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The Lower Semi-Continuity of the Plurigenera of Complex Varieties

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Introduction

This paper is an extension of [34] in which it was shown that the **Conjecture** L (see below) follows from the minimal model conjectures in the case of algebraic varieties. In this paper, we treat complex varieties.

Conjecture L. *Let* $\pi: X \to D$ *be a proper surjective morphism from a complex manifold* X *onto a unit disk* D. *Assume that* $\pi^{-1}(0) = \bigcup_{i \in I} \Gamma_i$, *where all the* Γ_i *are compact complex varieties in class* $\mathscr{C}$ *in the sense of Fujiki* [5]. *Then*

$$\sum_{i \in I} P_m(\Gamma_i) \leqq \operatorname{rank} \pi_* \mathcal{O}_X(mK_X) \qquad \text{for all } m \geqq 1,$$

where P_m *denotes the m-genus.*

Clearly, this induces the invariance of plurigenera under smooth deformations. The invariance of the plurigenera of compact complex surfaces was proved by Iitaka [16]. But we have many counterexamples without assuming that the Γ_i belong to the class $\mathscr{C}$ in the higher dimensional case or even in the case of degeneration of surfaces (see Nakamura [31], Nishiguchi [36]).

The theory of minimal models developed by Mori, Reid, Kawamata, Tsunoda, Shokurov, Benveniste, Kollár and others is not yet completed even in the case of algebraic varieties. In this paper we shall prove Conjecture L in the case of semi-stable relative minimal models. A relative good minimal model $\pi: X \to D$ is defined to be a proper surjective morphism from a variety X with only canonical singularities such that K_X is π-semi-ample. Conjecture L can be proved with the help of some kind of the theory of minimal models. In fact if π is a projective degeneration of surfaces with non-negative Kodaira dimensions, then it is proved (see (7.5)) by a result of Tsunoda [48]. The main technique of our paper is the same one as in Kawamata [21]. But since his arguments require some properties of projective varieties in some steps, we must modify the proofs.

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In Section 0 and Section 1, we fix the notations, and in Section 2 and Section 3, we prove the key theorems which might also be useful for other problems in complex analytic geometry. In Section 4, we discuss the relative version of the minimal model theory for projective morphisms using a result of Section 3. Section 5 is a slight modification of [21]. In Section 6, we shall prove a partial answer to Conjecture L, and in Section 7, we discuss the open problems arising from our discussion.

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Convention

(1) All complex spaces are Hausdorff spaces with countable open basis.

(2) For a real number m, by saying that for $m \gg 0$, we mean that there is a positive number m_0 such that for any $m \geqq m_0, \cdots$. Similarly by saying that for $0 < \delta \ll 1$, we mean that there is a $0 < \delta_0 < 1$ such that for any $0 < \delta \leqq \delta_0, \cdots$.

(3) For a coherent sheaf $\mathscr{E}$ on a complex space S, $\boldsymbol{P}_S(\mathscr{E})$ denotes Projan $\bigoplus_{d \geqq 0} \mathrm{Sym}^d(\mathscr{E})$.

(4) For a morphism $f: X \to Y$, X_s denotes the scheme theoretical fiber $f^{-1}(s)$, and if L is a Cartier divisor on X, then $L_s = L_{|X_s}$ is the restriction of L to X_s.

(5) A proper surjective morphism $f: X \to Y$ between normal varieties is called a *fiber space* if the general fibers of f are connected.

(6) A line bundle (or a Cartier divisor) L on a compact normal complex variety X is called *base point free* (or *free*) if L is generated by global sections. L is called *semi-ample* if mL is free for some positive integer m. Let $f: X \to Y$ be a proper surjective morphism a normal complex variety X onto a complex variety Y. A line bundle L on X is called *f-free* if $f^* f_* \mathcal{O}_X(L) \to \mathcal{O}_X(L)$ is surjective. L is called *f-semi-ample* if mL is f-free for a positive integer m.

(7) A compact complex variety in class $\mathscr{C}$ is a variety which is dominated by a compact Kähler manifold ([5]). It is known that X is in class $\mathscr{C}$ if and only if X is bimeromorphically equivalent to a compact Kähler manifold.

(8) A reduced divisor D on a complex manifold X is said to have only *normal crossings* if for every point $p \in X$, there exists an open neighborhood U with a system of local coordinates $(z_1, z_2, \cdots, z_n)$ such that $D \cap U = \{z_1 \cdot z_2 \cdots z_l = 0\}$ for some l. D is said to have only *simple normal crossings* if all the irreducible components of D are smooth and intersect transversally.

§ 0. Preliminaries

(A) Weakly 1-complete variety.

Let X be a complex space and $\mathscr{A}_X$ be the sheaf of C^∞-functions on X in the sense of Fujiki [5]. A C^∞-function φ on X is called plurisubharmonic (resp. strictly plurisubharmonic) if there exist an open covering $\{U_\alpha\}$ of X, a closed embedding $\eta_\alpha\colon U_\alpha \to D_\alpha$ to a domain $D_\alpha \subset \boldsymbol{C}^{N_\alpha}$, and a C^∞-function ψ_α on D_α such that $\psi_{\alpha|U_\alpha} = \varphi_{|U_\alpha}$ and that ψ_α is plurisubharmonic (resp. strictly plurisubharmonic) on D_α.

Definition 0.1. Let X be a complex space and Ψ be a real valued C^∞-function on X. (X, Ψ) is said to be *weakly* 1-*complete* if (X, Ψ) has the following two properties.

(1) Ψ is plurisubharmonic on X.

(2) $X_c := \{x \in X \mid \Psi(x) < c\}$ is a relatively compact open subset in X for every $c \in \boldsymbol{R}$.

The property (2) is equivalent to:

$$c_0 := \operatorname{Inf}\{\Psi(x) \mid x \in X\} > -\infty$$

and $\Psi\colon X \to [c_0, \infty)$ is proper.

A complex space X is called a *weakly* 1-*complete space* if there is a Ψ such that (X, Ψ) is weakly 1-complete. In this case, we denote the set $\{x \in X \mid \Psi(x) < c\}$ simply by X_c. For example, any Stein space is weakly 1-complete. Therefore if one has a proper morphism $X \to S$ to a Stein space S, then X is also weakly 1-complete.

Let X be a complex space and L be a line bundle on X. Then there exist an open covering $\{U_\alpha\}$ of X and isomorphisms $\varphi_\alpha\colon L_{|U_\alpha} \cong \mathcal{O}_{U_\alpha}$. Conversely, the set of functions $\{f_{\alpha\beta}\}$, where $f_{\alpha\beta} := \varphi_\alpha \circ \varphi^{-1}_{\beta|U_\alpha \cap U_\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$, defines L. Such a $(\{f_{\alpha\beta}\}, \{U_\alpha\})$ is called a *system of transition functions of L*.

A *metric* on L with respect to a system of transition functions $(\{f_{\alpha\beta}\}, \{U_\alpha\})$ of L is a collection of positive C^∞-functions $h = \{h_\alpha\}$, where $h_\alpha \in \Gamma(U_\alpha, \mathscr{A}_X)$ such that $h_\beta/h_\alpha = |f_{\alpha\beta}|^2$ on $U_\alpha \cap U_\beta$.

Definition 0.2. A line bundle L on X is said to be *positive* if there is a metric $\{h_\alpha\}$ on L such that $-\log h_\alpha$ is strictly plurisubharmonic on U_α for all α.

Then we have a vanishing theorem of Nakano [32], [33].

Theorem 0.3. *Let X be a weakly* 1-*complete manifold and let A be a positive line bundle on X. Then*

$$H^q(X, \Omega_X^p \otimes A) = 0 \qquad \textit{for } p+q > \dim X.$$

Fujiki [4] obtained the following generalizing [32], [33]:

Theorem 0.4 ([4, Lemma 3]). *Let X be a weakly 1-complete complex space and let L be a line bundle on X. Then the following conditions are equivalent.*

(1) *For any $c \in \boldsymbol{R}$, there exists a positive integer m_0 such that for any $m \geqq m_0$, one can find a finite number of sections $\varphi_0, \varphi_1, \cdots, \varphi_l \in \Gamma(X_c, L^{\otimes m})$ which generate $L^{\otimes m}$ and that the morphism $j := (\varphi_0 : \cdots : \varphi_l) : X_c \to \boldsymbol{P}^l$ is a locally closed embedding with $j^*\mathcal{O}_{\boldsymbol{P}^l}(1) \cong L^{\otimes m}_{|X_c}$.*

(2) *For any coherent sheaf $\mathcal{E}$ on X and for any $c \in \boldsymbol{R}$, there exists a positive integer m_1 such that for every $m \geqq m_1$, $\mathcal{E} \otimes L^{\otimes m}_{|X_c}$ is generated by a finite number of sections on X_c.*

(3) *For any coherent sheaf $\mathcal{E}$ on X and for any $c \in \boldsymbol{R}$, there exists a positive integer m_2 such that $H^i(X_c, \mathcal{E} \otimes L^{\otimes m}) = 0$ for every $m \geqq m_2$ and for $i \geqq 1$,*

(4) *L is positive on X.*

Remark 0.5. The condition (1) corresponds to ampleness, (2) to "Theorem A", and (3) to "Theorem B". If a weakly 1-complete variety X has a positive line bundle L, then L works as if it were an ample line bundle on a projective variety.

(B) D-canonical fibration.

We discuss the *relative D-canonical fibration.* Let $f: X \to S$ be a proper surjective morphism from a normal complex variety X onto a complex variety S, and let D be an effective Cartier divisor on X. Then $f_*\mathcal{O}_X(D) \neq 0$, and the homomorphism $f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ defines a proper meromorphic map $\Phi_D: X \cdots \to \boldsymbol{P}_S(f_*\mathcal{O}_X(D))$ over S. In this situation, there exists an open dense subset $S^{(1)}$ of S such that

(a) $f_*\mathcal{O}_X(D)_{|S^{(1)}}$ is locally free,

(b) X_s is a normal complex variety for all $s \in S^{(1)}$ and

(c) $f_*\mathcal{O}_X(D) \otimes \boldsymbol{C}(s) \cong H^0(X_s, \mathcal{O}_X(D) \otimes \mathcal{O}_{X_s})$ for all $s \in S^{(1)}$.

Therefore if $s \in S^{(1)}$, then $\Phi_D \otimes \boldsymbol{C}(s)$ is defined by $H^0(X_s, \mathcal{O}_{X_s}(D_s)) \otimes \mathcal{O}_{X_s} \to \mathcal{O}_{X_s}(D_s)$. For any positive integer m, let $S^{(m)}$ be an open dense subset of S which satisfies the conditions (a), (b), (c) for mD. By Baire's category theorem, $S^{(\infty)} := \bigcap_{m \geqq 1} S^{(m)}$ is a dense subset of S. If $s \in S^{(\infty)}$, then

(α) $f_*\mathcal{O}_X(mD)$ is free at s for all $m \geqq 1$,

(β) X_s is a normal complex variety, and

(γ) $f_*\mathcal{O}_X(mD) \otimes \boldsymbol{C}(s) \cong H^0(X_s, \mathcal{O}(mD_s))$ for any $m \geqq 1$.

Furthermore, by Baire's category theorem we can construct a dense subset U of $S^{(\infty)}$ which satisfies the following conditions.

(δ) $\dim \Phi_{mD_s}(X_s) = \dim (Z_m \times_S \boldsymbol{C}(s))$ for any $s \in U$ and for $m \geqq 1$, where $Z_m := \Phi_{mD}(X) \subset \boldsymbol{P}_S(f_*\mathcal{O}_X(mD))$,

(ε) $Z_m \to S$ is flat over U for any $m \geqq 1$.
We define $\kappa(X/S, D)$ to be $\max_{m\geqq 1}(\dim Z_m - \dim S)$. If S is a point, then $\kappa(X, D)$ is abbreviated as $\kappa(D)$. It was shown in [17] that for any $s \in U$, there exist a positive integer m and a bimeromorphic morphism μ: $Y \to X_s$ from a normal compact variety Y such that

(i) $h_s := \Phi_{mD_s} \circ \mu$: $Y \to \Phi_{mD_s}(X_s)$ is a morphism which defines a fiber space and

(ii) $\kappa(Y/\Phi_{mD_s}(X_s), \mu^*D_s) = 0$.

Thus $\kappa(X/S, D) = \kappa(X_s, D_s)$ for $s \in U$. Furthermore, there exist a positive integer m and a proper bimeromorphic morphism ν: $W \to X$ over S from a normal variety W such that

(i) $h := \Phi_{mD} \circ \nu$: $W \to Z_m$ is a morphism and is a fiber space, and

(ii) $\kappa(W/X, \nu^*D) = 0$.

Conversely, a proper surjective meromorphic map g: $B \cdots \to G$ over S which satisfies the following conditions (a) and (b) is proper bimeromorphically equivalent to Φ_{mD} over S for $m \gg 0$.

(a) $\dim G = \dim S + \kappa(X/S, D)$ and

(b) there exist proper bimeromorphic morphisms γ: $V \to B$ and δ: $V \to X$ over S from a normal variety V such that $g \circ \gamma$: $V \to G$ is a morphism and is a fiber space with $\kappa(V/G, \delta^*D) = 0$.

Such a map is called the *relative canonical fibration* of D over S.

(C) Divisors and singularities.

Let X be a normal complex variety. A *Weil divisor* is a locally finite formal sum $\sum a_i D_i$ of integers a_i and subvarieties D_i of codimension 1 in X. A Weil divisor D is called a *Cartier divisor* if there exists an open covering $\{U_\alpha\}$ of X and nonzero meromorphic functions f_α on U_α such that $D_{|U_\alpha} = \operatorname{div}(f_\alpha)$, where $\operatorname{div}(f_\alpha)$ is the principal divisor associated with f_α. To any open subset U of X, we attach the set $\{f \in \Gamma(U, \mathscr{M}_X^*) \mid \operatorname{div}(f) + D_{|U} \geqq 0\}$, where $\mathscr{M}_X^*$ is the sheaf of nonzero meromorphic functions on X. Then the correspondence $U \mapsto \{f\}$ defines a coherent $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ which is *reflexive*, i.e.,

$$\mathscr{H}om(\mathscr{H}om(\mathcal{O}_X(D), \mathcal{O}_X), \mathcal{O}_X) \cong \mathcal{O}_X(D).$$

Conversely, if $\mathscr{L}$ is a coherent reflexive sheaf of rank one on X, then locally $\mathscr{L}$ is represented by $\mathcal{O}_X(D)$ for some Weil divisor D. It is easy to see that a Weil divisor D is a Cartier divisor if and only if $\mathcal{O}_X(D)$ is invertible.

A ***Q**-divisor* is an element of (the group of Weil divisors on X)$\otimes \boldsymbol{Q}$, and a ***Q**-Cartier divisor* is an element of (the group of Cartier divisors on X)$\otimes \boldsymbol{Q}$. Note that it may be possible that D is a $\boldsymbol{Q}$-Cartier divisor and $\mathcal{O}_X(kD)$ is not invertible on X for any integer k. For a $\boldsymbol{Q}$-divisor $D = \sum d_i D_i$, we use the following symbols.

$$[D]:=\sum [d_i]D_i, \text{ where } [d_i] \text{ is the integral part of } d_i,$$
$$\ulcorner D \urcorner := -[-D]$$

and

$$\langle D \rangle := D-[D].$$

Then we have the following:

Proposition 0.6. *Let X be a complex manifold and D a $\boldsymbol{Q}$-divisor on X such that* Supp $\langle D \rangle$ *has only normal crossings, and let $\mu\colon Y \to X$ be a proper bimeromorphic morphism such that* Supp $\mu^*\langle D \rangle$ *has only normal crossings. Then we have*

$$\mu_*\mathcal{O}_Y(K_Y+\ulcorner \mu^*D \urcorner)=\mathcal{O}_X(K_X+\ulcorner D \urcorner).$$

Proof. Let $\langle D \rangle=\sum_j e_jE_j$ be the irreducible decomposition. Then

$$\mu^*\langle D \rangle=\sum_j e_j\mu^*E_j=\sum_j e_j(\sum_k b_{jk}F_k),$$

where $\mu^*E_j=\sum_k b_{jk}F_k$. By the log-ramification formula [17]

$$K_Y+\sum_k F_k=\mu^*(K_X+\sum_j E_j)+\bar{R}_\mu,$$

where $\bar{R}_\mu$ is a μ-exceptional effective divisor on Y, we obtain

$$\begin{aligned}
&K_Y+\ulcorner \mu^*D \urcorner-\mu^*(K_X+\ulcorner D \urcorner)\\
&\quad =K_Y+\ulcorner \mu^*\langle D \rangle \urcorner-\mu^*(K_X+\textstyle\sum_j E_j)\\
&\quad =\textstyle\sum_k \ulcorner((\sum_j e_jb_{jk})-1)\urcorner F_k+K_Y+\sum_k F_k-\mu^*(K_X+\sum_j E_j)\\
&\quad =\textstyle\sum_k \ulcorner((\sum_j e_jb_{jk})-1)\urcorner F_k+\bar{R}_\mu.
\end{aligned}$$

If F_k is not μ-exceptional, then F_k is a strict transform of some E_j, so $\ulcorner(\sum_j e_jb_{jk})-1\urcorner=\ulcorner e_j-1\urcorner=0$. Since $\sum_j e_jb_{jk}>0$ for any k, it follows that $\ulcorner(\sum_j e_jb_{jk})-1\urcorner\geqq 0$. Therefore $K_Y+\ulcorner \mu^*D \urcorner-\mu^*(K_X+\ulcorner D \urcorner)$ is a μ-exceptional effective divisor. Thus $\mu_*\mathcal{O}_Y(K_Y+\ulcorner \mu^*D \urcorner)=\mathcal{O}_X(K_X+\ulcorner D \urcorner)$. □

Let (X, p) be a germ of a d-dimensional normal complex variety. Then $\omega_X:=\mathcal{H}^{-d}(\omega_X^{\cdot})$ is a reflexive sheaf of rank one on X, where $\omega_X^{\cdot}$ is the dualizing complex of X. Therefore $\omega_X=\mathcal{O}_X(D)$ for some Weil divisor D. The linear equivalence class of such D is denoted by K_X which is called the *canonical divisor* of X. (X, p) is called a *$\boldsymbol{Q}$-Gorenstein* germ if K_X is a $\boldsymbol{Q}$-Cartier divisor.

Let $\mu\colon Y\to X$ be a resolution of singularities of (X, p) such that the μ-exceptional locus is a divisor $E=\sum E_j$ with only simple normal crossings. Then for a $\boldsymbol{Q}$-Gorenstein germ (X, p), there is a unique rational number a_j for each E_j such that $K_Y=\mu^*K_X+\sum a_jE_j$.

The singularities of the germ (X, p) of a normal $\boldsymbol{Q}$-Gorenstein variety is called *terminal*, *canonical*, or *log-terminal* according as $a_j > 0$, $a_j \geqq 0$, or $a_j > -1$ for all j.

Let Δ be a $\boldsymbol{Q}$-divisor on (X, p) such that Δ is effective and $[\Delta]=0$. The pair (X, Δ) is said to be *log-terminal* at p if and only if

(1) $K_X + \Delta$ is a $\boldsymbol{Q}$-Cartier divisor and

(2) there exist a proper bimeromorphic morphism μ: $Y \to X$ from a complex manifold Y and a reduced divisor $F = \sum F_j$ with only simple normal crossings on Y such that

$$K_Y = \mu^*(K_X + \Delta) + \sum b_j F_j \quad \text{with} \quad b_j > -1 \quad \text{for all } j.$$

Fujita [8] proved that if (X, Δ) is log-terminal for some Δ, then the singularity of (X, p) is rational.

§ 1. Projective morphisms

Definition 1.1. Let f: $X \to S$ be a proper morphism between complex spaces. A line bundle L on X is said to be *f-ample*, if there exists an open covering $\{U_\alpha\}$ of S such that each U_α is weakly 1-complete (hence $f^{-1}(U_\alpha)$ is also weakly 1-complete) and that L is positive on $f^{-1}(U_\alpha)$.

Definition 1.2. (1) A proper morphism f: $X \to S$ is called a *projective morphism*, if there exists an f-ample line bundle on X.

(2) A proper morphism f: $X \to S$ is called a *locally projective morphism*, if there exists an open covering $\{U_\alpha\}$ of S such that $f_{|f^{-1}(U_\alpha)}$ is projective for all α. (This definition is not the same as that in [7]).

Remark 1.3. Let f: $X \to Y$ and g: $Y \to Z$ be proper morphisms and let h denote the composite $g \circ f$: $X \to Z$. Even if f and g are projective morphisms, h is not necessarily a projective morphism, but is always locally projective.

Example. Let Z be a unit disk, $g = \mathrm{pr}_2$: $Y = \boldsymbol{P}^1 \times Z \to Z$, q_i $(1 \leqq i < \infty)$ a discrete sequence of mutually distinct points on Z, and $p_{i,j}$ $(1 \leqq i < \infty,$ $1 \leqq j \leqq i)$ mutually distinct points on Y such that $g(p_{i,j}) = q_i$. Let f: $X \to Y$ be the blowing up with center $\{p_{i,j}\}$. Then it is easy to show that there is no h-ample line bundle, where $h = g \circ f$.

Proposition 1.4. *Let f: $X \to S$ be a proper morphism, s a point of S and L a line bundle on X. Assume that L_s is ample. Then there exists an open neighborhood U of s such that $L_{|f^{-1}(U)}$ is f-ample.*

Proof. (cf. [13, (4.7.1)]). Let I be the ideal sheaf of X_s in X.

Step 1. *There exists a positive integer* m_0 *such that* $R^i f_*(I\otimes L^{\otimes m})_s=0$ *for* $i>0$ *and for* $m\geqq m_0$.

The formal function theorem [10] says that

$$R^i f_*(I\otimes L^{\otimes m})^{\wedge}_s \cong \operatorname{proj\,lim}_n H^i(X_s, I\otimes L^{\otimes m}\otimes \mathcal{O}_{X_s^{(n)}})$$
$$\cong \operatorname{proj\,lim}_n H^i(X_s, I/I^{n+1}\otimes L^{\otimes m}),$$

where $\mathcal{O}_{X_s^{(n)}} := \mathcal{O}_X/I^{n+1}$. Thus it is enough to prove that there exists a positive integer m_0 such that

$$H^i(X_s, I^n/I^{n+1}\otimes L^{\otimes m})=0 \quad \text{for any } n\geqq 1,\ m\geqq m_0 \quad \text{and } i>0.$$

Let $\mathrm{Gr}_I(\mathcal{O}_X) := \oplus_{n\geqq 0} I^n/I^{n+1}$, and $G := \oplus_{n\geqq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$, where $\mathfrak{m}$ is the maximal ideal defining $\{s\}$ in S. The natural surjective homomorphisms $f^*(\mathfrak{m}^n/\mathfrak{m}^{n+1})\to I^n/I^{n+1}$ induce a surjective ring homomorphism $\varphi: f^*G\to \mathrm{Gr}_I(\mathcal{O}_X)$. Let $g: V := \mathrm{Specan}_X(\mathrm{Gr}_I(\mathcal{O}_X))\to X_s\subset X$. Then we obtain the following commutative diagram:

$$\begin{array}{ccccc} g: V & \xrightarrow{\varphi^*} & X_s\times \mathrm{Spec}\, G & \longrightarrow & X_s \\ \downarrow{\scriptstyle\psi} & & \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, G & = & \mathrm{Spec}\, G & \longrightarrow & C(s). \end{array}$$

Since L_s is ample and φ^* is a closed embedding, g^*L is also ψ-ample. Therefore $R^i\psi_*(g^*L^{\otimes m})=0$ for $i>0$ and for $m\gg 0$. This implies that $H^i(V, g^*L^{\otimes m})=0$ for $i>0$ and $m\gg 0$. By the spectral sequence

$$H^p(X_s, R^q g_*(g^*L^{\otimes m}))\Longrightarrow H^{p+q}(V, g^*L^{\otimes m}),$$

we obtain

$$H^p(X_s, g_*(g^*L^{\otimes m}))=H^p(V, g^*L^{\otimes m}),$$

because g is an affine morphism. Therefore,

$$H^p(X_s, g_*g^*(L^{\otimes m}))=H^p(X_s, L^{\otimes m}\otimes g_*\mathcal{O}_V)=H^p(X_s, L^{\otimes m}\otimes \mathrm{Gr}_I(\mathcal{O}_X))=0$$

for $p>0$ and $m\gg 0$.

Step 2. *There exist an open neighborhood* U *of* s *and a positive integer* m *such that*

(1) $\varphi_m: f^*f_*L^{\otimes m}\to L^{\otimes m}$ *is surjective on* $f^{-1}(U)$ *and*
(2) φ_m *defines the closed embedding*

$$j: f^{-1}(U)\longrightarrow \boldsymbol{P}_U(f_*L^{\otimes m}_{|U}).$$

Since the last statement (2) asserts that L is f-ample on $f^{-1}(U)$, in order to complete the proof it suffices to prove Step 2.

Proof of Step 2. By Step 1, we get the following exact sequence for $m \geqq m_0$:

$$0 \longrightarrow f_*(I \otimes L^{\otimes m}) \longrightarrow f_*(L^{\otimes m}) \longrightarrow f_*(L^{\otimes m} \otimes \mathcal{O}_{X_s}) \longrightarrow 0.$$

Thus $f_*(L^{\otimes m}) \otimes C(s) \rightarrow f_*(L^{\otimes m} \otimes \mathcal{O}_{X_s})$ is surjective. Hence if we take m so large that $L^{\otimes m} \otimes \mathcal{O}_{X_s}$ is very ample on X_s, then the homomorphism $(f^* f_* L^{\otimes m}) \otimes \mathcal{O}_{X_s} \rightarrow L^{\otimes m} \otimes \mathcal{O}_{X_s}$ is surjective. Therefore there exists an open neighborhood U' of s such that U' and m satisfy the condition (1). By φ_m, we obtain a morphism $j: f^{-1}(U') \rightarrow \boldsymbol{P}_{U'}(f_* L^{\otimes m}|_{U'})$ over U'. Then by the following Lemma 1.5, we can find an open neighborhood U which satisfies (2). □

Lemma 1.5. *Let X, Y and S be complex spaces and let f: $X \rightarrow Y$, g: $Y \rightarrow S$ and S and $h := g \circ f$: $X \rightarrow S$ be proper morphisms. Assume that for a point $s \in S$, the fiber f_s: $X_s \rightarrow Y_s$ of the morphism f is a closed embedding. Then there is an open neighborhood U of s such that $f \times_S U$: $X \times_S U \rightarrow Y \times_S U$ is also a closed embedding.*

Proof. First of all we must show that we can take an open neighborhood U' so that $f \times_S U'$: $X \times_S U' \rightarrow Y \times_S U'$ is a finite morphism. But $X' := \{x \in X : x \text{ is isolated in } f^{-1}(f(x))\}$ is open. $U' := S \setminus f(X \setminus X')$ is not an empty set, since X_s is contained in X'. Then $f \times_S U'$: $X \times_S U' \rightarrow Y \times_S U'$ is finite, since its Stein factorization coincides with $X \times_S U'$. Next we shall prove that $f \times_S U$ is a closed embedding for some neighborhood $U \subset U'$. By the previous argument, we have $f_* \mathcal{O}_X \otimes \mathcal{O}_{Y_s} \cong f_{s*} \mathcal{O}_{X_s}$. Therefore the homomorphism $\mathcal{O}_{Y_s} \rightarrow f_* \mathcal{O}_X \otimes \mathcal{O}_{Y_s}$ is surjective. Thus Supp (Coker ($\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$)) $\cap Y_s = \phi$. Hence letting $U := U' \setminus g$(Supp (Coker ($\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$))), we see that $f \times_S U$ is a closed embedding. □

Corollary 1.6. *Let f: $X \rightarrow S$ be a proper morphism, and let L be a line bundle on X. Then L is f-ample if and only if L_s is ample for all $s \in S$.*

Definition 1.7. Let f: $X \rightarrow S$ be a projective morphism and let H be a line bundle on X. H is called *f-nef* if $L \cdot C \geqq 0$ for any irreducible curve C such that $f(C)$ is a point.

Corollary 1.8. *Let f: $X \rightarrow S$ be a projective morphism and let L and H be line bundles on X. If L is f-ample and H is f-nef, then $L \otimes H$ is f-ample.*

§ 2. Nef line bundles

Let Y be a d-dimensional compact Kähler manifold. We define the Kähler cone $\mathrm{KC}(Y)$ of Y to be the set

$$\{[\omega] \in H^{1,1}(Y, \boldsymbol{R});\ \omega \text{ is a Kähler form on } Y.\},$$

where $H^{1,1}(Y, \boldsymbol{R}) := H^2(Y, \boldsymbol{R}) \cap H^{1,1}(Y, \boldsymbol{C})$. Then $\mathrm{KC}(Y)$ is an open convex cone in $H^{1,1}(Y, \boldsymbol{R})$.

Lemma 2.1. *The closure* $\overline{\mathrm{KC}}(Y)$ *does not contain any linear subspace in* $H^{1,1}(Y, \boldsymbol{R})$.

Proof. Let z be an element of $H^{1,1}(Y, \boldsymbol{R})$ such that z and $-z$ are contained in $\overline{\mathrm{KC}}(Y)$. Then $z\cdot[\omega]^{d-1}=0$ for any Kähler form ω on Y, where the dot $\cdot$ denotes the cup product. Thus z is primitive with respect to ω. Moreover, $z^2\cdot[\omega]^{d-2}=0$, since $z\cdot z\cdot[\omega]^{d-2}$ and $z\cdot(-z)\cdot[\omega]^{d-2}$ are nonnegative. Therefore $z=0$. □

Lemma 2.2. *If* ω *is a Kähler form on* Y *and if* $z \in \overline{\mathrm{KC}}(Y)$, *then* $[\omega]+z \in \mathrm{KC}(Y)$.

Proof. Since ω is a Kähler form, there exists a positive number r such that $r[\omega]+z \in \mathrm{KC}(Y)$. We define $s(\omega) := \inf\{r \geqq 0;\ r[\omega]+z \in \mathrm{KC}(Y)\}$. We have only to prove that $s(\omega)=0$ for any Kähler form ω. Since $z \in \overline{\mathrm{KC}}(Y)$, there exists a Kähler form ω_1 such that $s(\omega_1)=0$. If $s(\omega)>0$ for some ω, then $(s(\omega)-\delta)[\omega]+z-(s(\omega)-\delta)[\omega-\varepsilon\omega_1]=\varepsilon\delta[\omega_1]+z \in \mathrm{KC}(Y)$, for any $\varepsilon>0$ and $0<\delta<s(\omega)$. If ε is sufficiently small, then $[\omega-\varepsilon\omega_1] \in \mathrm{KC}(Y)$; therefore $(s(\omega)-\delta)[\omega]+z \in \mathrm{KC}(Y)$, a contradiction. Thus $s(\omega)=0$ for all ω. □

Corollary 2.3. $\mathrm{KC}(Y)=\mathrm{Int}\,\overline{\mathrm{KC}}(Y)$, *the interior of the cone* $\overline{\mathrm{KC}}(Y)$.

Proof. $\mathrm{KC}(Y)\subset\mathrm{Int}\,\overline{\mathrm{KC}}(Y)$, since $\mathrm{KC}(Y)$ is an open set in $H^{1,1}(Y, \boldsymbol{R})$. On the other hand, if $z \in \mathrm{Int}\,\overline{\mathrm{KC}}(Y)$, then for any Kähler form ω on Y, there exists a positive number ε such that $z-\varepsilon[\omega] \in \overline{\mathrm{KC}}(Y)$. Therefore by (2.2), $z \in \mathrm{KC}(Y)$. □

Definition 2.4. Let L be a line bundle on a compact Kähler manifold Y. L is said to be nef if the real first Chern class $c_1(L)$ is contained in $\overline{\mathrm{KC}}(Y)$.

Remark 2.4.1. If Y is a projective manifold, then L is nef if and only if $L\cdot C\geqq 0$ for any irreducible curve C on Y.

Remark 2.4.2. If f: $Y \to X$ is a morphism between compact Kähler manifolds Y and X and if L is a nef line bundle on X, then f^*L is also nef, since $f^*\mathrm{KC}(X) \subset \overline{\mathrm{KC}}(Y)$.

Problem 2.5. *Suppose* f: $Y \to X$ *is a surjective morphism between compact Kähler manifolds* Y *and* X, *and* L *is a line bundle on* X *such that* f^*L *is nef. Then is* L *also nef on* X? *More generally, does the equality* $(f^*)^{-1}(\overline{\mathrm{KC}}(Y)) = \overline{\mathrm{KC}}(X)$ *hold, where* f^* *is the homomorphism* $H^{1,1}(X, \boldsymbol{R}) \to H^{1,1}(Y, \boldsymbol{R})$?

Definition 2.6. Let X be a compact complex variety in class $\mathscr{C}$. A line bundle L on X is called quasi-nef if there exists a bimeromorphic morphism μ: $Y \to X$ from a compact Kähler manifold Y such that μ^*L is nef.

We have a partial answer to (2.5).

Proposition 2.7. *In the situation of* (2.5), *if* X *is projective, then* L *is nef on* X.

Proof. Let C be an irreducible curve on X and let V be an irreducible component of $f^{-1}(C)$ such that $f(V) = C$. Put $d = \dim V$ and fix a Kähler form ω on Y. We have only to show that $L \cdot C \geqq 0$. Thus we may assume V to be smooth. Since f^*L if nef, we have $0 \leqq (f^*L) \cdot V \cdot \omega^{d-1}$. If we regard $V \cdot \omega^{d-1}$ as an element of $H_2(V, \boldsymbol{R})$ and consider f_*: $H_2(V, \boldsymbol{R}) \to H_2(X, \boldsymbol{R})$, then $L \cdot f_*(V \cdot \omega^{d-1}) \geqq 0$. Since f_* passes through $H_2(\tilde{C}, \boldsymbol{R})$, where $\tilde{C}$ denotes the normalization of C, there is a real number α such that $f_*(V \cdot \omega^{d-1}) = \alpha[C]$, where $[C]$ is the class of C in $H_2(X, \boldsymbol{R})$. Thus it remains to show that $\alpha > 0$. Take an ample A on X. Then $f^*A \cdot V \cdot \omega^{d-1} > 0$, because $f^*A \cdot V$ corresponds to the fibers of $V \to C$. Hence $A \cdot f_*(V \cdot \omega^{d-1}) = \alpha A \cdot C > 0$ and $\alpha > 0$. □

Corollary. *If* X *is a Moishezon variety, then* L *is quasi-nef if and only if* L *is nef, i.e.,* $L \cdot C \geqq 0$ *for any irreducible curve* C *on* X.

Lemma 2.8. *Let* D *be a nonzero effective Cartier divisor on* $X \in \mathscr{C}$. *Then* $-D$ *is not quasi-nef.*

Proof. If $-D$ is quasi-nef, then there exists a bimeromorphic morphism f: $Y \to X$ from a compact Kähler manifold Y such that $-f^*D$ is nef. Take a Kähler form ω on Y. Since f^*D is a nonzero effective divisor, we have $f^*D \cdot \omega^{d-1} > 0$, where $d = \dim Y$, a contradiction. □

Definition 2.9. Let L be a quasi-nef line bundle on $X \in \mathscr{C}$. Take a bimeromorphic morphism f: $Y \to X$ from a compact Kähler manifold Y

such that f^*L is nef. Then we define $\kappa_{\mathrm{hom}}(L) := \max\{l \geqq 0;\ 0 \neq c_1(f^*L)^l \in H^{l,l}(Y, \boldsymbol{R})\}$ and call it the *homological Kodaira dimension* of L. It is well-defined, because it is independent of the choice of Y.

Proposition 2.10. *Let L be a quasi-nef line bundle on a compact complex variety X in class $\mathscr{C}$. Then $\kappa(L) \leqq \kappa_{\mathrm{hom}}(L)$, where the left hand side is the usual Kodaira dimension (cf.* § 0, (B)).

Proof. We may assume that X is normal. If $\kappa(L) = -\infty$ or if $\kappa_{\mathrm{hom}}(L) = \dim X$, then there is nothing to prove. If $\kappa_{\mathrm{hom}}(L) = 0$ and $\kappa(L) \geqq 0$, then $mL = 0$ for some m. Indeed, if $|mf^*L|$ has an effective member D for some bimeromorphic morphism $f\colon Y \to X$ from a compact Kähler manifold Y such that f^*L is nef, then $f^*D \cdot \omega^{d-1} > 0$ for any Kähler form ω on Y, where $d = \dim Y$, a contradiction. So we may assume that $\kappa(L) \geqq 0$ and $0 < \kappa_{\mathrm{hom}}(L) < d$. Consider the canonical fibration $\Phi_{mL}\colon X \cdots \to Z$. By blowing ups, we may assume that $h := \Phi_{mL} \circ f\colon Y \to Z$ is a morphism. Then $f^*(mL) = h^*A + F$, where A is an ample divisor on Z and F is the fixed part of $|f^*(mL)|$. Let ω be a Kähler form on Y. Then

$$
\begin{aligned}
m^\kappa (f^*L)^\kappa \cdot \omega^{d-\kappa} &= (h^*A)^\kappa \cdot \omega^{d-\kappa} + (h^*A)^{\kappa-1} \cdot F \cdot \omega^{d-\kappa} \\
&\quad + (h^*A)^{\kappa-2} \cdot (h^*A + F) \cdot F \cdot \omega^{d-\kappa} + \cdots + (h^*A + F)^{\kappa-1} \cdot F \cdot \omega^{d-\kappa} > 0,
\end{aligned}
$$

where $\kappa = \kappa(L)$. Hence $c_1(f^*L)^\kappa \neq 0$. Therefore $\kappa(L) \leqq \kappa_{\mathrm{hom}}(L)$. □

Definition 2.11. Let L be a line bundle on a compact complex variety X in class $\mathscr{C}$. L is said to be *big* if $\kappa(L) = \dim X$. If L is quasi-nef and $\kappa(L) = \kappa_{\mathrm{hom}}(L)$, then L is called *good*.

Remark 2.11.1. If $f\colon X \to Z$ is a surjective morphism from a compact complex variety X in class $\mathscr{C}$ onto a projective variety Z and if H is a nef and big line bundle on Z, then f^*H is good.

Remark 2.11.2. If X is a projective variety and if L is a nef line bundle on X, then $\kappa(L) \leqq \nu(L) \leqq \kappa_{\mathrm{hom}}(L)$. Here $\nu(L) := \max\{l \geqq 0;\ L^l \not\approx_{\mathrm{num}} 0\}$ is called the *numerical Kodaira dimension* of L. By (2.11.1), $\kappa(L) = \kappa_{\mathrm{hom}}(L)$ if and only if $\kappa(L) = \nu(L)$. Therefore our "goodness' is the same as that in Kawamata [21] in the case of projective varieties.

Conjecture 2.12. *If L is a nef line bundle on a projective variety X, then $\nu(L) = \kappa_{\mathrm{hom}}(L)$.*

Remark 2.12.1. If $\nu(L) \leqq 1$ or $\kappa_{\mathrm{hom}}(L) \geqq \dim X - 1$, then we have $\nu(L) = \kappa_{\mathrm{hom}}(L)$.

Conjecture 2.13. *If L is a quasi-nef line bundle on a compact complex variety X in class $\mathscr{C}$ such that $\kappa_{\mathrm{hom}}(L)=\dim X$, then L is big.*

Remark 2.13.1. (2.13) is equivalent to the following statement: *If L is a nef line bundle on a compact Kähler manifold Y such that $L^{\dim Y}>0$, then $H^i(Y, \omega_Y \otimes L)=0$ for $i>0$.*

Proposition 2.14. *Let L be a quasi-nef and good line bundle on a compact complex variety X in class $\mathscr{C}$. Then there exists the following diagram*

$$X \xleftarrow{\ \mu\ } Y \xrightarrow{\ h\ } Z,$$

where

(a) *Y is a compact Kähler manifold and μ is a bimeromorphic morphism,*
(b) *Z is a projective variety, h is a fiber space, and*
(c) *there exists a nef and big $\mathbf{Q}$-divisor H on Z such that $\mu^*L=h^*H$.*

Proof. (cf. [21, Proposition 2.1]). Let $\Phi_{mL}: X \cdots \to Z_0$ be the canonical fibration of L. By blowing ups and flattening (see [15]), we have the following diagram:

$$\begin{array}{ccccccccc}
X & \xleftarrow{\mu_0} & Y_1 & \xleftarrow{\mu_1} & Y_2 & \xleftarrow{\nu} & Y_3 & \xleftarrow{d} & Y \\
\vdots\downarrow{\scriptstyle \Phi_{mL}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
Z_0 & \xleftarrow{\tau_0} & Z_1 & \xleftarrow{\tau_1} & Z & = & Z & = & Z,
\end{array}$$

where

(a) μ_0, μ_1, ν, d, τ_0 and τ_1 are bimeromorphic, and f_1, g, and h are fiber spaces,

(b) Y_1 and Y are compact Kähler manifolds and Z_1 and Z are non-singular projective varieties,

(c) f_2 is flat, ν is the normalization of Y_2, and d is the resolution of singularities of Y_3 and

(d) there exists an ample divisor A on Z_0 such that $\lambda^*(mL)= g^*(\tau_1^*\tau_2^*A)+F$, where $\lambda=\mu_0 \circ \mu_1 \circ \nu$ and F is the fixed part of $|\lambda^*(mL)|$.

Take a Kähler form ω on Y. Then

$$\begin{aligned}
m^{\kappa+1}(\mu^*L)^{\kappa+1}\cdot\omega^{d-\kappa-1} &= (h^*B)^{\kappa+1}\cdot\omega^{d-\kappa-1}+(h^*B)^{\kappa}\cdot d^*F\cdot\omega^{d-\kappa-1}+\cdots \\
&\quad +(h^*B)\cdot(h^*B+d^*F)^{\kappa-2}\cdot d^*F\cdot\omega^{d-\kappa-1} \\
&\quad +(h^*B+d^*F)^{\kappa-1}\cdot d^*F\cdot\omega^{d-\kappa-1} \\
&= 0,
\end{aligned}$$

where $\kappa=\kappa(L)$, $d=\dim X$, $B=\tau_1^*\tau_0^*A$, and $\mu=\lambda\circ\nu$. This implies that $g(F)\subsetneqq Z$, since $(h^*B)^\kappa\cdot d^*F\cdot\omega^{d-\kappa-1}=0$.

Lemma 2.15. *Let $g\colon V\to Z$ be a proper surjective morphism from a normal complex variety V onto a complex manifold Z and let F be an effective Cartier divisor on V. Assume that*

(1) *g is equi-dimensional with connected fibers,*

(2) *$g(F)\subsetneqq Z$, and*

(3) *if Γ is an irreducible component of a fiber of $F\to g(F)$, then $\Gamma\in\mathscr{C}$ and $F_{|\Gamma}$ is quasi-nef.*

Then there exists a $\boldsymbol{Q}$-divisor E on Z such that $F=g^*E$.

Proof of (2.15). By (1) and (2), we have only to prove $F=g^*E$, where $E=\min\{\Delta|\text{ a }\boldsymbol{Q}\text{-divisor on }Z\text{ such that }F\leqq g^*\Delta\}$. Thus we may assume that $\dim Z=1$. If $g^*E\neq F$, then there exists a component Γ of F such that $(g^*E-F)_{|\Gamma}$ is a nonzero effective divisor on Γ. This contradicts (2.8). Therefore $F=g^*E$. □

The proof of (2.14) *continued.* Applying (2.15) to the case $g\colon Y_3\to Z$ and $F\subset Y_3$, we obtain a $\boldsymbol{Q}$-divisor E on Z such that $F=g^*E$. Therefore $\mu^*(mL)=h^*(B+E)$. Hence let $H:=(1/m)(B+E)$. Then $\mu^*L=h^*H$. Since h^*H is nef, by (2.7), H is also nef. Therefore H is a nef and big $\boldsymbol{Q}$-divisor on Z. □

Corollary 2.16 ([21, Proposition 2.3]). *In the situation of* (2.14), *let L' be another quasi-nef $\boldsymbol{Q}$-Cartier divisor on X. Assume that $\kappa_{\text{hom}}(L+L')=\kappa_{\text{hom}}(L)$ and that $\kappa(L+L')\geqq 0$. Then there is a nef $\boldsymbol{Q}$-divisor H' on Z such that $\mu^*L'=h^*H'$.*

Proof. Let ω be a Kähler form on Y. Then

$$0=(\mu^*L'+\mu^*L)^{\kappa+1}\cdot\omega^{d-\kappa-1}\geqq(\kappa+1)(\mu^*L')\cdot(\mu^*L)^\kappa\cdot\omega^{d-\kappa-1}\geqq 0,$$

where $\kappa=\kappa(L)$. Thus if $\Delta\in|m\mu^*(L+L')|$ for a positive integer m, then $\mu(\Delta)\subsetneqq Z$. Hence by (2.15), $\Delta=g^*E$ for an effective $\boldsymbol{Q}$-divisor E on Z. Therefore $\mu^*L'=h^*H'$ for some $\boldsymbol{Q}$-divisor H' on Z and by (2.7), H' is nef. □

The following proposition is a relative version of (2.14) whose proof is omitted.

Proposition 2.17. *Let $\pi\colon X\to S$ be a proper surjective morphism from a normal complex variety X onto a complex variety S, and let L be a line bundle on X. Assume that*

(1) *every component Γ of any fiber of π is in class $\mathscr{C}$ and $L_{|\Gamma}$ is quasi-nef and*

(2) *L_1 is good for a general fiber X_t.*

Then for any point $s \in S$, there exist an open neighborhood S_1 of s and a commutative diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{\ \mu\ } & Y \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle h} \\ S_1 & \xleftarrow{\ g\ } & Z, \end{array}$$

where

(a) *$X_1 = \pi^{-1}(S_1)$ and $\pi_1 = \pi_{|X_1}$,*

(b) *Y and Z are complex manifolds and μ is a proper bimeromorphic morphism,*

(c) *g is a projective morphism and h is a proper fiber space, and*

(d) *there exists a g-nef $\boldsymbol{Q}$-divisor H on Z such that $H_{|Z_t}$ is nef and for general $t \in S$ and that $\mu^* L = h^* H$.*

§ 3. Covering lemma and vanishing theorems

Lemma 3.1. *Let X be an n-dimensional complex manifold, D a reduced divisor on X with only simple normal crossings, and let $D = \sum_{1 \leqq i \leqq k} D_i$ be the irreducible decomposition of D. Assume that there are smooth divisors H^i_j, line bundles $\mathscr{L}_i$ and positive integers m_i for $1 \leqq i \leqq k$, $0 \leqq j \leqq n$ such that*

(α) *$\mathcal{O}_X(H^i_j + D_i) \cong \mathscr{L}_i^{\otimes m_i}$ and*

(β) *$\sum_i D_i + \sum_{i,j} H^i_j$ is a divisor with simple normal crossings.*

Then there exists a finite Galois covering $\pi \colon Y \to X$ which satisfies the following conditions:

(1) *Y is smooth,*

(2) *$(\pi^* D)_{\mathrm{red}}$ has only simple normal crossings,*

(3) *there are divisors Δ_i $(1 \leqq i \leqq k)$ with only simple normal crossings such that $\pi^* D_i = m_i \Delta_i$.*

For the proof, see Kawamata [18, Theorem 17], [21, Lemma 3.1].

By a property of positive line bundles on a weakly 1-complete variety, we obtain:

Lemma 3.2. *Let X be an n-dimensional weakly 1-complete manifold with positive line bundles. Let $D = \sum_{i \in I} D_i$ be a reduced divisor with only simple normal crossings, where each D_i is an irreducible component of D, and let m_i be a positive integer for each $i \in I$. Then for any $c \in \boldsymbol{R}$, there exist smooth divisors H^i_j on X_c for $i \in I$, $1 \leqq j \leqq n$ and there exist line bundles*

$\mathscr{L}_i$ on X_c for $i \in I$ such that $\mathcal{O}_{X_c}(H^i_j + D_i) \cong \mathscr{L}_i^{\otimes m_i}$ for all i and j, and that $\sum_{i \in I} D_i + \sum_{i,j} H^i_j$ has only simple normal crossings.

Therefore combining these lemmas, we get the following:

Lemma 3.3 (cf. [21, Lemma 3.1]). *Let X be a weakly* 1*-complete manifold with positive line bundles, and let D be a* $\boldsymbol{Q}$*-divisor such that* Supp $\langle D \rangle$ *has only normal crossings. Then for any* $c \in \boldsymbol{R}$*, there exists a proper generically finite surjective morphism* $\pi\colon Y \to X_c$ *from a complex manifold Y such that*

(1) *$\pi^* D$ is a Cartier divisor,*

(2) *$\mathcal{O}_{X_c}(K_X + \ulcorner D \urcorner)$ is a direct summand of $\pi_* \mathcal{O}_Y(K_Y + \pi^* D)$.*

Proof. Take a proper bimeromorphic morphism $\mu\colon Z \to X_c$ from a complex manifold Z such that Supp $\mu^* \langle D \rangle$ has only simple normal crossings. Take a positive integer m such that $m\langle D \rangle$ is a Cartier divisor on X_c. Then by (3.1) and (3.2), we have a finite Galois covering $\tau\colon Y \to Z$ such that $\tau^* \mu^* \langle D \rangle$ is a Cartier divisor and Supp $\tau^* \mu^* \langle D \rangle$ has only simple normal crossings. By the same argument as in Lemma 3.1 of [21], we can show that $\mathcal{O}_Z(K_Z + \ulcorner \mu^* D \urcorner)$ is a direct summand of $\tau_* \mathcal{O}_Y(K_Y + \tau^* \mu^* D)$. Since $\mu_* \mathcal{O}_Z(K_Z + \ulcorner \mu^* D \urcorner) \cong \mathcal{O}_{X_c}(K_X + \ulcorner D \urcorner)$ by (0.6), we complete the proof. □

Theorem 3.4. *Let X be a weakly* 1*-complete manifold and let A be a* $\boldsymbol{Q}$*-divisor on X. Assume that*

(1) *there is a positive integer m such that mA is a Cartier divisor and that $\mathcal{O}_X(mA)$ is a positive line bundle on X, and*

(2) Supp $\langle A \rangle$ *has only normal crossings.*

Then $H^i(X_c, \mathcal{O}_X(K_X + \ulcorner A \urcorner)) = 0$ for $i > 0$ and for any $c \in \boldsymbol{R}$.

Proof. Step 1. The case where Supp $\langle A \rangle$ has only simple normal crossings. By (3.3), for any $c \in \boldsymbol{R}$, we obtain a finite surjective morphism $\pi\colon Y \to X_c$ such that $\pi^* A$ is a Cartier divisor and that $\mathcal{O}_{X_c}(K_X + \ulcorner A \urcorner)$ is a direct summand of $\pi_* \mathcal{O}_Y(K_Y + \pi^* A)$. Since π is finite, $\mathcal{O}_Y(\pi^* A)$ is a positive line bundle. Thus by (0.3), $H^i(Y, \mathcal{O}_Y(K_Y + \pi^* A)) = 0$ for $i > 0$. Therefore $H^i(X_c, \mathcal{O}_X(K_X + \ulcorner A \urcorner)) = 0$ for $i > 0$, because $\mathcal{O}_{X_c}(K_X + \ulcorner A \urcorner)$ is a direct summand of $\pi_* \mathcal{O}_Y(K_Y + \pi^* A)$.

Step 2. General case. Take a proper bimeromorphic morphism μ: $Z \to X_c$ from a complex manifold Z so that

(a) the μ-exceptional locus E is a divisor $\sum E_j$,

(b) Supp $\mu^* \langle A \rangle \cup E$ is a divisor with only simple normal crossings and

(c) $\mu^* A - \sum \delta_j E_j$ is positive for $0 < \delta_j \ll 1$.

Then by Step 1, we obtain

$$H^i(Z, \mathcal{O}_Z(K_Z+\ulcorner\mu^*A\urcorner))=0 \qquad \text{for } i>0,$$

and

$$R^i\mu_*\mathcal{O}_Z(K_Z+\ulcorner\mu^*A\urcorner)=0 \qquad \text{for } i>0,$$

because $\ulcorner\mu^*A-\sum\delta_jE_j\urcorner=\ulcorner\mu^*A\urcorner$ for $0<\delta_j\ll 1$. Therefore by (0.6),

$$H^i(X_c, \mu_*\mathcal{O}_Z(K_Z+\ulcorner\mu^*A\urcorner))=H^i(X_c, \mathcal{O}_X(K_X+\ulcorner A\urcorner))=0 \qquad \text{for } i>0. \qquad \square$$

Corollary 3.5. *Let f: $X\to S$ be a projective morphism from a complex manifold X onto a complex variety S and let A be an f-ample $\boldsymbol{Q}$-divisor on X such that* Supp $\langle A\rangle$ *has only normal crossings. Then* $R^if_*\mathcal{O}_X(K_X+\ulcorner A\urcorner)=0$ *for any* $i>0$.

Theorem 3.6 (cf. [12]). *Let f: $X\to S$ be a proper generically finite morphism from a complex manifold X onto a complex variety S and let H be a $\boldsymbol{Q}$-divisor on X such that*

(1) $H\cdot\Gamma\geqq 0$ *for any curve Γ such that $f(\Gamma)$ is a point,*

(2) Supp $\langle H\rangle$ *has only normal crossings.*

Then $R^if_*\mathcal{O}_X(K_X+\ulcorner H\urcorner)=0$ *for* $i>0$.

Proof. We may assume that f is bimeromorphic and that S is a Stein space. By relative Chow's lemma [15], there exists a proper bimeromorphic morphism g: $Y\to S$ from a smooth manifold Y which has the following properties.

(1) There is a morphism μ: $Y\to X$ such that $g=f\circ\mu$,

(2) $Y_c\to S_c$ is a finite succession of blowing ups.

Therefore $g_c=g_{|Y_c}$ and $\mu_c=\mu_{|Y_c}$ are projective morphisms. Furthermore, if Supp $\mu^*\langle H\rangle$ has only normal crossings, then $\mu_*\mathcal{O}_Y(K_Y+\ulcorner\mu^*H\urcorner)\cong\mathcal{O}_X(K_X+\ulcorner H\urcorner)$. Hence the Leray spectral sequence reduces the proof of the vanishing theorem for f to that for g and μ. Thus we may assume in what follows that f is a projective morphism.

Let A be an f-ample divisor on X_c. Since S_c is a Stein space, there is a nonzero section s of $f_*\mathcal{O}_{X_c}(-A)$, which also gives a section of $\mathcal{O}_{X_c}(-A)$; Therefore there is an effective divisor D on X_c such that $\mathcal{O}_{X_c}(A+D)\cong\mathcal{O}_{X_c}$. Let ν: $Z\to X_c$ be a projective bimeromorphic morphism from a smooth manifold Z such that Supp $\nu^*\langle H\rangle\cup$ Supp $\nu^*D\cup$ (ν-exceptional locus) is a divisor with only normal crossings. Here we denote by E the ν-exceptional locus $\sum E_i$. Then by (1.8), $-(1/m)\nu^*D-\sum\delta_iE_i+\nu^*H$ is $f\circ\nu$-ample for $0<\delta_i\ll 1$ and $m\gg 1$, if we replace S_c by $S_{c'}$ for some $0<c'<c$. Thus by (3.5),

$$R^i\nu_*\mathcal{O}_Z(K_Z+\ulcorner\nu^*H\urcorner)=0 \qquad \text{for } i>0$$

and

$$R^i(f\circ\nu)_*\mathcal{O}_Z(K_Z+\ulcorner\nu^*H\urcorner)=0 \qquad \text{for } i>0.$$

Since $\nu_*\mathcal{O}_Z(K_Z+\ulcorner\nu^*H\urcorner)\cong\mathcal{O}_{X_c}(K_X+\ulcorner H\urcorner)$, we obtain $R^if_*\mathcal{O}_X(K_X+\ulcorner H\urcorner)=0$ for $i>0$. □

Definition. Let $f\colon X\to S$ be a proper surjective morphism from a normal complex manifold X onto a complex variety S, and let H be a Cartier divisor on X. H is called *f-big* if $\kappa(X/S, H)=\dim X-\dim S$. Furthermore if $(H\cdot C)\geqq 0$ for every irreducible curve C such that $f(C)=$ point, then H is called *f-nef-big*.

We have the following theorem which was first formulated by Fujita.

Theorem 3.7. *Let $f\colon X\to S$ be a proper surjective morphism from a complex manifold X onto a complex variety S and let H be a $\mathbf{Q}$-divisor on X such that H is f-nef-big and that* Supp $\langle H\rangle$ *has only normal crossings. Then $R^if_*\mathcal{O}_X(K_X+\ulcorner H\urcorner)=0$ for $i>0$.*

Proof. Since the statement is local on S, we may assume S to be a Stein space. By the relative canonical fibration of H over S and by relative Chow's lemma [15], we may assume that there exists a projective morphism $\pi\colon Y\to S$ from a complex manifold Y such that π factors through f, i.e., $\pi=f\circ\mu$ for some $\mu\colon Y\to X$. We may also assume that Supp $\mu^*\langle H\rangle$ has only normal crossings. By (3.6), $\boldsymbol{R}\mu_*\mathcal{O}_Y(K_Y+\ulcorner\mu^*H\urcorner)=\mathcal{O}_X(K_X+\ulcorner H\urcorner)$, so we have only to prove that $R^i\pi_*\mathcal{O}_Y(K_Y+\ulcorner\mu^*H\urcorner)=0$ for $i>0$. Therefore from the beginning we may assume that f is a projective morphism. Let A be an f-ample divisor on X. Then there exists a positive integer m such that mH is a Cartier divisor and that $f_*\mathcal{O}_X(mH-A)$ is a nonzero sheaf. Since S is a Stein space, we obtain an effective divisor Δ on X such that $\mathcal{O}_X(mH)\cong\mathcal{O}_X(\Delta+A)$. Because H is f-nef, $mH-\varepsilon\Delta$ is f-ample for any $0<\varepsilon<1$ by (1.8). By blowing ups, we obtain a proper bimeromorphic morphism $\mu\colon Z\to X$ from a complex manifold Z such that the μ-exceptional locus is a divisor $E=\sum E_j$ and that Supp $\mu^*\langle H\rangle\cup$ Supp $\mu^*\Delta\cup$ Supp E is a divisor with only normal crossings. Thus $\mu^*(mH-\varepsilon\Delta)-\sum\delta_jE_j$ is $f\circ\mu$-ample for $0<\varepsilon\ll 1$, $0<\delta_j\ll 1$. Therefore by (3.5), $R^i(f\circ\mu)_*\mathcal{O}_Z(K_Z+\ulcorner\mu^*H\urcorner)=0$ for $i>0$. Thus by (3.6), $R^if_*\mathcal{O}_X(K_X+\ulcorner H\urcorner)=0$ for $i>0$. □

Lemma 3.8 (relative algebraic reduction for a divisor). *Let $f\colon X\to S$ be a proper surjective morphism from a complex manifold X onto a complex variety S, H a line bundle on S, and let D be a (not necessarily effective) Cartier divisor on X such that $\mathcal{O}_X(D)\cong f^*H$. Then there exist a proper* *bimeromorphic morphism* μ: $X' \to X$ *from a complex manifold* X', *a projective surjective morphism* λ: $S' \to S$, *a proper surjective morphism* f': $X' \to S'$ *such that* $\lambda \circ f' = f \circ \mu$, *and a Cartier divisor* Δ *on* S' *such that* $\mu^* D = f'^* \Delta$ *as divisors.*

Proof. Let $D = D_+ - D_-$ be the decomposition into the effective part D_+ and the negative part D_-. Take two sections $s_+ \in H^0(X, \mathcal{O}_X(D_+))$ and $s_- \in H^0(X, \mathcal{O}_X(D_-))$ such that $\operatorname{div}(s_+) = D_+$ and $\operatorname{div}(s_-) = D_-$. Then $s_+ : \mathcal{O}_X \to \mathcal{O}_X(D_+)$ and $s_- \otimes \mathcal{O}_X(D)$: $\mathcal{O}_X(D) \to \mathcal{O}_X(D_+)$ give a homomorphism

$$\varphi: f^*(\mathcal{O}_S \oplus H) \cong \mathcal{O}_X \oplus \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D_+).$$

By φ, we can construct a meromorphic map

$$\varphi^*: X \cdots \longrightarrow \boldsymbol{P}_S(\mathcal{O}_S \oplus H).$$

So take a suitable proper bimeromorphic morphism μ: $X' \to X$ such that $f' := \varphi^* \circ \mu$: $X' \to \boldsymbol{P}_S(\mathcal{O}_S \oplus H)$ is a morphism. Let $X' \to S'$ be the Stein factorization of f' and let λ: $S' \to S$ be the induced morphism. Then the image of $\mu^*(\varphi)$: $\mathcal{O}_{X'} \oplus \mathcal{O}_{X'}(\mu^* D) \to \mathcal{O}_{X'}(\mu^* D_+)$ is a line bundle M and the induced homomorphisms $\mathcal{O}_{X'} \to M$ and $\mathcal{O}_{X'}(\mu^* D) \to M$ give effective divisors $D'_+ \in |M|$ and $D'_- \in |M \otimes \mathcal{O}_{X'}(-\mu^* D)|$, respectively. Then there is an effective μ-exceptional divisor E such that $\mu^* D_+ = D'_+ + E$ and $\mu^* D_- = D'_- + E$. Therefore $\mu^* D = D'_+ - D'_-$.

On the other hand, M is the pull back of a λ-ample line bundle N on S'. Hence $D'_+ = f'^* \Delta_+$ for an effective Cartier divisor $\Delta_+ \in |N|$. Similarly, $D'_- = f'^* \Delta_-$ for an effective Cartier divisor $\Delta_- \in |N \otimes \lambda^* H^{-1}|$. Thus $D'_+ - D'_- = f'^*(\Delta_+ - \Delta_-)$. Therefore if we denote $\Delta_+ - \Delta_-$ by Δ, then $\mathcal{O}_S(\Delta) \cong \lambda^* H$ and $\mu^* D = f'^* \Delta$. □

Lemma 3.9 (Covering lemma). *Let* f: $X \to S$ *be a proper surjective morphism from a complex manifold* X *onto a complex variety* S, H *a line bundle on* S, *and let* D *be a* $\boldsymbol{Q}$-*divisor on* X. *Assume that*

(1) *S is a weakly 1-complete variety with positive line bundles,*

(2) *there is an isomorphism* $\mathcal{O}_X(kD) \cong f^* H$ *for some positive integer* k *such that* kD *is Cartier and*

(3) Supp $\langle D \rangle$ *has only normal crossings.*

Then for any $c \in \boldsymbol{R}$, *there exists a proper generically finite surjective morphism* π: $Y \to X_c$ *such that*

(a) *Y is smooth,*

(b) $\pi^* D$ *is a Cartier divisor and*

(c) $\mathcal{O}_{X_c}(K_X + \ulcorner D \urcorner)$ *is a direct summand of* $\pi_* \mathcal{O}_Y(K_Y + \pi^* D)$.

Proof. Applying (3.8) to the divisor kD, we obtain the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \lambda} \\ X & \xrightarrow{f} & S \end{array}$$

which has the following properties.

(1) X' and S' are complex manifolds,

(2) μ is proper bimeromorphic, λ is projective, and f' is surjective,

(3) there exists a Cartier divisor Δ on S' such that $\mu^*(kD)=f'^*\Delta$,

(4) Supp $\langle(1/k)\Delta\rangle$ and Supp $\mu^*\langle D\rangle$ have only simple normal crossings on S' and X', respectively.

Since λ is projective, S' is also a weakly 1-complete variety with positive line bundles. Therefore by (3.3), for any $c \in \boldsymbol{R}$, there exists a finite surjective morphism $\tau: T \to S'_c$ such that T is smooth and that $\tau^*((1/k)\Delta)$ is a Cartier divisor. Here we may assume that the divisors H^i_j defined in Lemma 3.2 are smooth divisors such that $f'^*H^i_j$ are also smooth and that $\bigcup f'^*H^i_j \cup \operatorname{Supp} \mu^*\langle D\rangle$ has only simple normal crossings. Then the normalization V of $X'_c \times_{S'_c} T$ has only rational singularities, since the branch locus for $p: V \to X'_c$ is a divisor with only simple normal crossings. Note that $p^*\mu^*D$ is a Cartier divisor. Since $p_*\mathcal{O}_V(K_V+p^*\mu^*D)$ is a reflexive sheaf on X and since $p: V \to X'_c$ is a cyclic covering in codimension one on X'_c, it is easy to see that $\mathcal{O}_{X'_c}(K_{X'}+\ulcorner\mu^*D\urcorner)$ is a direct summand of $p_*\mathcal{O}_V(K_V+p^*\mu^*D)$. Let $Y \to V$ be the resolution of singularities and let $\pi: Y \to X_c$ be the induced morphism. Then the three conditions of this lemma are satisfied. □

The following theorem was proved by Kollár [25].

Theorem 3.10. (A) *Let $f: X \to Z$ be a proper surjective morphism from a compact Kähler manifold X onto a projective variety Z. Then*

(i) *$R^if_*\omega_X$ is torsion free for $i \geqq 0$,*

(ii) *$H^p(Z, R^if_*\omega_X \otimes A)=0$ for $p>0$ and for any ample line bundle A on Z.*

(B) *Let X be a compact Kähler manifold, L a semi-ample line bundle on X, and let s be a global section of $L^{\otimes k}$ for some positive integer k. Then the natural homomorphisms*

$$\otimes s: H^i(X, \omega_X \otimes L^j) \longrightarrow H^i(X, \omega_X \otimes L^{j+k})$$

are injective for any $i \geqq 0$ and $j \geqq 1$.

Remark 3.10.1. It is easy to see that (A) and (B) are equivalent.

Remark 3.10.2. [25] treats only projective varieties, but his argument works also in the situation of (3.10).

Applying (3.9), we have two generalizations of (3.10). These are essentially the same as Theorems 3.2 and 3.3 of [21].

Theorem 3.11 (A generalization of B). *Let X be a compact Kähler manifold, L a quasi-nef and good $\boldsymbol{Q}$-divisor on X, and let D be an effective divisor on X. Assume that* Supp $\langle L\rangle$ *has only normal crossings and that there is an injection $\mathcal{O}_X(D)\to\mathcal{O}_X(mL)$ for some positive integer such that mL is a Cartier divisor. Then the natural homomorphisms*

$$+D\colon H^i(X, \mathcal{O}_X(K_X+\ulcorner L\urcorner))\longrightarrow H^i(X, \mathcal{O}_X(K_X+D+\ulcorner L\urcorner))$$

are injective for $i\geqq 0$.

Proof. It is enough to prove this when $D\in|mL|$. Since L is quasi-nef and good, by (2.14) we have a bimeromorphic morphism $\mu\colon X'\to X$ from a compact Kähler manifold X', a fiber space $h\colon X'\to Z$ onto a projective manifold Z, and a nef and good $\boldsymbol{Q}$-divisor H on Z such that $\mu^*L=h^*H$. Here we may assume that Supp $\mu^*\langle L\rangle$ has only simple normal crossings. Then by (3.9), there is a generically finite surjective morphism $\pi\colon Y\to X'$ from a compact Kähler manifold Y such that $\pi^*\mu^*L$ is a Cartier divisor and that $\mathcal{O}_{X'}(K_{X'}+\ulcorner\mu^*L\urcorner)$ is a direct summand of $\pi_*\mathcal{O}_Y(K_Y+\pi^*\mu^*L)$. Thus $\mathcal{O}_X(K_X+\ulcorner L\urcorner)$ is a direct summand of $\mu_*\pi_*\mathcal{O}_Y(K_Y+\pi^*\mu^*L)$. Similarly, $\mathcal{O}_X(K_X+\ulcorner L\urcorner+D)$ is also a direct summand of $\mu_*\pi_*\mathcal{O}_Y(K_Y+\pi^*\mu^*L+\pi^*\mu^*D)$. Thus by (3.6),

$$H^i(X, \mathcal{O}_X(K_X+\ulcorner L\urcorner))\longrightarrow H^i(X, \mathcal{O}_X(K_X+\ulcorner L\urcorner+D))$$

is a direct summand of

$$H^i(Y, \mathcal{O}_Y(K_Y+\pi^*\mu^*L))\longrightarrow H^i(Y, \mathcal{O}_Y(K_Y+\pi^*\mu^*L+\pi^*\mu^*D)).$$

Therefore we may assume that L is a Cartier divisor and that there exist a fiber space $h\colon X\to Z$ and nef and big $\boldsymbol{Q}$-divisor H on Z with $L=h^*H$. Since H is nef and big, there is an effective divisor Δ on Z such that $H-\delta\Delta$ is ample for $0<\delta\ll 1$. Then $\ulcorner h^*(H-\delta\Delta)\urcorner=L$, if δ is sufficiently small. By (3.6), we can replace X by its blowing ups. Thus we may assume that Supp $h^*\Delta$ has only normal crossings. Hence it is enough to prove in the case that L is a semi-ample $\boldsymbol{Q}$-divisor. Then for the same reason as above, we may assume that L is a semi-ample Cartier divisor. This is just the case (3.10.B). □

Theorem 3.12 (A generalization of A). *Let $f: X \to Z$ be a proper surjective morphism from a compact Kähler manifold X onto a projective variety Z, and let L be a quasi-nef and good $\boldsymbol{Q}$-divisor on X such that* Supp $\langle L \rangle$ *has only normal crossings. Then*

(a) *$R^i f_* \mathcal{O}_X(K_X + \ulcorner L \urcorner)$ is torsion free for all $i \geqq 0$, and*

(b) *if there is an injection $\mathcal{O}_X(f^*A) \to \mathcal{O}_X(nL)$, where A is an ample divisor on Z and n is a positive integer such that nL is Cartier, then $H^p(Z, R^i f_* \mathcal{O}_X(K_X + \ulcorner L \urcorner)) = 0$ for $p > 0$ and $i \geqq 0$.*

Proof. (a) Since L is quasi-nef and good, there exist a bimeromorphic morphism $\mu: Y \to X$ from a compact Kähler manifold Y, a fiber space $h: Y \to W$ onto a projective manifold W, and a nef and big $\boldsymbol{Q}$-divisor H on W such that $\mu^* L = h^* H$. We can also take an effective divisor Δ on W such that $H - \delta\Delta$ is ample for $0 < \delta \ll 1$. We may assume that Supp $\mu^*\langle L \rangle$ $\cup$ Supp $h^*\Delta$ is a divisor with only simple normal crossings. If δ is sufficinetly small, then $\ulcorner \mu^* L \urcorner = \ulcorner h^*(H - \delta\Delta) \urcorner$. Applying (3.9) to $h^*(H - \delta\Delta)$, we obtain a generically finite surjective morphism $\pi: Y' \to Y$ from a compact Kähler manifold Y' such that $\pi^* h^*(H - \delta\Delta)$ is a Cartier divisor and that $\mathcal{O}_Y(K_Y + \ulcorner \mu^* L \urcorner)$ is a direct summand of $\pi_* \mathcal{O}_{Y'}(K_{Y'} + \pi^* h^*(H - \delta\Delta))$. Thus we may assume that L is a semi-ample Cartier divisor. Then by taking a cyclic covering corresponding to a general section of some multiple of L, we are reduced to (3.10.A.(i)).

(b) By assumption we can apply (3.11) to L and f^*A. Then by the same argument as in [25, Theorem 2.1, Step 4], it is easily proved. □

The following theorem was derived from the argument of [44].

Theorem 3.13. *Let $\pi: X \to D$ be a proper surjective morphism from a complex manifold X onto a unit disk D. Suppose that X_t is smooth for any $t \in D^* := D \setminus \{0\}$, and that any irreducible component of X_0 is a variety in class $\mathscr{C}$. Then $R^i \pi_* \omega_X$ is free at 0 for $i \geqq 0$.*

Proof. By taking semi-stable reduction, we may assume that π is semi-stable. In this situation, Steenbrink [44] proved that

$$\boldsymbol{R}^i \pi_* \Omega^{\cdot}_{X/D}(\log X_0) \otimes \boldsymbol{C}(t) \cong \boldsymbol{H}^i(X_t, \Omega^{\cdot}_{X/D}(\log X_0) \otimes \mathcal{O}_{X_t})$$

for any $i \geqq 0$ and $t \in D$, and that $\boldsymbol{R}^i \pi_* \Omega^{\cdot}_{X/D}(\log X_0)$ is a locally free sheaf for all $i \geqq 0$. Since all the components of X_0 are in class $\mathscr{C}$, we can also obtain a result similar to Theorem (4.19) of [44]. In particular, the following spectral sequence degenerates at the E_1-term:

$$E_1^{p,q} = H^q(X_0, \Omega^p_{X/D}(\log X_0) \otimes \mathcal{O}_{X_0}) \Longrightarrow \boldsymbol{H}^{p+q}(X_0, \Omega^{\cdot}_{X/D}(\log X_0) \otimes \mathcal{O}_{X_0}).$$

Hence

$$\dim H^i(X_0, \Omega^{\cdot}_{X/D}(\log X_0)\otimes\mathcal{O}_{X_0}) = \sum_{p+q=i} \dim H^q(X_0, \Omega^p_{X/D}(\log X_0)\otimes\mathcal{O}_{X_0}).$$

Since the functions $D \ni t \mapsto \dim H^q(X_0, \Omega^p_{X/D}(\log X_0)\otimes\mathcal{O}_{X_t})$ are upper semi-continuous, $R^i\pi_*\Omega^p_{X/D}(\log X_0)$ are free at 0 for all q and p. Especially $R^i\pi_*\omega_{X/D}$ is free at 0 for $i \geqq 0$. □

Corollary 3.14. *Let π: $X \to D$ be a proper surjective morphism from a complex manifold X onto a unit disk D and let L be a $\boldsymbol{Q}$-divisor on X. Then $R^i\pi_*\mathcal{O}_X(K_X + \ulcorner L \urcorner)$ is free at 0 for all $i \geqq 0$, if the following conditions are satisfied:*

(1) Supp $\langle L \rangle$ *has only normal crossings,*

(2) *X_t is smooth and L_t is semi-ample for any $t \in D^*$, and*

(3) *every irreducible component Γ of X_0 is in class $\mathscr{C}$ and $L_{|\Gamma}$ is quasi-nef.*

Proof. By the same argument as in (2.17), we obtain the following commutative diagram after replacing D by a smaller disk.

$$\begin{array}{ccc} X & \stackrel{\mu}{\longleftarrow} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle h} \\ D & \stackrel{g}{\longleftarrow} & Z, \end{array}$$

where

(a) Y and Z are complex manifolds,

(b) μ is a proper bimeromorphic morphism, g is a projective morphism and h is a proper fiber space, and

(c) there exists a g-nef $\boldsymbol{Q}$-divisor H on Z such that $H_t := H_{|Z_t}$ is nef and big for general $t \in D$, and that $\mu^*L = h^*H$.

Since H_t is nef and big for general $t \in D$, there is an effective divisor Δ on Z such that $H - \delta\Delta$ is g-ample for $0 < \delta \ll 1$. Further we may assume that

(d) Supp $\mu^*\langle L \rangle \cup$ Supp $h^*\Delta$ is a divisor with only normal crossings.

Then by (3.6), $R^i(\pi\cdot\mu)_*\mathcal{O}_Y(K_Y + \ulcorner h^*(H - \delta\Delta) \urcorner) = R^i\pi_*\mathcal{O}_X(K_X + \ulcorner L \urcorner)$, if δ is sufficiently small. Thus we can replace X by Y and L by $H - \delta\Delta$, respectively. Then L is π-semi-ample and $L = h^*A$ for a g-ample $\boldsymbol{Q}$-divisor A on Z. By (3.9), after replacing D by a small disk, we obtain a proper generically finite surjective morphism τ: $T \to X$ from a complex manifold T such that τ^*L is a Cartier divisor and that $\mathcal{O}_X(K_X + \ulcorner L \urcorner)$ is a direct summand of $\tau_*\mathcal{O}_Y(K_Y + \pi^*D)$. Thus we may assume that L is a π-semi-ample Cartier divisor on X. Then by taking a cyclic covering, we are reduced to (3.13).

□

Corollary 3.15. *Let X be a normal complex variety with only log-terminal singularities and let $\pi: X \to D$ be a proper surjective morphism onto a unit disk D. Assume that every irreducible component of X_0 is a variety in class $\mathcal{C}$ and that K_X is π-semi-ample. Then $R^i\pi_*\mathcal{O}_X(mK_X)$ is free at* 0 *for $i \geqq 0$ and $m \geqq 1$.*

Proof. We may replace D by a smaller disk, if necessary. Since K_X is π-semi-ample, there exists a positive integer m such that mK_X is a Cartier divisor and there exists a section s of $\mathcal{O}_X(mK_X)$, whose cyclic covering $Y = \mathrm{Specan} \bigoplus_{0 \leqq i \leqq m-1} \mathcal{O}_X(-iK_X)$ has only rational Gorenstein singularities (see [21, Proposition 7.5]). Thus $R^i(\pi \cdot \tau)_*\mathcal{O}_Y(K_Y) = \bigoplus_{1 \leqq \nu \leqq m} R^i\pi_*\mathcal{O}_X(\nu K_X)$, where τ is the natural morphism $Y \to X$. Since $R^i(\pi \cdot \tau)_*\mathcal{O}_Y(K_Y)$ is free at 0 and since we can choose m large enough, we obtain (3.15). □

§ 4. Minimal model problem for projective morphisms

Let $f: X \to Y$ be a projective surjective morphism, and let W be a closed subset of Y.

Definition 4.1. $\mathrm{Pic}(X/Y; W)$ denotes the group $\mathrm{ind\,lim}\,\mathrm{Pic}(f^{-1}(U))$, where U runs through all the open neighborhoods of W in Y and $Z_1(X/Y; W)$ denotes the free abelian group generated by irreducible curves on X whose image by f is a point of W. Let

$$(\cdot): \mathrm{Pic}(X/Y; W) \times Z_1(X/Y; W) \longrightarrow \mathbf{Z}$$

be the natural intersection pairing. Then two elements L_1 and L_2 of $\mathrm{Pic}(X/Y; W)$ are said to be *numerically equivalent over* W if $(L_1 \cdot \Gamma) = (L_2 \cdot \Gamma)$ for all $\Gamma \in Z_1(X/Y; W)$. Similarly, two elements Γ_1 and Γ_2 of $Z_1(X/Y; W)$ are said to be *numerically equivalent over* W if $(L \cdot \Gamma_1) = (L \cdot \Gamma_2)$ for all $L \in \mathrm{Pic}(X/Y; W)$. We define $A^1(X/Y; W)$ and $A_1(X/Y; W)$ to be the quotient groups of $\mathrm{Pic}(X/Y; W)$ and $Z_1(X/Y; W)$ modulo the numerical equivalences over W, respectively. For simplicity, we denote $A^1(X/Y; Y)$ and $A_1(X/Y; Y)$ by $A^1(X/Y)$ and $A_1(X/Y)$, respectively.

Remark 4.2. $A^1(X/Y)$ need not be a finitely generated abelian group. For example, let Y be a 2-dimensional complex surface, p_i $(1 \leqq i < \infty)$ a discrete sequence of mutually distinct points of Y, and let $f: X \to Y$ be the blowing up with center $\{p_i\}$. Then the exceptional curves $E_i := f^{-1}(p_i)$ are linearly independent in $A^1(X/Y)$.

Proposition 4.3. *Let U be a relative compact open subset of Y. Then $A^1(f^{-1}(U)/U; W \cap U)$ is a finitely generated abelian group.*

Proof. We shall prove this by induction on dim Y.

If dim $Y=0$, then U is a finite set $\{p_1, p_2, \cdots, p_l\}$, hence

$$A^1(f^{-1}(U)/U;\ W\cap U)=\oplus_{p_i\in W} A^1(f^{-1}(p_i)).$$

Since $f^{-1}(p_i)$ are projective spaces, $A^1(f^{-1}(U)/U;\ W\cap U)$ is finitely generated.

Next we assume that dim $Y\geqq 1$. Let $X=U_{i\in I}X_i$ be the irreducible decomposition of X, $Y_i:=f(X_i)$ and let $J:=\{i\in I\,|\,Y_i\cap U\neq\phi\}$. Then J is a finite set and we have an injection

$$A^1(f^{-1}(U)/U;\ W\cap U)\longrightarrow\oplus_{i\in J} A^1(X_i\cap f^{-1}(U)/Y_i\cap U;\ W\cap Y_i\cap U).$$

Therefore we may assume that X and Y are varieties. By taking a resolution of singularities of X and taking the Stein factorization of f, we may further assume that X is a manifold, Y is normal and that f is a fiber space. Then there is a proper closed analytic subset T of U such that $f_{|f^{-1}(Y\setminus T)}$ is a smooth morphism. Hence we obtain an injection

$$\begin{aligned}A^1(f^{-1}(U)/U;\ W\cap U)\longrightarrow &A^1(f^{-1}(U\setminus T)/U\setminus T;\ (W\cap U)\setminus T)\\ &\oplus A^1(f^{-1}(U\cap T)/U\cap T;\ W\cap U\cap T).\end{aligned}$$

Since $\dim T<\dim Y$, $A^1(f^{-1}(U\cap T)/U\cap T;\ W\cap T)$ is finitely generated by induction hypothesis. Thus it is enough to prove the following:

Claim. *If $f\colon X\to Y$ is a projective smooth surjective morphism between complex manifolds X and Y, then the homomorphism $A^1(X/Y)\to A^1(X_y)$ is injective for all $y\in Y$.*

Proof of the claim. If dim X=dim Y, then there is nothing to prove. If dim X=dim $Y+1$, then for every $L\in A^1(X/Y)$,

$$L\in \mathrm{Ker}\,(A^1(X/Y)\longrightarrow A^1(X_y))\Longleftrightarrow \deg(L_{|X_y})=0.$$

Since f is flat, $\deg(L_{|X_y})$ is constant on Y. Thus $A^1(X/Y)\to A^1(X_y)$ is injective. Suppose $d:=\dim X-\dim Y\geqq 2$. Let A be an f-ample line bundle. Then by flatness, the intersection numbers $(L_{|X_y}\cdot(A_{|X_y})^{d-1})$ and $((L_{|X_y})^2\cdot(A_{|X_y})^{d-2})$ are independent of $y\in Y$. Thus by the following Lemma 4.4, we are done. □

Lemma 4.4. *Let L and A be line bundles on a normal projective variety X such that A is ample and that $n:=\dim X\geqq 2$. Then*

$$L\approx_{\mathrm{num}} 0\Longleftrightarrow(L\cdot A^{n-1})=(L^2\cdot A^{n-2})=0.$$

Proof. The implication $\Rightarrow$ is trivial. As for $\Leftarrow$, let C be an irreducible curve on X. Then by Bertini's theorem, there exist divisors $H_i \in |m_i A|$ for some positive integers m_i $(1 \leqq i \leqq n-2)$ such that $S := \bigcap_{1 \leqq i \leqq n-2} H_i$ is an irreducible and reduced surface containing C. Let $\mu: S' \to S$ be a resolution of singularities and let $C' \subset S'$ be an irreducible curve such that $\mu(C') = C$. Since $(\mu^*(L_{|S}) \cdot \mu^*(A_{|S}))_{S'} = (\mu^*(L_{|S}))^2_{S'} = 0$ and $(\mu^* A_{|S})^2 > 0$, the Hodge index theorem says that $\mu^*(L_{|S}) \approx_{\mathrm{num}} 0$. Thus $0 = (\mu^*(L_{|S}) \cdot C')_{S'} = (L \cdot C)$. Therefore $L \approx_{\mathrm{num}} 0$. □

Corollary 4.5. *If W is a compact subset of Y, then $A^1(X/Y; W)$ is finitely generated.*

Definition 4.6. Let $f: X \to Y$ be a projective surjective morphism and let W be a compact subset of Y. We define $N^1(X/Y; W) := A^1(X/Y; W) \otimes \boldsymbol{R}$ and $N_1(X/Y; W) := A_1(X/Y; W) \otimes \boldsymbol{R}$, which are dual to each other by the intersection pairing $(\ \cdot\ )$. The *Picard number of f at W* is defined to be $\rho(X/Y; W) := \dim N^1(X/Y; W)$. We denote by $NE(X/Y; W)$ the cone in $N_1(X/Y; W)$ generated by effective 1-cycles in $N_1(X/Y; W)$ and $\overline{NE}(X/Y; W)$ denotes the closure of $NE(X/Y; W)$ in $N_1(X/Y; W)$ with the usual topology as a finite dimensional $\boldsymbol{R}$-vector space. Also we define $P(X/Y; W)$ to be the cone in $N^1(X/Y; W)$ generated by line bundles L such that $L_{|f^{-1}(U)}$ is f-ample for some open neighborhood U of W. An element $L \in N^1(X/Y; W)$ is called *f-nef at W* if $L \geqq 0$ on $\overline{NE}(X/Y; W)$.

Proposition 4.7 (Kleiman's criterion, see [23]).
(1) *$P(X/Y; W)$ is an open subset of $N^1(X/Y; W)$.*
(2) *$\overline{NE}(X/Y; W)$ contains no lines of $N_1(X/Y; W)$.*
(3) *$P(X/Y; W) = \{\zeta \in N^1(X/Y; W) \mid \zeta > 0$ on $\overline{NE}(X/Y; W) \setminus \{0\}\}$.*

Proof. (1) Let L and M be line bundles on $f^{-1}(U)$ for some open neighborhood of U of W. Suppose that L is f-ample. Then by (0.4), for every relatively compact open subset V of U containing W, there is a positive integer m such that $mL + M_{|f^{-1}(V)}$ is f-ample. Thus $P(X/Y; W)$ is open.

(2) Let $\Gamma \in N_1(X/Y; W)$ such that Γ and $-\Gamma \in \overline{NE}(X/Y; W)$. Then $(A \cdot \Gamma) \geqq 0$ and $-(A \cdot \Gamma) \geqq 0$ for $A \in P(X/Y; W)$. Thus $(A \cdot \Gamma) = 0$. If $\Gamma \neq 0$, then there is an element $\zeta \in N^1(X/Y; W)$ such that $(\zeta \cdot \Gamma) > 0$. Since $P(X/Y; W)$ is open by (1), we take a positive number α so that $\alpha A - \zeta \in P(X/Y; W)$. Hence $a(A \cdot \Gamma) = (\zeta \cdot \Gamma) > 0$, a contradiction. Thus $\Gamma = 0$.

(3) If $\zeta \in P(X/Y; W)$, then $\zeta > 0$ on $\overline{NE}(X/Y) \setminus \{0\}$ by the above argument. If L is a line bundle on $f^{-1}(U)$ for some open neighborhood U of W, and if $L > 0$ on $\overline{NE}(X/Y; W) \setminus \{0\}$, then $L_{|X_s}$ is ample for all $s \in W$. Indeed, it suffices to show that if $0 \neq \Gamma \in \overline{NE}(X_s)$, then $\varphi_s(\Gamma) \neq 0$, where

$\varphi_s\colon N_1(X_s)\to N_1(X/Y;\ W)$ is the natural homomorphism. Take an f-ample line bundle A. Then $A>0$ on $\overline{NE}(X_s)\setminus\{0\}$ by Kleiman's criterion [23]. Thus $(A\cdot\Gamma)=(A\cdot\varphi_s(\Gamma))>0$. Hence $\varphi_s(\Gamma)\neq 0$. By (1.5), there is an open neighborhood V of W such that $L_{|V}$ is f-ample. Thus $L\in P(X/Y;\ W)$. Therefore

$$P(X/Y;\ W)\cap N^1(X/Y;\ W)_{\boldsymbol{Q}} \\ =\{\zeta\in N^1(X/Y;\ W)_{\boldsymbol{Q}}\,|\,\zeta>0 \quad \text{on } \overline{NE}(X/Y;\ W)\setminus\{0\}\},$$

where $N^1(X/Y;\ W)_{\boldsymbol{Q}}:=A^1(X/Y;\ W)\otimes\boldsymbol{Q}$. The above set is dense in $\{\zeta\in N^1(X/Y;\ W)\,|\,\zeta>0 \text{ on } \overline{NE}(X/Y;\ W)\setminus\{0\}\}$. □

Theorem 4.8. *Let $f\colon X\to Y$ be a proper surjective morphism from a normal variety X onto a complex variety Y, Δ a $\boldsymbol{Q}$-divisor on X, and let H be a Cartier divisor on X. Suppose that*

(1) *(X, Δ) is log-terminal,*
(2) *$H-(K_X+\Delta)$ is f-nef-big,*
(3) *H is f-nef.*

Then there exist a projective surjective morphism $g\colon Z\to Y$ from a normal complex variety Z, a proper surjective morphism $\varphi\colon X\to Z$, and a g-ample line bundle A on Z such that $f=g\circ\varphi$ and $\varphi^*A=H$.

Proof. By (3.7) and the argument of [20], for any point $y\in Y$, we find a positive integer m_0 such that $\mathcal{O}_X(mH)$ is f-free near X_y for every $m\geqq m_0$. □

Corollary 4.9. *Let $f\colon X\to Y$ be a proper bimeromorphic morphism from a normal complex variety X onto a complex variety Y. Assume that X has only canonical singularities and that K_X is f-nef. Then $\bigoplus_{m\geqq 0} f_*\mathcal{O}_X(mK_X)$ is a locally finitely generated $\mathcal{O}_Y$-algebra.*

Theorem 4.10. *Let $f\colon X\to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y, Δ a $\boldsymbol{Q}$-divisor on X, H a line bundle on X, and let W be a compact subset of Y. Suppose that*

(1) *(X, Δ) is log-terminal,*
(2) *H is f-nef at W, and*
(3) *$H-(K_X+\Delta)\in P(X/Y;\ W)$.*

Then there exist an open neighborhood U of W in Y, a projective surjective morphism $g\colon Z\to U$ from a normal complex variety Z, a projective surjective morphism $\varphi\colon f^{-1}(U)\to Z$, and a g-ample line bundle A on Z such that $f_{|f^{-1}(U)}=g\circ\varphi$ and that $\varphi^*A=H_{|f^{-1}(U)}$.

Proof. Let Y' be an open subset of Y over which $H-(K_X+\Delta)$ is f-ample and let $X':=f^{-1}(Y')$. Since H is f-nef at W, by (4.7) and (1.4), for any f-ample line bundle L on X' and for any positive rational number $0<\varepsilon\ll 1$, there exists an open dense subset U_ε of Y' such that $(H+\varepsilon L)_{|f^{-1}(U_\varepsilon)}$ is f-ample. By Baire's category theorem, $\bigcap_{(0<\varepsilon\ll 1)} U_\varepsilon$ is dense. Take a general point $y\in\bigcap_{(0<\varepsilon\ll 1)} U_\varepsilon$. Then the non-vanishing theorem [20], [22] holds on X_y. Thus by the same argument as in [20], [22], we can prove this theorem. □

The proof of the following rationality theorem and cone theorem are similar to those in [22, Chapter 4].

Theorem 4.11 (Rationality theorem). *Let $f: X\to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y, Δ a $\boldsymbol{Q}$-divisor on X, H an f-ample line bundle on X, k a positive integer and W a compact subset of Y. Suppose that*

(1) *(X, Δ) is log-terminal and $K_X+\Delta$ is not f-nef at W,*

(2) *$k(K_X+\Delta)$ is a Cartier divisor near $f^{-1}(W)$.*

Then $r:=\max\{t\in\boldsymbol{R}\mid H+t(K_X+\Delta)$ is f-nef at $W\}$ is a positive rational number. If the reduced expression for r/k is u/v with coprime positive integers u and v, then $v\leqq k(d+1)$, where $d:=\max_{y\in Y}\dim f^{-1}(y)$.

Theorem 4.12 (Cone theorem). *Let $f: X\to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y, Δ a $\boldsymbol{Q}$-divisor on X, and let W be a compact subset of Y. Assume that (X, Δ) is log-terminal. Then we have the following:*

(1) *If $K_X+\Delta$ is not f-nef at W, then*

$$\overline{NE}(X/Y; W)=\overline{NE}_{K_X+\Delta}(X/Y; W)+\sum \boldsymbol{R}_+[l_i],$$

where $\overline{NE}_{K_X+\Delta}(X/Y; W):=\{\Gamma\in\overline{NE}(X/Y; W)\mid ((K_X+\Delta)\cdot\Gamma)\geqq 0\}$, each $\boldsymbol{R}_+[l_i]$ is the half line through the class of an irreducible curve l_i in $N_1(X/Y; W)$. Furthermore, $\sum\boldsymbol{R}_+[l_i]$ is locally finite and for any $R=\boldsymbol{R}_+[l_i]$, there exists $L\in A^1(X/Y; W)$ such that $R=\{\Gamma\in\overline{NE}(X/Y; W)\setminus\{0\}\mid (L\cdot\Gamma)=0\}$ and that L is f-nef at W. Such an L is called a supporting function *of R and R is called an* extremal ray at *W with respect to $K_X+\Delta$.*

(2) *For an extremal ray R, there exist an open neighborhood U of W and a proper surjective morphism $\varphi: f^{-1}(U)\to Z$ over U onto a normal variety Z such that*

$$\varphi(C)=point \Longleftrightarrow [C]\in R$$

for any irreducible curve C of $f^{-1}(U)$ which is mapped to a point of W. This φ is denoted by cont_R *and called the contraction morphism associated with R.*

(3) *$\varphi=\mathrm{cont}_R$ has the following properties:*

(a) $-(K_X+\Delta)_{|f^{-1}(U)}$ is φ-ample,
(b) $\text{Image}(\varphi^*: \text{Pic}(Z) \longrightarrow \text{Pic}(f^{-1}(U)))$
$= \{D \in \text{Pic}(f^{-1}(U)) \mid (D \cdot \Gamma) = 0 \text{ for all } \Gamma \in R\}$.
(c) The following mutually dual sequences are exact.

$$0 \longrightarrow N_1(f^{-1}(U)/Z; g^{-1}(W)) \longrightarrow N_1(X/Y; W) \longrightarrow N_1(Z/U; W) \longrightarrow 0,$$
$$0 \longleftarrow N^1(f^{-1}(U)/Z; g^{-1}(W)) \longleftarrow N^1(X/Y; W) \longleftarrow N^1(Z/U; W) \longleftarrow 0.$$

Here $g: Z \to U$ is the structure morphism. In particular, $\rho(X/Y; W) = \rho(Z/U; W) + 1$.

Definition 4.13. Let $f: X \to Y$ be a projective surjective morphism from a normal complex variety X onto a complex variety Y and let W be a compact subset of Y. X is called $\boldsymbol{Q}$-factorial over W if for any Weil divisor D on $f^{-1}(U)$ for an open neighborhood U of W, there is a positive integer m such that mD is a Cartier divisor on $f^{-1}(W)$.

Let $R \subset \overline{NE}(X/Y; W)$ be an extremal ray with respect to K_X, where X has only terminal singularities and is $\boldsymbol{Q}$-factorial over W. Then one of the following three cases occurs for $\varphi := \text{cont}_R$:

(i) $\dim \varphi(X) < \dim X$.

(ii) φ is bimeromorphic and its exceptional set is a prime divisor. In this case $\varphi(X)$ is $\boldsymbol{Q}$-factorial over W with only terminal singularities. φ is then called a *good contraction.*

(iii) φ is isomorphic in codimension one. In this case $\varphi(X)$ is not $\boldsymbol{Q}$-Gorenstein but has only rational singularities. φ is then called a *bad contraction.*

Now we state the minimal model conjectures for a projective morphism $f: X \to Y$ with respect to a compact subset W of Y.

Flip Conjecture. *Let $\varphi: X \to Z$ be a projective bimeromorphic morphism from a normal complex variety X with only canonical singularities onto a normal variety Z such that φ is isomorphic in codimension one and that $-K_X$ is φ-ample. Then $\bigoplus_{m \geq 0} \mathcal{O}_Z(mK_Z)$ is a locally finitely generated $\mathcal{O}_Z$-algebra.*

In this situation, the proper bimeromorphic map $X \cdots \to X^+ := \text{Projan} \bigoplus_{m \geq 0} \mathcal{O}_Z(mK_Z)$ is called the *flip* associated to φ.

Minimal model conjecture. *Let $f: X \to Y$ be a projective surjective morphism from a complex manifold X onto a complex variety Y and let W be a compact subset of Y. Then after taking a finite number of good contractions and flips associated to bad contractions, one can obtain a proper bimeromorphic model $Z \to U$ of $f_{|f^{-1}(U)}: f^{-1}(U) \to U$ for some open neighborhood U of W such that Z is $\boldsymbol{Q}$-factorial over W with only terminal singularities and that either*

(α) *$K_Z \geqq 0$ on $\overline{NE}(Z/U; W)$, or*

(β) *Z has an extremal ray R in $\overline{NE}(Z/U; W)$ with* $\dim \mathrm{cont}_R(Z) < \dim Z$.

Definition. Let $f: X \to Y$ be a projective surjective morphism from a normal complex variety X with only canonical singularities onto a complex variety Y. X/Y is called a *minimal model* if K_X is f-nef.

If the minimal model conjecture is true, then $\oplus_{m \geqq 0} \mu_* \mathcal{O}_Y(mK_Y)$ is a locally finitely generated $\mathcal{O}_Z$-algebra for any complex variety Z and for any resolution $\mu: Y \to Z$ of singularities of Z.

§ 5. Semi-ampleness Theorems

The notations and the theorems of this section are almost the same as those in Kawamata [21, Section 4].

Definition 5.1. A reduced equi-dimensional complex space X is called a *generalized normal crossing variety* if for every point $P \in X$, the completion $\hat{\mathcal{O}}_{X,P}$ of the local ring is isomorphic to

$$\boldsymbol{C}[[x_{01}, \cdots, x_{0r_0}]] \hat{\otimes} (\hat{\otimes}_{1 \leqq i \leqq t} \boldsymbol{C}[[x_{i1}, \cdots, x_{ir_i}]]/(x_{i1} \cdots x_{ir_i})),$$

for some t and r_i, which depend on P.

A generalized normal crossing variety X is a local complete intersection, and hence has an invertible dualizing sheaf ω_X. Let X_0 be the normalization of X and let $X_{\bullet}$ be a simplicial complex space given by

$$\Delta_n \longmapsto X_n := X_0 \times_X \cdots \times_X X_0 \ ((n+1)\text{-times}).$$

We denote the natural projection $X_n \to X$ by ε_n. Note that the X_n's are smooth. The union B_n on X_n of the images of lower dimensional irreducible components of $X_{n'}$ $(n' > n)$ forms a divisor with only normal crossings on X_n. A Cartier divisor D on X is called *permissible* if the support of D does not contain any stratum of X locally. We denote by $\mathrm{Div}_0(X)$ the group of permissible Cartier divisors on X. A *generalized normal crossing divisor D* on X is defined to be a permissible Cartier divisor such that for any n the union $B_n \cup D_n$ is a reduced divisor with only normal crossings on X_n, where $D_n := \varepsilon_n^* D$. If D is an element of $\mathrm{Div}_0(X) \otimes \boldsymbol{Q}$ whose support is a generalized normal crossing divisor, then one can define a permissible Cartier divisor $\ulcorner D \urcorner$ by the system of divisors $\ulcorner D_n \urcorner$ on X_n.

Theorem 5.2 (cf. [21, Theorem (4.3)]). *Let X be a compact generalized normal crossing variety whose components are varieties in class $\mathscr{C}$, let $L \in$ $\mathrm{Div}_0(X) \otimes \boldsymbol{Q}$, and let $D \in \mathrm{Div}_0(X)$. Suppose that*

(1) *L is semi-ample, the support of L is a generalized normal crossing divisor,*

(2) *D is effective,*

(3) *there is an effective $D' \in \mathrm{Div}_0(X)$ such that $D+D' \in |mL|$ for some positive integer m with $mL \in \mathrm{Div}_0(X)$.*

Then the homomorphism

$$+D: H^i(X, \mathcal{O}_X(K_X+\ulcorner L \urcorner)) \longrightarrow H^i(X, \mathcal{O}_X(K_X+\ulcorner L \urcorner + D))$$

is injective for every i.

Proof. By the same argument as in [21], it is enough to prove that

$$H^q(X_p, \mathcal{O}_{X_p}(-\ulcorner L \urcorner)) \longrightarrow H^q(D_p, \mathcal{O}_{D_p}(-\ulcorner L \urcorner))$$

are zero for all p and q, which is nothing but (3.11). □

For the same reason we can prove the following:

Theorem 5.3. *In the situation of* (5.2), *let $f: X \to Z$ be a surjective morphism onto a projective variety Z such that $nL = f^*A$ for an ample line bundle A on Z and a positive integer n. Then $H^p(Z, R^q f_* \mathcal{O}_X(K_X+\ulcorner L \urcorner)) = 0$ for all $p \geqq 1$ and $q \geqq 0$.*

Theorem 5.4 (Non-vanishing theorem). *Let X be a compact generalized normal crossing variety whose components are varieties in class $\mathscr{C}$, $f: X \to Z$ a surjective morphism onto a projective variety Z, $H \in \mathrm{Div}_0(X)$, $A \in \mathrm{Div}_0(X) \otimes \boldsymbol{Q}$, and let q be a positive integer. Then there exist positive integers p and t_0 such that $H^0(X, \mathcal{O}_X(ptH+\ulcorner A \urcorner) \neq 0$ for all integers $t \geqq t_0$, if the following conditions are satisfied:*

(1) *f induces a surjective morphism from each irreducible component of X_n onto Z,*

(2) *The support of A is a generalized normal crossing divisor on X and $\ulcorner A \urcorner$ is effective,*

(3) *There is a nef Cartier divisor H_0 on Z such that $\mathcal{O}_X(qH) \cong f^*\mathcal{O}_Z(H_0)$,*

(4) *There is an ample Cartier divisor L_0 on Z such that $\mathcal{O}_X(q(H+A-K_X)) \cong f^*\mathcal{O}_Z(L_0)$, where $qA \in \mathrm{Div}_0(X)$.*

The following theorem is also easily proved if one follows the argument of the proof of [21, Theorem 6.1] using (2.14) and (5.4).

Theorem 5.5. *Let X be a compact normal complex variety in class $\mathscr{C}$, Δ a $\boldsymbol{Q}$-divisor on X, and H a $\boldsymbol{Q}$-Cartier divisor on X. Then H is semi-ample under the following conditions:*

(1) *(X, Δ) is log-terminal,*

(2) *H is quasi-nef,*

(3) *$H-(K_X+\Delta)$ is quasi-nef and good,*

(4) *$\kappa_{\mathrm{hom}}(aH-(K_X+\Delta))=\kappa_{\mathrm{hom}}(H-(K_X+\Delta))$ and $\kappa(X, aH-(K_X+\Delta))$ $\geqq 0$ for some $a \in \boldsymbol{Q}$ with $a>1$.*

Corollary 5.6. *Let X be a normal compact complex variety in class $\mathscr{C}$ which has only canonical singularities. If K_X is quasi-nef and good, then K_X is semi-ample.*

Definition 5.7. A compact complex variety X in class $\mathscr{C}$ is said to be a minimal model, if X has only canonical singularities and if K_X is quasi-nef. A minimal model X is said to be good, if K_X is semi-ample.

(5.6) is a partial answer to the following:

Conjecture G. *If X is a minimal model in class $\mathscr{C}$, then K_X is semi-ample.*

The purpose of the rest of this section is to prove the following:

Theorem 5.8 (cf. [34]). *Let π: $X \to D$ be a proper surjective morphism from a normal complex variety X onto a unit disk D, Δ an effective $\boldsymbol{Q}$-divisor on X. For a $\boldsymbol{Q}$-Cartier divisor H on X, there exist positive integers p and m_0 such that $\mathcal{O}_X(mpH)$ is f-free near X_0 for all $m \geqq m_0$, if the following conditions are satisfied.*

(1) *(X, Δ) is log-terminal.*

(2) *X_t is a normal complex variety for $t \neq 0$.*

(3) *$H_{|X_t}$ and $H-(K_X+\Delta)_{|X_t}$ are semi-ample, and $\kappa(aH-(K_X+\Delta)_{|X_t})$ $=\kappa(H-(K_X+\Delta)_{|X_t})$ for $t \neq 0$ and for a rational number $a>1$.*

(4) *Every component of Γ of X_0 is compact complex variety in class $\mathscr{C}$ and $H_{|\Gamma}$, $H-(K_X+\Delta)_{|\Gamma}$ are quasi-nef.*

Proof. Since the statement is local, we can replace S by an open neighborhood of 0 if necessary. By the same argument as in (2.16) and (2.17), we obtain the following diagram

$$\begin{array}{ccc} X & \stackrel{\mu}{\longleftarrow} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle h} \\ D & \stackrel{g}{\longleftarrow} & Z, \end{array}$$

where

(1) Y and Z are complex manifolds and μ is a proper bimeromorphic morphism,

(2) g is a projective morphism and h is a proper fiber space.

Moreover, there exist g-nef $\boldsymbol{Q}$-divisors M'' and H'' on Z such that

(3) $\mu^*(H-(K_X+\Delta))=h^*M''$, and

(4) $\mu^*H=h^*H''$.

We may assume that H'' and H are Cartier divisors. Since M'' is g-nef-big, we can take an effective $\boldsymbol{Q}$-divisor M_1 on Z such that $M''-\delta M_1$ is g-ample for $0<\delta\ll 1$. Since $H_{|X_t}$ is semi-ample for a general fiber X_t, there is a positive integer p_1 such that $\pi_*\mathcal{O}_X(p_1mH)$ is not zero for $m\gg 0$. From $\mu^*\mathcal{O}_X(H)\cong h^*\mathcal{O}_Z(H'')$, we have isomorphisms

$$\pi_*\mathcal{O}_X(mH)\cong\pi_*\mu_*\mathcal{O}_Y(m\mu^*H)\cong g_*\mathcal{O}_Z(mH''),$$

for all integers m.

We define $\Lambda(m)$ to be $\mathrm{Supp}\,(\mathrm{Coker}\,(\pi^*\pi_*\mathcal{O}_X(mH)\to\mathcal{O}_X(mH)))\cap X_0$ for a positive integers m such that $\pi_*\mathcal{O}_X(mH)\neq 0$. It is enough to show that $\Lambda(m)=\phi$ for some m. Fix a positive integer e_1 with $\Lambda(p_1e_1)\neq\phi$. By blowing ups, we may assume that the following conditions are satisfied.

(5) There is a divisor $F=\sum_{i\in I}F_i$ with only simple normal crossings on Y,

(6) $K_Y=\mu^*(K_X+\Delta)+\sum_{i\in I}a_iF_i$ with $a_i>-1$,

(7) $h^*M_1=\sum_{i\in I}b_iF_i$ with $b_i\geqq 0$,

(8) $\mu^*(p_1e_1H)=L+\sum_{i\in I}r_iF_i$ with $r_i\geqq 0$,

where

$$\pi_*\mathcal{O}_X(p_1e_1H)=\pi_*\mu_*\mathcal{O}_Y(L)\quad\text{and}\quad\mu^*\pi^*\pi_*\mu_*\mathcal{O}_Y(L)\to\mathcal{O}_Y(L)$$

is surjective.

Note that $\Lambda(p_1e_1)=\mu(\cup_{(r_i\neq 0)}F_i)$. Set $c:=\min\,(a_i+1-\delta b_i)/r_i$. Then $c>0$. Let $I_0=\{i\in I\,|\,a_i+1-\delta b_i=cr_i\}$. If we replace Y by its blowing up, then we choose a member $M_2\in|q(M''-\delta M_1)|$ for a positive integer q, where $q(M''-\delta M_1)$ is a Cartier divisor on Z, so that the following conditions (9) and (10) are satisfied.

(9) $h^*M_2=\sum_{i\in I}s_iF_i$ with $s_i\geqq 0$.

Set $c':=\min\,(a_i+1-\delta b_i)/(r_i+\delta's_i)$ for a sufficiently small positive δ'. Let

$$I_0':=\{i\in I\,|\,a_i+1-\delta b_i=c'(r_i+\delta's_i)\}$$
$$A:=\sum_{i\in I\setminus I_0'}(-c'(r_i+\delta's_i)+a_i-\delta b_i)F_i,$$

and

$$B:=\sum_{i\in I_0'}F_i.$$

(10) $h\colon B\to h(B)$ induces a surjective morphism from any nonempty intersection of F_i $(i\in I_0')$ onto $h(B)$ which is irreducible.

Consider a $\boldsymbol{Q}$-divisor

$$N := m\mu^*H + A - B - K_Y$$
$$= c'L + (m - (p_1e_1c' + 1))h^*H'' + (1 - c'\delta'q)h^*(M'' - \delta M_1)$$

on Y. If δ' is sufficiently small, then N is $\pi\cdot\mu$-semi-ample for $m \geqq c'p_1e_1 + 1$. Thus by (3.14), $R^1(\pi\cdot\mu)_*\mathcal{O}_Y(m\mu^*H + \ulcorner A \urcorner - B)$ is free at 0. Hence $R^1(\pi\cdot\mu)^*\mathcal{O}_Y(m\mu^*H + \ulcorner A \urcorner - B) \to R^1(\pi\cdot\mu)_*\mathcal{O}_Y(m\mu^*H + \ulcorner A \urcorner)$ is injective, because $\pi\cdot\mu(B) = \{0\}$. Therefore $\pi_*\mu_*\mathcal{O}_Y(m\mu^*H + \ulcorner A \urcorner) \to \pi_*\mu_*\mathcal{O}_B(m\mu^*H + \ulcorner A \urcorner_{|B})$ is surjective. On the other hand, $B \to h(B)$, $\mu^*H_{|B} \in \mathrm{Div}_0(B)$, and $A_{|B} \in \mathrm{Div}_0(B) \otimes \boldsymbol{Q}$ satisfy the hypothesis of (5.4). Thus there is a positive integer p_2 such that $\pi_*\mu_*\mathcal{O}_B(p_2m\mu^*H + \ulcorner A \urcorner_{|B}) \neq 0$ for $m \gg 0$. Since $\mu_*\ulcorner A \urcorner = 0$, we obtain $\mu(B) \not\subset \Lambda(p_2m)$ for $m \gg 0$. Hence $\Lambda(p_1e_1p_2e_2) \subsetneqq \Lambda(p_1e_1)$ for some positive integer e_2. Therefore there is a positive integer m such that $\Lambda(m) = \phi$. □

§ 6. The lower semi-continuity of the plurigenera

Lemma 6.1 (cf. [34, Lemma 1]). *Let X be a normal complex variety with only log-terminal singularities, $X_0 = \sum a_iD_i$ an effective Cartier divisor on X, where D_i are irreducible components of X_0. Moreover, let $D := \sum_{(a_i=1)} D_i$, and $\sigma: X_1 \to D$ the normalization of D. Then for each integer $m \geqq 1$, there exists a natural injection*

$$\psi_m: \sigma_*\mathcal{O}_{X_1}(mK_{X_1}) \longrightarrow \mathcal{O}_X(mK_X + mX_0) \otimes \mathcal{O}_{X_0}$$

which is isomorphic at general points of D.

Corollary 6.2. *Let X be a complex variety with X_0 having only canonical singularities. Then X has only log-terminal singularities and $\mathcal{O}_X(mK_X + mX_0) \otimes \mathcal{O}_{X_0} \cong \mathcal{O}_{X_0}(mK_{X_0})$ for $m \geqq 1$.*

Proof. By a result of Kollár [24], X has only log-terminal singularities. By (6.1), we have an injection

$$\mathcal{O}_{X_0}(mK_{X_0}) \longrightarrow \mathcal{O}_X(mK_X + mX_0) \otimes \mathcal{O}_{X_0}$$

for every $m \geqq 1$. Since $\mathcal{O}_{X_0}(mK_{X_0})$ is reflexive, we are done. □

The following theorem is a partial answer to the Conjecture L.

Theorem 6.3. *Let $\pi: X \to D$ be a proper surjective morphism from a normal complex variety X with only log-terminal singularities onto a unit disk D. Suppose that*

(1) *X_t is a variety with only canonical singularities and K_{X_t} is semi-ample for any $t \neq 0$,*

(2) *if Γ is a component of X_0, then $\Gamma \in \mathscr{C}$ and $K_{X|\Gamma}$ is quasi-nef.* *Then*

(A) *there exists a positive integer l such that lK_X is a Cartier divisor at X_0 and that $\mathcal{O}_X(lK_X)$ is π-free near X_0,*

(B) *for any integers $\nu \geqq 1$ and $i \geqq 0$, $R^i\pi_*\mathcal{O}_X(\nu K_X)$ is free at 0,*

(C) $\sum P_m(\Gamma_i) \leqq \operatorname{rank} \pi_*\mathcal{O}_X(mK_X)$, *for any positive integer $m \geqq 1$, where $\cup \Gamma_i = X_0$.*

Proof. (A) follows from (5.8) and (6.2). Thus there is an open neighborhood U of 0 such that $\mathcal{O}_X(lK_X)$ is f-free on $\pi^{-1}(U)$. Therefore (B) follows from (3.5). Hence

$$\pi_*\mathcal{O}_X(\nu K_X) \otimes \boldsymbol{C}(0) \cong H^0(X_0, \mathcal{O}_X(\nu K_X) \otimes \mathcal{O}_{X_0})$$

for any $\nu \geqq 1$, where $\boldsymbol{C}(0)$ is the residue field at 0. On the other hand by (6.1), we have

$$\sum P_m(\Gamma_i) \leqq h^0(X_0, \mathcal{O}_X(mK_X) \otimes \mathcal{O}_{X_0}).$$

Therefore $\sum P_m(\Gamma_i) \leqq \operatorname{rank} \pi_*\mathcal{O}_X(mK_X)$. □

The following statement is proved by Levine [27] when π is smooth.

Corollary 6.4. *If $\pi \colon X \to D$ is a proper surjective morphism from a complex variety X onto a unit disk D such that all the fibers of π are good minimal models in class $\mathscr{C}$, then $P_m(X_t)$ is independent of $t \in D$ for every $m \geqq 1$.*

Proof. By a result of Kollár [24], X has only log-terminal singularities. Therefore the assertion follows from (6.3). □

§ 7. Open problems

In Section 2, we introduced the Kähler cone $\mathrm{KC}(Y)$ of a compact Kähler manifold Y. If $h^{2,0}(Y)=0$, then $\mathrm{KC}(Y)$ is nothing but the ample cone of Y.

Problem 7.1. How can one construct a minimal model theory of compact Kähler manifolds?

For projective varieties, Kleiman's criterion [23] and Kawamata-Viehweg's vanishing theorem [19], [50] are essential and enough to prove the cone theorem [20]. But for compact Kähler manifolds, one does not have any results corresponding to the above two theorems.

For Problem 2.5 we need to represent the dual cone of $\overline{\mathrm{KC}}(Y)$ geometrically. The fact that the effective 1-cycles are contained in it is not enough, because there is a compact Kähler manifold with no curves.

Recently (Nov. 1985), K. Sugiyama proved Conjecture 2.13 using results of Demailly [2] and Yau [52].

Conjecture 2.12 is a problem in algebraic geometry. In fact, it is an easy exercise to show that the Hodge Conjecture implies Conjecture 2.12.

In Section 3, we obtained a generalization (3.7) of Kawamata-Viehweg's vanishing theorem. But we do not yet have the generalization of Kollár's theorem in the following formulation.

Conjecture 7.2. *Let $\pi: X \to S$ be a proper surjective morphism from a Kähler manifold X onto a complex variety S. Assume that S is a weakly 1-complete variety with a positive line bundle A. Then*

(1) *$R^i\pi_*\omega_X$ is torsion free for $i \geqq 0$,*

(2) *$H^p(S_c, R^i\pi_*\omega_X \otimes A) = 0$ for $p > 0$ and $i \geqq 0$.*

When $i > \dim X - \dim S$, (1) was proved by Takegoshi [46]. If π is a projective morphism, then (1) is proved by Moriwaki [30] and also by Morihiko Saito, independently.

By the same arguments as in [26], [35], one can derive (7.3) from (7.2).

Conjecture 7.3. *Let $\pi: X \to S$ be a proper surjective morphism from a Kähler manifold X onto a complex manifold S. Assume that there is an open subset S^0 of S such that*

(1) *$S \setminus S^0$ is a divisor with only normal crossings, and*

(2) *π is smooth over S^0.*

Then $R^i\pi_*\omega_{X/S} \cong F^d({}^u\mathscr{H}_S^{d+i})$ for all $i \geqq 0$, where $d = \dim X - \dim S$ and ${}^u\mathscr{H}_S^{d+i}$ is the upper canonical extension (see [26] or [30]) of the variation of Hodge structures $R^{d+i}\pi_*\boldsymbol{C}_{X|S^0}$.

In [30], (7.3) is proved in the projective case. But as in the arguments of M. Saito, these conjectures may follow from results in the theory of Hodge modules.

In Section 4, we formulated and proved the cone theorem (Theorem 4.12) for any projective morphism with respect to a compact subset of the base space.

Proposition 7.4. *Let $\pi: X \to D$ be a projective fiber space from a 3-dimensional manifold X onto a disk D such that π is smooth over D^* and that $\kappa(X_t) \geqq 0$ for all $t \in D^*$. Then replacing D by a small disk, we obtain a projective fiber space $f: Y \to D$ from a 3-dimensional manifold Y such that f is proper bimeromorphically equivalent to π and that the general fibers of f are minimal models.*

Proof. Let W be a closed disk in D. Apply Theorem 4.12 to $\overline{NE}(X/D; W)$.

If K_X is not π-nef near X_0, then we have an extremal ray R and the contraction morphism $\varphi := \mathrm{cont}_R: X \to X_1$ over some neighborhood of W. Then $X_1 \to D$ is smooth over D^* by [34].

If φ is a good contraction, then $\rho(X_t) > \rho(X_{1,t})$ for any $t \in D^*$. In this case, we replace X_1 by its resolution $X^{(1)}$.

If φ is a bad contraction, then X and X_1 are isomorphic over D^*. If all the exceptional rays in $\overline{NE}(X/D; W)$ induce bad contractions, then consider the homomorphism

$$\lambda: \overline{NE}(X/D; W_1) \longrightarrow \overline{NE}(X/D; W),$$

where W_1 is a connected compact subset of W which does not contain 0. If $K_X \geqq 0$ on $\overline{NE}(X/D; W_1)$, then there is nothing to prove. Otherwise, there exists an element $A \in P(X/D; W)$ such that $K_X + A \geqq 0$ on $\overline{NE}(X/D; W_1)$ and that $K_X + A \notin P(X/D; W_1)$. Then $(K_X + A)_{|X_t}$ is nef and not ample for any $t \in W \setminus \{0\}$. Indeed if $(K_X + A)_{|X_t}$ is not nef, then take a connected compact subset W_2 of $W \setminus \{0\}$ which contains $W_1 \cup \{t\}$. Note that $1 < r := \min\{s \in \boldsymbol{R} \mid K_X + sA \geqq 0 \text{ on } \overline{NE}(X/D; W_2)\}$. Hence $K_X + rA$ defines a contraction morphism $\mu: X' = \pi^{-1}(U') \to Z'$ such that $\mu^* L = K_X + rA$ for a $\boldsymbol{Q}$-divisor L on Z', where U' is an open neighborhood of W_2. Since μ is not isomorphic in codimension one, there exists a μ-exceptional divisor E on X'. But μ is isomorphic near W_1. Thus $\pi(E)$ reduces to a point $P \in U'$. But since $\dim Z'_P = 2$, we see that E is a fiber, a contradiction. Therefore $(K_X + A)_{|X_t}$ is nef for all $t \in W \setminus \{0\}$, and $(K_X + A)_{|X_t}$ is not ample for the same reason. Therefore the $(K_X + A)$-canonical fibration $\varphi: X \cdots \to Z$ over some open neighborhood U of W is a morphism over $U \setminus \{0\}$, Z_t is also smooth, and $\rho(X_t) > \rho(Z_t)$ for all $t \in U \setminus \{0\}$. Let $X^{(1)}$ be a resolution of singularities of Z.

Combining the above, we have a sequence of proper bimeromorphic maps

$$X \cdots \longrightarrow X^{(1)} \cdots \longrightarrow \cdots \quad \cdots \longrightarrow X^{(n)}$$

over some neighborhood of W such that $\rho(X_t^{(i)}) > \rho(X_t^{(i+1)})$ for all $t \in W$ and for all i. Therefore this sequence terminates. □

Theorem 7.5. *If $\pi: X \to D$ is a projective surjective morphism from a 3-dimensional complex manifold X onto a disk D, then Conjecture L is true.*

Proof. When $\kappa(X_t) = -\infty$, this was already proved by Ueno [49]. Otherwise, using a result of Tsunoda [48] and (7.4), we are reduced to the situation in (6.3). □

Finally, we pose the following two problems which are related to Theorem 6.3.

Problem 7.6. *Is any small deformation of canonical singularities also canonical?*

Problem 7.7. *Is any small deformation of a minimal model in class $\mathscr{C}$ also a minimal model in class $\mathscr{C}$?*

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