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Variation of Mixed Hodge Structure and the Torelli Problem

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Dedicated to Professor Masayoshi Nagata on the occasion of his sixtieth birthday

Contents

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Conventions

We use the following abbreviations:

HS: Hodge Structure,
VHS: Variation of Hodge Structure,
PHS: Polarized Hodge Structure,
VPHS: Variation of Polarized Hodge Structure,
IVHS: Infinitesimal Variation of Hodge Structure,
MHS: Mixed Hodge Structure,
VMHS: Variation of Mixed Hodge Structure,
GPMHS: Gradedly Polarized Mixed Hodge Structure,
VGPMHS: Variation of Gradedly Polarized Mixed Hodge Structure,
SNC: Simple Normal Crossing,
SNCD: Simple Normal Crossing Divisor.

We add the adjective "*mixed*" to the variants in the VMHS theory for the usual concepts in the VHS theory. For instance,

mixed period map,
mixed lattice,
mixed Torelli theorem,
infinitesimal *mixed* Torelli theorem,

generic *mixed* Torelli theorem,
"*mixed* Clemens-Schmid sequence".
The varieties concerned in this article are always those defined over $\boldsymbol{C}$.

Introduction

This article is a survey including some new results, on the Torelli problem in the frame of VGPMHS (Variation of gradedly polarized mixed Hodge structure).

Let (X, Y) be a pair consisting of a smooth projective variety X of dimension n and a smooth divisor Y on X. Then we can consider the mixed period map as well as the period maps:

$$\begin{array}{ccc} \mathscr{M}=\left\{\begin{matrix}\text{isomorphism}\\ \text{classes } [X, Y]\end{matrix}\right\} & \xrightarrow{\quad\Phi\quad} & \Gamma\backslash D \\ \downarrow & & \downarrow \\ \mathscr{M}_n\times\mathscr{M}_{n+1}=\left\{\begin{matrix}\text{isomorphism}\\ \text{classes } [X]\end{matrix}\right\}\times\left\{\begin{matrix}\text{isomorphism}\\ \text{classes } [Y]\end{matrix}\right\} & \xrightarrow{\Phi_n\times\Phi_{n+1}} & \Gamma_n\backslash D_n\times\Gamma_{n+1}\backslash D_{n+1} \end{array}$$

where

$$\Phi([X, Y])=(\text{GPMHS on } H^n(X-Y)) \bmod \Gamma,$$
$$\Phi_n([X])=(\text{PHS on } \mathrm{Gr}_n^W H^n(X-Y)) \bmod \Gamma_n, \quad \text{and}$$
$$\Phi_{n+1}([Y])=(\text{PHS on } \mathrm{Gr}_{n+1}^W H^n(X-Y)) \bmod \Gamma_{n+1}.$$

We hope to investigate these maps and the relationship among them.

In Part II, Section 1, we explain the motivation for the notion of VGPMHS in conformity with examples of certain canonical surfaces with $p_g=1$, which were extensively studied by Kŭnev, Catanese, Todorov and Usui.

The infinitesimal mixed Torelli theorem for (X, Y) with sufficiently ample Y, proved by Green and Griffiths (cf. Theorem (7.1)), is an encouraging result for the enlarged frame of VGPMHS.

In order to show the generic (mixed) Torelli theorem, there are two approaches:

One is to make use of general points of $\mathscr{M}$ or $\mathscr{M}_k$. In this direction, IVHS (Infinitesimal VHS) theory developed by Griffiths, Donagi and others obtained good results. Especially the generic Torelli theorem for sufficiently ample hypersurfaces Y on a fixed X, proved recently by Green, fits nicely in our context and has a possibility to be completed further.

The other is to make use of special points of $\mathscr{M}$ or $\mathscr{M}_k$. Kummer surfaces were used as these points in the proof of the Torelli theorem for

$K3$ surfaces by Piateckii-Shapiro and Shafarevich and others. In this case the density property played an essential role. If we cannot hope for a density property, the compactification of the (mixed) period map becomes an inevitable problem. Inverting this point, Friedman made positive use of the boundary points in his proof of the Torelli theorem for $K3$ surfaces. This method seems to have a possibility of generalization.

As a survey, this article includes résumés of known results, discussions and a lot of problems as well as some new results. We give proofs only for new results and the corrections of published ones. For the known results we only indicate references.

Part I, Sections 1, 2, 3 and 5 are résumés of [U. 4], Carlson [Car. 2] and [S.S.U].

Section 4 is a résumé of Kawamata [Kaw. 1].

Section 6 is a systematic treatment of the deformation theory of a pair of a variety and a line bundle. An observation of Griffiths in [Gri. 5] is included. Welters [W] contains some related results.

Sections 7 and 8 are résumés of Green [Gre] and Griffiths [Gri. 5].

Section 9 is a rewriting of the result of Mumford in [K.K.M.S-D] in our context.

Section 10 is a slight generalization of Steenbrink and Zucker [S.Z] (see also Elzein's [E]).

Section 11 includes generalizations of some results of Friedman [F.1].

Section 12 consists of problems and discussions.

Part II, Section 1, (1.2), (1.4) and (1.5) are résumés of Catanese [Cat. 1], [Cat. 2], [Cat. 3] and Oliverio [O]. We give a correction for [O] in (1.4.2).

(1.6) is a résumé of Todorov [To. 1], [To. 2] and [U. 1], [U. 2].

(1.7) is a résumé of [U. 4], [U. 5].

(1.8) is devoted to some discussion.

Section 2 includes a correction of Letizia's result in [L] and a new result for Todorov surfaces with $c_1^2=2$ and $\pi_1=\boldsymbol{Z}/2\boldsymbol{Z}$.

Section 3 is devoted to a new result. Friedman [F. 3] is related.

Section 4 includes some new results, discussions and problems.

The contributions of the three authors to the new results are as follows:

Saito: Part I, Section 9; Part II, Sections 2 and 4.

Shimizu: Part I, Sections 10 and 11.

Usui: Part I, Section 6 and (12.6); Part II, (1.4.2), (1.8), Sections 2, 3 and 4.

The problems included here have various range of difficulties.

I. General theory

1. Variation of gradedly polarized mixed Hodge structure

(1.1) **Definition.** A *variation of gradedly polarized mixed Hodge structure* (VGPMHS for short) is a quintuplet $(S, H_{\boldsymbol{Z}}, W, F, Q)$ consisting of

S: a connected complex manifold,

$H_{\boldsymbol{Z}}$: a local system of $\boldsymbol{Z}$-free modules of finite rank on S,

W: an increasing filtration of $H_{\boldsymbol{Z}}$ by primitive local subsystems,

F: a decreasing filtration of $H_{\mathcal{O}} := H_{\boldsymbol{Z}} \otimes \mathcal{O}_S$ by holomorphic subbundles, and

Q: a collection of locally constant $(-1)^k$-symmetric bilinear forms Q_k on $\mathrm{Gr}_k^W H_{\boldsymbol{Q}} := W_{k,\boldsymbol{Q}}/W_{k-1,\boldsymbol{Q}}$ with values in $\boldsymbol{Q}$,

which satisfy the following conditions:

(E.H) The Gauss-Manin connection ∇ for $H_{\mathcal{O}}$ corresponding to the local system $H_{\boldsymbol{Z}}$ satisfies $\nabla F^p H_{\mathcal{O}} \subset \Omega_S^1 \otimes F^{p-1} H_{\mathcal{O}}$ for all p.

(M.H) For every point $s \in S$, the fibre $(H_{\boldsymbol{Z}}, W, F)(s)$ is a mixed Hodge structure (MHS for short), i.e., $\mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q \mathrm{Gr}_k^W H_{\boldsymbol{C}}(s) = 0$ unless $p+q = k$.

(G.P) Q_k is a polarization of the variation of Hodge structure (VHS for short) $(S, \mathrm{Gr}_k^W H_{\boldsymbol{Z}}, F\mathrm{Gr}_k^W H_{\mathcal{O}})$ for all k, i.e., at every point $s \in S$,

$$Q_k((F^p \mathrm{Gr}_k^W H_{\mathcal{O}})(s),\quad (F^{k-p+1}\mathrm{Gr}_k^W H_{\mathcal{O}})(s)) = 0 \qquad \text{for all } p \text{ and}$$

$$Q_k(Cu, \bar{u}) > 0 \qquad \text{for all nonzero } u \in (\mathrm{Gr}_k^W H_{\mathcal{O}})(s),$$

where C is the Weil operator determined by $(F\mathrm{Gr}_k^W H_{\mathcal{O}})(s)$.

2. Classifying spaces

Let $(H_{\boldsymbol{Z}}, W, F(0), Q)$ be a reference GPMHS. Set

$f^p = \dim F(0)^p H_{\boldsymbol{C}}$

$f_k^p = \dim F(0)^p \mathrm{Gr}_k^W H_{\boldsymbol{C}}$

$\mathscr{F} = \{F \in \mathrm{Flag}(H_{\boldsymbol{C}}; \cdots, f^p, \cdots) \mid \dim F^p \mathrm{Gr}_k^W H_{\boldsymbol{C}} = f_k^p, \text{ for } \forall p \text{ and } \forall k\}$

$\mathrm{GL}_W(H_{\boldsymbol{C}}) = \{g \in \mathrm{GL}(H_{\boldsymbol{C}}) \mid gW_k = W_k \text{ for } \forall k\}$

$\pi_k\colon \mathscr{F} \to \mathscr{F}_k := \mathrm{Flag}(\mathrm{Gr}_k^W H_{\boldsymbol{C}}; \cdots, f_k^p, \cdots),\ FH_{\boldsymbol{C}} \mapsto F\mathrm{Gr}_k^W H_{\boldsymbol{C}}$

$\check{D}_k = \{F \in \mathscr{F}_k \mid Q_k(F^p, F^{k-p+1}) = 0 \text{ for } \forall p\}$

$D_k = \{F \in D_k \mid Q_k(Cu, \bar{u}) > 0 \text{ for } 0 \neq \forall u \in \mathrm{Gr}_k^W H_{\boldsymbol{C}}\}$.

Define

$\check{D} = \bigcap_k \pi_k^{-1}(\check{D}_k) \subset F$

$D = \bigcap_k \pi_k^{-1}(D_k) \subset \check{D}$

$\check{\pi}\colon \check{D}\to\prod_k \check{D}_k$ by $\check{\pi}(F):=(\cdots,\pi_k(F),\cdots)$
$\pi\colon D\to\prod_k D_k$ as the restriction of $\check{\pi}$ to D
$G_{\boldsymbol{C}}=\{g\in \mathrm{GL}_W(H_{\boldsymbol{C}})\mid \mathrm{Gr}_k^W g$ preserves Q_k for $\forall k\}$ $\quad G_{k,\boldsymbol{C}}=\mathrm{Gr}_k^W G_{\boldsymbol{C}}$
$G_{\boldsymbol{R}}=\{g\in G_{\boldsymbol{C}}\mid gH_{\boldsymbol{R}}=H_{\boldsymbol{R}}\}$ $\quad G_{k,\boldsymbol{R}}=\mathrm{Gr}_k^W G_{\boldsymbol{R}}$
$G_{\boldsymbol{Z}}=\{g\in G_{\boldsymbol{R}}\mid gH_{\boldsymbol{Z}}=H_{\boldsymbol{Z}}\}$ $\quad G_{k,\boldsymbol{Z}}=\mathrm{Gr}_k^W G_{\boldsymbol{Z}}$
$G=G'_{\boldsymbol{C}}\cdot(G_{\boldsymbol{R}}\cap G''_{\boldsymbol{C}})$ where $G_{\boldsymbol{C}}=G'_{\boldsymbol{C}}\cdot G''_{\boldsymbol{C}}$ is a Levi decomposition with the unipotent radical $G'_{\boldsymbol{C}}$ and a semi-simple part $G''_{\boldsymbol{C}}$.

The following theorem can be found in [U.4, II], [Car. 2] and [S.S.U]:

(2.1) **Theorem** (Usui, Carlson).

(2.1.1) *$\check{\pi}\colon \check{D}\to\prod_k \check{D}_k$ is an algebraic homogeneous vector bundle with respect to $G_{\boldsymbol{C}}$.*

(2.1.2) *G acts transitively on D, while $G_{\boldsymbol{R}}$ does not.*

(2.1.3) *$G_{\boldsymbol{Z}}$ acts on D properly discontinuously.*

(2.1.4) *There is an extended horizontal subbundle $T^{eh}_{\check{D}}$ of $T_{\check{D}}$, which is compatible with the horizontal subbundle $\oplus_k T^h_{\check{D}_k}$ on $\prod_k \check{D}_k$ via $\check{\pi}$.*

(2.1.5) *The mixed period map $\Phi\colon S\to\Gamma\backslash D$, where $\Gamma:=\mathrm{Im}\,(\pi_1(S,0)\to G_{\boldsymbol{Z}})$, associated to the* VGPMHS *$(S, H_{\boldsymbol{Z}}, W, F, Q)$ has extended horizontal local liftings with respect to T^{eh}_D and is compatible via π with the period maps $\Phi_k\colon S\to\Gamma_k\backslash D_k$, where $\Gamma_k:=\mathrm{Gr}_k^W\Gamma$, associated to the* VPHS *$(S, \mathrm{Gr}_k^W H_{\boldsymbol{Z}}, F\mathrm{Gr}_k^W H_{\mathcal{O}}, Q_k)$ for all k.*

3. Some results from hyperbolicity

We use the notation S, D and $G_{\boldsymbol{Z}}$ in Section 1 and Section 2. We can derive easily from the hyperbolicity of the horizontal subbundle ([G.S.1]) the following (see [U.4, II]):

(3.1) *Let Γ be a subgroup of $G_{\boldsymbol{Z}}$ and $\Phi\colon S\to\Gamma\backslash D$ a holomorphic map with extended horizontal local liftings. If the universal cover of S is compact (resp. a Euclidian space), then $\Phi(S)$ is one point (resp. contained in one fibre of $\Gamma\backslash D\to\prod(\Gamma_k\backslash D_k)$).*

(3.2) *Let S' be a subvariety of S of codimension $\geqq 2$ and $\Phi\colon (S-S')\to\Gamma\backslash D$ a map as in* (3.1) *above. Then Φ extends to the whole S.*

(3.3) *Let Δ^* be the punctured open unit disc and $\Phi\colon \Delta^*\to\Gamma\backslash D$ with $\Gamma:=\mathrm{Im}\,(\pi_1(\Delta^*, s_0)\to G_{\boldsymbol{Z}})$. Then every $\gamma\in\Gamma$ is quasi-unipotent.*

4. Deformation theory of smooth pairs

This section is a summary of the results of Kawamata [Kaw. 1].

(4.1) **Definition.** (4.1.1) A pair (X, Y) is called a *smooth pair* if X is a compact complex manifold and Y is a simple normal crossing divisor on X.

(4.1.2) A *smooth family of pairs* is a quadruplet $(\mathscr{X}, \mathscr{Y}, f, S)$ consisting of a connected complex manifold S, a connected, proper, smooth morphism $f: \mathscr{X} \to S$ of a complex manifold $\mathscr{X}$, and a simple normal crossing divisor $\mathscr{Y} = \bigcup \mathscr{Y}_i$ on $\mathscr{X}$ s.t. $\mathscr{Y}_{i_1} \cap \cdots \cap \mathscr{Y}_{i_k}$ for all $i_1, \cdots, i_k$ with $k \geq 1$ are smooth over S.

Known Results (e.g. [Kaw. 1])

(4.2) *For a smooth pair* (X, Y) *define*

$$T_X(-\log Y) = \{\theta \in T_X \,|\, \theta \mathscr{I}_Y \subset \mathscr{I}_Y\},$$

where $\mathscr{I}_Y$ *is the sheaf of ideals for* Y *in* X. *Then we have*:

(4.2.1) $T_X(-\log Y)$ *is the sheaf of infinitesimal automorphisms of* (X, Y).

$H^1(T_X(-\log Y))$ *is the set of the infinitesimal deformations of* (X, Y).

$H^2(T_X(-\log Y))$ *is the set of obstructions.*

(4.2.2) *There exist exact sequences*:

$$0 \longrightarrow T_X(-Y) \longrightarrow T_X(-\log Y) \longrightarrow T_Y \longrightarrow 0$$

$$0 \longrightarrow T_X(-\log Y) \longrightarrow T_X \longrightarrow N_{Y/X} \longrightarrow 0,$$

where $T_Y := \mathrm{Der}(\mathscr{O}_Y)$ *and* $N_{Y/X} := \mathrm{Coker}(T_Y \to T_X \otimes \mathscr{O}_Y)$.

(4.2.3) *There exists a semi-universal family of deformations of* (X, Y).

(4.3) *For a smooth family of pairs* $(\mathscr{X}, \mathscr{Y}, f, S)$, *we can define the Kodaira-Spencer map* $\rho_s: T_S(s) \to H^1(T_{X_s}(-\log Y_s))$ *at* $s \in S$ *in the usual way.*

5. VGPMHS arising from geometry

The following theorem can be found in [U. 4] and [S.S.U]:

(5.1) **Theorem.** *Let* $(\mathscr{X}, \mathscr{Y}, f, S)$ *be a smooth family of pairs. Assume that* f *factors as* $\mathscr{X} \hookrightarrow \boldsymbol{P}^N \times S \xrightarrow{\mathrm{pr}} S$. *Then we have*:

(5.1.1) *The spectral sequences for the hypercohomology of the relative logarithmic de Rham complex* $\Omega^{\cdot}_f(\log \mathscr{Y})$ *with respect to the weight filtration* W *and the Hodge filtration* F *degenerate in* ${}_WE_2 = {}_WE_\infty$ *and* ${}_FE_1 = {}_FE_\infty$. *Thus we get a* VGPMHS $(S, R^n_{\mathbf{Z}}(\mathring{f}), W[n], F, Q)$, *where* $R^n_{\mathbf{Z}}(\mathring{f})$ *is* $R^n f_* \mathbf{Z}_{\mathscr{X}-\mathscr{Y}}$ *modulo torsion and* $(W[n]_k)_{\mathbf{Z}}$ *is the primitive span* $(W[n]_k)_{\mathbf{Q}} \cap R^n_{\mathbf{Z}}(f)$.

(5.1.2) *Let* $\Phi: S \to \Gamma \backslash D$ *be the mixed period map associated to the* VGPMHS *in* (5.1.1) *and* $\tilde{\Phi}$ *a local lifting of* Φ *at* $s \in S$. *Then we have a diagram which is commutative up to* $\oplus(-1)^p$:

$$\begin{array}{ccc}
T_S(s) & \xrightarrow{\quad d\tilde{\Phi}(s) \quad} & T_D^{eh}(\tilde{\Phi}(s)) \subset T_D(\tilde{\Phi}(s)) \\
\text{Kodaira-Spencer map} \Big\downarrow \rho_s & & \Big\uparrow \wr \\
H^1(T_{X_s}(-\log Y_s)) & \xrightarrow{\text{contraction}} & \bigoplus_p \mathrm{Hom}_{(W,Q)}(H^{n-p}(\Omega^p_{X_s}(\log Y_s)), \\
 & & H^{n-p+1}(\Omega^{p-1}_{X_s}(\log Y_s))).
\end{array}$$

6. Deformation theory of polarized varieties

In [Gri. 5] and [Gre] cited in the next two sections, the sheaf $D_1(L, L)$ of first order differential operators on sections of a line bundle L plays the key role in computation. We would like to point out that $D_1(L, L)$ is not merely an assistant but a substantial object in our context (see also (12.6)).

Let L be a line bundle on a compact complex manifold X. First we recall a geometric construction of $D_1(L, L)$ and its dual $J^1(L)\otimes L^{-1}$, where $J^1(L)$ is the sheaf of 1-jets of sections of L. Denote by $\pi: L^{-1}\to X$ the projection of the dual line bundle and by X the 0-section by abuse of notation. For the natural $\boldsymbol{G}_m$-action on L^{-1}, we have:

(6.1) **Lemma.**

(6.1.1) $(\pi_* T_{L^{-1}}(-\log X))^{G_m} = D_1(L, L)$ *and* $(\pi_*\Omega^1_{L^{-1}}(\log X))^{G_m} = J^1(L)\otimes L^{-1}$.

(6.1.2) *Taking the direct image and then the $\boldsymbol{G}_m$-invariant part, the exact sequences*

$$0\longrightarrow T_\pi(-\log X)\longrightarrow T_{L^{-1}}(-\log X)\longrightarrow \pi^* T_X\longrightarrow 0 \quad \textit{and}$$
$$0\longrightarrow \pi^*\Omega^1_X\longrightarrow \Omega^1_{L^{-1}}(\log X)\longrightarrow \Omega^1_\pi(\log X)\longrightarrow 0$$

yield the exact sequences

$$0\longrightarrow \mathcal{O}_X\longrightarrow D_1(L, L)\longrightarrow T_X\longrightarrow 0 \quad \textit{and}$$
$$0\longrightarrow \Omega^1_X\longrightarrow J^1(L)\otimes L^{-1}\longrightarrow \mathcal{O}_X\longrightarrow 0$$

with extension classes $-2\pi i c_1(L)$ and $2\pi i c_1(L)$, respectively.
The connecting homomorphisms of the cohomology sequences

$$H^r(T_X)\longrightarrow H^{r+1}(\mathcal{O}_X) \quad \textit{and} \quad H^r(\mathcal{O}_X)\longrightarrow H^{r+1}(\Omega^1_X)$$

are given by the contraction with these extension classes.

The proof is easy. We only mention here that if $(x_1, \cdots, x_n)$ are local coordinates on X and ξ a fibre coordinate of L^{-1}, then $(\xi\partial/\partial\xi, \partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $(d\xi/\xi, dx_1, \cdots, dx_n)$ give local frames of $D_1(L, L)$ and $J^1(L)\otimes L^{-1}$, respectively.

(6.2) **Proposition.** *Let* (X, L) *be as above. Then:*

(6.2.1) $D_1(L, L)$ *is the sheaf of infinitesimal automorphisms of* (X, L). $H^1(D_1(L, L))$ *is the set of infinitesimal deformations of* (X, L). $H^2(D_1(L, L))$ *is the set of obstructions.*

(6.2.2) *For a smooth family* $(\mathscr{X}, \mathscr{L}, f, S)$ *of deformations of* $(X, L) = (X_0, L_0)$ $(0 \in S)$, *the Kodaira-Spencer map* $\rho_0: T_S(0) \to H^1(D_1(L, L))$ *at* $0 \in S$ *is defined in the usual way, and the involution*

$$\iota: S \longrightarrow S, \qquad (X_s, L_s) \longrightarrow (X_s, L_s^{-1})$$

gives a commutative diagram

$$\begin{array}{ccc} T_S(s) & \xrightarrow{\rho_s} & H^1(D_1(L_s, L_s)) \\ {\scriptstyle \iota_*}\downarrow \wr & & \downarrow \wr \\ T_S(\iota s) & \xrightarrow{\rho_{\iota s}} & H^1(D_1(L_s^{-1}, L_s^{-1})), \end{array}$$

where the right vertical map is induced by $D_1(L_s, L_s) \xrightarrow{\sim} D_1(L_s^{-1}, L_s^{-1})$ *sending* $\xi\partial/\partial\xi$ *to* $-\xi^*\partial/\partial\xi^*$ *and* $\partial/\partial x_i$ *to* $\partial/\partial x_i$ *for all* i.

(6.2.3) *Assume that* X *is a Kähler manifold. Then there exists a semi-universal family of deformations of* (X, L).

(6.2.4) *Assume that there exists* $s \in H^0(L)$ *such that* $Y = \{s = 0\}$ *is smooth. Then the contraction with* $j(s) \in H^0(J^1(L))$ *yields exact sequences*

$$0 \longrightarrow T_X(-\log Y) \longrightarrow D_1(L, L) \xrightarrow{\cdot j(s)} L \longrightarrow 0 \quad \text{(Griffiths [Gri. 5])}$$

as well as

$$0 \longrightarrow T_X(-Y - \log Y) \longrightarrow T_X(-\log Y) \longrightarrow D_1(L|_Y, L|_Y) \longrightarrow 0.$$

Proof. The assertion on the sheaf $D_1(L, L)$ in (6.2.1) follows from the following observation which is easy to check: For an automorphism $\tilde{\sigma}$ of L^{-1} as a complex manifold, $\tilde{\sigma}$ induces an automorphism of the line bundle $L^{-1} \to X$ if and only if $\tilde{\sigma}$ is $\boldsymbol{G}_m$-equivariant. The other assertions in (6.2.1) and (6.2.2) are standard and easy to verify.

In order to see (6.2.3), let $(\mathscr{X}, f, S)$ be the Kuranishi family for the deformations of $X = X_0$ $(0 \in S)$ and γ the section of $R^2 f_* \boldsymbol{Z}_{\mathscr{X}}$ determined by $\gamma(0) = c_1(L)$ in $H^2(X, \boldsymbol{Z})$. Set

$$S^\gamma = \{s \in S \mid \omega(s) \cup \gamma(s) = 0 \text{ for any section } \omega \text{ of } R^2 f_* \mathcal{O}_{\mathscr{X}}\},$$

where $\cup$ means the cup product on the cohomology of the fibres of f. Denote by $(P^\gamma, \mathscr{L}^\gamma)$ the pair of the variety P^γ over S^γ and the universal family $\mathscr{L}^\gamma$ which represents the relative Picard functor $\mathrm{Pic}^\gamma(\mathscr{X}/S)$ for the

Chern class γ (cf. [Bi, (6.2)]). Then it is easy to see that $(\mathscr{X}\times_S P^r, \mathscr{L}^r, f', P^r)$, where f' is the induced morphism $\mathscr{X}\times_S P^r \to P^r$, is a semi-universal family of the deformations of $(X, L)=(X_0, L_0)$ for $0 \in P^r$.

As for the first sequence in (6.2.4), it can be checked easily, by using the local frames mentioned just after Lemma (6.1), that

$$\operatorname{Ker}(\cdot j(s);\, D_1(L, L) \longrightarrow L) \xrightarrow{\sim} T_X(-\log Y)$$

via the symbol map $D_1(L, L) \to T_X$. $T_X(-\log Y) \to D_1(L|_Y, L|_Y)$ in the second sequence sends $s\partial/\partial s$ to $-\xi\partial/\partial\xi|_Y$ and $\partial/\partial x_i$ to $\partial/\partial x_i|_Y$ for $i \geqq 2$, where we take $x_1 = s, x_2, \cdots, x_n$ as local coordinates on X, and it is easily seen to be exact. Q.E.D.

(6.3) **Remark.** After writing up the manuscript, we found that Welters had already recognized and used the infinitesimal versions in this section ((6.2.1) and the latter half of (6.1.2)) in his article [W] (see also [K.S]).

7. Infinitesimal mixed Torelli theorem for smooth pairs with sufficiently ample divisor

The result in this section is due to Griffiths ([Gri. 5], cf. also Green [Gre]).

Let (X, Y) be a smooth pair with $\dim X = n \geqq 2$. Set $\mathcal{O}_X(1) = \mathcal{O}_X(Y)$. Denote by $\Sigma = D_1(\mathcal{O}_X(1), \mathcal{O}_X(1))$ the sheaf of first order differential operators of sections of $\mathcal{O}_X(1)$. Let Δ be the diagonal $\Delta_X \subset X\times X$ and let $p_i: \Delta \to X$ be the i-th projection for $i=1, 2$. Then we have the following by essentially the same argument as in [Gre]:

(7.1) **Theorem** (Green, Griffiths). *Assume*:

(7.1.1) *Y is smooth.*

(7.1.2) $H^q((\bigwedge^{q+1}\Sigma)(-q))=0$ *for* $1\leqq q\leqq n-1$.

(7.1.3) $H^1(\mathscr{I}_\Delta \otimes p_1^*\omega_X(1)\otimes p_2^*\omega_X(n-1))=0$.

Then the map

$$H^1(T_X(-\log Y)) \longrightarrow \operatorname{Hom}(H^0(\Omega_X^n(\log Y)), H^1(\Omega_Y^{n-2}))$$

induced by contraction and the Poincaré residue is injective.

(7.2) **Remark.** Let $f: \mathscr{X}\to S$ be a connected, proper, smooth morphism of quasi-projective varieties with a factorization $f: \mathscr{X} \hookrightarrow P^N\times S \xrightarrow{\text{pr}} S$. Set $L=(p_1^*\mathcal{O}_{P^N}(1)\otimes p_2^*\mathcal{O}_S(1))|\mathscr{X}$. Then there exists an integer m_0 such that the conditions (7.1.2) and (7.1.3) are satisfied for $(X_s, \mathcal{O}_{X_s}(m))$ for all $s\in S$ and all $m\geqq m_0$.

8. Generic Torelli theorem for sufficiently ample divisors for a fixed (X, L)

Green developed the technique of IVHS, symmetrizer and polynomial structure ([Gri. 3], [C.G], [C.G.G.H], [Do]) and obtained in [Gre] the following:

(8.1) **Theorem** (Green). *Let X be a smooth projective variety of dimension $n \geqq 2$ and L a sufficiently ample line bundle on X. Assume, furthermore, that the canonical line bundle K_X is very ample. Then the period map*

$$\Phi_{n+1}\colon |L|_{\mathrm{reg}}/\mathrm{Aut}\,(X, L) \longrightarrow G_{n+1,\mathbf{Z}} \backslash D_{n+1}$$

has degree 1 *over its image, where $|L|_{\mathrm{reg}}$ is the set of smooth members of the linear system $|L|$.*

(8.2) **Remark.** Besides Green's theorem, several generic Torelli theorems were recently proved in this direction:

(8.2.1) *Certain hypersurfaces in weighted projective space* (Saito [Sa], Donagi and Tu [D.T]).

(8.2.2) *Most hypersurfaces in Kähler C-spaces with the second Betti number* $=1$ (K. Konno, to appear, see also M.-H. Saito, to appear).

9. Semi-stable reduction theorem for pairs

The same proof as in Mumford [K.K.M.S-D, Chap. II] works in our context and we get:

(9.1) **Theorem.** *Let Δ be the open unit disc in $\mathbf{C}$ and set $\Delta^* = \Delta - \{0\}$. Let $f\colon \mathscr{X} \to \Delta \times \Delta^r$ be a proper holomorphic map of a complex manifold $\mathscr{X}$ and let $\mathscr{Y}$ be an f-flat divisor on $\mathscr{X}$. Assume that $(\mathscr{X}, \mathscr{Y}, \Delta \times \Delta^r)|_{\Delta^* \times \Delta^r}$ is a smooth family of pairs (see* (4.1.2)*). We assume further that $\mathscr{X}_0 := f^{-1}(0 \times \Delta^r)$ is flat over $0 \times \Delta^r$ and $\mathscr{Y} \cup \mathscr{X}_0$ is an* SNCD *on $\mathscr{X}$ (not necessarily reduced).*

Then there exist a base extension

$$\Delta_d \times \Delta^r \longrightarrow \Delta \times \Delta^r, \qquad (s, t) \longmapsto (s^d, t)$$

and a diagram

$$\begin{array}{ccccc} \mathscr{X} & \xleftarrow{\;q\;} & \mathscr{X} \times_\Delta \Delta_d & \xleftarrow{\;p\;} & \mathscr{X}' \\ {\scriptstyle f}\downarrow & & \downarrow & \swarrow {\scriptstyle f'} & \\ \Delta \times \Delta^r & \longleftarrow & \Delta_d \times \Delta^r & & \end{array}$$

such that

(9.1.1) *p is proper and is an isomorphism over $\Delta_d^* \times \Delta^r$,*

(9.1.2) *p is obtained by blowing-up a sheaf of ideals $\mathscr{I}$ with*

$$\mathscr{I}|_{\Delta_d^* \times \Delta^r} \simeq \mathscr{O}_{\mathscr{X} \times_\Delta \Delta_d}|_{\Delta_d^* \times \Delta^r},$$

(9.1.3) *$\mathscr{X}'$ is smooth and $\mathscr{X}_0' := f'^{-1}(0 \times \Delta^r)$ is a reduced* SNCD, *and*

(9.1.4) *$\mathscr{Y}' \cup \mathscr{X}_0'$ is also an* SNCD, *where $\mathscr{Y}' := (q \circ p)^{-1}(\mathscr{Y})$ is the proper transform.*

We call the resulting family $(\mathscr{X}', \mathscr{Y}', f', \Delta_d \times \Delta^r)$ a *semi-stable degeneration of pairs.*

10. Degeneration of the GPMHS associated to semi-stable degeneration of pairs

The results in this section are only slight generalizations of those in Steenbrink and Zucker [S.Z] (see also Elzein [E]).

Let $(\mathscr{X}, \mathscr{Y}, f, \Delta \times \Delta^r)$ be a semi-stable degeneration of pairs (see Section 9). We use the notation:

$$\mathring{\mathscr{X}} = \mathscr{X} - \mathscr{Y}, \quad \mathscr{X}_0 = f^{-1}(0 \times \Delta^r), \quad \mathring{\mathscr{X}}_0 = \mathscr{X}_0 - (\mathscr{Y} \cap \mathscr{X}_0),$$
$$\mathring{\mathscr{X}}' = \mathring{\mathscr{X}} - \mathring{\mathscr{X}}_0, \quad f_0 = f|_{\mathscr{X}_0}, \quad \mathring{f}_0 = f|_{\mathring{\mathscr{X}}_0}, \quad \mathring{f}' = f|_{\mathring{\mathscr{X}}'}.$$

The following lemma can be proved in the same way as (5.3) in [S.Z];

(10.1) **Lemma.** *The locally free $\mathscr{O}_{\Delta^* \times \Delta^r}$-module $\mathscr{V} := R^n \mathring{f}'_* \mathbf{C} \otimes \mathscr{O}_{\Delta^* \times \Delta^r}$ has $\tilde{\mathscr{V}} := R^n f_* \Omega_f^\cdot(\log(\mathscr{Y} + \mathscr{X}_0))$ as its canonical extension over $\Delta \times \Delta^r$.*

Proof. We have to check:

(10.1.1) $\tilde{\mathscr{V}}$ is locally free.

(10.1.2) The Gauss-Manin connection ∇ on $\mathscr{V}$ extends to a connection $\tilde{\nabla}$ on $\tilde{\mathscr{V}}$ with logarithmic poles along $0 \times \Delta^r$.

(10.1.3) $\mathrm{Res}_{0 \times \Delta^r}(\tilde{\nabla})$ is nilpotent.

Recall that $\tilde{\nabla}$ is the connecting homomorphism of the hypercohomology of the exact sequence

$$0 \longrightarrow f^* \Omega^1_{\Delta \times \Delta^r}(\log(0 \times \Delta^r)) \otimes \Omega_f^\cdot(\log(\mathscr{Y} + \mathscr{X}_0))[-1]$$
$$\longrightarrow \Omega_{\mathscr{X}}^\cdot(\log(\mathscr{Y} + \mathscr{X}_0)) \longrightarrow \Omega_f^\cdot(\log(\mathscr{Y} + \mathscr{X}_0)) \longrightarrow 0.$$

(10.1.2) follows from this. The formation of $\tilde{\mathscr{V}}$ and $\tilde{\nabla}$ commutes with restriction of the base $\Delta \times t \hookrightarrow \Delta \times \Delta^r$, so that (10.1.1) and (10.1.3) are reduced to the 1-dimensional case [S.Z]. Q.E.D.

(10.2) **Corollary.** *$\tilde{\mathscr{V}}|_{0\times\Delta^r}\simeq R^n f_*(\Omega_f^{\cdot}(\log(\mathscr{Y}+\mathscr{X}_0))\otimes\mathcal{O}_{\mathscr{X}_0})$ holds and it is a locally free $\mathcal{O}_{0\times\Delta^r}$-module.*

Consider the double complex $(A^{\cdot\cdot}, d', d'')$ defined by

$$A^{p,q}=\Omega_{\mathscr{X}}^{p+q+1}(\log(\mathscr{Y}+\mathscr{X}_0))/W(\mathscr{X}_0)_q,$$

where $d'=d$ is the exterior differentiation, while

$$d''=\cdot\wedge f^*(ds/s).$$

The filtrations $W(\mathscr{Y})$, $W(\mathscr{Y}+\mathscr{X}_0)$ and F on $\Omega_{\mathscr{X}}^{\cdot}(\log(\mathscr{Y}+\mathscr{X}_0))$ induce

$$W_k A^{p,q}=\text{the image of } W(\mathscr{Y})_k\Omega_{\mathscr{X}}^{p+q+1}(\log(\mathscr{Y}+\mathscr{X}_0)) \quad \text{in } A^{p,q},$$
$$M_m A^{p,q}=\text{the image of } W(\mathscr{Y}+\mathscr{X}_0)_{2q+m+1}\Omega_{\mathscr{X}}^{p+q+1}(\log(\mathscr{Y}+\mathscr{X}_0)) \quad \text{in } A^{p,q},$$
$$F^pA^{\cdot\cdot}=\bigoplus_{p'\geq p} A^{p',\cdot}.$$

Define $A^{\cdot}$ to be the simple complex associated to $A^{\cdot\cdot}$.

By the argument in [St, (4.15)], we can generalize [S.Z, (5.5)] as follows:

(10.3) **Lemma.** *The morphism*

$$\theta\colon (\Omega_f^{\cdot}(\log(\mathscr{Y}+\mathscr{X}_0))\otimes\mathcal{O}_{\mathscr{X}_0}, W(\mathscr{Y}), F)\longrightarrow(A^{\cdot}, W, F)$$

induced by the exterior product with $f^*(ds/s)$ is a quasi-isomorphism of bifiltered complexes.

[S.Z, (5.6)] can be rewritten as follows:

(10.4) **Theorem.** *$(A^{\cdot}, W, M, F)$ is a filtered mixed Hodge complex (for the definition, see* [E] *or* [S.Z, § 6]*). Therefore $\tilde{\mathscr{V}}|_{0\times\Delta^r}=R^n f_{0*}A^{\cdot}$ carries a* VGPMHS.

For the proof of this theorem, we can construct a complex, which gives a $\boldsymbol{Q}$-structure of $A^{\cdot}$ in a way similar to that in [S.Z, § 5], and prove that $\mathrm{Gr}_m^M A^{\cdot}$ is a cohomological pure Hodge complex of weight m. (The formula in [S.Z, (5.22)] can be generalized to fit our context.)

Let $\Phi\colon \Delta^*\times\Delta^r\to\langle T\rangle\backslash D$ be the mixed period map associated to the VGPMHS $(\Delta^*\times\Delta^r, R_{\boldsymbol{Z}}^n(\mathring{f}'), W[n], F, Q)$, where T is the local monodromy. Since $\mathscr{X}_0$ is reduced, T is unipotent. Set $N=\log T$ and

$$\Psi(s,t)=\exp(-(\log s)N/2\pi i)\Phi(s,t) \qquad \text{for } (s,t)\in\Delta^*\times\Delta^r.$$

Then in the notation in Section 2, we have the following in the same way as in [S.Z, (5.7)]:

(10.5) **Corollary** (cf. [S.Z, (3.13)]).

(10.5.1) *$NM_m \subset M_{m-2}$ for all m.*

$$N^\ell\colon \mathrm{Gr}^M_{k+\ell}\mathrm{Gr}^{W[n]}_k R^n f_{0*}A^\cdot \xrightarrow{\sim} \mathrm{Gr}^M_{k-\ell}\mathrm{Gr}^{W[n]}_k R^n f_{0*}A^\cdot \qquad \text{for all } k \text{ and } \ell.$$

(10.5.2) *$\Psi\colon \Delta^*\times\Delta^r \to D$ extends to a holomorphic map $\tilde{\Psi}\colon \Delta\times\Delta^r \to \check{D}$.*

(10.5.3) *$\tilde{\Psi}(0, t) =: F_\infty(t) \in \check{D}$ satisfies:*

(10.5.3.1) *$((W[n]_k)_Z, M, F_\infty(t))$ is a* MHS *for each k and N gives a morphism of type* $(-1, -1)$.

(10.5.3.2) *$((\mathrm{Gr}^{W[n]}_k)_Z, M, F_\infty(t), Q_k) = \tilde{\Psi}_k(0, t)$ for each k, where $\tilde{\Psi}_k$ is the extension of the Ψ_k associated to the period map Φ_k.*

11. Abstract log complex for *d*-semi-stable pairs

In this section we will introduce an analogue in our context of the abstract log complex for a *d*-semi-stable variety in Friedman [F.1].

(11.1) **Definition.** A *pair (X, Y) with simple normal crossing* (SNC *pair* for short) consists of a compact connected reduced variety $X = \bigcup X_i$ with SNC of pure dimension n and a reduced Cartier divisor Y on X satisfying the following conditions:

(11.1.1) At every point of Y, there exist an open neighborhood U in the classical topology of an ambient space of X and a reduced subvariety $\mathscr{Y}$ of U with SNC of pure dimension n such that $(X\cap U)\cup\mathscr{Y}$ is also a variety with SNC and $Y\cap U = \mathscr{Y}\cap X\cap U$.

(11.1.2) Every component of Y is smooth.

For a variety $Z = \bigcup_{1\leqq i\leqq r} Z_i$ with SNC of pure dimension, we use the notation:

$$Z^{(p)} = \bigcup_{1\leqq i_0<\cdots<i_p\leqq r} (Z_{i_0}\cap\cdots\cap Z_{i_p}) \subset Z$$

$$a\colon Z^{[p]} = \coprod (Z_{i_0}\cap\cdots\cap Z_{i_p}) \longrightarrow Z^{(p)} \subset Z \quad \text{the normalization.}$$

Denote, in paricular, $\tilde{Z} = Z^{[0]}$.

(11.2) **Definition.** (11.2.1) (Friedman [F.1, (1.9) and (1.13)]. Let $Z = \bigcup Z_i$ be an SNC variety of pure dimension. The *infinitesimal normal bundle $\mathcal{O}_D(Z)$ of the double locus* $D := \mathrm{Sing}(Z)$ in Z is defined as the dual to $\mathcal{O}_D(-Z) := \otimes_i(\mathscr{I}_{Z_i}\otimes\mathcal{O}_D)$, where $\mathscr{I}_{Z_i}$ is the sheaf of ideals of Z_i in Z. Z is *d-semi-stable* if $\mathcal{O}_D(Z) = \mathcal{O}_D$.

(11.2.2) An SNC pair (X, Y) is called *d-semi-stable* if X and $Y^{(p)}$ for all p are *d*-semi-stable.

For an SNC pair (X, Y), set $Y^i = Y \cap X_i$, which is an SNC divisor on a component X_i, and set $\bar{Y} = \coprod Y^i$, which is an SNC divisor on $\tilde{X}$. Denote also $D^i = X_i \cap (\bigcup_{j \neq i} X_j)$, which is an SNC divisor on X_i, and $\bar{D} = \coprod D^i$, which is an SNC divisor on $\tilde{X}$.

We will define a subcomplex $\Lambda_X^{\cdot}(\log Y)$ of $a_*\Omega_{\tilde{X}}^{\cdot}(\log(\bar{Y}+\bar{D}))$ which coincides with $\Omega_f^{\cdot}(\log(\mathscr{Y}+X))|_X$ (cf. (10.3)) when (X, Y) is the central fibre of a semi-stable degeneration of pairs $(\mathscr{X}, \mathscr{Y}, f, \Delta)$ (cf. § 9). As in [F.1], we consider first the model case; (X, Y) is the central fibre. By definition, the partial weight filtration $W(\bar{Y})$ on $a_*\Omega_{\tilde{X}}^1(\log(\bar{Y}+\bar{D}))$ satisfies

(11.3) $W(\bar{Y})_0 = a_*\Omega_{\tilde{X}}^1(\log \bar{D})$ and $\mathrm{res}_Y\colon \mathrm{Gr}_1^{W(\bar{Y})} \xrightarrow{\sim} a_*\mathcal{O}_{\bar{Y}}$.

(11.4) **Lemma.** (11.4.1) *The partial weight filtration* $W(Y)$ *on* $\Omega_f^1(\log(\mathscr{Y}+X))|_X$ *satisfies*

$$W(Y)_0 = \Omega_f^1(\log X)|_X \quad \textit{and} \quad \mathrm{res}_Y\colon \mathrm{Gr}_1^{W(Y)} \xrightarrow{\sim} \mathrm{Ker}\,(a_*\mathcal{O}_{\bar{Y}} \to a_*\mathcal{O}_{Y^{[1]}}).$$

(11.4.2) *There is a natural injection*

$$r\colon \Omega_f^1(\log(\mathscr{Y}+X))|_X \hookrightarrow a_*\Omega_{\tilde{X}}^1(\log(\bar{Y}+\bar{D}))$$

and the isomorphisms (11.3) *and* (11.4.1) *fit in the commutative exact diagram*

$$\begin{array}{ccccccc}
0 \to \Omega_f^1(\log X)|_X & \longrightarrow & \Omega_f^1(\log(\mathscr{Y}+X))|_X & \longrightarrow & \mathrm{Ker}\,(a_*\mathcal{O}_{\bar{Y}} \longrightarrow a_*\mathcal{O}_{Y^{[1]}}) \to 0 \\
\downarrow r & & \downarrow r & & \downarrow \\
0 \to a_*\Omega_{\tilde{X}}^1(\log \bar{D}) & \longrightarrow & a_*\Omega_{\tilde{X}}^1(\log(\bar{Y}+\bar{D})) & \longrightarrow & a_*\mathcal{O}_{\bar{Y}} \to 0.
\end{array}$$

Proof. The first equality in (11.4.1) is trivial. The second is also easy: If $(\psi_i) \in a_*\mathcal{O}_{\bar{Y}}$ belongs to $\mathrm{Ker}\,(a_*\mathcal{O}_{\bar{Y}} \to a_*\mathcal{O}_{Y^{[1]}})$, then a prolongation over a neighborhood of X in $\mathscr{X}$ of a suitable lifting $(\phi_i) \in a_*\Omega_{\tilde{X}}^1(\log(\bar{Y}+\bar{D}))$ can be defined, i.e., $\phi_i \in \Omega_{X_i}^1(\log(Y^i+D^i))$ has a prolongation $\tilde{\phi}_i$ over a neighborhood of X_i in $\mathscr{X}$ and $\tilde{\phi}_i|_{X_i \cap X_j} = \tilde{\phi}_j|_{X_j \cap X_i}$, so that $(\tilde{\phi}_i)$ can be glued together to induce a section of $\Omega_f^1(\log(\mathscr{Y}+X))|_X$. (11.4.2) can be proved in a way similar to [F.1, Lemma (3.1)].

(11.5) *Description of* $\mathrm{Im}\, r$ *in* (11.4.2) *in terms of local coordinates*: Let $(x_1, \cdots, x_{n+1})$ be local coordinates on X such that $f\colon X \to \Delta$ is given by $f(x) = x_1 \cdots x_\ell$, X_i is defined by $x_i = 0$ on X, D^i is defined by $x_1 \cdots \hat{x}_i \cdots x_l = 0$ on X_i, and Y^i is defined by $x_{\ell+1} \cdots x_{\ell+m} = 0$ on X_i.

Then $(\phi_i) \in a_*\Omega_{\tilde{X}}^1(\log(\bar{Y}+\bar{D}))$ can be written as

$$\phi_i = \sum_{k \neq i, k \leq \ell} b_{ik}\, dx_k/x_k + \sum_{\ell < k \leq \ell+m} b_{ik}\, dx_k/x_k + \sum_{k > \ell+m} b_{ik}\, dx_k,$$

and we can verify similarly as in [F. 1, § 3] that

$(\phi_i) \in r(\Omega^1_f(\log(\mathscr{Y}+X))|_X) \subset a_*\Omega^1_{\bar{X}}(\log(\bar{Y}+\bar{X}))$ *if and only if the following three conditions are satisfied*:

(11.5.1) $b_{ij}+b_{ji}=0$ *on* $X_i \cap X_j$

$b_{ij}+b_{jk}+b_{ki}=0$ *on* $X_i \cap X_j \cap X_k \quad (1 \leqq i, j, k \leqq \ell)$

(11.5.2) $b_{is}-b_{it}=b_{js}-b_{jt}$ *on* $X_i \cap X_j$

$b_{is}-b_{it}=-b_{st}$ *on* $X_i \cap X_s \quad (1 \leqq i, j, s, t \leqq \ell$ *and* $s, t \neq i, j)$

$b_{ik}=b_{jk}$ *on* $X_i \cap X_j \quad (k > \ell+m)$

(11.5.3) $b_{ik}=b_{jk}$ *on* $Y^i \cap Y^j$ *if* $Y^i \cap Y^j \neq \phi$.

(11.5.4) **Remark.** (11.5.3) corresponds to the *satisfied* isomorphism in (11.4.1).

(11.6) **Definition.** Let (X, Y) be a d-semi-stable pair (11.2.2). Define $\Lambda^1_X(\log Y)=\{(\phi_i) \in a_*\Omega^1_{\bar{X}}(\log(Y+D)) \mid (\phi_i)$ satisfies (11.5.1), (11.5.2) and (11.5.3)$\}$ and

$$\Lambda^{\cdot}_X(\log Y)=\Lambda^{\cdot}\Lambda^1_X(\log Y) \subset a_*\Omega^{\cdot}_{\bar{X}}(\log(\bar{Y}+\bar{D})).$$

We call the latter the *abstract log complex for a d-semi-stable pair* (X, Y). We denote by W and M the *weight filtrations on* $\Lambda^{\cdot}_X(\log Y)$ induced by $W(\bar{Y})$ and $W(\bar{Y}+\bar{D})$, respectively.

We can prove in the same way as in [F. 1, § 3] that:

(11.7) **Proposition**

(11.7.1) $\Lambda^1_X(\log Y)$ *is a locally free* $\mathcal{O}_X$*-module.*

(11.7.2) $W_0\Lambda^m_X(\log Y)=\Lambda^m_X$ *and* $\mathrm{res}_Y: \mathrm{Gr}^W_k\Lambda^m_X(\log Y) \xrightarrow{\sim} \Lambda^{m-k}_{Y(k-1)}$ *for* $k \geq 1$.

(11.7.3) $M_0\Lambda^m_X(\log Y)=\mathrm{Ker}(a_*\Omega^m_{\bar{X}} \to a_*\Omega^m_{\bar{X}[1]})$.

$$\mathrm{res}_{Y+D}: \mathrm{Gr}^M_\ell\Lambda^m_X(\log Y) \xrightarrow{\sim} \mathrm{Ker}(a_*\Omega^{m-\ell}_{(\bar{Y}+\bar{D})[\ell-1]} \longrightarrow a_*\Omega^{m-\ell}_{(\bar{Y}+\bar{D})[\ell]}) \text{ for } \ell \geqq 1.$$

12. Problems and discussion

(12.1) **Problem.** *Compactify the mixed period map by extending it over points with finite local monodromy.*

This would be a generalization of (9.5), (9.6), and (9.11) in [Gri. 1, III]. In the case of the mixed period map arising from geometry, the extension over the points with finite local monodromy is already obtained by the results in Sections 9 and 10 and (3.2).

(12.2) **Problem.** *Generalize the Schmid theory* ([Sc], *see also* [C.K.S], [Kas]) *into the context of* VGPMHS.

(12.3) **The mixed Clemens-Schmid sequence:**

Let $(\mathscr{X}, \mathscr{Y}, f, \Delta)$ be a semi-stable degeneration of pairs with $\dim \mathscr{X}= n+1$ (cf. § 9). Set

$$\mathring{\mathscr{X}}=\mathscr{X}-\mathscr{Y}\hookleftarrow \mathring{X}_0=X_0-Y_0,$$

$$\mathring{\mathscr{X}}^*=\mathring{\mathscr{X}}-\mathring{X}_0\hookleftarrow \mathring{X}_t=X_t-Y_t \quad \text{for some } t\in\Delta^*.$$

T: the local monodromy on $H^\cdot(\mathring{X}_t)$, and $N=\log T$.

Note that $\mathring{\mathscr{X}}\hookleftarrow\mathring{X}_0$ has a retraction. The local cohomology sequence and the Wang sequence give rise to the diagram:

$$\begin{array}{ccccccccc}
H^q(\mathring{X}_0) & \dashrightarrow & H^q(\mathring{X}_t) & \xrightarrow{N} & H^q(\mathring{X}_t) & \dashrightarrow & H^{q+2}_{\mathring{X}_0}(\mathring{\mathscr{X}}) & \longrightarrow & H^{q+2}(\mathring{X}_0)\\
& \searrow & & \nearrow & & \searrow & & \nearrow & \\
& & H^q(\mathring{\mathscr{X}}^*) & & & & H^{q+1}(\mathring{\mathscr{X}}^*) & & \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \\
H^{q-1}(\mathring{X}_t) & \dashrightarrow & H^{q+1}_{\mathring{X}_0}(\mathring{\mathscr{X}}) & \rightarrow & H^{q+1}(\mathring{X}_0) & \dashrightarrow & H^{q+1}(\mathring{X}_t) & \xrightarrow{N} & H^{q+1}(\mathring{X}_t)\\
& & \wr\| & & \wr\| & & & & \\
& & H^{q+1}(\mathring{\mathscr{X}},\mathring{\mathscr{X}}^*) & & H^{q+1}(\mathring{\mathscr{X}}) & & & & \\
& & \wr\| & & & & & & \\
& & H_{2n+2-q-1}(\mathring{X}_0) & & & & & &
\end{array}$$

(12.3.1) **Problem.** *Show the exactness of the horizontal sequences in the above diagram.*

(12.4) **Problem.** *Study the deformation theory for d-semi-stable pairs* (cf. (11.2.2) and [F. 1], [P]).

(12.5) **Problem.** *Prove the infinitesimal mixed Torelli theorem for d-semi-stable pairs.*

(12.6) **Comparison between the mixed period map and the period maps:**

Let $P(x)\in \boldsymbol{Q}[x]$ be a polynomial of degree n with integral values. Let $\mathscr{M}^P$ be the set of isomorphism classes $[X, M]$ of pairs (X, M) consisting of a smooth projective variety X and an ample line bundle M on X satisfying $P(a)=\chi(M^{\otimes a})$ for all integers a. Assume that $\mathscr{M}^P$ is nonempty. Fix a positive integer m and denote by $\mathscr{M}$ the set of isomorphism classes $[X, Y]$ of pairs (X, Y) consisting of a smooth projective variety X and a smooth divisor Y for which there exists $[X, M]\in\mathscr{M}^P$ such that $\mathscr{O}_X(Y)= M^{\otimes m}$. We denote also by $\mathscr{M}_n^{\mathrm{pol}}$ (resp. $\mathscr{M}_n$, $\mathscr{M}_{n+1}^{\mathrm{pol}}$, $\mathscr{M}_{n+1}$) the set of isomorphism classes $[X, \mathscr{O}_X(Y)]$ (resp. $[X]$, $[Y, \mathscr{O}_Y(Y)]$, $[Y]$) for $[X, Y]\in\mathscr{M}$.

Then we have natural surjections

(12.6.1) $$\mathcal{M}_n \overset{p''_n}{\twoheadleftarrow} \mathcal{M}_n^{\mathrm{pol}} \overset{p'_n}{\twoheadleftarrow} \mathcal{M} \overset{p'_{n+1}}{\twoheadrightarrow} \mathcal{M}_{n+1}^{\mathrm{pol}} \overset{p''_{n+1}}{\twoheadrightarrow} \mathcal{M}_{n+1}.$$

The infinitesimal versions are given by the cohomology diagrams of the commutative exact diagrams as in Diagram 1 (cf. §§ 4 and 6).

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X & = & \mathcal{O}_X & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & T_X(-\log Y) & \to & D_1(L, L) & \to & L & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & T_X(-\log Y) & \to & T_X & \to & N_{Y/X} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{O}_Y & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & T_X(-Y -\log Y) & \to & T_X(-\log Y) & \to & D_1(L|_Y, L|_Y) & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & T_X(-Y) & \to & T_X(-\log Y) & \to & T_Y & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{O}_Y & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Diagram 1

(12.6.2) **Lemma.** *For the maps in* (12.6.1), *the following hold*:

(12.6.2.1) $p_n'^{-1}([X, L]) = |L|_{\mathrm{reg}}/\mathrm{Aut}\,(X, L)$ (cf. (8.1)).

(12.6.2.2) p''_n *is injective if* M *of* $[X, M] \in \mathcal{M}^P$ *is the canonical bundle of* X.

(12.6.2.3) p'_{n+1} *is injective if the integer* m *is sufficiently large* (*see* (12.6.2.5) *below*), $H_1(X, \boldsymbol{Z}) = 0$ *and if* $n = \dim X \geqq 3$.

(12.6.2.4) p''_{n+1} *is injective if* M *of* $[X, M] \in \mathcal{M}^p$ *is the canonical bundle of* X, $H_1(X, \boldsymbol{Z}) = 0$ *and if* $n = \dim X \geqq 3$.

Proof. (12.6.2.1) and (12.6.2.2) are trivial.

In order to prove (12.6.2.3), take positive integers k, ℓ and m satisfying:

(12.6.2.5) $f_{|M^{\otimes k}|}: X \to \boldsymbol{P} = \boldsymbol{P}^N$ $(N = h^0(M^{\otimes k}) - 1)$ *is an embedding for all* $[X, M] \in \mathcal{M}^P$. $H^0(\mathcal{I}_X \otimes \mathcal{O}_{\boldsymbol{P}}(\ell)) \otimes H^0(\mathcal{O}_{\boldsymbol{P}}(a)) \to H^0(\mathcal{I}_X(\ell + a))$ *is surjective for all* $a \geq 0$ *and all* $[X, M] \in \mathcal{M}^P$, *where* $\mathcal{I}_X$ *is the sheaf of ideals of* X *in* $\boldsymbol{P}$. $m > \ell^{N-n+1}/k^{n-1}d$, *where* $d/n!$ *is the leading coefficient of the Hilbert polynomial* $P(x)$.

The existence of the integer k above is the assertion of Matsusaka's Big Theorem ([Ma. 2], see also [L.M]). The existence of the integer ℓ above can be seen as follows. Let H be the Hilbert scheme of the embedded $X \subset \boldsymbol{P}$ and $\mathcal{X} \subset \boldsymbol{P} \times H \xrightarrow{p} H$ the universal family ([Gro. 2]). Take a positive integer ℓ_1 so that $R^i p_*(\mathcal{I}_{\mathcal{X}}(a)) = 0$, $p^* p_*(\mathcal{I}_{\mathcal{X}}(a)) \twoheadrightarrow \mathcal{I}_{\mathcal{X}}(a)$ is surjective and $R^i p_*(\mathcal{O}_{\mathcal{X}}(a)) = 0$ for all $i > 0$ and all $a \geq \ell_1$ (cf. [Gro. 1]). Then we have an exact sequence

$$0 \longrightarrow p_*(\mathcal{I}_{\mathcal{X}}(a)) \longrightarrow p_*(\mathcal{O}_{\boldsymbol{P} \times H}(a)) \longrightarrow p_*(\mathcal{O}_{\mathcal{X}}(a)) \longrightarrow 0 \quad \text{for all } a \geq \ell_1.$$

Since $\mathcal{O}_{\boldsymbol{P} \times H}(a)$ and $\mathcal{O}_{\mathcal{X}}(a)$ are p-flat, we see by the Continuity Theorem ([Gro. 1]) that these sheaves are cohomologically flat in dimension 0. In particular, $p_*(\mathcal{O}_{\boldsymbol{P} \times H}(a))$ and $p_*(\mathcal{O}_{\mathcal{X}}(a))$ are locally free, therefore so is $p_*(\mathcal{I}_{\mathcal{X}}(a))$ for all $a \geq \ell_1$. Hence $\mathcal{I}_{\mathcal{X}}$ is p-flat by a corollary to the Base Change Theorem ([Gro. 1], see also [Mu, p. 52, Cor. 3]). Since

$$H^i(\mathcal{I}_{X_t}(a)) = 0 \quad \text{for all } i > 0, \text{ all } a \geq \ell_1 \text{ and all } t \in H$$

by another corollary to the Base Change Theorem ([Gro. 1], see also [Mu, p. 52, Corollary $2\frac{1}{2}$]), the function

$$t \longmapsto h^0(\mathcal{I}_{X_t}(a)) = \chi(\mathcal{I}_{X_t}(a))$$

is locally constant for all $a \geq \ell_1$. Again by the Continuity Theorem, $\mathcal{I}_{\mathcal{X}}(a)$ is cohomologically flat in dimension 0 for all $a \geq \ell_1$. Denote by $\mathcal{K}$ the kernel of the canonical homomorphism $p^* p_*(\mathcal{I}_{\mathcal{X}}(\ell_1)) \to \mathcal{I}_{\mathcal{X}}(\ell_1)$. Take a positive integer ℓ_2 such that $R^1 p_*(\mathcal{K}(a)) = 0$ for all $a \geq \ell_2$. Then we have a surjection

$$p_*(p^* p_*(\mathcal{I}_{\mathcal{X}}(\ell_1)) \otimes \mathcal{O}_{\boldsymbol{P} \times H}(a)) \twoheadrightarrow p_*(\mathcal{I}_{\mathcal{X}}(\ell_1 + a)) \quad \text{for all } a \geq \ell_2.$$

This yields, by the projection formula and the cohomological flatness, a surjection

$$H^0(\mathcal{I}_{X_t}(\ell_1)) \otimes H^0(\mathcal{O}_P(a)) \twoheadrightarrow H^0(\mathcal{I}_{X_t}(\ell_1 + a)) \quad \text{for all } a \geq \ell_2 \text{ and all } t \in H.$$

This fits in the commutative diagram:

$$\begin{array}{ccc} H^0(\mathscr{I}_{X_t}(\ell_1))\otimes H^0(\mathcal{O}_P(a)) & \twoheadrightarrow & H^0(\mathscr{I}_{X_t}(\ell_1+a)) \\ \uparrow\!\!\uparrow & & \uparrow \\ H^0(\mathscr{I}_{X_t}(\ell_1))\otimes H^0(\mathcal{O}_P(1))^{\otimes a} & \longrightarrow & H^0(\mathscr{I}_{X_t}(\ell_1+\ell_2))\otimes H^0(\mathcal{O}_P(a-\ell_2)) \end{array}$$

for all $a \geqq \ell_2$ and all $t \in H$.

Thus we can take $\ell_1+\ell_2$ as the integer ℓ in (12.6.2.5).

Now we will prove (12.6.2.3) for the integer m in (12.6.2.5). Let $[X, Y]$, $[X', Y'] \in \mathscr{M}$ and suppose there exists an isomorphism $g\colon Y \cong Y'$ such that $g^*(\mathcal{O}_{Y'}(Y')) = \mathcal{O}_Y(Y)$. Then $g^*(\mathcal{O}_{Y'}(m)) = g^*(\mathcal{O}_{Y'}(kY')) = \mathcal{O}_Y(kY) = \mathcal{O}_Y(m)$. Hence $g^*(\mathcal{O}_{Y'}(1)) \otimes \mathcal{O}_Y(-1)$ is a torsion sheaf. But the assumption in (12.6.2.3) implies that Pic Y has no torsion by the Lefschetz Hyperplane Theorem and the Universal Coefficient Theorem. Hence $g^*(\mathcal{O}_{Y'}(1)) = \mathcal{O}_Y(1)$ and g comes from a projective transformation of $\boldsymbol{P}$. Now suppose that there exist X and X' containing Y in $\boldsymbol{P}$. Choose a maximal regular sequence $f_1, \cdots, f_{N-n+1}$ in $H^0(\mathscr{I}_X \otimes \mathcal{O}_P(\ell)) + H^0(\mathscr{I}_{X'} \otimes \mathcal{O}_P(\ell))$ and set $Z = \{f_1 = \cdots = f_{N-n+1} = 0\}$. Since $Z \supset Y$, we have $\ell^{N-n+1} = \deg Z \geqq \deg Y = mk^{n-1}d$ which contradicts the choice of m in (12.6.2.5).

We can prove (12.6.2.4) in a similar way. Q.E.D.

(12.6.3) **Problem.** *Improve Lemma* (12.6.2) *by using* [Sh] *and* [Kaw. 3].

(12.6.4) *Case* $n = \dim X = 2$: Consider the set $\mathscr{M}^P = \{[X, \omega_X]\}$ of isomorphism classes of pairs of minimal surfaces of general type and their canonical bundles with a fixed Hilbert polynomial $P(x)$. In this case, (12.6.1) can be regarded as a diagram in the category of quasi-projective schemes (cf. [Gi]) and for the integer k in (12.6.2.5) we can take $k=5$, i.e., $f_{|\omega_X^{\otimes 5}|}\colon X \to \boldsymbol{P}$ is now a birational embedding (cf. [Bo]). Take positive integers ℓ and m as in (12.6.2.5). Then, as in the proof of (12.6.2), $(Y, \mathcal{O}_Y(1)) = (Y', \mathcal{O}_{Y'}(1))$ implies $(X, Y) = (X', Y')$.

(12.6.4.1) **Problem.** *In the above situation, compute the degree of* $p_3 := p_3'' \circ p_3'$ *in* (12.6.1). *Is it true that the mixed period map* $\Phi\colon \mathscr{M} \to G_Z \backslash D$ *is injective*?

(12.6.5) **Problem.** *Investigate the relationship between the mixed period map* $\Phi\colon \mathscr{M} \to G_Z \backslash D$ *and the period maps* $\Phi_k\colon \mathscr{M}_k \to G_{k,Z} \backslash D_k$.

(12.6.6) **Problem.** *Develop a "Hodge theory" which corresponds to* $\mathscr{M}_k^{\mathrm{pol}}$ *in* (12.6.1).

II. Examples: Surfaces with $p_g=c_1^2=1$ and surfaces with $p_g=1$, $c_1^2=2$ and $\pi_1=Z/2Z$

1. Motivation for VGPMHS

Kŭnev first constructed a certain surface with $p_g=c_1^2=1$ ([Kŭ]), and Todorov then constructed some surfaces with $p_g=1$ and $2\leqq c_1^2\leqq 8$ ([To. 2]). These surfaces give counterexamples to both the infinitesimal and the global Torelli theorems for surfaces of general type in the sense of Griffiths ([Gri, 2]). The following names are being fixed (cf. [Morr]):

(1.1) **Definition.** (1.1.1) A *canonical surface* X is a surface which has at most canonical singularities (i.e., rational double points in the 2-dimensional case) and whose canonical sheaf ω_X is ample.

(1.1.2) A *Todorov surface* is a canonical surface X with $\chi(\mathcal{O}_X)=2$ which has an involution σ such that the quotient X/σ is a $K3$ surface with rational double points whose bi-canonical map $f_{|\omega_X^{\otimes 2}|}$ factors through X/σ.

(1.1.3) A *Kŭnev surface* is a Todorov surface with $c_1^2=1$.

Morrison showed that Todorov surfaces form an irreducible subvariety of the coarse moduli space of surfaces with $p_g=1$ in case $c_1^2=1$, 5, 6, 7, 8 and are divided into two irreducible components in case $c_1^2=2$, 3, 4 ([Morr], see also [C.D]). We are concerned here mainly with the surfaces with $p_g=c_1^2=1$ and surfaces with $p_g=1$, $c_1^2=2$ and $\pi_1=Z/2Z$ from the Hodge-theoretic view-point.

(1.1.4) We denote by $\mathcal{M}_{(i)}$ the coarse moduli space of canonical surfaces with $p_g=1$, $q=0$ and $c_1^2=i$ for $i=1, 2, \cdots, 8$ (cf. [Gi]). Throughout this chapter we fix this notation as well as the following:

$$\mathcal{M}^a_{(i)}=\{X\in\mathcal{M}_{(i)}\,|\,X \text{ is smooth}\}.$$
$$\mathcal{M}^s_{(i)}=\{X\in\mathcal{M}_{(i)}\,|\,\text{the canonical divisor of } X \text{ is smooth}\}.$$
$$\mathcal{M}^{as}_{(i)}=\mathcal{M}^a_{(i)}\cap\mathcal{M}^s_{(i)}.$$
$$\mathcal{T}_{(i)}=\{X\in\mathcal{M}_{(i)}\,|\,X \text{ is a Todorov surface}\}.$$
$$\mathcal{T}^a_{(i)}=\mathcal{T}_{(i)}\cap\mathcal{M}^a_{(i)}.$$
$$\mathcal{M}'_{(2)}=\{X\in\mathcal{M}_{(2)}\,|\,X \text{ is simply cennected}\}.$$
$$\mathcal{M}''_{(2)}=\{X\in\mathcal{M}_{(2)}\,|\,\pi_1(X)=Z/2Z\}.$$
$$\mathcal{T}'_{(2)}=\mathcal{T}_{(2)}\cap\mathcal{M}'_{(2)}.$$
$$\mathcal{T}''_{(2)}=\mathcal{T}_{(2)}\cap\mathcal{M}''_{(2)}.$$

Notice that the canonical divisor of a surface in $\mathcal{T}^a_{(i)}$ is a component of the fixed point locus of the involution, hence is automatically smooth.

Known Results

(1.2) **Description of the canonical rings.**

(1.2.1) *Case* $X \in \mathcal{M}_{(1)}$ (Catanese [Cat. 1]). Every $X \in \mathcal{M}_{(1)}$ can be represented as a weighted complete intersection of type (6, 6) in $\boldsymbol{P}=\boldsymbol{P}(1, 2, 2, 3, 3)$ with partially normalized equations:

$$(1.2.1.1) \qquad f=z_3^2+f^{(1)}z_4x_0+f^{(3)}, \quad g=z_4^2+g^{(1)}z_3x_0+g^{(3)},$$

where $(x_0, y_1, y_2, z_3, z_4)$ are weighted homogeneous coordinates of $\boldsymbol{P}(1, 2, 2, 3, 3)$, and $f^{(i)}$ and $g^{(i)}$ are homogeneous polynomials of degree i in $y_0:=x_0^2$, y_1 and y_2, i.e., there exist $f_i, g_i, f_{ijk}, g_{ijk} \in \boldsymbol{C}$ such that

$$f^{(1)}=\sum f_i y_i, \quad f^{(3)}=\sum f_{ijk}y_iy_jy_k, \quad g^{(1)}=\sum g_i y_i, \quad g^{(3)}=\sum g_{ijk}y_iy_jy_k.$$

Conversely, if X is a weighted complete intersection of type (6, 6) in $\boldsymbol{P}$ with at most rational double points, then $X \in \mathcal{M}_{(1)}$. Moreover, two pairs (f, g) and (f', g') as in (1.2.1.1) give rise to isomorphic surfaces if and only if there exists a projective transformation $\sigma\colon \boldsymbol{P}\to\boldsymbol{P}$ such that

$$(1.2.1.2) \qquad \begin{aligned}&\sigma(x_0, y_1, y_2, z_3, z_4)\\ &\quad=(d_0x_0, d_{10}x_0^2+d_{11}y_1+d_{12}y_2, d_{20}x_0^2+d_{21}y_1+d_{22}y_2, d_3z_3, d_4z_4)\end{aligned}$$

with $f'=\sigma f/d_3^2$ and $g'=\sigma g/d_4^2$, or

$$\begin{aligned}&\sigma(x_0, y_1, y_2, z_3, z_4)\\ &\quad=(d_0x_0, d_{10}x_0^2+d_{11}y_1+d_{12}y_2, d_{20}x_0^2+d_{21}y_1+d_{22}y_2, d_3z_4, d_4z_3)\end{aligned}$$

with $g'=\sigma f/d_3^2$ and $f'=\sigma g/d_4^2$.

(1.2.2) *Case* $X \in \mathcal{M}''_{(2)}$ (Catanese and Debarre [C.D]). Every $X \in \mathcal{M}''_{(2)}$ can be represented as the quotient $\tilde{X}/\tilde{\tau}$ of a weighted complete intersection $\tilde{X}$ of type (4, 4) in $\boldsymbol{P}=\boldsymbol{P}(1, 1, 1, 2, 2)$ with partially normalized equations:

$$(1.2.2.1) \qquad \begin{cases} f=z_3^2+f^{(1)}wz_4+f^{(0)}w^4+f^{(2)}w^2+f^{(4)}\\ g=z_4^2+g^{(1)}wz_3+g^{(0)}w^4+g^{(2)}w^2+g^{(4)}\end{cases}$$

where (w, x_1, x_2, z_3, z_4) are weighted homogeneous coordinates of $\boldsymbol{P}(1, 1, 1, 2, 2)$ and

$f^{(i)}$ and $g^{(i)}$ are homogeneous polynomials of degree i in x_1 and x_2,

$f^{(4)}$ and $g^{(4)}$ are mutually prime,

$f^{(0)}$ and $g^{(0)}$ are not both zero, and

$\tilde{\tau}$ is the involution $\tilde{\tau}(w, x_1, x_2, z_3, z_4)=(w, -x_1, -x_2, -z_3, -z_4)$.

Conversely, if $\tilde{X}$ is a weighted complete intersection of type (4, 4) in $\boldsymbol{P}$ which has at most rational double points and does not meet the fixed point locus of $\tilde{\tau}$, then the quotient $\tilde{X}/\tilde{\tau} \in \mathcal{M}''_{(2)}$. Moreover, two pairs of (f, g) and (f', g') as in (1.2.2.1) give rise to isomorphic surfaces if and only if there exists a projective transformation $\tilde{\sigma}$: $\boldsymbol{P}\to\boldsymbol{P}$ such that

(1.2.2.2) $\quad \tilde{\sigma}(w, x_1, x_2, z_3, z_4)=(d_0w, d_{11}x_1+d_{12}x_2, d_{21}x_1+d_{22}x_2, d_3z_3, d_4z_4)$

with $f'=\tilde{\sigma}f/d_3^2$ and $g'=\tilde{\sigma}g/d_4^2$, or

$$\tilde{\sigma}(w, x_1, x_2, z_3, z_4)=(d_0w, d_{11}x_1+d_{12}x_2, d_{21}x_1+d_{22}x_2, d_3z_4, d_4z_3)$$

with $g'=\tilde{\sigma}f/d_3^2$ and $f'=\tilde{\sigma}g/d_4^2$.

(1.3) **Hodge numbers and moduli numbers.** *For $X \in \mathcal{M}^a_{(i)}$, we have:*

(1.3.1) $\quad h^{2,0}(X)=h^{0,2}(X)=1, \quad h^{1,1}_{\mathrm{prim}}(X)=19-i.$

$\quad H^0(T_X)=0, \ \chi(T_X)=-(20-2i).$

(1.3.2) $\quad H^2(T_X)=0 \quad$ *for* $X \in \mathcal{M}^a_{(1)} \cup \mathcal{M}^a_{(2)}$.

(1.3.3) *In case $X \in \mathcal{T}^a_{(i)}$ with the involution σ, we see as for the σ-invariant part that:*

$$h^{2,0}(X)^\sigma=h^{0,2}(X)^\sigma=1, \quad h^{1,1}_{\mathrm{prim}}(X)^\sigma=11-i.$$
$$h^1(T_X)^\sigma=12, \quad h^2(T_X)^\sigma=0.$$

Indication. (1.3.1) follows from the Riemann-Roch formula. For (1.3.2), see, e.g., [U.1], [U.2] and [C.D]. As for (1.3.3), we can calculate the desired numbers in a way similar to [U.5] with the aid of the double cover $X\to X/\sigma$.

(1.3.4) **Remark-Problem.** For $X \in \mathcal{T}^a_{(i)}$, construct the diagram:

(1.3.4.1)
$$\begin{array}{ccc} X & \xleftarrow{\ p\ } & \hat{X} \\ \downarrow & & \downarrow q \\ X':=X/\sigma & \xleftarrow{\ p'\ } & \hat{X}':=\hat{X}/\hat{\sigma} \end{array}$$

where p' is the minimal resolution, p is the blowing-up of the isolated fixed points of σ, and $\hat{\sigma}$ is the induced involution. Denote by R the ramification divisor of the double cover q in (1.3.4.1) and by L the line bundle on $\hat{X}'$ such that $q^*L=\mathcal{O}_{\hat{X}'}(R)$. Dualizing the exact sequence

$$0\longrightarrow q^*\Omega^1_{\hat{X}'}\longrightarrow \Omega^1_{\hat{X}}\longrightarrow \Omega^1_q\simeq N_{R/\hat{X}}\longrightarrow 0,$$

we get

$$0\longrightarrow T_{\hat{X}}\longrightarrow q^*T_{\hat{X}'}\longrightarrow \mathrm{Ext}^1(N_{R/\hat{X}}, \mathcal{O}_{\hat{X}})\simeq \Omega^1_R\longrightarrow 0.$$

This yields the cohomology sequence

$$\begin{array}{ccccccc} H^1(q^*T_{\hat{X}'}) & \longrightarrow & H^1(\Omega^1_R) & \longrightarrow & H^2(T_{\hat{X}}) & \longrightarrow & H^2(q^*T_{\hat{X}'})\longrightarrow 0. \\ \wr\,\big| & \nearrow & & & & & \\ H^1(\Omega^1_{\hat{X}'})\oplus H^1(\Omega^1_{\hat{X}'}\otimes L^{-1}) & & & & & & \end{array}$$

Hence

$$h^2(T_X)=h^2(T_{\hat{X}})=h^2(q^*T_{\hat{X}'})=h^2(T_{\hat{X}'})+h^2(T_{\hat{X}'}\otimes L^{-1})=h^0(\Omega^1_{\hat{X}'}\otimes L).$$

On the other hand, (1.3.1) and (1.3.3) yield $h^2(T_X)\geqq \chi(T_X)+h^1(T_X)^\sigma=2i-8$. *The problem is to calculate* $h^0(\Omega^1_{\hat{X}'}\otimes L)$.

(1.4) **Generic infinitesimal Torelli theorem.**

(1.4.1) *Case* $\mathcal{M}^a_{(1)}$ (Catanese [Cat. 1]). Let $X\in\mathcal{M}^a_{(1)}$. The Kuranishi space S of the deformations of $X=X_0$ $(0\in S)$ is smooth by (1.3.2). Let

(1.4.1.1) $$\phi_2\colon S\longrightarrow D_2$$

be the period map of the second cohomology. By using the representation (1.2.1), the *defining equation of the ramification locus of* ϕ_2 can be calculated as

(1.4.1.2) $$\Delta:=\det\left|\begin{array}{cc|cc|cc} f_1 & & 3f_{111} & f_{112} & & \\ f_2 & f_1 & 2f_{112} & 2f_{122} & & \\ & f_2 & f_{122} & 3f_{222} & & \\ \hline g_1 & & & & 3g_{111} & g_{112} \\ g_2 & g_1 & & & 2g_{112} & 2g_{122} \\ & g_2 & & & g_{122} & 3g_{222} \end{array}\right|=0$$

for suitable local coordinates of S and D_2 (see also [U.1]). It is easy to see that $\Delta\neq 0$ *for general* $X=\{f=g=0\}$ *but* $\Delta=0$ *for special* X.

(1.4.2) *Case* $\mathcal{M}''^a_{(2)}$ (cf. Oliverio [O]). Let $X\in\mathcal{M}''^a_{(2)}$. We claim:

(1.4.2.1) *The infinitesimal period map*

$$d\phi_2(0)\colon H^1(T_X)\longrightarrow \mathrm{Hom}\,(H^0(\Omega_X^2),\ H^1_{\mathrm{prim}}(\Omega_X^1))$$

is injective for general X but not injective for special X.

Since there seems to be a gap in [O], we will give here an outline of the proof of the correction (1.4.2.1). We use the notation $\boldsymbol{P}$, $\tilde{X}$, f, g etc. in (1.2.2). Notice that $\Omega^2_{\tilde{X}}\simeq \mathcal{O}_{\tilde{X}}(1)$. The exact sequences

$$0\longrightarrow T_{\tilde{X}}\longrightarrow T_{\boldsymbol{P}}|_{\tilde{X}}\longrightarrow N_{\tilde{X}/\boldsymbol{P}}\longrightarrow 0 \quad \text{and}$$
$$0\longrightarrow \mathcal{O}_{\tilde{X}}\longrightarrow \bigoplus_i \mathcal{O}_{\tilde{X}}(e_i)\longrightarrow T_{\boldsymbol{P}}|_{\tilde{X}}\longrightarrow 0,$$

where $(e_0, e_1, e_2, e_3, e_4)=(1, 1, 1, 2, 2)$, give the commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & & & H^0(N_{\tilde{X}/\boldsymbol{P}}) & \longrightarrow & H^1(T_{\tilde{X}}) & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
0\longrightarrow H^0(\mathcal{O}_{\tilde{X}}(1)) & \longrightarrow & \oplus H^0(\mathcal{O}_{\tilde{X}}(e_i+1)) & \longrightarrow & H^0(N_{\tilde{X}/\boldsymbol{P}}(1)) & \longrightarrow & H^1_{\mathrm{prim}}(\Omega^1_{\tilde{X}}) & \longrightarrow & 0
\end{array}$$

Here the vertical maps are induced by the pairing with a basis of $H^0(\Omega_X^2)$ $=H^0(\Omega^2_{\tilde{X}})^{\tilde{\tau}}$. Writing the zero-th cohomology in terms of the weighted homogeneous coordinate ring $\boldsymbol{C}[w, x_1, x_2, z_3, z_4]/(f, g)=:R/I$ (see also [U. 1]) and taking the $\tilde{\tau}$-invariant part, we get the commutative exact diagram:

$$\text{(1.4.2.2)}\qquad \begin{array}{ccccccccc}
 & & & & (R_4^+)^{\oplus 2} & \longrightarrow & H^1(T_X) & \longrightarrow & 0\\
 & & & & \gamma\downarrow & & \mu\downarrow & & \\
0\longrightarrow R_1^+ & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & (R_5^+)^{\oplus 2} & \longrightarrow & H^1_{\mathrm{prim}}(\Omega^1_X) & \longrightarrow & 0
\end{array}$$

where $R^\pm$ stands for the (± 1)-eigensubspaces of R with respect to $\tilde{\tau}$,

$$\alpha(w)=w(-4, 0, 0, -4, w, x_1, x_2, 2z_3, 2z_4)$$
$$\in H:=(R_1^+)^{\oplus 4}\oplus R_2^+\oplus(R_2^-)^{\oplus 2}\oplus(R_3^-)^{\oplus 2},$$
$$\beta(a_1, a_2, b_1, b_2, c_0, c_1, c_2, c_3, c_4)=\begin{pmatrix} a_1f+b_1g+\sum c_i\partial_i f\\ a_2f+b_2g+\sum c_i\partial_i g\end{pmatrix}\in (R_5^+)^{\oplus 2},$$

γ is the multiplication by w, and $\mu(\theta)=\theta\cdot\omega$ for $\omega\in H^0(\Omega_X^2)$ corresponding to w. Set

$$U=\left\{u=\begin{pmatrix} f^{(1)}wz_4+f^{(0)}w^4+f^{(2)}w^2+f^{(4)}\\ g^{(1)}wz_3+g^{(0)}w^4+g^{(2)}w^2+g^{(4)}\end{pmatrix}\in (R_4^+)^{\oplus 2}\ \middle|\ \begin{matrix}X_u \text{ is a smooth}\\ \text{surface in } \boldsymbol{P}.\end{matrix}\right\}.$$

Then U is a Zariski open set in $\boldsymbol{C}^{22}$. Take a point $u\in U$ such that $\tilde{X}_u/\tilde{\tau}$ $\simeq X$ and identify $T_U(u)\subset(R_4^+)^{\oplus 2}$. Set $E_1=E_{f1}\oplus E_{g1}=wT_U(u)\subset(R_5^+)^{\oplus 2}$ and let E_{f2} (resp. E_{g2}) be the subspace of R_5^+ spanned by the monomials in R_5^+

which do not appear in E_{f1} (resp. E_{g1}). Further, let $E_2=E_{f2}\oplus E_{g2}\subset(R_5^+)^{\oplus 2}$, and

$$\beta_i\colon H\xrightarrow{\beta}(R_5^+)^{\oplus 2}\xrightarrow{\mathrm{pr}_i}E_i \qquad \text{for } i=1,2.$$

Substituting the above in (1.4.2.2), we have

$$\text{(1.4.2.3)}\qquad \begin{array}{ccccccccccc} & & & & T_U(u) & \xrightarrow{\delta} & H^1(T_X) & \longrightarrow & 0 & & \\ & & & & \downarrow\gamma & & \downarrow\mu & & & & \\ 0\longrightarrow R_1^+ & \xrightarrow{\alpha} & H & \xrightarrow{\beta=(\beta_1,\beta_2)} & E_1\oplus E_2 & \longrightarrow & H^1_{\mathrm{prim}}(\Omega^1_X) & \longrightarrow & 0. & & \end{array}$$

Since $\operatorname{Im}\gamma=E_1$ by definition, we have an isomorphism

$$\gamma\colon \operatorname{Ker}(\mu\circ\delta)\xrightarrow{\sim}E_1\cap\operatorname{Im}\beta=\beta_1(\operatorname{Ker}\beta_2).$$

Hence, using $\dim T_U(u)=\dim E_1=22$, $h^1(T_X)=16$ ((1.3.1)+(1.3.2)), $\dim H=32$ and $\dim E_2=26$, we have

$$\begin{aligned}\dim\operatorname{Ker}\mu+(22-16)&=\dim\operatorname{Ker}(\mu\circ\delta)=\dim\operatorname{Ker}\beta_2-\dim\operatorname{Ker}\beta\\ &=\operatorname{corank}\beta_2+(32-26)-\operatorname{corank}\beta.\end{aligned}$$

Therefore

$$\text{(1.4.2.4)}\qquad \dim\operatorname{Ker}d\phi(0)=\dim\operatorname{Ker}\mu=\operatorname{corank}\beta_2-\operatorname{corank}\beta,$$

where corank means (the maximal rank)$-$(rank).

As in [O], by matrix representation of β_2, we have corank $\beta_2\geqq 1$, and the equality holds for general X. On the other hand, from (1.4.2.3), we have corank $\beta_2\geqq$corank $\beta=1$.
(The gap in [O, p. 568] is the assertion corank $\beta=0$ for general X.) Thus we get our claim (1.4.2.1).

(1.4.3) *Case* $\mathcal{M}'^{a}_{(2)}$ (Catanese [Cat. 3]). *Also in this case, the infinitesimal Torelli theorem holds for general* $X\in\mathcal{M}'^{a}_{(2)}$ *but does not hold for special* X. The proof in [Cat. 3] is based on the description of the canonical model in $\boldsymbol{P}(1,2,2,2,3,3,3,3)$ and the geometry developed in [C.D].

(1.5) **Counterexample to the generic Torelli theorem** (Catanese [Cat. 2]).

[Cat. 2] pointed out:

(1.5.1) *For any choice of monodromy group* Γ_2, *the period map of the second cohomology* $\Phi_2\colon\mathcal{M}_{(1)}\to\Gamma_2\backslash D_2$ *has degree* $\geqq 2$.

The assertion follows from the existence of $X \in \mathscr{M}^a_{(1)}$ satisfying the conditions:

(1.5.2) The differential of a local lifting $\tilde{\Phi}_2$ of Φ_2 at X has 1-dimensional kernel which is not tangent to the ramification locus $\{\Delta=0\}$ in (1.4.1.2).

(1.5.3) $\operatorname{Aut}(X)=\{\mathrm{id}\}$.

Indeed, $\tilde{\Phi}_2$ is a morphism of the manifolds of the same dimension 18 by (1.5.3), (1.3.1) and (1.3.2). Therefore (1.5.2) yields $\deg \Phi_2 \geqq \deg \tilde{\Phi}_2 \geqq 2$.

The surface $X \in \mathscr{M}^a_{(1)}$ satisfying the conditions (1.5.2) and (1.5.3) is given by, e.g.,

$$f=z_3^2+y_1x_0z_4+y_1^3+y_2^3+y_1x_0^4, \quad g=z_4^2+y_2x_0z_3+y_1^3+y_2x_0^4$$

(for the verification, see also [U. 1]).

(1.6) **Positive dimensional fibres of the period map.**

(1.6.1) *Case* $\mathscr{T}^a_{(i)}$ (Todorov [To. 1], [To. 2], and Usui [U. 1], [U. 2]). The period map of the second cohomology

$$\Phi_2\colon \mathscr{M}^a_{(i)} \longrightarrow \Gamma_2 \backslash D_2 \tag{1.6.1.1}$$

has fibres of dimension $i+1$ at every point $X \in \mathscr{T}^a_{(i)}$ $(i=c_1^2=1, 2, \cdots, 8)$.

(1.6.2) **Geometric reasoning** ([To. 1], [To. 2]): *We observe from the diagram* (1.3.4.1) *that the periods of the holomorphic 2-forms on X and on $\hat{X}'$ are equivalent data. Therefore Φ_2 in* (1.6.1.1) *distinguishes only the $K3$ surface $\hat{X}'$, and the moduli of the branch locus B of q for a fixed X' appears as the fibre of Φ_2, which has dimension $h^0(N_{B/X'})=g(C)=i+1$.*

(1.6.3) **Reasoning by the effect of automorphism on VHS** ([U.2]): $\operatorname{Aut}(X)$ *has the induced actions on the Kuranishi space S, by its universality* (1.3.1), *and on the classifying space D_2. The local period map $\phi_2\colon S \to D_2$ is* $\operatorname{Aut}(X)$*-equivariant. In particular, the fixed point loci S^σ and D_2^σ by $\sigma \in \operatorname{Aut}(X)$ must be $\phi_2(S^\sigma) \subset D_2^\sigma$. Therefore from* (1.3.3)

$$\begin{aligned}\dim_X \Phi_2^{-1}(\Phi_2(X)) &= \dim_0 \phi_2^{-1}(\phi_2(0)) = \dim S^\sigma - \dim D_2^\sigma = h^1(T_X)^\sigma - h^{1,1}_{\mathrm{prim}}(X)^\sigma \\ &= g(C) = i+1.\end{aligned}$$

(1.6.2) *Case* $\mathscr{M}^a_{(1)}$ ([U. 1], [U. 2]). [U. 2] contains a table of the classification of the automorphisms together with their actions on $H^1(T_X)$ and on $H^{p,q}(X)$ for $X \in \mathscr{M}^a_{(1)}$. By the observation (1.6.1.3), we can conclude from this table the following in the notation of (1.2.1) and (1.6.1).

(1.6.2.1) *If there exists a* $\sigma \in \operatorname{Aut}(X)$ *conjugate in* (1.2.1.2) *to the* *projective transformation*

$$\sigma_3(x_0, y_1, y_2, z_3, z_4)=(x_0, y_1, y_2, -z_3, -z_4) \qquad on\ \boldsymbol{P}(1, 2, 2, 3, 3),$$

then the period map Φ_2 *in* (1.6.1.1) *has a* 2*-dimensional fibre through* X. *These* X *form a* 12*-dimensional subvariety of* $\mathcal{M}^a_{(1)}$. *This is the case of Kŭnev surfaces already mentioned in* (1.6.1.1).

(1.6.2.2) *If there exists a* $\sigma \in \mathrm{Aut}(X)$ *conjugate in* (1.2.1.2) *to the projective transformation*

$$\sigma_1(x_0, y_1, y_2, z_3, z_4)=(x_0, y_1, y_2, z_3, -z_4)$$

$$(resp.\ \sigma_8(x_0, y_1, y_2, z_3, z_4)=(x_0, y_1, \omega y_2, z_3, z_4),\ where\ \omega=\exp(2\pi i/3))$$

on $\boldsymbol{P}(1, 2, 2, 3, 3)$, *then* Φ_2 *has a positive dimensional fibre through* X. *These* X *form a* 15 (*resp.* 9)*-dimensional subvariety of* $\mathcal{M}^a_{(1)}$.

[U. 1] gives a characterization of Kŭnev surfaces by the period map Φ_2 in (1.6.1.1) for $i=1$:

(1.6.2.3) *For* $X \in \mathcal{M}^a_{(1)}$, *we have* $X \in \mathcal{T}^a_{(1)}$ *if and only if* $\dim_X \Phi_2^{-1}(\Phi_2(X))=2$.

The idea of the proof of (1.6.2.3) is as follows in the notation of (1.4.1): We already know the defining equation Δ of the ramification locus of ϕ_2, and can calculate Ker $d\phi_2$ explicitly. Let

$$S^1=\{s \in S \mid \dim \mathrm{Ker}\ d\phi_2(s)\neq 0\},$$
$$\theta \in \mathrm{Ker}\ d\phi_2 \subset H^0(T_S|_{S^1}):\ \text{nowhere vanishing},$$
$$S^2=\{s \in S^1 \mid (\theta\Delta)(s)=0\},$$
$$S^3=\{s \in S^2 \mid (\theta(\theta\Delta))(s)=0\},\quad \text{etc.}$$

Then θ induces a nowhere vanishing vector field on $S':=\bigcap_i S^i$. Therefore, if $\dim S'>0$, the integral curve of θ through $0 \in S'$ is in the fibre of ϕ_2 through 0. Actually since the S^i have singularities, we should be more careful (for detail, see [U. 1]).

(1.6.3) *Case* $\mathcal{M}''^a_{(2)}$ ([U. 5]). [U. 5] contains a table of the classification of the automorphisms together with their actions on $H^1(T_X)$ and on $H^{p,q}(X)$ for $X \in \mathcal{M}''^a_{(2)}$. By the observation (1.6.1.3), we can conclude from this table the following in the notation of (1.2.2) and (1.6.1):

(1.6.3.1) *If there eixsts a* $\sigma \in \mathrm{Aut}(X)$ *which has a lifting* $\tilde{\sigma} \in \mathrm{Aut}(\tilde{X})$ *conjugate in* (1.2.2.2) *to the projective transformation*

$$\tilde{\sigma}_2(w, x_1, x_2, z_3, z_4)=(w, x_1, x_2, -z_3, -z_4) \qquad on\ \boldsymbol{P}(1, 1, 1, 2, 2),$$

then the period map Φ_2 *in* (1.6.1.1) *for* $i=2$ *has a* 3-*dimensional fibre through* X. *These* X *form a* 12-*dimensional subvariety of* $\mathcal{M}''^{a}_{(2)}$. *This is the case of Todorov surfaces already mentioned in* (1.6.1).

(1.6.3.2) *If there exists a* $\sigma \in \mathrm{Aut}(X)$ *which has a lifting* $\tilde{\sigma} \in \mathrm{Aut}(\tilde{X})$ *conjugate in* (1.2.2.2) *to the projective transformation*

$$\tilde{\sigma}_1(w, x_1, x_2, z_3, z_4)=(w, x_1, x_2, z_3, -z_4)$$

$$(\textit{resp.}\ \tilde{\sigma}_8(w, x_1, x_2, z_3, z_4)=(w, x_1, \sqrt{-1}\,x_2, z_3, z_4)) \quad \textit{on}\ \boldsymbol{P}(1, 1, 1, 2, 2),$$

the period map Φ_2 *has positive dimensional fibre through* X. *These* X *form a* 14 (*resp.* 6)-*dimensional subvariety of* $\mathcal{M}''^{a}_{(2)}$.

(1.6.3.3) **Problem.** *Characterize* $\mathcal{T}''^{a}_{(2)}$ *in* $\mathcal{M}''^{a}_{(2)}$ *in terms of the period map* Φ_2 *as in* (1.6.2.3).

(1.7) **Infinitesimal mixed Torelli theorem** ([U. 4], [U. 5]).

The failure of the infinitesimal Torelli theorems (1.4) and (1.5) and especially the existence of positive dimensional fibres of the period map forced us to enlarge the frame of VHS into VMHS, and we get:

(1.7.1) *The infinitesimal mixed Torelli theorem holds for smooth pairs* $(X, C) \in \mathcal{M}^{as}_{(1)} \cup \mathcal{M}''^{as}_{(2)}$ *where* $C \in |K_X|$, *i.e.*,

$$\mu\colon H^1(T_X(-\log C)) \longrightarrow \mathrm{Hom}_{(W,Q)}(H^0(\Omega^2_X(\log C)), H^1(\Omega^1_X(\log C)))$$

is injective.

In both cases we used the commutative exact diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^1(T_X(-C)) & \xrightarrow{\ \mu_2\ } & \mathrm{Hom}_{Q_2}(H^0(\Omega^2_X(\log C)), H^1_{\mathrm{prim}}(\Omega^1_X)) \\
\downarrow & & \downarrow \\
H^1(T_X(-\log C)) & \xrightarrow{\ \mu\ } & \mathrm{Hom}_{(W,Q)}(H^0(\Omega^2_X(\log C)), H(\Omega^1_X(\log C))) \\
\downarrow & & \downarrow \\
\mathrm{Im}\,\{H^1(T_X(-\log C)) \longrightarrow H^1(T_C)\} & \xrightarrow{\ \mu_3\ } & \mathrm{Hom}_{Q_3}(H^0(\Omega^1_C), H^1(\mathcal{O}_C)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

and proved the injectivity of μ_2 and μ_3 with the aid of (1.2).

(1.7.2) **Example** ([U. 4]). Let $X \in \mathcal{T}^{a}_{(1)}$ (resp. the universal cover $\tilde{X}$

of $X \in \mathcal{T}''^{a}_{(2)}$) be defined by

$$\begin{cases} f = z_3^2 + f^{(3)} \\ g = z_4^2 + g^{(3)} \end{cases} \left(\text{resp. } \begin{cases} f = z_3^2 + f^{(0)} w^4 + f^{(2)} w^2 + f^{(4)} \\ g = z_4^2 + g^{(0)} w^4 + g^{(2)} w^2 + g^{(4)} \end{cases} \right) \quad \text{(see (1.2)).}$$

In contrast to (1.7.1), *the infinitesimal Torelli theorem for* $(\Phi_2, \Phi_3) = \pi \circ \Phi$ (see (I. 1.2)) *does not hold if*

$$\frac{\partial}{\partial y_0}(f^{(3)} g^{(3)}) \equiv 0 \bmod (y_0) \qquad \textit{where } y_0 = x_0^2$$

$$\left(\textit{resp. } \frac{\partial}{\partial y_0}((f^{(0)} y_0^2 + f^{(2)} y_0 + f^{(4)})(g^{(0)} y_0^2 + g^{(2)} y_0 + g^{(4)})) \equiv 0 \bmod (y_0), \right.$$
$$\left. \textit{where } y_0 = w^2 \right).$$

For instance take $f^{(3)}$ and $g^{(3)}$ (resp. $f^{(0)} y_0^2 + f^{(2)} y_0 + f^{(4)}$ and $g^{(0)} y_0^2 + g^{(2)} y_0 + g^{(4)}$) to be Fermat type polynomials.

(1.7.3) **Problem.** *Verify the infinitesimal mixed Torelli theorem for other* $\mathcal{M}^{as}_{(i)}$.

(1.8) **Explanation for the relation among (1.6.1), (1.7.1) and (1.7.2).**
Let $X \in \mathcal{T}^{a}_{(i)}$. We use the diagram (1.3.4.1). Let C be the unique canonical curve of X, $\hat{C}$ be the proper transform of C by p, and let $\hat{C}' = q(\hat{C})$. We denote by $\hat{E}$ the union of the exceptional curves for p. Set $\hat{E}' = q(\hat{E})$, $\hat{D} = \hat{C} + \hat{E}$ and $\hat{D}' = \hat{C}' + \hat{E}'$.

Then the exact sequences

$$0 \longrightarrow p^* \Omega_X^2(\log C) \longrightarrow \Omega_{\hat{X}}^2(\log \hat{C}) \longrightarrow N_{\hat{E}/\hat{X}} \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \Omega_{\hat{X}}^2(\log \hat{C}) \longrightarrow q^* \Omega_{\hat{X}'}^2(\log \hat{D}') \longrightarrow \Omega_{\hat{E}}^1 \longrightarrow 0$$

give

(1.8.1) $$H^0(\Omega_X^2(\log C)) = H^0(\Omega_X^2(\log C))^{\sigma} \xrightarrow{\sim} H^0(p^* \Omega_X^2(\log C))^{\hat{\sigma}}$$
$$\xrightarrow{\sim} H^0(q^* \Omega_{\hat{X}'}^2(\log \hat{D}'))^{\hat{\sigma}} \xleftarrow{\sim} H^0(\Omega_{\hat{X}'}^2(\log \hat{D}')).$$

The exact sequences

$$0 \longrightarrow p^* \Omega_X^1(\log C) \longrightarrow \Omega_{\hat{X}}^1(\log \hat{C}) \longrightarrow \Omega_{\hat{E}}^1 \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \Omega_{\hat{X}}^1(\log \hat{C}) \longrightarrow q^* \Omega_{\hat{X}'}^1(\log \hat{D}') \longrightarrow \mathcal{O}_{\hat{E}} \longrightarrow 0$$

give

(1.8.2) $$H^1(\Omega_X^1(\log C))^{\sigma} \xrightarrow{\sim} H^1(p^* \Omega_X^1(\log C))^{\hat{\sigma}} \xrightarrow{\sim} H^1(q^* \Omega_{\hat{X}'}^1(\log \hat{D}'))^{\hat{\sigma}}$$
$$\xleftarrow{\sim} H^1(\Omega_{\hat{X}'}^1(\log \hat{D}')),$$

since the composite $H^0(\mathcal{O}_{\hat{E}})\to H^1(\Omega^1_{\hat{X}}(\log \hat{C}))\to H^1(\Omega^1_{\hat{E}})$ is an isomorphism. The exact sequences

$$0\longrightarrow T_{\hat{X}}(-\log \hat{C})\longrightarrow p^*T_X(-\log C)\longrightarrow \check{N}_{\hat{E}/\hat{X}}\longrightarrow 0 \quad \text{and}$$
$$0\longrightarrow q^*T_{\hat{X}'}(-\log \hat{D}')\longrightarrow T_{\hat{X}}(-\log \hat{C})\longrightarrow N_{\hat{E}/\hat{X}}\longrightarrow 0$$

yield

$$(1.8.3)\quad H^1(T_X(-\log C))^\sigma \xrightarrow{\sim} H^1(p^*T_X(-\log C))^{\hat{\sigma}} \xleftarrow{\sim} H^1(T_{\hat{X}}(-\log \hat{C}))^{\hat{\sigma}} \xleftarrow{\sim} H^1(q^*T_{\hat{X}'}(-\log \hat{D}')) \xleftarrow{\sim} H^1(T_{\hat{X}'}(-\log \hat{D}')).$$

Thus, from (1.8.1), (1.8.2) and (1.8.3), we get the commutative diagram:

$$\begin{array}{ccc}
H^1(T_X(-\log C))^\sigma & \xrightarrow{\mu} & \mathrm{Hom}_{(W,Q)}(H^0(\Omega^2_X(\log C)),\ H^1(\Omega^1_X(\log C))^\sigma) \\
\downarrow\wr & & \downarrow\wr \\
H^1(T_{\hat{X}'}(-\log \hat{D}')) & \xrightarrow{\mu'} & \mathrm{Hom}_{(W,Q)}(H^0(\Omega^2_{\hat{X}'}(\log \hat{D}')),\ H^1(\Omega^1_{\hat{X}'}(\log \hat{D}')))
\end{array}$$

where μ is the restriction to the Todorov part of the infinitesimal mixed period map for a smooth pair (X, C) and μ' is the infinitesimal mixed period map for a smooth pair $(\hat{X}', \hat{C}')$.

By using the exact sequence in (I. 4.2.2), the exact sequence of normal bundle and the residue exact sequences for $\Omega^p_{\hat{X}'}((\log \hat{D}')$ $(p=1, 2)$, μ' is divided into the commutative exact Diagram 2.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(T_{\hat{X}'}|_{\hat{D}'}) & \xrightarrow{\mu'_f} & \mathrm{Hom}(H^0(\Omega^1_{\hat{D}'}),\ H^1(\Omega^1_{\hat{X}'})^{\perp\{\hat{D}'\}}) \\
\alpha\downarrow & & \downarrow \\
H^1(T_{\hat{X}'}(-\log \hat{D}')) & \xrightarrow{\mu'} & \mathrm{Hom}_{(W,Q)}(H^0(\Omega^2_{\hat{X}'}(\log \hat{D}')),\ H^1(\Omega^1_{\hat{X}}(\log \hat{D}'))) \\
\downarrow & & \downarrow \\
T & \xrightarrow{\mu'_b} & \mathrm{Hom}_{Q_2}(H^0(\Omega^2_{\hat{X}'}),\ H^1(\Omega^1_{\hat{X}'})^{\perp\{\hat{D}'\}})\times \mathrm{Hom}_{Q_3}(H^0(\Omega^1_{\hat{D}'}),\ H^1(\mathcal{O}_{\hat{D}'})) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Diagram 2

In Diagram 2, T stands for $\mathrm{Im}\{H^1(T_{\hat{X}'}(-\log \hat{D}'))\to H^1(T_{\hat{X}'})\times H^1(T_{\hat{D}'})\}$ and $H^1(\Omega^1_{\hat{X}'})^{\perp\{\hat{D}'\}}$ is the subspace of $H^1(\Omega^1_{\hat{X}'})$ perpendicular to all the cohomology classes of components of $\hat{D}'$ with respect to the cup product. μ'_b is well-defined because, in $H^1(\Omega^1_{\hat{X}'})$, $\theta\omega\cup\xi=-\omega\cup\theta\xi=-\omega\cup 0=0$ for

$\theta \in H^1(T_{\hat{X}'}(-\log \hat{D}'))$, $\omega \in H^0(\Omega^2_{\hat{X}'})$ and the cohomology class ξ of a component of $\hat{D}'$, where $\cup$ is the cup product. α is the composed map

$$H^0(T_{\hat{X}'}|_{\hat{D}'}) \longrightarrow H^0(N_{\hat{D}'/\hat{X}'}) \longrightarrow H^1(T_{\hat{X}'}(-\log \hat{D}')),$$

and μ'_f is defined as the factorization of $\mu' \circ \alpha$.

Diagram 2 is very illustrative:

(1.6.1) *says the map* $\pi_2 \circ \mu'$, *which is essentially the infinitesimal period map for* X, *has* $(i+1)$*-dimensional kernel for* $X \in \mathcal{T}^a_{(i)}$.

(1.7.2) *says even the map* μ'_b, *which is essentially the product of the infinitesimal period maps for* (X, C), *is not injective for some* $X \in \mathcal{T}^a_{(i)}$.

(1.7.1) *says the map* μ', *which is essentially the infinitesimal mixed period map for* (X, C), *is injective for* $X \in \mathcal{T}^a_{(1)} \cup \mathcal{T}''^a_{(2)}$.

(1.8.4) **Remark-Problem.** μ'_f *in Diagram* 2 *is the infinitesimal version of the map from* (*the displacements of* $\hat{D}'$ *in a fixed* $\hat{X}'$ *without changing the moduli*) *to* (*the extension data of* GPMHS *on* $H^2(\hat{X}'-\hat{D}')$ *with fixed* $\mathrm{Gr}^W_\bullet F^\bullet$). *This has the meaning in the general set-up. Can* μ'_f *be defined directly*? *Can one prove the injectivity of* μ'_f?

2. Generic mixed Torelli theorem for Kǔnev surfaces and Todorov surfaces with $c_1^2=2$ and $\pi_1=Z/2Z$

Just after we had obtained Theorem (2.2) below, we found Letizia [L] in November, 1984. Nevertheless we would like to include here the results partly because there seems to be a gap on the monodromy in [L] and partly because we can prove the generic mixed Torelli theorem for $\mathcal{T}''_{(2)}$ (see (1.1.4)) as well.

(2.1) Recall that, by Definition (1.1.2), the bicanonical map of $X \in \mathcal{T}_{(1)}$ (resp. $\mathcal{T}''_{(2)}$) with involution σ yields a Galois cover over $\boldsymbol{P}^2$ (resp. a quadric cone $\boldsymbol{Q} \subset \boldsymbol{P}^3$) which factors through a $K3$ surface $X' := X/\sigma$ with rational double points (cf. [Cat. 1], [C.D]):

$$(2.1.1) \qquad f_{|2K|}: X \xrightarrow{q'} X' \xrightarrow{r} \boldsymbol{P}^2 \quad (\text{resp. } \boldsymbol{Q}).$$

We consider

(2.1.2) $\boldsymbol{Q}$: the complete weighted projective space $\boldsymbol{Q}(2, 1, 1)$.

Here the branch locus of $f_{|2K|}$ consists of two cubics (resp. two curves of degree 4) F and G and of a line (resp. a curve of degree 2) L in $\boldsymbol{P}^2$ (resp. $\boldsymbol{Q}$). Then r is the double cover branched over $F+G$ and q' is the double cover branched over $r^{-1}(L+F \cap G)$ (cf. (1.2)). Therefore X (resp. X') can be determined by (F, G, L) (resp. (F, G)), and we denote by

$X_{(F,G,L)}$ (resp. $X'_{(F,G)}$) the corresponding Todorov surface (resp. $K3$ surface).

Set

(2.1.3)

$$T_{(1)}=\{(F, G, L)\in S^2\boldsymbol{P}H^0(\mathcal{O}_{\boldsymbol{P}^2}(3))\times \boldsymbol{P}H^0(\mathcal{O}_{\boldsymbol{P}^2}(1))\,|\,X_{(F,G,L)} \text{ is smooth}\},$$

$$\text{and}\quad T_{(2)}=\{(F, G, L)\in S^2\boldsymbol{P}H^0(\mathcal{O}_Q(4))\times \boldsymbol{P}H^0(\mathcal{O}_Q(2))\,|\,X_{(F,G,L)} \text{ is smooth}\},$$

where $\boldsymbol{P}A$ means the set of lines through 0 in a vector space A and $S^2\boldsymbol{P}A$ the second symmetric product of $\boldsymbol{P}A$.

Using the geometric monodromy (cf. [U. 3]), we have for both $T_{(1)}$ and $T_{(2)}$ ($T=T_{(1)}$ or $T_{(2)}$):

(2.2) **Theorem.** *Let $t_i\in T$ $(i=1, 2)$. Denote X_{t_i} by X_i and let C_i be the canonical curve on X_i. Assume that t_1 is generic and that there exists a path γ in T joining t_1 and t_2 which induces an isomorphism γ^* of the* PHS *on* $\mathrm{Gr}^W_\bullet H^2(X_i-C_i)^\sigma$ $(i=1, 2)$. *Then there exists an isomorphism τ of X_1 to X_2 inducing γ^*; and such τ is uniquely determined up to composition with an element of $\langle\sigma\rangle$.*

Proof. We use the diagram (1.3.4.1) for X_i $(i=1, 2)$ or more precisely the relative version of (1.3.4.1) over T. We also use the notations $\hat{C}$, $\hat{C}'$, $\hat{E}$, $\hat{E}'$, $\hat{D}$, $\hat{D}'$ etc. in (1.8).

Claim 1. *There exists a unique isomorphism $\hat{\tau}'$ of the $K3$ surfaces $\hat{X}'_1$ and $\hat{X}'_2$ which yields the isometry on $H^2(\hat{X}'_i, \boldsymbol{Z})$ $(i=1, 2)$ induced from γ.*

Indeed, by assumption, γ induces an isomorphism of the commutative exact diagrams of GPMHS $(i=1, 2)$:

$$\begin{array}{ccccccccc}
0\longrightarrow & (H^2(X_i, \boldsymbol{C})^{\perp\{C_i\}})^\sigma & \longrightarrow & H^2(X_i-C_i, \boldsymbol{C})^\sigma & \longrightarrow & H^1(C_i, \boldsymbol{C}) & \longrightarrow 0\\
 & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & \\
0\longrightarrow & (H^2(\hat{X}_i, \boldsymbol{C})^{\perp\{\hat{D}_i\}})^{\hat{\sigma}} & \longrightarrow & H^2(\hat{X}_i-\hat{D}_i, \boldsymbol{C})^{\hat{\sigma}} & \longrightarrow & H^1(\hat{D}_i, \boldsymbol{C}) & \longrightarrow 0\\
 & \uparrow{\scriptstyle q^*/2} & & \uparrow{\scriptstyle q^*/2} & & \uparrow{\scriptstyle (q|\hat{D}_i)^*} & \\
0\longrightarrow & H^2(\hat{X}'_i, \boldsymbol{C})^{\perp\{\hat{D}'_i\}} & \longrightarrow & H^2(\hat{X}'_i-\hat{D}'_i, \boldsymbol{C}) & \longrightarrow & H^1(\hat{D}'_i, \boldsymbol{C}) & \longrightarrow 0
\end{array}\tag{2.2.1}$$

In particular, γ induces an isomorphism of the PHS on $H^2(\hat{X}'_i)$ $(i=1, 2)$ preserving ample classes, e.g., $[4\hat{C}'_i-\hat{E}'_i]$. Therefore we obtain Claim 1 by the strong Torelli theorem for $K3$ surfaces (cf. [P-S, S], [B.P.V]).

Claim 2. *The isomorphism $\hat{\tau}'\colon \hat{X}'_1\xrightarrow{\sim}\hat{X}'_2$ in Claim* 1 *sends $\hat{D}'_1$ to $\hat{D}'_2$ and yields the isometry on $H^1(\hat{D}'_i, \boldsymbol{Z})$ induced from γ.*

Indeed since $\hat{\tau}'$ preserves the cohomology classes of $\hat{E}'_i$ $(i=1, 2)$ and the homological equivalence and the linear equivalence coincide on $\hat{X}'_i$, we see $\hat{\tau}'(\hat{E}'_1)=\hat{E}'_2$. In order to see $\hat{\tau}'(\hat{C}'_1)=\hat{C}'_2$, set

$$A=\begin{cases} S^2PH^0(\mathcal{O}_{P^2}(3)) & \text{in case } T=T_{(1)}, \\ S^2PH^0(\mathcal{O}_Q(4)) & \text{in case } T=T_{(2)}, \end{cases}$$
$$B=\begin{cases} PH^0(\mathcal{O}_{P^2}(1)) & \text{in case } T=T_{(1)}, \\ PH^0(\mathcal{O}_Q(2)) & \text{in case } T=T_{(2)}. \end{cases}$$

Let $C_{(F,G,L)}$ be the canonical curve of $X_{(F,G,L)}$ of genus 2 (resp. 3), and denote by $\mathcal{M}$ the coarse moduli space of curves of genus 2 (resp. 3) in case $T=T_{(1)}$ (resp. $T=T_{(2)}$).

Define a rational map

(2.2.2) $$\psi_{(F,G)}: B \dashrightarrow \mathcal{M}, \quad L \longmapsto [C_{(F,G,L)}]$$

for each fixed $(F, G)\in A$. From (2.2.1), γ induces an isomorphism of the PHS on $H^1(\hat{C}'_i)$ $(i=1, 2)$. Therefore by the strong Torelli theorem for curves (cf. [Ma. 1]), there exists a unique isomorphism of $\hat{C}'_1$ and $\hat{C}'_2$ which yields the same isometry between $H^1(\hat{C}'_i, Z)$ induced from γ. Thus, replacing $\hat{C}'_2$ by $\hat{\tau}'^{-1}\hat{C}'_2$, Claim 2 is reduced to:

Claim 3. *For generic $(F, G)\in A$, the degree of $\psi_{(F,G)}$ is 1 over its image.*

Since this claim in case $T=T_{(1)}$ is covered by Proposition 1 in [L], which is correct, we will give a proof only for the case $T=T_{(2)}$. Notice that the following argument works also for the case $T=T_{(1)}$ and this gives another proof of Proposition 1 in [L].

Let $I:=\{(F, G)\times(L_1, L_2)\in A\times(B\times B-\Delta)\mid C_{(F,G,L_1)}\simeq C_{(F,G,L_2)}\}$,

where Δ is the diagonal. The projections give rise to the diagram

(2.2.3) $$A \xleftarrow{p_1} I \xrightarrow{p_2} B\times B-\Delta.$$

Set

$$A'=\{(F, G)\in A \mid \text{degree of } \psi_{(F,G)} \text{ is } \geqq 2 \text{ over its image}\}.$$

It is easy to see:

(2.2.4) $$\dim A=2\cdot 8=16.$$

(2.2.5) $$\text{rel. dim } (p_1|_{A'})\geqq \dim B=3.$$

We will show in Claim 4 below that:

(2.2.6) $$\dim I = 17.$$

From (2.2.4), (2.2.5) and (2.2.6), we have the codimension estimate for $A' \subset A$:

$$\dim A' + 3 \leqq \dim p_1^{-1}(A') \leqq \dim I = 17.$$

Hence

$$\dim A' \leqq 17 - 3 = 14 < \dim A = 16.$$

Therefore, for $(F, G) \in A \backslash A'$, the degree of $\psi_{(F,G)}$ is 1 over its image.

Claim 4. $\dim I = 17$.

Let (y_0, x_1, x_2) be a weighted homogeneous coordinate system of $Q = Q(2, 1, 1)$ and let

(2.2.7) $$\begin{cases} F = f_0 y_0^2 + \sum_{1 \leqq i \leqq j \leqq 2} f_{ij} y_0 x_i x_j + \sum_{1 \leqq i \leqq j \leqq k \leqq \ell \leqq 2} f_{ijk\ell} x_i x_j x_k x_\ell, \\ G = g_0 y_0^2 + \sum g_{ij} y_0 x_i x_j + \sum g_{ijk\ell} x_i x_j x_k x_\ell. \end{cases}$$

Notice that $H := \mathrm{Aut}\, Q$ induces an action on the diagram (2.2.3) and that with respect to this action the p_i are H-equivariant and $B' := H \cdot (y_0, y_0 - x_1 x_2)$ is a Zariski open orbit in $B \times B - \Delta$. Hence

(2.2.8) $$\text{rel.} \dim (p_2|_{B'}) = \dim p_2^{-1}(y_0, y_0 - x_1 x_2).$$

Therefore in order to get $\dim I$ it is enough to compute the right-hand-side of (2.2.8). Substituting $y_0 = 0$ and $y_0 = x_1 x_2$ in (2.2.7), we have:

(2.2.9) $$\begin{cases} F' := F(0, x_1, x_2) = \sum f_{ijk\ell} x_i x_j x_k x_\ell. \\ G' := G(0, x_1, x_2) = \sum g_{ijk\ell} x_i x_j x_k x_\ell. \\ F'' := F(x_1 x_2, x_1, x_2) = f_{1111} x_1^4 + (f_{1112} + f_{11}) x_1^3 x_2 \\ \qquad + (f_{1122} + f_{12} + f_0) x_1^2 x_2^2 + (f_{1222} + f_{22}) x_1 x_2^3 + f_{2222} x_2^4. \\ G'' := G(x_1 x_2, x_2, x_2) = g_{1111} x_1^4 + (g_{1112} + g_{11}) x_1^3 x_2 \\ \qquad + (g_{1122} + g_{12} + g_0) x_1^2 x_2^2 + (g_{1222} + g_{22}) x_1 x_2^3 + g_{2222} x_2^4. \end{cases}$$

Assuming $f_{2222} = g_{2222} = 1$ and decomposing the above into linear factors, we get:

(2.2.10) $$\begin{cases} F' = \sum_{1 \leqq i \leqq 4} (x_2 - \alpha_i' x_1). \\ G' = \prod (x_2 - \beta_i' x_1). \\ F'' = (x_2 - \alpha_i'' x_1). \\ G'' = \prod (x_2 - \beta_i'' x_1). \end{cases}$$

Notice that $C_{(F,G,y_0)}$ and $C_{(F,G,y_0-x_1x_2)}$ are hyperelliptic curves expressed respectively as

$$\{u^2-F'\cdot G'=0\} \quad\text{and}\quad \{u^2-F''\cdot G''=0\} \quad\text{in } \boldsymbol{P}(1,1,4),$$

where u is a variable with $\deg u=4$, and that the roots in (2.2.10) give the branch points. Hence

$$(F,G)\in p_2^{-1}(y_0, y_0-x_1x_2) \quad\text{if and only if } C_{(F,G,y_0)}\simeq C_{(F,G,y_0-x_1x_2)}.$$

This is the case if and only if there exists a $\nu\in \operatorname{Aut}\boldsymbol{P}^1$ such that

$$\nu\left(\sum(1:\alpha_i')+\sum(1:\beta_i')\right)=\sum(1:\alpha_i'')+\sum(1:\beta_i'') \quad \text{as 0-cycles on } \boldsymbol{P}^1. \tag{2.2.11}$$

Now consider the finite cover $\tilde{p}\colon \tilde{A}\to A^\circ$, where

$$\tilde{A}:=\{(f_0, \alpha_1', \cdots, \alpha_4', \alpha_1'', \cdots, \alpha_4'', g_0, \beta_1', \cdots, \beta_4', \beta_1'', \cdots, \beta_4'')\mid \textstyle\sum\alpha_i' =\sum\alpha_i'', \sum\beta_i'=\sum\beta_i''\}$$

and $A^\circ:=\{f_{2222}=g_{2222}=1\}$ Zariski open $\subset A=\{(F,G)\}$ (see (2.2.9)). Then

$$\tilde{p}^{-1}p_2^{-1}(y_0, y_0-x_1x_2)=\{\tilde{a}\in\tilde{A}\mid \tilde{a} \text{ satisfies the condition (2.2.11)}\}.$$

Therefore, from (2.2.11), (2.2.3) and (2.2.8), we have

$$\dim p_2^{-1}(y_0, y_0-x_1x_2)=\dim \tilde{p}^{-1}p_2^{-1}(y_0, y_0-x_1x_2)=(8+2)+3-2=11.$$

$\dim I=2\cdot 3+11=17$ and we conclude the proof of Claim 4.

Now Theorem follows easily from Claims 1 and 2. Q.E.D.

(2.3) **Remark.** In the notation of (1.3.4.1), [L, p. 1145] claims the following:

(2.3.1) The given isomorphism of GPMHS $\alpha\colon H^2(X_1-C_1)\xrightarrow{\sim} H^2(X_2-C_2)$ for generic X_1 induces an isomorphism of PHS $\alpha_2\colon H^2(X_1)^\sigma\xrightarrow{\sim} H^2(X_2)^\sigma$, and α_2 has a lifting $\hat{\alpha}_2\colon H^2(\hat{X}_1)^{\hat{\sigma}}\xrightarrow{\sim} H^2(\hat{X}_2)^{\hat{\sigma}}$.

(2.3.2) A suitable lifting $\hat{\alpha}_2$ descends via $q^*/2$ to an isomorphism of PHS preserving ample classes $\hat{\alpha}_2'\colon H^2(\hat{X}_1')\xrightarrow{\sim} H^2(\hat{X}_2')$.

But (2.3.2) *is not clear*. In the situation (2.3.1), we have the diagram:

$$\begin{array}{ccc} H^2(X_1)^\sigma & \xrightarrow{\alpha_2} & H^2(X_2)^\sigma \\ \big\downarrow{\scriptstyle (q_*/2)\circ p^*} & & \big\downarrow{\scriptstyle (q_*/2)\circ p^*} \\ H^2(\hat{X}_1')^{\perp\{\hat{E}_1'\}} & & H^2(\hat{X}_2')^{\perp\{\hat{E}_2'\}} \end{array} \tag{2.3.3}$$

It is easy to see that the maps $(q_*/2)\circ p^*$ in (2.3.3) are embeddings of lattices but these embeddings are not primitive, i.e., not surjective in the present case. So *in general* α_2 *does not descend to an isometry of the* *bottoms in* (2.3.3). Moreover, *even if* α_2 *would descend to an isomorphism of* PHS *preserving positive structure* α_2' *of the bottoms in* (2.3.3), α_2' *does not come from an isomorphism of* X_1' *to* X_2'. Such a phenomenon is precisely studied in [M.S], from which we derive the following:

Put $X=X_1$. We denote by Γ_2 (resp. Γ_2') the subgroup of $\mathrm{Aut}(H^2(X, \boldsymbol{Z})^\sigma)$ (resp. $\mathrm{Aut}(H^2(\hat{X}', \boldsymbol{Z})^{\perp\{\hat{E}'\}})$) consisting of those elements which preserve the bilinear form and the positive structure. Let Γ_2^d be the subgroup of Γ_2 consisting of those elements which descend to isometries of $H^2(\hat{X}', \boldsymbol{Z})^{\perp\{\hat{E}'\}}$. Then Γ_2^d can be considered to be a subgroup of Γ_2' as well. Set $b=[\Gamma_2: \Gamma_2^d]$ and $c=[\Gamma_2': \Gamma_2^d]$. Let $\Phi_2: \mathscr{T}_{(1)}^a \to \Gamma_2 \backslash D_2$ be the period map and consider its Stein factorization

$$\bar{\Phi}_2: \bar{\mathscr{T}}_{(1)}^a \longrightarrow \Gamma_2 \backslash D_2. \tag{2.3.4}$$

Then $\bar{\mathscr{T}}_{(1)}^a$ can be naturally identified with the coarse moduli space of the $K3$ surfaces X' with ordinary double points, and the *degree of* $\bar{\Phi}_2$ *over its image* is $960b/c$. However we cannot yet calculate these indices b and c.

(2.4) **Problem.** *Prove the generic mixed Torelli theorem for other Todorov surfaces.*

(2.5) **Example.** The following example shows that, *even if the period map* $(\Phi_2, \Phi_3): \mathscr{T}_{(1)}^a \to \Gamma_2 \backslash D_2 \times \Gamma_3 \backslash D_3$ *has degree* 1 *over its image, it is not necessarily injective, where* Γ_k $(k=2, 3)$ *are the geometric monodromies coming from* $\pi_1(T_{(1)}, 0)$ *in* (2.1.3).

Let (y_0, y_1, y_2) be homogeneous coordinates of $\boldsymbol{P}^2$. Take

$$\begin{cases} F=y_0y_1\ell(y_0, y_1, y_2)+y_0p(y_0, y_2)+y_1q(y_1, y_2)+ay_2^3 \\ G=y_0y_1m(y_0, y_1, y_2)+y_0q(y_0, y_2)+y_1p(y_1, y_2)+ay_2^3. \end{cases} \tag{2.5.1}$$

Then

$$\begin{cases} F(0, y_1, y_2)\cdot G(0, y_1, y_2)=(y_1q(y_1, y_2)+ay_2^3)(y_1p(y_1, y_2)+ay_2^3) \\ F(y_0, 0, y_2)\cdot G(y_0, 0, y_2)=(y_0p(y_0, y_2)+ay_2^3)(y_0q(y_0, y_2)+ay_2^3). \end{cases}$$

Hence

$$C_{(F,G,y_0)} \simeq C_{(F,G,y_1)}. \tag{2.5.2}$$

On the other hand, it is easy to see that curve $\{F\cdot G=0\}\subset \boldsymbol{P}^2$ has no non-trivial projective automorphisms for general choice of ℓ, m, p, q and a in (2.5.1). From this follows

(2.5.3) $\psi_{(F,G)}$ in (2.2.2) has degree 1 over its image.

(2.6) **Problem.** *Does the phenomenon as in* (2.5) *not occur for the mixed period map* $\Phi: \mathscr{T}_{(1)}^a \to \Gamma \backslash D$?

3. Characterization of smoothness of canonical surfaces by GPMHS

(3.1) **Dual one-motif.**
Let $H=(H_{Z}, W, F, Q)$ be a GPMHS with

$$0=W_1\subset W_2\subset W_3=H \quad \text{and} \quad H=F^0\supset F^1\supset F^2\supset F^3=0.$$

Denote by

$$0\longrightarrow A\longrightarrow H\longrightarrow B\longrightarrow 0 \tag{3.1.1}$$

the exact sequence of GPMHS $0\to W_2\to H\to H/W_2\to 0$, and take its dual

$$0\longrightarrow \check{B}\longrightarrow \check{H}\xrightarrow{\pi} \check{A}\longrightarrow 0. \tag{3.1.2}$$

Then on $\check{H}$ we have

$$\check{H}=W_{-2}\supset W_{-3}\supset W_{-4}=0 \quad \text{and} \quad 0=F^1\subset F^0\subset F^{-1}\subset F^{-2}=\check{H}.$$

Consider the weak one-motif associated to the separated extension of MHS (3.1.2) (cf. [Car. 1, Proposition 3]):

$$\check{u}=u_{\check{H}}\colon L^{-1}\check{A}\longrightarrow J^{-1}\check{B}, \tag{3.1.3}$$

where $L^{-1}\check{A}=\check{A}_{C}^{-1,-1}\cap\check{A}_{Z}$ and $J^{-1}\check{B}=\check{B}_{C}/(F^{-1}\check{B}+\check{B}_{Z})$. The map $\check{u}$ is defined by $\check{u}(\gamma)=s_{Z}(\gamma)-s_{F}(\gamma)$ modulo $F^{-1}\check{B}+\check{B}_{Z}$ for $\gamma\in L^{-1}\check{A}$, where s_{Z} (resp. s_F) is a section of π in (3.1.2) preserving the Z-structure (resp. the Hodge filtration).

When H comes from geometry, i.e., H is the GPMHS on $H^2(X-C)$ for a smooth pair (X, C) consisting of a surface X and a smooth divisor C, the graded polarization Q yields

$$L^{-1}\check{A}\xleftarrow[\sim]{Q_2} H^1(X, Z)\cap H^{1,1}(X)^{\perp[C]} \quad \text{and} \quad J^{-1}\check{B}\xleftarrow[\sim]{Q_3} J^2B:=B_C/(F^2B+B_{Z}).$$

These fit in the commutative Diagram 3.

$$\begin{array}{ccc}
L^{-1}\check{A} & \xrightarrow{\quad\quad\check{u}\quad\quad} & J^{-1}\check{B}\\
\uparrow & & \beta\uparrow \\
 & & J^2B \\
\cup & & \delta\downarrow\cap \\
H^2(X, Z)\cap H^{1,1}(X)^{\perp[C]} & & H^1(C, C)/(H^{1,0}(C)+H^1(C, Z)) \\
\alpha\uparrow & & \| \\
\mathrm{Pic}\,(X)^{\perp[C]} & \xrightarrow{\quad\text{pull-back}\quad} & \mathrm{Pic}^0(C)
\end{array}$$

Diagram 3

(3.1.4) **Remark.** After we used Diagram 3 to get Proposition (3.2) below, we received a preprint [F.3] from Friedman where he obtained the same notion as Diagram 3 (cf. [F.3, (3.8)]). We also found the same notion in the recent paper of Carlson [Car. 3, § 15] during the proof.

(3.2) **Proposition.** *Let X be the minimal resolution of a canonical surface. Assume*:

(3.2.1) $H_1(X, \mathbf{Z})=0$.

(3.2.2) *There exists a smooth member* $C \in |K_X|$.

Then α, β *and* δ *in Diagram* 3 *are isomorphic, and the following are equivalent*:

(3.2.3) K_X *is not ample.*

(3.2.4) *There exists* $\gamma \in H^2(X, \mathbf{Z}) \cap H^{1,1}(X)^{\perp[C]}$ *such that* $Q_2(\gamma, \gamma)=-2$ *and* γ *goes to zero by the associated dual one-motif* $\check{u}$ *in Diagram* 3.

Proof. The assertion on α, β and δ follows from (3.2.1) and the fact that $B_{\mathbf{Z}}=H^1(C, \mathbf{Z})$, which is a unimodular lattice.

It is well-known that (3.2.3) is equivalent to the existence of (-2)-curves on X. This, in turn, implies that there exists a line bundle L on X such that $L^2=-2$ and $L|_C=\mathcal{O}_C$, which is equivalent to (3.2.4) by Diagram 3.

Now suppose that there exists an $L \in \mathrm{Pic}(X)$ such that $L^2=-2$ and $L|_C=\mathcal{O}_C$. We claim $H^0(L)\neq 0$ or $H^0(L^{-1})\neq 0$. Indeed, by the Riemann-Roch theorem, we have

$$\chi(L)=L\cdot(L\otimes\omega_X^{-1})/2+\chi(\mathcal{O}_X)=-1+1+p_g(X)=p_g(X).$$

Hence, if $h^0(L)=0$, then

$$h^0(\omega_X\otimes L^{-1})\geqq p_g(X). \tag{3.2.5}$$

Tensoring $\omega_X\otimes L^{-1}$ (resp. ω_X) to

$$0\longrightarrow\mathcal{O}_X(-C)\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_C\longrightarrow 0,$$

we get

$$0\longrightarrow L^{-1}\longrightarrow\omega_X\otimes L^{-1}\longrightarrow(\omega_X\otimes L^{-1})|_C\longrightarrow 0$$
$$(\text{resp. } 0\longrightarrow\mathcal{O}_X\longrightarrow\omega_X\longrightarrow\omega_X|_C\longrightarrow 0).$$

By using $(\omega_X\otimes L^{-1})|_C=\omega_X|_C$ and (3.2.1), we have

$$\begin{array}{l} 0\longrightarrow H^0(L^{-1})\longrightarrow H^0(\omega_X\otimes L^{-1})\longrightarrow H^0(\omega_X|_C) \quad \text{and} \\ 0\longrightarrow H^0(\mathcal{O}_X)\longrightarrow H^0(\omega_X)\longrightarrow H^0(\omega_X|_C)\longrightarrow 0. \end{array} \tag{3.2.6}$$

(3.2.5) and (3.2.6) imply $h^0(L^{-1})+h^0(\omega_X|_C)\geqq h^0(\omega_X\otimes L^{-1})\geqq p_g(X)$ and $h^0(\omega_X|_C)=p_g(X)-1$, hence $h^0(L^{-1})\geqq 1$. Q.E.D.

4. Toward mixed Torelli theorem for surfaces with $p_g = c_1^2 = 1$

(4.1) **First approach: By the Kŭnev locus.**

Let $S = \boldsymbol{C}[x_0, y_1, y_2, z_3, z_4]$ be the weighted homogeneous coordinate ring of $\boldsymbol{P} = \boldsymbol{P}(1, 2, 2, 3, 3)$. Set $S_6 = \{f \in S \mid f$ is weighted homogeneous of degree $6\}$. Let U (resp. V) be the set of 2-dimensional subspaces u of the $\boldsymbol{C}$-vector space S_6 satisfying that $X_u := \{x \in \boldsymbol{P} \mid f(x) = 0$ for all $f \in u\}$ is a canonical surface (resp. a canonical Kŭnev surface) with a smooth canonical curve C_u. Denote by U^a (resp. V^a) the subset of U (resp. V) consisting of those points u for which X_u is smooth.

In the notation (1.1.4), since $\omega_{X_u} \simeq \mathcal{O}_{X_u}(1)$ $(u \in U)$, we see $\mathcal{M}^s_{(1)} = U/H$ by (1.2.1), $\mathcal{M}^{as}_{(1)} = U^a/H$, $\mathcal{T}_{(1)} = V/H$ and $\mathcal{T}^a_{(1)} = V^a/H$, where $H = \operatorname{Aut} \boldsymbol{P}$. Take a base point $0 \in V^a$ and set $G_{\boldsymbol{Z}} := \operatorname{Aut}(H^2(X_0 - C_0, \boldsymbol{Z}), W, Q)$ and $G_{k,\boldsymbol{Z}} := \operatorname{Gr}^W_k G_{\boldsymbol{Z}}$ for $k = 2, 3$ (see I. Section 2). We denote

$$(4.1.1)\quad \begin{aligned} &\Gamma_{U^a} := \operatorname{Im}\{\pi_1(U^a, 0) \longrightarrow G_{\boldsymbol{Z}}\}, \quad \Gamma_{V^a} := \operatorname{Im}\{\pi_1(V^a, 0) \longrightarrow G_{\boldsymbol{Z}}\}, \\ &\Gamma_{k,U^a} := \operatorname{Im}\{\Gamma_{U^a} \longrightarrow G_{k,\boldsymbol{Z}}\}, \quad \text{and} \quad \Gamma_{k,V^a} := \operatorname{Im}\{\Gamma_{V^a} \longrightarrow G_{k,\boldsymbol{Z}}\}. \end{aligned}$$

The next lemma follows easily from the discreteness of $H^2(X_u - C_u, \boldsymbol{Z})$ and the path-connectedness of H:

(4.1.2) **Lemma.** *In the above notation, let $u_1, u_2 \in U^a$ and $\tau \in H$ such that $\tau u_1 = u_2$. Take any path $\tau(t)$ in H with $\tau(0) = \mathrm{id}$ and $\tau(1) = \tau$, and denote by γ the path $\tau(t) u_1$ in U^a. Then we have*

$$\tau^* = \gamma^* : H^2(X_{u_2} - C_{u_2}, \boldsymbol{Z}) \xrightarrow{\sim} H^2(X_{u_1} - C_{u_1}, \boldsymbol{Z}).$$

Hence we can define the mixed period map

$$\Phi : \mathcal{M}^{as}_{(1)} \longrightarrow \Gamma_{U^a} \backslash D.$$

By the existence of a local simultaneous resolution of rational double points (e.g. [Ty]) and the connectedness of H, Φ can be extended to

$$(4.1.3)\qquad \Phi : \mathcal{M}^s_{(1)} \longrightarrow \Gamma_{U^a} \backslash D.$$

(4.1.4) **Lemma.** *In the above notation, there exists a Zariski open subset T of $\mathcal{T}^a_{(1)}$ satisfying $\Phi^{-1}(\Phi(T)) = T$.*

Proof. First notice that, by Proposition (3.2), we see $\Phi^{-1}(\Phi(\mathcal{M}^{as}_{(1)})) = \mathcal{M}^{as}_{(1)}$. Set

$$\Phi_2 : \mathcal{M}^{as}_{(1)} \xrightarrow{\Phi} \Gamma_{U^a} \backslash D \xrightarrow{\operatorname{Gr}^W_2} \Gamma_{2,U^a} \backslash D_2.$$

We use the characterization (1.6.2.3) of the Kŭnev locus $\mathcal{T}^a_{(1)}$ in $\mathcal{M}^{as}_{(1)}$ by the period map Φ_2. Let $\Phi^{-1}(\Phi(\mathscr{T}^a_{(1)}))=\mathscr{T}^a_{(1)}\cup T^1\cup\cdots\cup T^r$ be the decomposition into irreducible components.

Claim. $\dim T^i\leqq 11$ for all i.

Indeed, if $\dim T^i\geqq 12$, then $\dim \Phi_2(T^i)\geqq 12-1=11$ because of $T^i\not\subset \mathscr{T}^a_{(1)}$ and (1.6.2.3). On the other hand, $\dim \Phi_2(T^i)=\dim \Phi_2(\mathscr{T}^a_{(1)})=12-2=10$ by (1.6.2.3), a contradiction.

Now by the infinitesimal mixed Torelli theorem (1.7.1), we have $\dim \Phi(\mathscr{T}^a_{(1)})=12$. Therefore $T=\Phi^{-1}(\Phi(\mathscr{T}^a_{(1)})-\bigcup_i \Phi(T^i))$ is the desired Zariski open subset. Q.E.D.

(4.1.5) **Problem.** *Extend the mixed period map* (4.1.3) *to* $\Phi\colon {}^{fc}\mathscr{M}^s_{(1)}\to\Gamma_{U^a}\backslash D$ *through extension over the points with finite local monodromy. This is possible by the comment just after Problem* (I.12.1). *Show that* $\Phi({}^{fc}\mathscr{M}^s_{(1)}-\mathscr{M}^s_{(1)})\not\supset\Phi(T)$, *where T is in* (4.1.4).

(4.1.6) **Problem.** *Do the monodromies* Γ_{U^a} *and* Γ_{V^a} *in* (4.1.1) *coincide?*

After solving Problems (4.15) and (4.16), we shall be able to arrive at the generic mixed Torelli theorem for $\mathscr{M}_{(1)}$ by Theorem (2.2) and Lemma (4.1.4). Indeed by Lemma (4.1.2) and the argument in the proof of Theorem (2.2), we see that the mixed period map $\Phi\colon\mathscr{T}_{(1)}\to\Gamma_{V^a}\backslash D$ has degree 1 over its image.

(4.1.7) **Remark.** For Problem (4.1.5), the degenerate curves C_0 which we should be concerned with are of types $[I_{0-0-0}]$ and $[I^*_{0-0-0}]$ in [N.U], and the following should be carried out:

(4.1.7.1) Explicit description of semi-stable reduction of pairs.

(4.1.7.2) Computation of the limit of the MHS by "the mixed Clemens-Schmid sequence" (Problem (I.12.3)) or by the abstract log complex for d-semi-stable pairs (I.11).

(4.2) **Second approach: By boundary.**

In [F. 2], Friedman gave a proof of the Torelli theorem for $K3$ surfaces by using a general point of type II degeneration. We hope that his argument will go through in our context, but we have not yet gone far in this approach. I. Section 10, Problems (I.12.3), (I.12.4), (I.12.5) etc. are related.

(4.3) **Third approach: By "IVMHS"**

This is only a program at the moment.

(4.3.1) **Problem.** *Rewrite* IVHS *theory* ([C.G], [C.G.G.H], [G.H], [Gri. 4], [Do], [Gre] etc.) *in the context of* "IVMHS".

Added in Proof. The following article is closely related to our Prblem (II. 1.8.4):

Hain, R.M. and Zncker. S., Unipotent variation of mixed Hodge structure, Invent. Math., **88–1** (1987), 83–124.

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