* V_Q \simeq W_0 K_Q$ holds, and the pairing 3) is nothing but the self-duality pairing of the intersection complex $j_* V_Q$ (cf. (4.2.1), [GM]).

Since $W_{-1}L_Q = \mathrm{Ker}\, N[-1]$ by (3.2.4) and $K_Q/W_0K_Q = V_Y/\mathrm{Im}\, N[-1] = \mathrm{Coker}\, N[-1]$ by (3.1.5), 4) can be identified with the pairing $\mathrm{Ker}\, N \otimes \mathrm{Coker}\, N \to \boldsymbol{Q}_Y(-m-1)$ induced from S_Y on V_Y. $W_{-1}L$ (resp. K/W_0K) carries the filtration $W.$, which is induced from $W_{Y,.}$ (resp. $Z_{Y,.}$) on V_Y. They satisfy the orthogonality relation $S_Y(W_k, W_\ell)=0$ for $k+\ell<0$. $\mathrm{Gr}_{-k}^W \mathrm{Ker}\, N$ and $\mathrm{Gr}_k^W \mathrm{Coker}\, N$ are dual to each other through S_Y. By the identification N^k: $\mathrm{Gr}_k^Z \mathrm{Coker}\, N \simeq P_{k-1}(-1) \to \mathrm{Gr}_{-k}^W \mathrm{Ker}\, N$ in (1.2) for $k>0$, the pairing $\mathrm{Gr}_{-k}^W \mathrm{Ker}\, N \otimes \mathrm{Gr}_k^Z \mathrm{Coker}\, N \to \boldsymbol{Q}_Y(-m-1)$ is nothing but the self-duality pairing

$$P_{k-1}(k-1) \otimes P_{k-1}(-1) \longrightarrow \boldsymbol{Q}_Y(-m-1).$$

Consider now the weight spectral sequences for K and L, whose E_1-terms are given by the following:

$$\begin{aligned} {}_W E_1^{p,q}(L) &= H^{p+q}(X, \mathrm{Gr}_{-p}^W L) = H^{p+q-1}(Y, P_{p-1})(-p) \text{ for } p>0, \\ {}_W E_1^{-p,r}(K) &= H^{-p+r}(X, \mathrm{Gr}_p^W K) = H^{-p+r-1}(Y, P_{p-1})(-1). \end{aligned}$$

Then the self-duality pairing for $j_* V$ and P_{p-1} induces $S_Y^{(p)}$: ${}_W E_1^{p,q}(L) \otimes {}_W E_1^{-p,r}(K) \to \boldsymbol{Q}_Y(-n-m)$ for $q+r=2n$.

Note that, by [D1, III (8.1.9)], the above spectral sequences degenerate at E_2, i.e., $E_2 = E_\infty$. We must prove that the duality on E_1-terms passes to one on E_2-terms. This can be done by the sublemma below if we show the following relation:

5) $$S_Y^{(p+1)}(\alpha(x), y) = \pm S_Y^{(p)}(x, \beta(y))$$

for $x \in E_1^{p,q}(L)$, $y \in E_1^{-p,r}(K)$, where α: $E_1^{p,q}(L) \to E_1^{p+1,q}(L)$ β: $E_1^{-p-1,r}(K) \to E_1^{-p,r}(K)$ are the differentials of the spectral sequences.

To see this relation, recall that the differentials on E_1-terms are the connecting homomorphisms of the long exact sequence associated to the short exact sequences:

$$\begin{array}{c} 0 \longrightarrow \mathrm{Gr}_{-p-1}^W L \longrightarrow W_{-p}/W_{-p-2}(L) \longrightarrow \mathrm{Gr}_{-p}^W L \longrightarrow 0 \\ 0 \longleftarrow \mathrm{Gr}_{p+1}^W K \longleftarrow W_{p+1}/W_{p-1}(K) \longleftarrow \mathrm{Gr}_p^W K \longleftarrow 0. \end{array}$$

The terms in the same column are to be paired.

Take a representative $\tilde{x}$ of x in $W_{-p}/W_{-p-2}(L)$ (resp. $\tilde{y}$ of y in $W_{p+1}/W_{p-1}(K)$). Then, by definition, $S_Y^{(p)}(x, \beta(y)) = S_Y(\tilde{x}, d(\tilde{y}))$, $S_Y^{(p+1)}(\alpha(x), y) = S_Y(d(\tilde{x}), \tilde{y})$ and the formula 5) follows from the relation $S_Y(\tilde{x}, d(\tilde{y})) = \pm S_Y(d(\tilde{x}), \tilde{y})$.

The following sublemma is easy to check.

Sublemma. *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *(resp.* $A' \xleftarrow{f'} B' \xleftarrow{g'} C'$*) be a complex in an abelian category, i.e.,* $g \cdot f = 0$ *(resp.* $f' \cdot g' = 0$*). Suppose that perfect pairings* $\langle\ ,\ \rangle$ *are given on* $A \times A'$, $B \times B'$ *and* $C \times C'$, *and satisfy the relations* $\langle f(a), b' \rangle = \langle a, f'(b') \rangle$, $\langle g(b), c' \rangle = \langle b, g'(c') \rangle$ *for* $a \in A$, $b' \in B'$ *etc. Then* $\langle\ ,\ \rangle$ *induces a perfect pairing on* $(\operatorname{Ker} g/\operatorname{Im} f) \times (\operatorname{Ker} f'/\operatorname{Im} g')$.

§ 4. Relation with $\mathscr{D}$-Modules

In this paragraph, we will interpret the mixed Hodge structures constructed in Sections 2 and 3 in terms of $\mathscr{D}$-Modules. The general program in this direction was proposed by Brylinski [Br1]. After recalling the relevant part of his conjecture, we verify some of his conjectures in our setting (0.1).

(4.1) Brylinski's conjecture

In Introduction, we recalled the Riemann-Hilbert correspondence $DR_X := R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{O}_X,\ )$. Note that the shift [dim X] is necessary in order that the category of regular holonomic $\mathscr{D}_X$-Modules corresponds to the category of perverse sheaves $\operatorname{Perv}(X, \boldsymbol{C})$ defined in [BBD], which is self-dual under the Verdier duality. (cf. [Br2])

The intersection complex is an example of perverse sheaves; to a closed subvariety Z of X and a local system L defined on an open dense subset $Z^{\cdot}$ of Z, we can associate the intersection complex $\mathrm{IC}^{\cdot}(Z, L)$ on Z in the notation of [GM] (cf. [BBD]). Then $Ri_* IC^{\cdot}(Z, L)[-\dim Z]$ is an object of $\operatorname{Perv}(X, \boldsymbol{C})$, where $i: Z \to X$ is the inclusion.

We denote by $\mathscr{L}(Z, X; L)$ the $\mathscr{D}_X$-Module corresponding to it under $DR_X[\dim X]$. The intersection cohomology $IH^i(Z, L)$ is by definition $H^i(Z, IC^{\cdot}(Z, L)[-2\dim Z]) = H^i(X, DR_X \mathscr{L}(Z, X; L)[\operatorname{codim} Z])$. Note that $IH^i = 0$ for $i \notin [0, 2\dim Z]$.

When L underlies a polarized VHS V of weight m as in (0.3), following Brylinski, we can define a good filtration on $\mathscr{L}(Z, X; V)$, something like a convolution of the Hodge filtration of V and the order filtration [KK1, (5.1.14)] of $\mathscr{L}(Z, X; V)$. This is the purified holonomic $\mathscr{D}_X$-Module of Brylinski [Br1, II (2.1)].

Conjecture 1 ([Br1, I, § 3; II, (2.1)]). *$IH^i(Z, V)$ has a HS of weight $m+i$ and its Hodge filtration is induced from this filtered $\mathscr{D}_X$-Module $\mathscr{L}(Z, X; V)$.*

We next recall another conjecture. For this purpose, following [Br1, II (2.3)], we call a holonomic $\mathscr{D}_X$-Module $\mathscr{M}$ a mixed holonomic $\mathscr{D}_X$-Module if the following conditions hold: Denote $n=\dim X$.

(i) $DR_X(\mathscr{M})[n]$ has a $\boldsymbol{Q}$-structure $D_{\boldsymbol{Q}} \in \mathrm{Perv}(X, \boldsymbol{Q})$, i.e., $D_{\boldsymbol{Q}} \otimes \boldsymbol{C} \simeq DR_X(\mathscr{M})[n]$.

(ii) $D_{\boldsymbol{Q}}$ has a finite increasing filtration $\{W_k D_{\boldsymbol{Q}}\}$ in $\mathrm{Perv}(X, \boldsymbol{Q})$. Let $W_k\mathscr{M}$ be the $\mathscr{D}_X$-subModule of $\mathscr{M}$ such that $DR_X(W_k\mathscr{M})[n]=W_k D_{\boldsymbol{Q}} \otimes \boldsymbol{C}$.

(iii) $\mathscr{M}$ has a good filtration $\{\mathscr{M}_\ell\}$ such that each $\mathrm{Gr}_k^W \mathscr{M}$ with the induced filtration is a direct sum of filtered $\mathscr{D}_X$-Modules of the type $\mathscr{L}(Z, X; V)$.

Let Y be a hypersurface of X, Z a closed subvariety of X, $j\colon U=X-Y \rightarrow X$ the inclusion, and consider a $\mathscr{D}_U$-Module $\mathscr{L}=\mathscr{L}(Z \cap U, U; V)$ for a polarized VHS V. We denote by $j_*^m \mathscr{L}$ (resp. $j_!^m \mathscr{L}$) the $\mathscr{D}_X$-Module such that $DR_X(j_*^m \mathscr{L})=Rj_* DR_U \mathscr{L}$ (resp. $DR_X(j_!^m \mathscr{L})=Rj_! DR_U \mathscr{L}$).

Conjecture 2 ([Br1, II (2.3)]). *For these $\mathscr{D}_X$-Modules, $j_*^m \mathscr{L}$ (resp. $j_!^m \mathscr{L}$) has a structure of a mixed holonomic $\mathscr{D}_X$-Module.*

(4.2) Some results on the mixed $\mathscr{D}$-Modules

We return to the situation (0.1). We shall be concerned with two conjectures in (4.1) in case $Z=X$.

We consider three $\mathscr{D}_X$-Modules which extend $\mathscr{V}=\mathscr{O}_U \otimes V_{\boldsymbol{C}}$ to X: $\mathscr{L}=\mathscr{L}(X, X; V)$, $\mathscr{M}=j_*^m \mathscr{V}$ (cf. (4.1)) and its dual $\mathscr{M}^*=j_!^m \mathscr{V}$ (cf. [K3]). These correspond to $j_* V$, $Rj_* V$ and $j_! V$ under DR_X. Note that $IC^\cdot(X, V)=j_* V[2n]$ in the situation (0.1) with $n=\dim X$.

The inclusion $\Omega(V)^\cdot_{(2)} \rightarrow DR_X \mathscr{L}$ is F-filtered and is known to be a filtered quasi-isomorphism if $n=1$ (cf. [Br1, II, (2.2)]. Then by the reduction to the case $n=1$ in (0.5), Theorem (2.5) implies:

Proposition (4.2.1). $(\mathrm{IC}^\cdot(X, V)[-2n], (DR_X \mathscr{L}, F))$ *is a cohomological Hodge complex of weight m.*

Thus Conjecture 1 in (4.1) holds for our $\mathscr{L}$.

Next we will construct the weight filtration on $\mathscr{M}$ and $\mathscr{M}^*$ to verify Conjecture 2 in (4.1) in the setting (0.1). Consider the following exact sequences of $\mathscr{D}_X$-Modules, which are dual to each other.

(4.2.2) (i) $0 \longrightarrow \mathscr{L} \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}/\mathscr{L} \longrightarrow 0$

(ii) $0 \longrightarrow (\mathscr{M}/\mathscr{L})^* \longrightarrow \mathscr{M}^* \longrightarrow \mathscr{L}^* \longrightarrow 0$

They correspond to the following triangles in $D_c^b(X, \boldsymbol{C})$ under DR_X, which give rise to the exact sequences in (3.3.2).

(4.2.3) (i)
$$\begin{tikzcd} j_*V \arrow[r] & Rj_*V \arrow[dl] \\ R^1j_*V[-1] \arrow[u] & \end{tikzcd}$$
(ii)
$$\begin{tikzcd} i_*i^*j_*V[-1] \arrow[r] & j_!V \arrow[dl] \\ j_*V \arrow[u] & \end{tikzcd}$$

In Section 1, we constructed a local system V_Y on Y, its filtrations W_Y, Z_Y and an endomorphism N on V_Y. Then, by comparison of (4.2.2) and (4.2.3), we have

(4.2.4)
$$DR_X(\mathcal{M}/\mathcal{L})=R^1j_*V[-1]=\operatorname{Coker} N[-1]$$
$$DR_X((\mathcal{M}/\mathcal{L})^*)=i_*i^*j_*V[-1]=\operatorname{Ker} N[-1].$$

Let $\mathcal{B}_{Y|X}:=\int_i \mathcal{O}_Y$ be the "integration" of the $\mathcal{D}_Y$-Module $\mathcal{O}_Y$ by the inclusion i (cf. [K3]), or equivalently, $\mathcal{B}_{Y|X}=\mathcal{O}_X(*Y)/\mathcal{O}_X$. By the commuting relation $DR_X\cdot\int_i=Ri_*[-1]\cdot DR_Y$ (cf. [K2]) and (4.2.4), we have

$$\mathcal{M}/\mathcal{L}=\mathcal{B}_{Y|X}\otimes\operatorname{Coker} N,\ (\mathcal{M}/\mathcal{L})^*=\mathcal{B}_{Y|X}\otimes\operatorname{Ker} N.$$

From this description, we define filtrations on $\mathcal{M}$ and $\mathcal{M}^*$ as follows.

Definition (4.2.5). 1) We put

$$W_k(\mathcal{M}/\mathcal{L});=\mathcal{B}_{Y|X}\otimes Z_k(\operatorname{Coker} N)\qquad(k\geqq 1)$$
$$W_{-k}(\mathcal{M}/\mathcal{L})^*:=\mathcal{B}_{Y|X}\otimes W_{-k-1}(\operatorname{Ker} N).$$

Here $Z_k(\operatorname{Coker} N)$ (resp. $W_k(\operatorname{Ker} N)$) is induced from $Z_k(V_Y)$ (resp. $W_k(V_Y)$). Then we define $W_k\mathcal{M}$ (resp. $W_{-k}\mathcal{M}^*$) to be the inverse images of $W_k(\mathcal{M}/\mathcal{L})$ (resp. the image of $W_k(\mathcal{M}/\mathcal{L})^*$) (4.2.2) for $k\geqq 1$, and we put $W_0\mathcal{M}=\mathcal{L}$, $W_{-1}\mathcal{M}=0$, and $W_0\mathcal{M}^*=\mathcal{M}^*$.

2) We remark that $\mathcal{M}=j_*^m\mathcal{V}=\mathcal{O}_X(*Y)\otimes\bar{\mathcal{V}}$ by [KK1, (2.2.1)] (resp. $\mathcal{M}^*=j_!^m\mathcal{V}=\mathcal{D}_XH_Y\otimes\bar{\mathcal{V}}$ by [Sc, Chap. I, (4.2.2)]), where $\mathcal{O}_X(*Y)$ denotes the sheaf of meromorphic functions with poles only along Y (resp. H_Y denotes the Heaviside function as in [Sc, Chap. I, (4.2)]). Define a (good) filtration on $\mathcal{M}$ (resp. $\mathcal{M}^*$) as follows:

$$\mathcal{M}_\ell:=\sum_{i+j=\ell}\mathcal{O}_X(iY)\otimes\bar{\mathcal{F}}^{-j}$$
$$\mathcal{M}_\ell^*:=\sum_{i+j=\ell}\mathcal{D}_X(i)H_Y\otimes\bar{\mathcal{F}}^{-j}.$$

Here $\mathcal{O}_X(iY)$ denotes the subsheaf of $\mathcal{O}_X(*Y)$ consisting of functions with poles of order $\leqq i$, and $\mathcal{D}_X(i)$ denotes the sheaf of differential operators of order $\leqq i$.

These filtrations induce on $\mathcal{L}$ the filtration considered in (4.1) and (4.2.1). These also induce the following filtrations on $\mathcal{M}/\mathcal{L}$ and $(\mathcal{M}/\mathcal{L})^*$:

$$(\mathcal{M}/\mathcal{L})_\ell := \sum_{i+j=\ell} \mathcal{B}_{Y|X}(i) \otimes \mathcal{F}^{-j}(\mathrm{Coker}\, N)$$
$$(\mathcal{M}/\mathcal{L})_\ell^* := \sum_{i+j=\ell} \mathcal{B}_{Y|X}(i) \otimes \mathcal{F}^{-j}(\mathrm{Ker}\, N).$$

Here $\mathcal{B}_{Y|X}(i) := \mathcal{O}_X(iY)/\mathcal{O}_X$ and $\mathcal{F}^{-j}$ are the Hodge filtrations on Coker N and Ker N induced by that on V_Y.

Now we come to the main result in Section 4.

Theorem (4.2.6). 1) *The inclusion* $K_C = \Omega_X^\cdot(\log Y) \otimes \bar{\mathcal{V}} \to DR_X(\mathcal{M}) = \Omega_X^\cdot \otimes \mathcal{M}$ *is a bifiltered quasi-isomorphism.*

2) *The morphism* $L_C \to DR_X(\mathcal{M}^*) = \Omega_X^\cdot \otimes \mathcal{M}^*$ *defined below is a bifiltered isomorphism in* $D_c^b(X, \boldsymbol{C})$.

$$L_C = \underline{\underline{s}}[W_0 K_C \longrightarrow \Omega_Y^\cdot \otimes \mathrm{Ker}\, N] \simeq K_{!,C} \longrightarrow \Omega_X^\cdot \otimes \mathcal{M}^*.$$

Proof. 1) The fact that the inclusion respects the filtrations W, F is easy to see. Note $\mathcal{M} = \mathcal{O}_X(*Y) \otimes \bar{\mathcal{V}}$. For $k=0$, $W_0 K_C \to \Omega_X^\cdot \otimes \mathcal{L}$ is an F-filtered quasi-isomorphism, because so are $M_C \to W_0 K_C$ by (3.1.5) and $M_C \to \Omega_X^\cdot \otimes \mathcal{L}$ by (4.2.2).

For $k>0$, we use the Poincaré residue map (3.1.5, 2). We have Res: $\mathrm{Gr}_k^W K_C \to \Omega_Y^\cdot \otimes P_{k-1}(-1)[-1]$. On the other hand, $\mathrm{Gr}_k^W \mathcal{M} = \mathcal{B}_{Y|X} \otimes \mathrm{Gr}_k^Z V_Y = \mathcal{B}_{Y|X} \otimes P_{k-1}$ and the map

$$\Omega_X^\cdot \otimes (\mathcal{B}_{Y|X} \otimes P_{k-1}) = (\Omega_X^\cdot \otimes \mathcal{B}_{Y|X}) \otimes P_{k-1} \longrightarrow \Omega_Y^\cdot \otimes P_{k-1}(-1)[-1]$$

is also given by the residue. So the F-filtered quasi-isomorphy follows from the commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_k^W K_C \arrow[rr] \arrow[dr, "\mathrm{Res}"'] & & \Omega_X^\cdot \otimes \mathrm{Gr}_k^W \mathcal{M} \arrow[dl, "\mathrm{Res}"] \\ & \Omega_Y^\cdot \otimes P_{k-1}(-1)[-1] & \end{tikzcd}$$

2) Look at the vertical morphisms between (horizontal) distinguished triangles (cf. (4.2.2, ii)):

$$\begin{tikzcd} K_{!,C} \arrow[r] \arrow[d] & W_0 K_C \arrow[r] \arrow[d] & W_0 K_C / K_{!,C} \arrow[d, dotted] \\ \Omega_X^\cdot \otimes \mathcal{M}^* \arrow[r] & \Omega_X^\cdot \otimes \mathcal{L}^* \arrow[r] & \Omega_X^\cdot \otimes (\mathcal{M}/\mathcal{L})^*[1] \simeq \Omega_Y^\cdot \otimes \mathrm{Ker}\, N(-1) \end{tikzcd}$$

We used the relation Res: $\Omega_X^\cdot \otimes (\mathcal{M}/\mathcal{L})^* = (\Omega_X^\cdot \otimes \mathcal{B}_{Y|X}) \otimes \mathrm{Ker}\, N(-1) \to \Omega_Y^\cdot \otimes \mathrm{Ker}\, N(-1)[-1]$, and we get the dotted arrow. The vertical maps in the right square represent the map $L_C \to \Omega_X^\cdot \otimes \mathcal{M}^*$.

Let us take Gr_k^W of this map. For $k=0$, $W_0K_C \to \Omega_X^{\cdot} \otimes \mathcal{M}^*$ is an F-filtered quasi-isomorphism as is seen in 1). For $k>0$, we get Gr_k^W of $W_0K_C/K_{1,C} \to \Omega_Y^{\cdot} \otimes \mathrm{Ker}\, N(-1)$, which is an F-filtered quasi-isomorphism by (3.2.5). q.e.d.

Remark (4.2.7). 1) We defined a $\boldsymbol{Q}$-structure on W_kK_C in (3.1.3) (resp. W_kL_C in (3.2.4)), and the above theorem shows that $\mathcal{M}$ (resp. $\mathcal{M}^*$) is a mixed holonomic $\mathcal{D}_X$-Module in the sense defined in (4.1), i.e., Brylinski's Conjecture 2 in (4.1) holds in our setting (0.1).

2) We might as well define a good filtration on $\mathcal{M}^* = j_!^m \mathcal{V}$ as the dual of that on $\mathcal{M}$. To carry it out, we must make use of the theory of filtered $\mathcal{D}$-Modules more systematically.

3) The above theorem may be regarded as a generalization of [Gr 2, § 8] when $V=\boldsymbol{Q}_U$, $X=\boldsymbol{P}^n$.

Added in proof. The author has recently extended Theorem 2 to the general case admitting Y to be a divisor with normal crossings, using a recent theory of VMHS by Prof. Kashiwara.

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Mathematical Institute
Tohoku University
Sendai 980, Japan