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Cremona Transformations and Degrees of Period Maps for $K3$ Surfaces with Ordinary Double Points

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Let X be a $K3$ surface with n ordinary double points on which an ample line bundle $\mathscr{L}$ has been fixed. If $\mu: S \rightarrow X$ is the minimal desingularization, the orthogonal complement in $H^2(S, \boldsymbol{Z})$ of $\mu^*\mathscr{L}$ and the rational curves $\mu^{-1}(P)$ (for $P \in \operatorname{Sing} X$) carries a Hodge structure with $h^{2,0}=h^{0,2}=1$ and $h^{1,1}=19-n$. The *period map* for these surfaces is the natural map from the moduli space to the classifying space for such Hodge structures; this generalizes the classical period maps for polarized $K3$ surfaces [24].

In contrast to the classical case, these period maps, while always étale, tend to have degree greater than one. In this paper, we study this phenomenon for two particular kinds of $K3$ surfaces with ordinary double points: the "$K3$ surfaces of Cremona type" in which the degree of $\mathscr{L}$ is 2 and the branch locus of the induced map to $\boldsymbol{P}^2$ is irreducible, and the "$K3$ surfaces of Todorov type" whose study was begun in [20]; these latter surfaces arise as quotients of certain surfaces of general type constructed by Todorov [26]. (We omit one of the families of $K3$ surfaces of Todorov type here, as to consider it would take us too far afield.)

Our main results are a computation of the degrees of the period maps (Corollaries (5.2) and (5.6)), a demonstration that for our families, two $K3$ surfaces with the same periods are birationally (but not always biregularly) isomorphic (Theorem (6.1)), and finally a consideration of the geometric consequences of these birational isomorphisms when the degree of $\mathscr{L}$ is small (Theorems (7.1), (7.3), and (8.5)). The geometric consequences we find involve the behavior of sets of points in $\boldsymbol{P}^2$ or $\boldsymbol{P}^3$ under the Cremona group of birational automorphisms of $\boldsymbol{P}^2$ or $\boldsymbol{P}^3$ (hence the name "Cremona type"), and we obtain modern proofs of some classical results of Coble [3], [4], [5] on this topic. We should mention that Coble's work has recently been studied from a different point of view by Cossec and Dolgachev [8], [6], whose results we use in our interpretation.

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The plan of the paper is as follows: we begin with some algebraic preliminaries on finite quadratic forms, their orthogonal groups, and the "Cremona and Todorov lattices" in the first three sections. Section 4 is devoted to the construction of moduli spaces for $K3$ surfaces of Cremona type. (The analogous construction for $K3$ surfaces of Todorov type was given in [20]). In Section 5 we compute the degrees of the period maps; we then find a link between some of these period maps and birational geometry in $\boldsymbol{P}^2$ and $\boldsymbol{P}^3$ in the last three sections.

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§ 1. Finite quadratic forms

A *finite quadratic form* is a pair (G, q) (sometimes denoted simply by G) consisting of a finite abelian group G together with a map $q\colon G\to \boldsymbol{Q}/2\boldsymbol{Z}$ which satisfies (1) $q(nx)\equiv n^2q(x) \bmod 2\boldsymbol{Z}$ for all $n\in \boldsymbol{Z}$, $x\in G$, and (2) the map $b\colon G\times G\to \boldsymbol{Q}/\boldsymbol{Z}$ defined by $b(x, x')\equiv(q(x+x')-q(x)-q(x'))/2 \bmod \boldsymbol{Z}$ is symmetric and $\boldsymbol{Z}$-bilinear. b is called the *associated bilinear form* of (G, q).

If (G, q) is a finite quadratic form and H is a subgroup of G, we define $H^\perp=\{x\in G\,|\,b(x, u)\equiv 0 \bmod \boldsymbol{Z}$ for all $u\in H\}$. $G^\perp$ is called the *radical* of G, and denoted by $\mathrm{Rad}(G)$; the *q-radical of G* is the subgroup $\mathrm{Rad}_q(G)=\{x\in \mathrm{Rad}(G)\,|\,q(x)\equiv 0 \bmod 2\boldsymbol{Z}\}$. The form (G, q) is *nondegenerate* if $\mathrm{Rad}(G)=0$, and *quasi-nondegenerate* if $\mathrm{Rad}_q(G)=0$.

Let G_i be a subgroup of G with induced quadratic form $q_i=q|_{G_i}$ for $i=1, 2$. If $G_2=G_1^\perp$ and $G=G_1\oplus G_2$, we say that (G, q) is the *orthogonal direct sum* of (G_1, q_1) and (G_2, q_2), and denote this by writing $(G, q)=(G_1, q_1)\oplus(G_2, q_2)$.

Lemma (1.1). *Let (G, q) be a finite quadratic form, and let H be a subgroup of G such that $q|_H$ is a nondegenerate form. Then (G, q) is the orthogonal direct sum of $(H, q|_H)$ and $(H^\perp, q|_{H^\perp})$.*

Proof. First note that $H\cap H^\perp=\mathrm{Rad}(H)$, so that $H\cap H^\perp=0$. To show that $H+H^\perp$ coincides with G, consider the homomorphism $H\to \mathrm{Hom}(H, \boldsymbol{Q}/\boldsymbol{Z})$ defined by the adjoint map of the bilinear form b. This is injective since H is nondegenerate; on the other hand, it is a homomorphism between two groups of the same order, so it must be an isomorphism.

Now given any $x\in G$, the map $u\mapsto b(u, x)$ defines an element of $\mathrm{Hom}(H, \boldsymbol{Q}/\boldsymbol{Z})$. There must then be some $y\in H$ such that the map $u\mapsto b(u, y)$ defines the same element of $\mathrm{Hom}(H, \boldsymbol{Q}/\boldsymbol{Z})$. But this means that $x-y\in H^\perp$; hence, $G=H+H^\perp$. Q.E.D.

We introduce some notation for quadratic forms on cyclic groups. If a and l are natural numbers with $2|al$ and $(a, l)=1$, we let z_l^a denote the form $(Z/lZ, q)$, where $q(x)=a/l \bmod 2Z$ for some generator x of Z/lZ. Following Brieskorn [1], when $\varepsilon \in \{1, 3, 5, 7\}$ we let $w_{2,k}^{\varepsilon}=z_{2^k}^{\varepsilon}$, and when p is an odd prime and $\varepsilon=\pm 1$ we let $w_{p,k}^{\varepsilon}=z_{p^k}^{a}$ for some even integer a with $\left(\frac{a}{p}\right)=\varepsilon$ (where $\left(\frac{-}{-}\right)$ denotes the Legendre symbol).

Let (G, q) be a nondegenerate quadratic form. The form (G, q) is *indecomposable* if (G, q) cannot be written as the orthogonal direct sum of two nontrivial forms. The cyclic forms $w_{p,k}^{\varepsilon}$ provide examples of indecomposable forms. Other examples are given by u_k and v_k (again following Brieskorn's notation), which are forms on $Z/2^kZ \times Z/2^kZ$, where on a generating set x, y we have $q(x)\equiv q(y)\equiv 0 \bmod 2Z$ and $b(x, y)\equiv 2^{-k} \bmod Z$ for u_k, and $q(x)\equiv q(y)\equiv 2^{1-k} \bmod 2Z$ and $b(x, y)\equiv 2^{-k} \bmod Z$ for v_k.

The following proposition is well-known (cf. [27], [9], [22]).

Proposition (1.2). (1) *Let (G, q) be a nondegenerate quadratic form, let $G=\oplus G_p$ be the decomposition of G into its p-Sylow subgroups, and let $q_p=q|_{G_p}$. Then $(G, q)=\oplus(G_p, q_q)$ is an orthogonal direct sum decomposition.*

(2) *Every nondegenerate finite quadratic form is isomorphic to an orthogonal direct sum of indecomposable forms.*

(3) *A nondegenerate finite quadratic form is indecomposable if and only if it is isomorphic to one of the forms u_k, v_k, or $w_{p,k}^{\varepsilon}$; in particular, these forms generate the semigroup* qu(Z) *of isomorphism classes of nondegenerate finite quadratic forms, where orthogonal direct sum is the semigroup operation.*

(4) *For an odd prime p, $(w_{p,k}^{-1})^{\oplus 2}\cong(w_{p,k}^{1})^{\oplus 2}$.*

(5) *The following relations hold among indecomposable forms on 2-groups. (We identify $\{1, 3, 5, 7\}$ with $(Z/8Z)^{\times}$.)*

(I) $$w_{2,1}^{5}\cong w_{2,1}^{1} \quad \textit{and} \quad w_{2,1}^{3}\cong w_{2,1}^{7}.$$

(II) $$w_{2,k}^{\varepsilon_1}\oplus w_{2,k}^{\varepsilon_2}\oplus w_{2,k}^{\varepsilon_3}\cong\begin{cases} u_k\oplus w_{2,k}^{s_1(\varepsilon)} & \textit{if } s_2(\varepsilon)=7 \\ v_k\oplus w_{2,k}^{s_1(\varepsilon)} & \textit{if } s_2(\varepsilon)=3 \end{cases}$$

where $s_1(\varepsilon)=\varepsilon_1+\varepsilon_2+\varepsilon_3$ and $s_2(\varepsilon)=\varepsilon_1\varepsilon_2+\varepsilon_2\varepsilon_3+\varepsilon_3\varepsilon_1$.

(III) $$u_k^{\oplus 2}\cong v_k^{\oplus 2}.$$

Remark. There are more relations among indecomposable forms than those listed in (4) and (5) above; for a more complete account, see [10] or [17].

We now use the relations in (5) above to derive a nice description of quadratic forms on 2-groups.

Proposition (1.3) (cf. [17], [18]). *Let* (G, q) *be a nondegenerate finite quadratic form with p-Sylow decomposition* $(G, q)=\oplus(G_p, q_p)$.

(1) *If p is odd, then* (G_p, q_p) *has an orthogonal direct sum decomposition*

$$(G_p, q_p)\cong \bigoplus_{k\geq 1}((w_{p,k}^{1})^{\oplus a(p,k)}\oplus(w_{p,k}^{-1})^{\oplus b(p,k)})$$

with $b(p, k)\leq 1$.

(2) (G_2, q_2) *has an orthogonal direct sum decomposition*

$$(G_2, q_2)\cong \bigoplus_{k\geq 1}(u_k^{\oplus n(k)}\oplus v_k^{\oplus m(k)}\oplus w(k))$$

with the following properties:

(a) $m(k)\leq 1$,

(b) $\operatorname{rank}(w(k))\leq 2$,

(c) $w(k)$ *is a sum of forms of type* $w_{2,k}^{\varepsilon}$,

(d) $w(1)$ *is a sum of forms of type* $w_{2,1}^{1}$ *and* $w_{2,1}^{7}$.

Proof. We decompose (G, q) as an orthogonal direct sum of indecomposable forms; (1) is a direct consequence of Proposition (1.2) (4). To prove (2), we apply the relations from Proposition (1.2) (5): first, for each k, by repeated applications of relation (II) we may reduce to the case in which there are at most 2 summands of type $w_{2,k}^{\varepsilon}$. Next, a repeated application of relation (III) will reduce the number of summands of type v_k to at most 1. Finally, by applying relation (I) we may ensure that condition (d) holds. Q.E.D.

We say that a decomposition satisfying the conditions in Proposition (1.3) is in *normal form*. Such a "normal form" is not in general unique, but the first terms in the normal form decomposition are unique in the following sense:

Proposition (1.4). *Fix natural numbers n and m, and let*

$$(G, q)\cong u_1^{\oplus n}\oplus v_1^{\oplus m}\oplus(G', q')$$

be a nondegenerate finite quadratic form. If H is any subgroup of G such that $(H, q|_H)\cong u_1^{\oplus n}\oplus v_1^{\oplus m}$, *then G is the orthogonal direct sum of H and* $H^{\perp}$, *and* $(H^{\perp}, q|_{H^{\perp}})\cong(G', q')$.

Proof. We use the signature invariants $\sigma_r(G, q)$ of a nondegenerate finite quadratic form[3] introduced by Kawauchi and Kojima [10] (cf. also [17]). These invariants, which are defined for $r\geq 1$, take values in the

[3] In [10] and [17], these invariants are defined for bilinear rather than quadratic forms; we are implicitly using a technique of Wall [27; Theorem 5], which associates a bilinear form to each quadratic form in such a way as to embed qu($\boldsymbol{Z}$) in the semigroup of isomorphism classes of nondegenerate finite bilinear forms.

semigroup $\bar{Z}_8=(Z/8Z)\cup\{\infty\}$ (in which addition mod 8 is extended by the rules $i+\infty=\infty+i=\infty+\infty=\infty$) and $\sigma_r\colon qu(Z)\to\bar{Z}_8$ is a semigroup homomorphism. The values of these invariants for the indecomposable forms are given in Table 1.

The fundamental theorem about the signature invariants is this: if G_i is a 2-group and (G_i, q_i) is a nondegenerate finite quadratic form for $i=1,2$ such that $G_1\cong G_2$ (as groups) and $\sigma_r(G_1, q_1)=\sigma_r(G_2, q_2)$ for all $r\geq 1$, then $(G_1, q_1)\cong(G_2, q_2)$. Applied to the present situation, by using the Sylow decomposition, we may reduce to the case in which G is a 2-group. It is clear from Lemma (1.1) that $(G, q)\cong(H, q|_H)\oplus(H^\perp, q|_{H^\perp})$ and that $H^\perp\cong G'$ as groups. Now since $\sigma_r(u_1^{\oplus n}\oplus v_1^{\oplus m})=n\sigma_r(u_1)+m\sigma_r(v_1)\in\{0, 4\}$ always has an additive inverse in $\bar{Z}_8$ we see that

$$\begin{aligned}\sigma_r(H^\perp, q|_{H^\perp})&=\sigma_r(G, q)-\sigma_r(H, q|_H)\\&=\sigma_r(G, q)-\sigma_r(u_1^{\oplus n}\oplus v_1^{\oplus m})\\&=\sigma_r(G', q').\end{aligned}$$

Hence, $(H^\perp, q|_{H^\perp})\cong(G', q')$. Q.E.D.

Table 1. Signature invariants of indecomposable forms

(G, q)	p or r	$\sigma_r(G, q)$
$w_{p,k}^\varepsilon$	$p\neq 2$	$(-1)^{(r-1)(p^2-1)/8}(k^2(1-p)+2k(1-\varepsilon))$
$w_{2,k}^\varepsilon$	$r\leq k$	$\{(-1)^{(\varepsilon-1)/2}(1+(-1)^{k+r})+\varepsilon(1-(-1)^{k+r})\}/2$
$w_{2,k}^\varepsilon$	$r=k+1$	∞
$w_{2,k}^\varepsilon$	$r\geq k+2$	0
u_k	all r	0
v_k	$r\leq k$	$2(1+(-1)^{k+r})$
v_k	$r\geq k+1$	0

§ 2. Finite orthogonal groups

For a finite quadratic form (G, q), we define the *orthogonal group of* (G, q) by $O(G, q)=\{g\in\mathrm{Aut}(G)\,|\,q(g(x))=q(x)\text{ for all }x\in G\}$. In this section we study the structure of $O(G, q)$ (and in particular calculate its order) in several special cases. We begin with some examples.

Examples (2.1). (1) *If $(G, q)=\oplus(G_p, q_p)$ is the decomposition of G into its p-Sylow subgroups, then there is a natural isomorphism $O(G, q)\cong\prod O(G_p, q_p)$.*

(2) *Let a and l be natural numbers with $2 \mid al$ and $(a, l)=1$. Then*

$$O(z_l^a) \cong \begin{cases} \{\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times \mid \alpha^2 \equiv 1 \bmod l\} & \text{if } l \text{ is odd} \\ \{\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times \mid \alpha^2 \equiv 1 \bmod 2l\} & \text{if } l \text{ is even.} \end{cases}$$

In particular,

$$O(w_{p,k}^\varepsilon) \cong \begin{cases} \{1\} & \text{if } p=2,\ k=1 \\ \mathbf{Z}/2\mathbf{Z} & \text{otherwise.} \end{cases}$$

(3)

$$O(w_{2,1}^\varepsilon \oplus w_{2,1}^\phi) \cong \begin{cases} \{1\} & \text{if } \varepsilon \equiv -\phi \bmod 4 \\ \mathbf{Z}/2\mathbf{Z} & \text{if } \varepsilon \equiv \phi \bmod 4. \end{cases}$$

Proof. Part (1) follows directly from Proposition (1.2) (1). To prove (2), let x be a generator of z_l^a such that $q(x)=a/l$. Any $\gamma \in O(z_l^a)$ must be of the form $\gamma(x)=\alpha x$ for some $\alpha \in (\mathbf{Z}/l\mathbf{Z})^\times$. Since $q(\alpha x)=a\alpha^2/l$, such a γ belongs to $O(z_l^a)$ if and only if $a\alpha^2/l \equiv a/l \bmod 2\mathbf{Z}$, i.e., $a\alpha^2 \equiv a \bmod 2l$. If l is odd then a is even and this is equivalent to $\alpha^2 \equiv 1 \bmod l$; if l is even then $(a, 2l)=1$ so that this is equivalent to $\alpha^2 \equiv 1 \bmod 2l$.

For (3), let x and y be generators of $w_{2,1}^\varepsilon$ and $w_{2,1}^\phi$ respectively. The nonzero elements in the group are then x, y and $x+y$ and on these elements the quadratic form takes values $q(x) \equiv \varepsilon/2$, $q(y) \equiv \phi/2$ and $q(x+y) \equiv (\varepsilon+\phi)/2 \bmod 2\mathbf{Z}$. Since $q(x+y) \equiv 0 \bmod \mathbf{Z}$, the only possible nontrivial automorphism would be the one interchanging x and y; this is an automorphism exactly when $q(x) \equiv q(y) \bmod 2\mathbf{Z}$, i.e., when $\varepsilon \equiv \phi \bmod 4$.

Q.E.D.

A finite quadratic form (G, q) is *special* if every $x \in G$ with $2x=0$ satisfies $q(x) \equiv 0 \bmod \mathbf{Z}$. If (G, q) is a special quadratic form on a 2-elementary group, we may regard (G, q) as a quadratic form over the field with two elements $\mathbf{F}_2$ by giving G its natural $\mathbf{F}_2$-vector space structure and identifying $\mathbf{Z}/2\mathbf{Z}$ with $\mathbf{F}_2$. The orthogonal group of (G, q) is then related to the classical orthogonal and symplectic groups over $\mathbf{F}_2$.

Proposition (2.2). *Let (G, q) be a special quadratic form on a 2-elementary group of rank k.*

(1) *If (G, q) is nondegenerate, then $k=2s$ is even and $(G, q) \cong u_1^{\oplus n} \oplus v_1^{\oplus m}$ with $m \leq 1$. The orthogonal groups in these cases have been studied classically: $O(u_1^{\oplus n})$ is usually called the even orthogonal group $O^+(2n, 2)$, and $O(u_1^{\oplus n} \oplus v_1)$ is the odd orthogonal group $O^-(2n+2, 2)$.*

(2) *If (G, q) is degenerate and quasi-nondegenerate, then $k=2n+1$ is odd and $O(G, q)\cong Sp(2n, 2)$, the symplectic group over the field $\boldsymbol{F}_2$.*

(3) *In general let $\bar{G}=G/\mathrm{Rad}_q(G)$, let $r=\mathrm{rank}(\mathrm{Rad}_q(G))$, and let $s=k-r$. Then $\bar{G}$ inherits a quasi-nondegenerate special quadratic form $\bar{q}$ from G, and there is an exact sequence*

$$1\longrightarrow T(r, s)\longrightarrow O(G, q)\longrightarrow O(\bar{G}, \bar{q})\longrightarrow 1$$

where $T(r, s)=\left\{\begin{pmatrix} A & 0\\ B & I\end{pmatrix} \in \mathrm{GL}(r+s, \boldsymbol{F}_2)\right\}$, *I being the $s\times s$ identity matrix.*

Proof. (1) Let $(G, q)\cong u_1^{\oplus n}\oplus v_1^{\oplus m}\oplus w(1)$ be a normal form decomposition. Since (G, q) is special, $w(1)$ must be zero; G then has rank $2n+2m$. For the identification of the orthogonal groups, see for example [25; pp. 380–382].

(2) If $x \in \mathrm{Rad}(G)$ and $x\neq 0$ then $q(x)\equiv 1 \bmod 2\boldsymbol{Z}$. Since q restricts to a linear function on $\mathrm{Rad}(G)$, this implies $\mathrm{Rad}(G)\equiv \boldsymbol{Z}/2\boldsymbol{Z}$; we let x_0 be the non-trivial element.

Let $\tilde{G}=G/\mathrm{Rad}(G)$ and define a bilinear form $\tilde{b}\colon \tilde{G}\times\tilde{G}\to\boldsymbol{Z}/2\boldsymbol{Z}$ by $\tilde{b}(\tilde{x}, \tilde{y})\equiv 2b(x, y) \bmod 2\boldsymbol{Z}$ for x and y in G mapping to $\tilde{x}$ and $\tilde{y}$ respectively. $\tilde{b}$ is clearly nondegenerate; moreover $\tilde{b}(\tilde{x}, \tilde{x})\equiv 2b(x, x)\equiv 2q(x)\equiv 0 \bmod 2\boldsymbol{Z}$ so that $\tilde{b}$ is an alternating form. In particular, $\tilde{G}$ has even rank $2n$ so that $k=2n+1$. Moreover, there is a natural map $f\colon O(G, q)\to O(\tilde{G}, \tilde{b})=Sp(2n, 2)$.

Let $x_1', \cdots, x_{2n}'$ be elements of G mapping to a basis $\tilde{x}_1, \cdots, \tilde{x}_{2n}$ of $\tilde{G}$, and let $x_i=x_i'+q(x_i')x_0$ so that $q(x_i)\equiv 0 \bmod 2\boldsymbol{Z}$. Then

$$(*)\qquad q(\textstyle\sum a_i x_i)\equiv\sum a_i a_j \tilde{b}(\tilde{x}_i, \tilde{x}_j).$$

Now given $\sigma \in O(\tilde{G}, \tilde{b})$, let $\sigma(\tilde{x}_i)=\sum s_{ij}\tilde{x}_j$ and define $g(\sigma)\colon G\to G$ by $g(\sigma)(x_0)=x_0$, $g(\sigma)(x_i)=\sum s_{ij}x_j$. By (*), since σ preserves $\tilde{b}$, $g(\sigma)$ preserves q so that g gives a homomorphism $g\colon O(\tilde{G}, \tilde{b})\to O(G, q)$. Since $f\circ g$ is the identity, $O(G, q)$ is isomorphic to $O(\tilde{G}, \tilde{b})=Sp(2n, 2)$.

(3) Let $f\colon O(G, q)\to O(\bar{G}, \bar{q})$ be the natural homomorphism. Choose a subgroup H of G such that $G\cong \mathrm{Rad}_q(G)\oplus H$ as groups. (This is possible since G is 2-elementary.) Then there is a natural isomorphism $p\colon (H, q|_H)\xrightarrow{\sim}(\bar{G}, \bar{q})$; in particular, for any $\tau \in O(\bar{G}, \bar{q})$ we have $1\oplus p^*\tau \in O(G, q)$ and $f(1\oplus p^*\tau)=\tau$ which shows that f is surjective.

If $\sigma \in \mathrm{Ker} f$ then $\sigma(y)-y \in \mathrm{Rad}_q(G)$ for any $y \in H$; moreover $\sigma(\mathrm{Rad}_q(G))=\mathrm{Rad}_q(G)$. This shows that $\mathrm{Ker} f\subset T(r, s)$; conversely, if $\sigma \in T(r, s)$, $x \in \mathrm{Rad}_q(G)$ and $y \in H$ then

$$q(\sigma(x+y))\equiv q((\sigma(x)+\sigma(y)-y)+y)\equiv q(y)\equiv q(x+y)$$

since x, $\sigma(x)$ and $\sigma(y)-y$ all lie in $\mathrm{Rad}_q(G)$. Thus, $T(r, s)$ is contained in $O(G, q)$, and it clearly lies in the kernel of f. Q.E.D.

Proposition (2.3). *Let $(G, q)=u_1^{\oplus n}\oplus v_1^{\oplus m}\oplus(G', q')$ be a nondegenerate quadratic form, and define $K=\{x \in G \mid 2x=0$ and $q(x) \in \mathbf{Z}/2\mathbf{Z}\}$. Then K is a subgroup of G, and $(K, q|_K)$ is a special 2-elementary form. If $K\cap G'$ has order less than or equal to 2, then there is an exact sequence*

$$1\longrightarrow O(G', q')\longrightarrow O(G, q)\longrightarrow O(K, q|_K)\longrightarrow 1.$$

Proof. If $x, y \in K$ then $2b(x, y)\equiv b(2x, y)\equiv 0 \bmod \mathbf{Z}$. Hence, $q(x+y)\equiv q(x)+q(y)+2b(x, y)\equiv 0 \bmod \mathbf{Z}$ so that K is indeed a subgroup of G.

Suppose that $K\cap G'$ has order at most 2. Any $\rho \in O(G, q)$ must preserve K, so there is a natural restriction homomorphism $r\colon O(G, q)\to O(K, q|_K)$. Since $u_1^{\oplus n}\oplus v_1^{\oplus m}\subset K$, any $\rho \in O(G, q)$ which acts trivially on K must have the form $\rho=1\oplus\nu$ with $\nu \in O(G', q')$. Moreover, any such element acts trivially on K: the induced action on $K\cap G'$ is trivial since $K\cap G'$ has order at most 2 (and hence has no non-trivial automorphisms). We conclude that the kernel of r is $O(G', q')$.

It remains to show that r is surjective. Let $\sigma \in O(K, q|_K)$ and let $H=\sigma(u_1^{\oplus n}\oplus v_1^{\oplus m})$. By Proposition (1.4), $(G, q)=(H, q|_H)\oplus(H^{\perp}, q|_{H^{\perp}})$ and there is an isomorphism $\tau\colon (G', q')\to(H, q|_H)$. Let $\rho=(\sigma|_{u_1^{\oplus n}\oplus v_1^{\oplus m}})\oplus\tau$; identifying G with $H\oplus H^{\perp}$, we may regard ρ as an element of $O(G, q)$. But then $\rho|_K\circ\sigma^{-1}$ acts trivially on $u_1^{\oplus n}\oplus v_1^{\oplus m}$; it thus induces an automorphism of $K\cap G'$ which must be trivial as well. Hence, $\rho|_K\circ\sigma^{-1}=1_K$ which implies that σ is in the image of r. Q.E.D.

Corollary (2.4). *Let $(G, q)=u_1^{\oplus n}\oplus v_1^{\oplus m}\oplus(G', q')$ be a nondegenerate quadratic form.*

(1) *If (G', q') is trivial, or $(G', q')\cong w_{2,1}^{\varepsilon}$, then*

$$O(G, q)\cong O(u_1^{\oplus n}\oplus v_1^{\oplus m}).$$

(2) *If $(G', q')\cong w_{2,2}^{\varepsilon}$ or $(G', q')\cong(w_{2,1}^{\varepsilon})^{\oplus 2}$, then there is an exact sequence*

$$1\longrightarrow \mathbf{Z}/2\mathbf{Z}\longrightarrow O(G, q)\longrightarrow Sp(2n+2m, 2)\longrightarrow 1.$$

(3) *If $(G', q')\cong w_{2,1}^{1}\oplus w_{2,1}^{7}$, then*

$$O(G, q)\cong(\mathbf{Z}/2\mathbf{Z})^{2n+2m}\ltimes O(u_1^{\oplus n}\oplus v_1^{\oplus m}).$$

(4) *If $(G', q')\cong w_{2,k}^{\varepsilon}$ with $k\geq 3$, then there is an exact sequence*

$$1\longrightarrow(\mathbf{Z}/2\mathbf{Z})\longrightarrow O(G, q)\longrightarrow(\mathbf{Z}/2\mathbf{Z})^{2n+2m}\ltimes O(u_1^{\oplus n}\oplus v_1^{\oplus m})\longrightarrow 1.$$

Proof. Let K be as in Proposition (2.3); we will show that in each case, $K \cap G'$ has order at most 2, and we will compute $O(G', q')$ and $O(K, q|_K)$. The statement is obvious when (G', q') is trivial.

Suppose first that $(G', q') \cong w^{\varepsilon}_{2,k}$ and let x be a generator of G' so that $q(2^{k-1}x) = 2^{k-2}\varepsilon$. If $k=1$ then $K \cap G'$ is trivial and $(K, q|_K) = u_1^{\oplus n} \oplus v_1^{\oplus m}$ is nondegenerate. By Example (2.1) (2), $O(G', q')$ is trivial, so the stated isomorphism follows from Proposition (2.3).

If $k=2$ then K is degenerate and quasi-nondegenerate with $\mathrm{Rad}(K) = K \cap G'$ generated by $2x$. The exact sequence follows from Example (2.1) (2) and Proposition (2.2) (2).

If $k \geq 3$ then $\mathrm{Rad}_q(K) = K \cap G'$ is generated by $2^{k-1}x$, and the induced form on $K/\mathrm{Rad}_q(K)$ is isomorphic to $u_1^{\oplus n} \oplus v_1^{\oplus m}$. The exact sequence follows from Example (2.1) (2) and Proposition (2.2) (3).

Suppose now that $(G', q') \cong w^{\varepsilon}_{2,1} \oplus w^{\phi}_{2,1}$, and let x and y generate $w^{\varepsilon}_{2,1}$ and $w^{\phi}_{2,1}$ respectively. Then $K \cap G'$ is generated by $x+y$, and $q(x+y) = (\varepsilon + \phi)/2$. If $\varepsilon \equiv \phi \bmod 4$ then K is degenerate and quasi-nondegenerate with $\mathrm{Rad}(K) = K \cap G'$. On the other hand, if $\varepsilon \equiv -\phi \bmod 4$ then $\mathrm{Rad}_q(K) = K \cap G'$ and the induced form on $K/\mathrm{Rad}_q(K)$ is isomorphic to $u_1^{\oplus n} \oplus v_1^{\oplus m}$. In both cases, the structure of $O(G, q)$ now follows from Example (2.1) (3) and Proposition (2.2). Q.E.D.

To finish the computation of the order of $O(G, q)$ in the cases covered by Corollary (2.4), we recall (from [25; pp. 382, 392], for example) the orders of $O^+(2n, 2)$, $O^-(2n+2, 2)$, and $Sp(2n, 2)$:

Lemma (2.5).

(1) $|O(u_1^{\oplus n})| = |O^+(2n, 2)| = 2 \cdot 2^{n(n-1)} \cdot (2^n - 1) \cdot \prod_{i=1}^{n-1} (2^{2i} - 1)$.

(2) $|O(u_1^{\oplus n} \oplus v_1)| = |O^-(2n+2, 2)| = 2 \cdot 2^{n(n+1)} \cdot (2^{n+1} + 1) \cdot \prod_{i=1}^{n} (2^{2i} - 1)$.

(3) $|Sp(2n, 2)| = 2^{n^2} \cdot \prod_{i=1}^{n} (2^{2i} - 1)$.

Table 2 shows the orders of these groups for low values of n. These can be combined with Corollary (2.4) to calculate $|O(G, q)|$ in a number of cases; we will carry out the calculations in the cases of interest to us in the next section.

Table 2. Orders of certain finite orthogonal and symplectic groups

k	$\lvert O^+(2k,2)\rvert$	$\lvert O^-(2k,2)\rvert$	$\lvert Sp(2k,2)\rvert$
1	2	$2\cdot 3$	$2\cdot 3$
2	$2^3\cdot 3^2$	$2^3\cdot 3\cdot 5$	$2^4\cdot 3^2\cdot 5$
3	$2^7\cdot 3^2\cdot 5\cdot 7$	$2^7\cdot 3^4\cdot 5$	$2^9\cdot 3^4\cdot 5\cdot 7$
4	$2^{13}\cdot 3^5\cdot 5^2\cdot 7$	$2^{13}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$2^{16}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$
5	$2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$	$2^{21}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17$	$2^{25}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17\cdot 31$

§ 3. Cremona lattices and Todorov lattices

A *lattice* is a free finitely generated $\boldsymbol{Z}$-module M together with a bilinear form $b\colon M\times M\to\boldsymbol{Z}$; the lattice is *even* if $b(x,x)\in 2\boldsymbol{Z}$ for all $x\in M$. A lattice M is *nondegenerate* if $\operatorname{Ad} b\colon M\to\operatorname{Hom}(M,\boldsymbol{Z})$ is injective, and *unimodular* if $\operatorname{Ad} b$ is an isomorphism. If M is a nondegenerate even lattice, the *discriminant-form* of M is the finite quadratic form (G_M, q_M) where $G_M=\operatorname{Coker}(\operatorname{Ad} b)$ and q_M is induced by the natural embedding of $\operatorname{Hom}(M,\boldsymbol{Z})$ into $M\otimes\boldsymbol{Q}$. We let $O(G_M)$ denote the orthogonal group of (G_M, q_M).

We single out several lattices for special attention. The *K*3 *lattice* Λ is an even unimodular lattice of signature (3, 19); it is unique up to isomorphism by a theorem of Milnor [15; Section II.5]. The *Cremona lattice* M_n (for $0\le n\le 10$) is the lattice generated by the elements $\lambda, e_1,\cdots,e_n$, where $\lambda\cdot\lambda=2$, $\lambda\cdot e_i=0$, $e_i\cdot e_j=-2\delta_{ij}$.

Let E_k be the lattice generated by the elements $e_1,\cdots,e_k$ where $e_i\cdot e_j=-2\delta_{ij}$. The *double point lattice* $L_{\alpha,k}$ is the lattice defined as the saturation $E_k\otimes\boldsymbol{Q}\cap j^{-1}(\Lambda)$ for some embedding $j\colon E_k\hookrightarrow\Lambda$, where the index $[L_{\alpha,k}\colon E_k]$ is equal to 2^α. The double point lattice $L_{\alpha,k}$ exists if and only if $\alpha\le 5$ and $2^{4-\alpha}(2^\alpha-1)\le k\le\alpha+11$ and it is uniquely determined by (α,k) up to isometries [20, Sections 1 and 2]. The *Todorov lattice* $M_{\alpha,k}$, defined for $\alpha\le 4$, $k\ge 9$ and $2^{4-\alpha}(2^\alpha-1)\le k\le\alpha+11$ but $(\alpha,k)\ne(1,9)$, is the lattice generated by $L_{\alpha,k}$ and the elements λ and μ, where $\lambda\cdot\lambda=2k-16$, $\lambda\cdot L_{\alpha,k}=0$ and $\mu=(\lambda+\sum e_i)/2$. For details, see [20, Section 6]. (The case $(\alpha,k)=(5,16)$ was also considered in [20], but we stress that *in this paper, all Todorov lattices have* $k\le 15$.) We let (G_n, q_n) and $(G_{\alpha,k}, q_{\alpha,k})$ denote the discriminant-forms of M_n and $M_{\alpha,k}$ respectively, and let $O(G_n)$ and $O(G_{\alpha,k})$ denote the orthogonal groups of these discriminant-forms.

Proposition (3.1). *The normal form decompositions and the orders of* *the orthogonal groups of* (G_n, q_n) *and* $(G_{\alpha,k}, q_{\alpha,k})$ *are those shown in Tables* 3 *and* 4, *respectively.*

Table 3. Normal form decomposition of (G_n, q_n), and the order of its orthogonal group

n	(G_n, q_n)	$\|O(G_n, q_n)$
0	$w^1_{2,1}$	$\|O^+(0,2)\|=1$
1	$w^1_{2,1}\oplus w^7_{2,1}$	$\|O^+(0,2)\|=1$
2	$u_1\oplus w^7_{2,1}$	$\|O^+(2,2)\|=2$
3	$u_1\oplus(w^7_{2,1})^{\oplus 2}$	$2\cdot\|Sp(2,2)\|=2^2\cdot 3$
4	$u_1\oplus v_1\oplus w^1_{2,1}$	$\|O^-(4,2)\|=2^3\cdot 3\cdot 5$
5	$u_1\oplus v_1\oplus w^1_{2,1}\oplus w^7_{2,1}$	$2^4\cdot\|O^-(4,2)\|=2^7\cdot 3\cdot 5$
6	$u_1^{\oplus 2}\oplus v_1\oplus w^7_{2,1}$	$\|O^-(6,2)\|=2^7\cdot 3^4\cdot 5$
7	$u_1^{\oplus 2}\oplus v_1\oplus(w^7_{2,1})^{\oplus 2}$	$2\cdot\|Sp(6,2)\|=2^{10}\cdot 3^4\cdot 5\cdot 7$
8	$u_1^{\oplus 4}\oplus w^1_{2,1}$	$\|O^+(8,2)\|=2^{13}\cdot 3^5\cdot 5^2\cdot 7$
9	$u_1^{\oplus 4}\oplus w^1_{2,1}\oplus w^7_{2,1}$	$2^8\cdot\|O^+(8,2)\|=2^{21}\cdot 3^5\cdot 5^2\cdot 7$
10	$u_1^{\oplus 5}\oplus w^7_{2,1}$	$\|O^+(10,2)\|=2^{21}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31$

Table 4. Normal form decomposition of $(G_{\alpha,k}, q_{\alpha,k})$, and the order of its orthogonal group

α	k	$(G_{\alpha,k}, q_{\alpha,k})$	$\|O(G_{\alpha,k}, q_{\alpha,k})\|$
0	9	$u_1^{\oplus 4}$	$\|O^+(8,2)\|=2^{13}\cdot 3^5\cdot 5^2\cdot 7$
0	10	$u_1^{\oplus 3}\oplus v_1\oplus w^3_{2,2}$	$2\cdot\|Sp(8,2)\|=2^{17}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$
0	11	$u_1^{\oplus 4}\oplus v_1\oplus w^{-1}_{3,1}$	$\|O^-(10,2)\|\cdot 2=2^{22}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17$
1	10	$u_1^{\oplus 2}\oplus v_1\oplus w^3_{2,2}$	$2\cdot\|Sp(6,2)\|=2^{10}\cdot 3^4\cdot 5\cdot 7$
1	11	$u_1^{\oplus 3}\oplus v_1\oplus w^{-1}_{3,1}$	$\|O^-(8,2)\|\cdot 2=2^{14}\cdot 3^4\cdot 5\cdot 7\cdot 17$
1	12	$u_1^{\oplus 3}\oplus v_1\oplus w^5_{2,3}$	$2^9\cdot\|O^-(8,2)\|=2^{22}\cdot 3^4\cdot 5\cdot 7\cdot 17$
2	12	$u_1^{\oplus 2}\oplus v_1\oplus w^5_{2,3}$	$2^7\cdot\|O^-(6,2)\|=2^{14}\cdot 3^4\cdot 5$
2	13	$u_1^{\oplus 3}\oplus v_1\oplus w^{-1}_{5,1}$	$\|O^-(8,2)\|\cdot 2=2^{14}\cdot 3^4\cdot 5\cdot 7\cdot 17$
3	14	$u_1^{\oplus 3}\oplus w^5_{2,2}\oplus w^1_{3,1}$	$2\cdot\|Sp(6,2)\|\cdot 2=2^{11}\cdot 3^4\cdot 5\cdot 7$
4	15	$u_1^{\oplus 3}\oplus w^1_{7,1}$	$\|O^+(6,2)\|\cdot 2=2^8\cdot 3^2\cdot 5\cdot 7$

Proof. Since M_n is given in diagonal form, it is easy to see that (G_n, q_n) is isomorphic to $w^1_{2,1} \oplus (w^7_{2,1})^{\oplus n}$. The normal forms shown in Table 3 now follow directly from the relations in qu($\boldsymbol{Z}$) given in Proposition (1.2) (5) (cf. the proof of Proposition (1.3)).

[20; Proposition (6.2)] shows that $(G_{\alpha,k}, q_{\alpha,k}) \cong z^a_l \oplus (G'_{\alpha,k}, q'_{\alpha,k})$, with $(G'_{\alpha,k}, q'_{\alpha,k})$ a special 2-elementary form of rank 2s, where

$$(2s, l, a) = \begin{cases} (k-2\alpha-1, k-8, 2) & \text{if } k \text{ is odd} \\ (k-2\alpha-2, 2k-16, k-7) & \text{if } k \text{ is even } (k<16). \end{cases}$$

If $k \neq 14$, l is a prime power so that z^a_l is indecomposable; if $k=14$ then $z^a_l = z^7_{12} \cong w^5_{2,2} \oplus w^1_{3,1}$.

Since $(G'_{\alpha,k}, q'_{\alpha,k})$ is special and 2-elementary of rank $2s$, we have $(G'_{\alpha,k}, q'_{\alpha,k}) \cong u_1^{\oplus(s-m)} \oplus v_1^{\oplus m}$ with $m=0$ or 1: it remains to determine m. To accomplish this, we use Milgram's theorem [15; Appendix 4], which says that the first signature invariant $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$ (which was introduced in the proof of Proposition (1.4)) coincides mod 8 with the signature of $M_{\alpha,k}$. The signature of $M_{\alpha,k}$ is $1-k$; on the other hand, $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$ can be computed from Table 1. We have carried out this computation in Table 5; the conclusion we draw is that to get $\sigma_1(G_{\alpha,k}, q_{\alpha,k}) \equiv 1-\mathrm{k} \bmod 8$, we must have $m=0$ for $k=9$ and $k \geq 14$, while $m=1$ for $10 \leq k \leq 13$. This gives the normal forms shown in Table 4.

The orders of the orthogonal groups are computed as follows: by Example (2.1) (1) we only need to compute the orders for the p-Sylow subgroups, and these orders when p is odd follow from Example (2.1) (2). For the 2-Sylow subgroups, we use the normal form decomposition together with Corollary (2.4) and Table 2 to obtain the orders given in Tables 3 and 4. Q.E.D.

Table 5. Computation of $\sigma_1(G_{\alpha,k}, q_{\alpha,k})$

k	$(G_{\alpha,k}, q_{\alpha,k})$	$\sigma_1(G_{\alpha,k}, q_{\alpha,k})$
9	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m}$	$4m$
10	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^3_{2,2}$	$4m+3$
11	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^{-1}_{3,1}$	$4m+2$
12	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^5_{2,3}$	$4m+1$
13	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^{-1}_{5,1}$	$4m+0$
14	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^5_{2,2} \oplus w^1_{3,1}$	$4m+5-2$
15	$u_1^{\oplus(s-m)} \oplus v_1^{\oplus m} \oplus w^1_{7,1}$	$4m-6$

A map $\phi\colon M \to L$ between nondegenerate lattices is called a *primitive embedding* if ϕ preserves the bilinear forms, the kernel of ϕ is trivial, and the cokernel of ϕ is free. A surjective primitive embedding is called an *isometry*. The group of self-isometries of a nondegenerate lattice L is called the *orthogonal group of* L and denoted by $O(L)$. Any $\gamma \in O(L)$ induces a natural automorphism of (G_L, q_L), denoted by $G_\gamma \in O(G_L)$.

The *signature* of a nondegenerate lattice L is the pair (r_+, r_-) describing the number of positive and negative eigenvalues of the induced real quadratic form on $L \otimes \boldsymbol{R}$. A (*positive*) *sign structure* (cf. [13]) on a nondegenerate lattice L of signature (r_+, r_-) is a choice of one of the connected components of the set of oriented r_+-planes in $L \otimes \boldsymbol{R}$ on which the form is positive-definite; the sign structure containing the oriented plane ν is denoted by $[\nu]$. $O_-(L)$ is the subgroup of $O(L)$ of isometries preserving a sign structure; it can also be described as

$$O_-(L) = \{\gamma \in O(L) \mid \det \gamma = \operatorname{spin} \gamma\}$$

where $\operatorname{spin} \gamma$ denotes the (real) spinor norm.

Theorem (3.2). *Suppose that* $2 \le n \le 10$.

(1) *Let* $\phi_1, \phi_2\colon M_n \to \Lambda$ *be two primitive embeddings of the Cremona lattice into the K3 lattice. Then there is some* $\gamma \in O_-(\Lambda)$ *such that* $\gamma \circ \phi_1 = \phi_2$.

(2) *Let* $\phi\colon M_n \to \Lambda$ *be a primitive embedding, and let* N_n *be the orthogonal complement of the image of* ϕ. *If* $\psi_1, \psi_2\colon N_n \to \Lambda$ *are two primitive embeddings, then there is some* $\gamma \in O_-(\Lambda)$ *such that* $\gamma \circ \psi_1 = \psi_2$.

Proof. (1) Since N_n has signature $(2, 19-n)$, it is indefinite and has rank at least 3. Since $n \ge 2$, by Proposition (3.1) (G_n, q_n) contains an orthogonal summand of type u_1 or v_1, and the p-Sylow subgroup of G_n is cyclic for $p \ne 2$. The statement now follows from Theorem (A.1) of [20] (which is a variant of a theorem of Nikulin [21]).

(2) We again use Theorem (A.1) of [20]. This time we must check that M_n (which has signature $(1, n)$) is indefinite and of rank at least 3; this holds for $n \ge 2$. By a standard argument [22; Section 1.6], there is an isomorphism $(G_{N_n}, q_{N_n}) \cong (G_{M_n}, -q_{M_n}) = (G_n, -q_n)$. Since the isomorphism types of u_1 and v_1 do not change when the signs on the quadratic forms are reversed, Proposition (3.1) again implies that (G_{N_n}, q_{N_n}) contains an orthogonal summand of type u_1 or v_1, and that the p-Sylow subgroup of G_{N_n} is cyclic for $p \ne 2$; the theorem follows. Q.E.D.

For Todorov lattices, the statement analogous to Theorem (3.2) (1) was proved in [20; Theorem 6.3].

Theorem (3.3). *Let $M_{\alpha,k}$ be a Todorov lattice, let $\phi: M_{\alpha,k} \to \Lambda$ be an embedding into the K3 lattice, and let $N_{\alpha,k}$ be the orthogonal complement of the image of ϕ. If $\psi_1, \psi_2: N_{\alpha,k} \to \Lambda$ are two primitive embeddings, then there is some $\gamma \in O_-(\Lambda)$ such that $\gamma \circ \psi_1 = \psi_2$.*

Proof. $M_{\alpha,k}$ has signature $(1, k-1)$; since $k \geq 9$, $M_{\alpha,k}$ is therefore indefinite and has rank at least 3. Since $(G_{N_{\alpha,k}}, q_{N_{\alpha,k}}) \cong (G_{\alpha,k}, -q_{\alpha,k})$, Proposition (3.1) implies that $(G_{N_{\alpha,k}}, q_{N_{\alpha,k}})$ contains an orthogonal summand of type u_1 or v_1, and that the p-Sylow subgroup of $G_{N_{\alpha,k}}$ is cyclic for $p \neq 2$. We may then apply Theorem (A.1) of [20] one more time, and the theorem follows. Q.E.D.

§ 4. The moduli of double sextics with *n* nodes

A *double sextic with n nodes* is a double cover X of $\boldsymbol{P}^2$ branched along a sextic curve C in $\boldsymbol{P}^2$ which has exactly n nodes as its singularities. Any double sextic X with n nodes is a $K3$ surface with n ordinary double points (lying over the nodes of C). Let $\mu: S \to X$ be the minimal desingularization of X, and let $\lambda: V \to \boldsymbol{P}^2$ be the blow up of the n nodes of C. We then get a Cartesian diagram

$$\begin{array}{ccc} S & \xrightarrow{\mu} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\lambda} & \boldsymbol{P}^2. \end{array}$$

Let $M_n(X)$ be the sublattice of $H^2(S, \boldsymbol{Z})$ generated by $c_1(\mathscr{L})$, $c_1(E_1)$, $\cdots$, $c_1(E_n)$, where $\mathscr{L} = (g \circ \lambda)^* \mathcal{O}_{\boldsymbol{P}^2}(1)$ and $\{E_i\}$ are the exceptional curves of μ. Then $M_n(X) \otimes \boldsymbol{Q}$ coincides with $g^* H^2(V, \boldsymbol{Q})$. The intersection numbers are given by $\mathscr{L} \cdot \mathscr{L} = 2$, $\mathscr{L} \cdot E_i = 0$, $E_i \cdot E_j = -2\delta_{ij}$, so that $M_n(X)$ is isomorphic to the Cremona lattice M_n.

Lemma (4.1). *Let X be a double sextic with n nodes, and let S be its minimal desingularization. Then the inclusion $M_n(X) \subset H^2(S, \boldsymbol{Z})$ is a primitive embedding if and only if the branch curve C is irreducible.*

Proof. Let $\iota: S \to S$ be the involution such that $S/\iota = V$. We first claim that if $x \in (M_n(X) \otimes \boldsymbol{Q}) \cap H^2(S, \boldsymbol{Z})$, then $\iota^*(x) = x$ and $2x \in M_n(X)$. For suppose that $kx \in M_n(X)$ for some $k \in \boldsymbol{Z}$. Then $k\iota^*(x) = \iota^*(kx) = kx$ so that $k(\iota^*(x) - x) = 0$; since $H^2(S, \boldsymbol{Z})$ is torsion free, we have $\iota^*(x) = x$. Moreover, $g^* g_*(x) = x + \iota^*(x) = 2x \in M_n(X)$, proving our claim.

Let $C = \sum_l C_l$ be the decomposition into irreducible components, and let $\tilde{C}_l$ be the proper transform of C_l on V. Then it is easy to see that

$\mathcal{O}_S(g^*(\tilde{C}_l)+\sum_{i\in I}E_i+\sum_{j\in J}2E_j)=\mathcal{L}^{\otimes d}$, where $d=\deg C_l$, I is the set of indices corresponding to the intersection points $C_l\cap(\sum_{k\neq l}C_k)$, and J is the set of indices corresponding to the double points of C_l. If C is reducible, then $C_l\cap(\sum_{k\neq l}C_k)\neq\emptyset$; since $g^*(\tilde{C}_l)=2\tilde{C}'_l$ for some curve $\tilde{C}'_l$ on S, we have $c_1(\tilde{C}'_l)\notin M_n(X)$ but $c_1(2\tilde{C}'_l)\in M_n(X)$, so that the inclusion $M_n(X)\subset H^2(S, \boldsymbol{Z})$ is not a primitive embedding (its cokernel is not free).

Suppose now that C is irreducible, and take any $x\in H^2(S, \boldsymbol{Z})$ such that $2x\in M_n(X)$. It suffices to show that $x\in M_n(X)$. By replacing (if necessary) x by $x+c_1(g^*(\mathcal{H}))$ for a sufficiently ample line bundle $\mathcal{H}$, we may assume that $x=c_1(D')$ for some effective divisor D'. Since $\iota^*(x)=x$, the involution ι^* acts on the complete linear system $|D'|$. Now ι^* has finite order, so its action on $|D'|$ must have fixed points. In other words, there is an effective divisor $D\in|D'|$ such that $\iota(D)=D$.

Let C' be the branch divisor of g in S. Since $\iota(D)=D$, D can be written as a sum of irreducible divisors as follows:

$$D=aC'+\textstyle\sum_i b_i(F_i+\iota(F_i))+\sum_k c_kG_k.$$

Here F_i and G_k denote irreducible divisors such that $\iota(F_i)\neq F_i$ and $\iota(G_k)=G_k$ for all i and k. Clearly, $c_1(F_i+\iota(F_i))\in M_n(X)$ and $c_1(G_k)\in M_n(X)$. It is easy to check that $\mathcal{L}^{\otimes 6}=\mathcal{O}_S(2C'+\sum_{j=1}^n 2E_j)$ which implies that $c_1(C')\in M_n(X)$. Thus, $x=c_1(D)\in M_n(X)$. Q.E.D.

A consequence of Lemma (4.1) is the existence of a primitive embedding of the Cremona lattice M_n into the $K3$ lattice Λ for $0\leq n\leq 10$, since there exist irreducible sextic curves with exactly n nodes in those cases. We fix once and for all a primitive embedding of M_n into Λ, and identify M_n with its image in Λ. We let N_n be the orthogonal complement of M_n in Λ, and fix a positive sign structure $[\nu_n]$ on N_n. The *period space* D_n is then defined by

$$D_n=\{\omega\in \boldsymbol{P}(N_n\otimes\boldsymbol{C})\,|\,\omega\cdot\omega=0,\ \omega\cdot\bar{\omega}>0,\ \mathrm{Re}\,(\omega)\wedge\mathrm{Im}\,(\omega)\in[v_n]\}.$$

(The last condition ensures that D_n is connected.) The *integral automorphism group* of D_n is the group $\mathrm{Aut}_{\boldsymbol{Z}}(D_n)=O_-(N_n)/(\pm 1)$.

A $K3$ *surface of Cremona type* n is a pair $(X, \mathcal{L})$ consisting of a $K3$ surface X with n ordinary double points and a line bundle $\mathcal{L}$ on X with $\mathcal{L}\cdot\mathcal{L}=2$ whose associated linear system is free, and maps X to $\boldsymbol{P}^2$ with branch locus an irreducible sextic curve with n nodes. If $(X, \mathcal{L})$ is a $K3$ surface of Cremona type n, and μ: $S\to X$ is the minimal desingularization, a *special marking* of $(X, \mathcal{L})$ is an isometry ϕ: $H^2(S, \boldsymbol{Z})\to\Lambda$ together with an isomorphism ψ: $\mathrm{Sing}\,X\to\{1, \cdots, n\}$ with the following properties:

(1) $c_1(\mu^*\mathcal{L})=\phi^{-1}(\lambda)$, and $c_1(\mu^{-1}(P))=\phi^{-1}(e_{\psi(P)})$ for each $P\in\mathrm{Sing}\,X$;

(2) for any nonzero holomorphic 2-form ω on S, $\mathrm{Re}(\phi(\omega))\wedge \mathrm{Im}(\phi(\omega)) \in [\nu_n]$ (regarding $\phi(\omega)$ as an element of $N_n \otimes \boldsymbol{C}$);

(3) If x in $\mathrm{NS}(S)$ satisfies the conditions that $x\cdot x=-2$ and $x\cdot\phi^{-1}(\lambda)=0$, then x is in $\phi^{-1}(M_n)$. (Here, $\mathrm{NS}(S)=H^2(S, \boldsymbol{Z})\cap H^{1,1}(S)$ is the Néron-Severi group.)

(4) There are no elements y in $\mathrm{NS}(S)$ with $y\cdot y=0$ and $y\cdot\phi^{-1}(\lambda)=1$.

Lemma (4.2). *Every $K3$ surface of Cremona type n has a special marking.*

Proof. Choose an isomorphism ψ: $\mathrm{Sing}\, X\to\{1, \cdots, n\}$ and an isometry η: $M_n(X)\to M_n$ such that $\eta^{-1}(\lambda)=c_1(\mu^*\mathscr{L})$ and $\eta^{-1}(e_{\psi(P)})=c_1(\mu^{-1}(P))$. Also choose an isometry ϕ': $H^2(S, \boldsymbol{Z})\to\Lambda$ such that $\phi'[c_1(\mu^*\mathscr{L})\wedge\mathrm{Re}(\omega)\wedge \mathrm{Im}(\omega)]=[\lambda\wedge\nu_n]$ for a nonzero holomorphic 2-form ω on S. Since the branch locus of the map f: $X\to\boldsymbol{P}^2$ is irreducible, Lemma (4.1) guarantees that $\phi'\circ\eta^{-1}$ determines a primitive embedding of M_n into Λ; by Theorem (3.2) (1), there is some $\gamma\in O_-(\Lambda)$ with $\gamma\circ\phi'\circ\eta^{-1}=1_{M_n}$. Let $\phi=\gamma\circ\phi'$; then $\phi^{-1}(\lambda)=c_1(\mu^*\mathscr{L})$ and $\phi^{-1}(e_{\psi(P)})=c_1(\mu^{-1}(P))$ for each $P\in\mathrm{Sing}\, X$. Since γ preserves the sign structure $[\lambda\wedge\nu_n]$ on Λ, $[\lambda\wedge\mathrm{Re}(\phi(\omega))\wedge\mathrm{Im}(\phi(\omega))]= \phi[c_1(\mu^*\mathscr{L})\wedge\mathrm{Re}(\omega)\wedge\mathrm{Im}(\omega)]=[\lambda\wedge\nu_n]$ so that $\mathrm{Re}(\phi(\omega))\wedge\mathrm{Im}(\phi(\omega))\in[\nu_n]$ as sign structures on N_n.

If x in $\mathrm{NS}(S)$ satisfies the conditions $x\cdot x=-2$ and $x\cdot\phi^{-1}(\lambda)=0$, then $\pm x$ is the first Chern class of an effective divisor E with $\mathscr{L}\cdot E=0$. The support of E is contained in $\mu^{-1}(\mathrm{Sing}\, X)$, so that $\pm x=c_1(E)$ is in the inverse image $\phi^{-1}(M_n)$.

Suppose y in $\mathrm{NS}(S)$ satisfies the conditions $y\cdot y=0$ and $y\cdot\phi^{-1}(\lambda)=1$. By Riemann-Roch, $\pm y$ is the first Chern class of an effective divisor F which moves in a linear system of (projective) dimension at least 1; since $y\cdot\mu^*\mathscr{L}>0$ and $\mu^*\mathscr{L}$ is nef, we have $y=c_1(F)$. Since irreducible curves of negative self-intersection on a $K3$ surface do not move, there must be some irreducible component F_0 of some member of the complete linear systme $|F|$ with $F_0\cdot F_0\geq 0$. Now $F_0\cdot\mu^*\mathscr{L}\neq 0$, for otherwise F_0 would have self-intersection -2; hence, $F_0\cdot\mu^*\mathscr{L}=1$. The Hodge index theorem now implies that $F_0\cdot F_0\leq 0$, so that F_0 is an effective irreducible curve with $F_0\cdot F_0=0$ and $F_0\cdot\mu^*\mathscr{L}=1$. By Mayer [14], the linear system $|\mu^*\mathscr{L}|$ cannot be free (it has the form $\mu^*\mathscr{L}=\mathscr{O}_S(2F_0+C)$ for some rational curve C which serves as its base locus), a contradiction. Hence, no such y exists. Q.E.D.

Note that if $\gamma\in O(\Lambda)$ satisfies $\gamma(\lambda)=\lambda$ and $\gamma(M_n)=M_n$ then $\gamma|_{N_n}$ preserves the sign structure $[\nu_n]$ if and only if $\gamma\in O_-(\Lambda)$. Thus, if (ϕ, ψ) is a special marking of $(X, \mathscr{L})$, it follows directly from the definitions that (ϕ', ψ') is a special marking of the same $(X, \mathscr{L})$ if and only if $(\phi', \psi')= (\gamma\circ\phi, \sigma\circ\psi)$ for some $(\gamma, \sigma)\in\tilde{\Gamma}_n$, where

$$\tilde{\Gamma}_n = \{(\gamma, \sigma) \in O_-(\Lambda) \times \mathfrak{S}_n \mid \gamma(\lambda) = \lambda, \gamma(e_i) = e_{\sigma(i)}\}$$

and $\mathfrak{S}_n$ denotes the symmetric group on n letters.

If $(\gamma, \sigma) \in \tilde{\Gamma}_n$ then $\gamma|_{N_n}$ preserves the sign structure $[\nu_n]$ so that $\gamma|_{N_n}$ acts on D_n; we define $\Gamma_n = \text{Image}(\tilde{\Gamma}_n \to \text{Aut}_{\boldsymbol{Z}}(D_n))$. The set of special markings of $(X, \mathscr{L})$ thus determines a point in D_n/Γ_n.

Define the open set D_n° of D_n as follows:

$$D_n^\circ = \{\omega \in D_n \mid \text{for all } \xi, \eta \text{ in } \Lambda, \text{ if } \xi \cdot \xi = -2, \ \xi \cdot \lambda = 0, \text{ and } \xi \notin M_n \text{ then } \xi \cdot \omega \neq 0; \text{ if } \eta \cdot \eta = 0, \text{ and } \eta \cdot \lambda = 1, \text{ then } \eta \cdot \omega \neq 0\}.$$

Note that D_n° is stable under the action of Γ_n, and that a special marking of $(X, \mathscr{L})$ determines a point in D_n° (the marking sends x, y to ξ, η).

Theorem (4.3). *D_n°/Γ_n is a coarse moduli space for K3 surfaces of Cremona type n, and for irreducible sextic curves in $\boldsymbol{P}^2$ with n nodes.*

Proof. First note that there is a one-to-one correspondence between $K3$ surfaces of Cremona type n and irreducible sextic curves in $\boldsymbol{P}^2$ with n nodes obtained by sending the surface $(X, \mathscr{L})$ to the branch locus of the map defined by the linear system $|\mathscr{L}|$ (and inversely sending a sextic curve to the double cover of $\boldsymbol{P}^2$ branched on that curve.)

Suppose that $(X_i, \mathscr{L}_i)$ for $i=1, 2$ are $K3$ surfaces of Cremona type n which are assigned to the same point in D_n°/Γ_n, and let $\mu_i\colon S_i \to X_i$ be the minimal desingularizations. There are then special markings $\phi_i\colon H^2(S_i, \boldsymbol{Z}) \to \Lambda$, $\psi_i\colon \text{Sing}\, X_i \to \{1, \cdots, n\}$ and we may assume (by changing one of the markings by an element of $\tilde{\Gamma}_n$ if necessary) that $\phi = \phi_2^{-1} \circ \phi_1\colon H^2(S_1, \boldsymbol{Z}) \to H^2(S_2, \boldsymbol{Z})$ is an isomorphism of Hodge structures; note that $\phi(c_1(\mu_1^*(\mathscr{L}_1))) = c_1(\mu_2^*(\mathscr{L}_2))$. Let $\psi = \psi_2^{-1} \circ \psi_1\colon \text{Sing}\, X_1 \to \text{Sing}\, X_2$ be the induced isomorphism. An easy computation shows that $\phi(c_1(\mu_1^{-1}(P))) = c_1(\mu_2^{-1}(\psi(P)))$ so that ϕ sends the effective curves $\mu_1^{-1}(P)$ to effective curves $\mu_2^{-1}(\psi(P))$.

We now use the weakly polarized global Torelli theorem [19; p. 329] (cf. also [24]): since $\phi(c_1(\mu_1^*\mathscr{L}_1)) = c_1(\mu_2^*\mathscr{L}_2)$ and since each irreducible curve $\mu_1^{-1}(P)$ which is contracted by μ_1 is mapped to an effective curve by ϕ, there is an isomorphism $\Phi\colon X_2 \to X_1$ such that $\Phi^* = \phi$. Since $\Phi^*(\mathscr{L}_1) = \mathscr{L}_2$, Φ induces an isomorphism between the pairs $(X_i, \mathscr{L}_i)$ for $i = 1, 2$; we conclude that the natural map from the moduli space of such pairs to D_n°/Γ_n is injective.

To prove that the map is surjective, take an arbitrary point $\omega \in D_n^\circ$. We use the surjectivity of the period map for algebraic $K3$ surfaces in the form given in [19; p. 325] (and due essentially to Kulikov [11]): there is a $K3$ surface with rational double points X, an ample line bundle $\mathscr{L}$ on X and an isometry $\phi\colon H^2(S, \mathbf{Z})\to\Lambda$ with $\phi^{-1}(\omega)\in H^{2,0}(S)$ and $\phi^{-1}(\lambda)=c_1(\mu^*(\mathscr{L}))$ where $\mu\colon S\to X$ is the minimal desingularization. By composing ϕ with reflections in some of the classes e_i if necessary, we may assume that each $\phi^{-1}(e_i)$ is the first Chern class of an effective divisor.

If E is the class of an irreducible curve contracted by μ, then $c_1(E)$ lies in $\phi^{-1}(M_n)$ (since we selected a point from D_n°) so that $c_1(E)$ must coincide with $\phi^{-1}(e_i)$ for some $i\in\{1,\cdots,n\}$. Hence, X has exactly n nodes; let $\psi\colon \operatorname{Sing} X\to\{1,\cdots,n\}$ be the isomorphism compatible with ϕ.

Suppose that the linear system $|\mathscr{L}|$ is not free; then $|\mu^*\mathscr{L}|$ is also not free. Since $\mu^*\mathscr{L}$ is nef, by Mayer [14], $\mu^*\mathscr{L}$ has the form $\mu^*\mathscr{L}=\mathcal{O}_S(2F+C)$ for some elliptic curve F and rational curve C with $F\cdot\mu^*\mathscr{L}=1$; this cannot happen since we selected a point from D_n°.

Hence, $|\mathscr{L}|$ is free, $(X, \mathscr{L})$ is a $K3$ surface of Cremona type n, and (ϕ, ψ) is a special marking sending it to the class of the chosen point of D_n°.

Q.E.D.

§ 5. Degrees of period maps

In the moduli problems for sextic curves with nodes studied in the last section, and in the moduli problem for Todorov surfaces studied in [20], we have encountered families of polarized $K3$ surfaces whose moduli spaces have a similar type of description. In each case, there is a lattice M primitively embedded in the $K3$ lattice Λ with orthogonal complement N such that $M\otimes\mathbf{Q}$ is generated by classes λ and $\{e_i\mid i\in I\}$ with $\lambda\cdot\lambda>0$, $\lambda\cdot e_i=0$, and $e_i\cdot e_j=-2\delta_{ij}$. The natural period space for such a moduli problem is

$$D=\{\omega\in \mathbf{P}(N\otimes\mathbf{C})\mid \omega\cdot\omega=0,\ \omega\cdot\bar{\omega}>0,\ \operatorname{Re}(\omega)\wedge\operatorname{Im}(\omega)\in[\nu]\}$$

once a sign structure $[\nu]$ has been fixed on N; D classifies polarized Hodge structures on N with the given sign structure. The group $\operatorname{Aut}_{\mathbf{Z}}(D)=O_-(N)/(\pm 1)$ acts on D and the quotient $D/\operatorname{Aut}_{\mathbf{Z}}(D)$ parametrizes equivalence classes of Hodge structures on N, where two Hodge structures are equivalent if there is an automorphism of N preserving the bilinear form and sign structure which induces an isomorphism from one Hodge structure to the other.

Associated to M is a group $\mathfrak{S}$ which is the set of permutations σ of I such that the automorphism $\lambda\mapsto\lambda$, $e_i\mapsto e_{\sigma(i)}$ of $M\otimes\mathbf{Q}$ induces an automorphism of M; this formula defines a natural homomorphism $j\colon\mathfrak{S}\to O_-(M)$. If we define

$$\tilde{\Gamma}=\{(\gamma,\sigma)\in O_-(\Lambda)\times\mathfrak{S}\mid \gamma(\lambda)=\lambda,\ \gamma(e_i)=e_{\sigma(i)}\}$$

and $\Gamma=\operatorname{Image}(\tilde{\Gamma}\to\operatorname{Aut}_{\mathbf{Z}}(D))$ then the moduli space of the $K3$ surfaces considered was described as $M = D^\circ/\Gamma$, where D° is a certain open subset of D.

Note that in the case of $K3$ surfaces of Cremona type n, M_n itself is generated by λ and $\{e_i\}$. Thus, the group $\mathfrak{S}$ is the full symmetric group $\mathfrak{S}_n$; we denote the corresponding group for $K3$ surfaces of Todorov type (α, k) by $\mathfrak{S}_{\alpha,k}$. Note also that the index set I is $\{1, \cdots, n\}$ in the Cremona case, and $\operatorname{Sing} \Sigma_{\alpha,k}$ in the Todorov case, where $\Sigma_{\alpha,k}$ is the fixed $K3$ surface of Todorov type (α, k) chosen in [20; Section 2].

Since $D/\operatorname{Aut}_{\mathbf{Z}}(D)$ is the natural classifying space for the Hodge structures being considered, there is a natural "period map" $p\colon M \to D^\circ/\operatorname{Aut}_{\mathbf{Z}}(D)$, which we will now study. In the two cases of primary interest to us, we denote the period maps by

$$p_n\colon M_n \to D_n^\circ/\operatorname{Aut}_{\mathbf{Z}}(D_n) \quad \text{and} \quad p_{\alpha,k}\colon M_{\alpha,k} \to D_{\alpha,k}/\operatorname{Aut}_{\mathbf{Z}}(D_{\alpha,k}).$$

Theorem (5.1). *Let $p\colon M \to D^\circ/\operatorname{Aut}_{\mathbf{Z}}(D)$ be a period map built from $M \subset \Lambda$ as above, and let $\mathscr{I}$ be the image of the natural homomorphism i: $\mathfrak{S} \to O(G_M)/(\pm 1)$. Suppose that M satisfies*:

(1) *for any two primitive embeddings $\phi_1, \phi_2\colon M \to \Lambda$, there is some $\gamma \in O_-(\Lambda)$ such that $\gamma \circ \phi_1 = \phi_2$, and*

(2) *the natural map $O(M) \to O(G_M)$ is surjective.*

Then $\deg p = |O(G_M)/(\pm 1)|/|\mathscr{I}|$.

Proof. Let $N = M^\perp$. We first claim that

$$\Gamma = \{\beta \in O_-(N) \mid \pm G_\beta \in \mathscr{I}\}/(\pm 1).$$

For if $\beta \in O_-(N)$ then $\pm\beta \in \Gamma$ if and only if there exists some $(\gamma, \sigma) \in \tilde{\Gamma}$ with $\gamma|_N = (-1)^k\beta$ for some k; in that case, $G_{(\gamma|M)} = i(\sigma)$ in $O(G_M)$. If we use the standard isomorphism $(G_M, q_M) \cong (G_N, -q_N)$ to identify $O(G_M)$ with $O(G_N)$ then for a given $\sigma \in \mathfrak{S}$ and $\beta \in O_-(N)$, the sum $j(\sigma) \oplus (-1)^k\beta \in O_-(M \oplus N)$ comes from some $\gamma \in O_-(\Lambda)$ if and only if $G_{j(\sigma)} = (-1)^k G_\beta$. Since $i(\sigma) = G_{j(\sigma)}$, a given $\pm\beta$ thus lies in Γ if and only if $\pm G_\beta \in \operatorname{Image}(i)$, proving the claim.

Since $\deg p = [\operatorname{Aut}_{\mathbf{Z}}(D)\colon \Gamma]$, it only remains to show that the natural map $O_-(N) \to O(G_N)/(\pm 1)$ is surjective. Let $\tau \in O(G_N) \cong O(G_M)$; there is then some $\beta \in O(M)$ with $G_\beta = \tau$. Now $-1_M \notin O_-(M)$ so that $(-1)^k\beta \in O_-(M)$ for some k. Let $\phi\colon M \subset \Lambda$ be the inclusion; then $\phi \circ (-1)^k\beta\colon M \to \Lambda$ gives another primitive embedding. There is then some $\gamma \in O_-(\Lambda)$ such that $\gamma \circ \phi = \phi \circ (-1)^k\beta$, in other words, $\gamma|_M = (-1)^k\beta$. But then $G_{(\gamma|N)} = (-1)^k\beta$ in $O(G_N)$; hence, $(\gamma|_N) \in O_-(N)$ is the required element mapping to $\pm\beta$.

Q.E.D.

Corollary (5.2). *If* $2 \leq n \leq 10$, *the degree of the period map* p_n: $M_n \to D_n^\circ/\mathrm{Aut}_Z(D_n)$ *is given by* $\deg p_n = |O(G_n)|/n!$.

These degrees are computed explicitly in Table 6.

Table 6. Degree of p_n

n	$\lvert O(G_n)\rvert$	$n!$	$\deg p_n$
2	2	2	1
3	$2^2 \cdot 3$	$2 \cdot 3$	2
4	$2^3 \cdot 3 \cdot 5$	$2^3 \cdot 3$	5
5	$2^7 \cdot 3 \cdot 5$	$2^3 \cdot 3 \cdot 5$	2^4
6	$2^7 \cdot 3^4 \cdot 5$	$2^4 \cdot 3^2 \cdot 5$	$2^3 \cdot 3^2$
7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^2$
8	$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^3 \cdot 5$
9	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3 \cdot 5$
10	$2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	$2^{13} \cdot 3 \cdot 17 \cdot 31$

Proof. By Theorem (3.2) (1), the embedding $M_n \subset \Lambda$ is unique up to action by $O_-(\Lambda)$. Moreover, [22; Theorem 1.14.2] together with our computation of (G_n, q_n) in Section 3 shows that the map $O(M_n) \to O(G_n)$ is surjective, so we may apply Theorem (5.1). Since G_n is a 2-elementary group, -1 is the trivial element of $O(G_n)$ and $|O(G_n)/(\pm 1)| = |O(G_n)|$. Now G_n is generated by $\lambda/2, e_1/2, \cdots, e_n/2 \bmod M_n$, so that every nontrivial permutation σ of $\{e_i\}$ induces a nontrivial permutation of $\{e_i/2 \bmod M_n\}$. This implies that the map $\mathfrak{S}_n \to O(G_n)$ is injective, so that $\deg p_n = |O(G_n)|/|\mathfrak{S}_n|$.

Q.E.D.

To compute the degree of $p_{\alpha,k}$ we must study the finite group $\mathfrak{S}_{\alpha,k}$. Since $k \leq 15$, by [20; Proposition 6.1], the Todorov lattice $M_{\alpha,k}$ is generated by the double point lattice $L_{\alpha,k}$ and $\mu = (\lambda + \sum e_P)/2$. Thus, a permutation σ of $\mathrm{Sing}\,\Sigma_{\alpha,k}$[4] induces an automorphism of $M_{\alpha,k}$ by the rule $\lambda \mapsto \lambda$, $e_P \mapsto e_{\sigma(P)}$ if and only if σ induces an automorphism of the double point code $\mathscr{C}_{\alpha,k}$. Hence, $\mathfrak{S}_{\alpha,k} \cong \mathrm{Aut}(\mathscr{C}_{\alpha,k})$; by [20; Lemma 1.3] there is an exact sequence

[4] In [20], $\Sigma_{\alpha,k}$ was introduced as a $K3$ surface with k ordinary double points $p_1, \cdots, p_k$ whose double point lattice is isomorphic to $L_{\alpha,k}$ and which has a line bundle $\mathscr{L}$ such that $(\pi^*(\mathscr{L}))^2 = 2k - 16$ and $(\pi^*(\mathscr{L}) + \sum \pi^{-1}(p_i))/2 \in H^2(S, \boldsymbol{Z})$, where π: $S \to \Sigma_{\alpha,k}$ is the minimal desingularization. An example of $\Sigma_{\alpha,k}$ is given by a partial desingularization of the Kummer surface of a principally polarized abelian surface.

$$1 \longrightarrow (\mathfrak{S}_{2^{4-\alpha}})^{2^\alpha-1} \times \mathfrak{S}_l \longrightarrow \mathfrak{S}_{\alpha,k} \longrightarrow \mathrm{GL}(\alpha, \boldsymbol{F}_2) \longrightarrow 1$$

where $l = k - 2^{4-\alpha}(2^\alpha - 1)$. We have used this exact sequence to compute the orders of the groups $\mathfrak{S}_{\alpha,k}$ in Table 7. (The computation involves the order of $\mathrm{GL}(\alpha, \boldsymbol{F}_2)$, which can be found for example in [25; p. 81].)

Table 7. Order of $\mathfrak{S}_{\alpha,k}$

α	k	l	$\lvert\mathfrak{S}_{\alpha,k}\rvert$
0	9	9	$9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7$
0	10	10	$10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$
0	11	11	$11! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
1	10	2	$8! \cdot 2! = 2^8 \cdot 3^2 \cdot 5 \cdot 7$
1	11	3	$8! \cdot 3! = 2^8 \cdot 3^3 \cdot 5 \cdot 7$
1	12	4	$8! \cdot 4! = 2^{10} \cdot 3^3 \cdot 5 \cdot 7$
2	12	0	$6 \cdot (4!)^3 \cdot 0! = 2^{10} \cdot 3^4$
2	13	1	$6 \cdot (4!)^3 \cdot 1! = 2^{10} \cdot 3^4$
3	14	0	$168 \cdot (2!)^7 \cdot 0! = 2^{10} \cdot 3 \cdot 7$
4	15	0	$20160 \cdot (1!)^{15} \cdot 0! = 2^6 \cdot 3^2 \cdot 5 \cdot 7$

Corollary (5.3). *The degree of the period map $p_{\alpha,k}\colon M_{\alpha,k} \to D_{\alpha,k}/\mathrm{Aut}_Z(D_{\alpha,k})$ is given by*

$$\deg p_{\alpha,k} = \begin{cases} \dfrac{\lvert O(G_{\alpha,k})\rvert}{\lvert\mathfrak{S}_{\alpha,k}\rvert} & \text{if } (\alpha, k) = (0, 9) \text{ or } (1, 10) \\[2ex] \dfrac{\lvert O(G_{\alpha,k})\rvert}{2\lvert\mathfrak{S}_{\alpha,k}\rvert} & \text{otherwise.} \end{cases}$$

These degrees are computed explicitly in Table 8.

Proof. By [20; Theorem (6.3)] the embedding $M_{\alpha,k} \subset \Lambda$ is unique up to action by $O_-(\Lambda)$. Moreover, [22; Theorem 1.14.2] together with our computation of $(G_{\alpha,k}, q_{\alpha,k})$ in Section 3 shows that the natural map $O(M_{\alpha,k}) \to O(G_{\alpha,k})$ is surjective, so that we may apply Theorem (5.1).

Let $\mathscr{K}_{\alpha,k}$ be the kernel of the homomorphism $i\colon \mathfrak{S}_{\alpha,k} \to O(G_{\alpha,k})/(\pm 1)$. If $\sigma \in \mathscr{K}_{\alpha,k}$ then σ acts as $(-1)^m$ on $G_{\alpha,k}$ for some m; thus, there is some $(\sigma, \gamma) \in \tilde{\Gamma}_{\alpha,k}$ with $\gamma|_{N_{\alpha,k}} = (-1)^m$. By [20; Lemma 7.5], the natural map $\tilde{\Gamma}_{\alpha,k} \to O_-(N_{\alpha,k})$ is injective, and -1 is in the image of that map if and only

Table 8. Degree of $p_{\alpha,k}$

α	k	$\lvert O(G_{\alpha,k})\rvert$	$\lvert\mathfrak{S}_{\alpha,k}\rvert$	$\deg p_{\alpha,k}$
0	9	$2^{13}\cdot 3^5\cdot 5^2\cdot 7$	$2^7\cdot 3^4\cdot 5\cdot 7$	$2^6\cdot 3\cdot 5$
0	10	$2^{17}\cdot 3^5\cdot 5^2\cdot 7\cdot 17$	$2^8\cdot 3^4\cdot 5^2\cdot 7$	$2^8\cdot 3\cdot 17$
0	11	$2^{22}\cdot 3^6\cdot 5^2\cdot 7\cdot 11\cdot 17$	$2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11$	$2^{13}\cdot 3^2\cdot 17$
1	10	$2^{10}\cdot 3^4\cdot 5\cdot 7$	$2^8\cdot 3^2\cdot 5\cdot 7$	$2^2\cdot 3^2$
1	11	$2^{14}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$2^8\cdot 3^3\cdot 5\cdot 7$	$2^5\cdot 3\cdot 17$
1	12	$2^{22}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$2^{10}\cdot 3^3\cdot 5\cdot 7$	$2^{11}\cdot 3\cdot 17$
2	12	$2^{14}\cdot 3^4\cdot 5$	$2^{10}\cdot 3^4$	$2^3\cdot 5$
2	13	$2^{14}\cdot 3^4\cdot 5\cdot 7\cdot 17$	$2^{10}\cdot 3^4$	$2^3\cdot 5\cdot 7\cdot 17$
3	14	$2^{11}\cdot 3^4\cdot 5\cdot 7$	$2^{10}\cdot 3\cdot 7$	$3^3\cdot 5$
4	15	$2^8\cdot 3^2\cdot 5\cdot 7$	$2^6\cdot 3^2\cdot 5\cdot 7$	2

if $(\alpha, k)=(0, 9)$ or $(1, 10)$. Thus, if σ is nontrivial then $m=1$ and $(\alpha, k)=(0, 9)$ or $(1, 10)$. Moreover, in these cases, there is a unique $\sigma_0\in\mathfrak{S}_{\alpha,k}$ which acts as -1 on $G_{\alpha,k}$. Since $G_{0,9}$ is 2-elementary, -1 acts trivially on it; this implies that σ_0 is the identity when $(\alpha, k)=(0, 9)$. Since $G_{1,10}$ is *not* 2-elementary, the action of -1 on it is nontrivial, so that σ_0 is nontrivial when $(\alpha, k)=(1, 10)$. We conclude that $\mathscr{K}_{\alpha,k}$ is trivial when $(\alpha, k)\neq(1, 10)$ and that $\mathscr{K}_{1,10}$ has order 2.

To compute the degree of $p_{\alpha,k}$, suppose first that $(\alpha, k)\neq(0,9)$ or $(1, 10)$. Then $\mathscr{I}_{\alpha,k}=\mathfrak{S}_{\alpha,k}/\mathscr{K}_{\alpha,k}=\mathfrak{S}_{\alpha,k}$. Moreover, since $G_{\alpha,k}$ is not 2-elementary, $|O(G_{\alpha,k})/(\pm 1)|=|O(G_{\alpha,k})|/2$. Thus, by Theorem (5.1),

$$\deg p_{\alpha,k}=\frac{|O(G_{\alpha,k})/(\pm 1)|}{|\mathscr{I}_{\alpha,k}|}=\frac{|O(G_{\alpha,k})|}{2|\mathfrak{S}_{\alpha,k}|}.$$

If $(\alpha,k)=(0,9)$ then $G_{0,9}$ is 2-elementary so that $O(G_{0,9})/(\pm 1)=O(G_{0,9})$. In addition, $\mathscr{I}_{0,9}=\mathfrak{S}_{0,9}/\mathscr{K}_{0,9}=\mathfrak{S}_{0,9}$, so that

$$\deg p_{0,9}=\frac{|O(G_{0,9})/(\pm 1)|}{|\mathscr{I}_{0,9}|}=\frac{|O(G_{0,9})|}{|\mathfrak{S}_{0,9}|}.$$

Finally, if $(\alpha,k)=(1,10)$ then $|O(G_{1,10})/(\pm 1)|=|O(G_{1,10})|/2$. This time, however, $|\mathscr{I}_{1,10}|=|\mathfrak{S}_{1,10}/\mathscr{K}_{1,10}|=|\mathfrak{S}_{1,10}|/2$. Thus,

$$\deg p_{1,10}=\frac{|O(G_{1,10})/(\pm 1)|}{|\mathscr{I}_{1,10}|}=\frac{|O(G_{1,10})|}{|\mathfrak{S}_{1,10}|}.$$

Q.E.D.

§ 6. Cremona transformations and the period maps

Let $M_n = D_n^\circ/\Gamma_n$ and $M_{\alpha,k} = D_{\alpha,k}/\Gamma_{\alpha,k}$; these are coarse moduli spaces which parametrize the projective equivalence classes of certain plane sextic curves with nodes or polarized $K3$ surfaces with rational double points (the polarization comes from the line bundle $\mathscr{L}$). Since the period maps p_n and $p_{\alpha,k}$ have finite degree greater than one, in contrast to the case of the usual global Torelli theorem for polarized $K3$ surfaces, the period point does not determine the projective equivalence class. A natural question arises: *What is determined by the period maps p_n and $p_{\alpha,k}$?* The following theorem provides a partial answer to this question.

Theorem (6.1). *Let $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ be two polarized $K3$ surfaces coming from M_n $(n \geq 2)$ or $M_{\alpha,k}$, and let $\mu\colon S \to X'$ and $\mu'\colon S' \to X'$ be the minimal desingularizations. If $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ give the same point in the period space, then there is an isomorphism $\Phi\colon S \cong S'$. In particular, the surfaces X and X' are birationally isomorphic.*

Proof. Let $M(X)$ and $M(X')$ denote the natural sublattices of $H^2(S, \boldsymbol{Z})$ and $H^2(S', \boldsymbol{Z})$ respectively, which are isomorphic to one of the Cremona or Todorov lattices. Let $N(X)$ and $N(X')$ denote the orthogonal complements of $M(X)$ and $M(X')$ in $H^2(X, \boldsymbol{Z})$ and $H^2(X', \boldsymbol{Z})$, respectively. By the definition of the period map, if $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ give the same point in the period space, there is an isometry $\gamma\colon N(X) \to N(X')$ which preserves the Hodge structure and positive sign structure. We first claim that the isometry γ extends to an isometry $\tilde{\gamma}\colon H^2(S, \boldsymbol{Z}) \to H^2(S', \boldsymbol{Z})$ which preserves the sign structures. Since $H^2(S, \boldsymbol{Z})$ and $H^2(S', \boldsymbol{Z})$ are isomorphic to the $K3$ lattice Λ, this follows from Theorem (3.2) (2) in the Cremona cases, and Theorem (3.3) in the Todorov cases. Now the extended isometry $\tilde{\gamma}$ maps $M(X)$ to $M(X')$; since $M(X)$ and $M(X')$ are contained in $\mathrm{NS}(S)$ and $\mathrm{NS}(S')$ respectively, $\tilde{\gamma}$ preserves the Hodge structures. But then by the weak global Torelli theorem for $K3$ surfaces [12], S and S' are isomorphic.

The second statement is an obvious consequence of the first. Q.E.D.

Note that this theorem does not guarantee that the isomorphism Φ between S and S' preserves the polarization; typically, $\Phi^*(\mu'^*(\mathscr{L}'))$ and $\mu^*(\mathscr{L})$ are distinct line bundles on S.

Let $(X, \mathscr{L})$ be a polarized $K3$ surface coming from M_n or $M_{\alpha,k}$, and let $U \subset \boldsymbol{P}^N$ be the image of X under the morphism defined by the complete linear system $|\mathscr{L}|$. In the Cremona cases, $U = \boldsymbol{P}^2$ and the map $X \to U$ always has degree 2. In the Todorov cases, if $(\alpha, k) \neq (0, 9)$ or $(1, 10)$ and X is generic, then the map $X \to U$ is an isomorphism; if $(\alpha, k) = (0, 9)$ then $U = \boldsymbol{P}^2$ and the map $X \to U$ has degree 2; and if $(\alpha, k) = (1, 10)$ then $U = \bar{F}^2$

is a quadric cone in $\boldsymbol{P}^3$ and the map $X \to U$ again has degree 2. (See [20; Lemma 5.4 and Corollary 7.6]; the fact that the image is a quadric *cone* when $(\alpha, k)=(1, 10)$ is due to Catanese and Debarre [2].)

If $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ are two polarized $K3$ surfaces coming from $\boldsymbol{M}_n$ or $\boldsymbol{M}_{\alpha,k}$ which are sent to the same point in period space, we have seen above that X and X' are birational. We now refine our previous question and ask: *Is the induced birational map $U \dashrightarrow U'$ given by a Cremona transformation of the ambient projective space $\boldsymbol{P}^N$?*

We address this question in two of our cases in the next two sections. We will in fact deal with a restricted version of the question, in which the only Cremona transformations considered are the "regular" Cremona transformations. We begin by reviewing the facts about these transformations, following Dolgachev [8]. Let Cr_N denote the group of all Cremona transformations of $\boldsymbol{P}^N$.

Let us choose projective coordinates $Z_0, Z_1, \cdots, Z_N$ on the source $\boldsymbol{P}^N$ (resp. $Z'_0, Z'_1, \cdots, Z'_N$ on the target $\boldsymbol{P}^N$) and let $p_0, \cdots, p_N$ be the coordinate vertices of $\boldsymbol{P}^N$. Consider the Cremona transformation $T_{p_0 \cdots p_N}$ defined by the following formula:

$$Z'_i = Z_0 \cdot Z_1 \cdot \cdots \cdot Z_{i-1} \cdot Z_{i+1} \cdot \cdots \cdot Z_N \qquad \text{for } i=0, 1, \cdots, N.$$

A composition of such a transformation $T_{p_0 \cdots p_N}$ with a projective automorphism of $\boldsymbol{P}^N$ is called a *standard* Cremona transformation of $\boldsymbol{P}^N$. Together with the projective automorphism group, the standard Cremona transformations generate a subgroup $\mathrm{Cr}_N^{\mathrm{reg}}$ of Cr_N, called the group of *regular* Cremona transformations (cf. [3], [4], [8]). A regular Cremona transformation $T\colon \boldsymbol{P}^N \dashrightarrow \boldsymbol{P}^N$ has a fundamental set which is completely determined by a finite set of points $p_1, p_2, \cdots, p_r$ (called the *fundamental points* of T) in the sense that all other base conditions follow from assigning certain multiplicities at the given set of points. The inverse transformation T^{-1} has the same property, and it has the same number of fundamental points $q_1, q_2, \cdots, q_r$ as does T. Let

$$\lambda_1\colon V_1 = B(p_1, \cdots, p_r) \longrightarrow \boldsymbol{P}^N, \qquad \lambda_2\colon V_2 = B(q_1, \cdots, q_r) \longrightarrow \boldsymbol{P}^N$$

be the blow ups of the fundamental points of T and T^{-1} respectively. Then there exists a pair of birational morphisms $\nu_1\colon V \to V_1$ and $\nu_2\colon V \to V_2$ such that as birational mappings, $T \circ \lambda_1 \circ \nu_1 = \lambda_2 \circ \nu_2$.

$$\begin{array}{ccccc} V_1 & \xleftarrow{\nu_1} & V & \xrightarrow{\nu_2} & V_2 \\ \downarrow{\scriptstyle \lambda_1} & & & & \downarrow{\scriptstyle \lambda_2} \\ \boldsymbol{P}^N & & \dashrightarrow & & \boldsymbol{P}^N \end{array}$$

In the case that $N=2$, Max Noether's theorem on the decomposition of birational automorphisms of $\boldsymbol{P}^2$ says that $\mathrm{Cr}_2=\mathrm{Cr}_2^{\mathrm{reg}}$, that is, all Cremona transformations of $\boldsymbol{P}^2$ are regular. However, if N is greater than 2, it is well known that Cr_N is a very big group which contains $\mathrm{Cr}_N^{\mathrm{reg}}$ as a proper subgroup.

§ 7. Cremona equivalence of irreducible sextic curves with nodes

Let C be an irreducible sextic curve in $\boldsymbol{P}^2$ with n nodes $p_1, \cdots, p_n$. Consider the standard Cremona transformation $T_{p_1p_2p_3}$: $\boldsymbol{P}^2 \dashrightarrow \boldsymbol{P}^2$ whose fundamental points are the first three points p_1, p_2, p_3. The transformation $T_{p_1p_2p_3}$ is nothing but the usual quadratic transformation and it is easy to see that it maps the curve C to another irreducible sextic C' with n nodes.[5] (See Figure 1.) Since an arbitrary Cremona transformation $T \in \mathrm{Cr}_2$ can be expressed as a composite of standard Cremona transformations, T has the same property: it maps any sextic curve with nodes at its fundamental points to a sextic curve with nodes. We call two irreducible sextics C and C' *Cremona equivalent* if there exists a Cremona transformation $T \in \mathrm{Cr}_2$ whose fundamental points are contained among the nodes of C, such that $T(C)=C'$.

Let $\boldsymbol{M}_n^{\mathrm{Cr}}$ denote the set of Cremona equivalence classes of irreducible sextics with n nodes for $2 \leq n \leq 10$. Recall that $\boldsymbol{M}_n$ is the moduli space of projective equivalence classes of irreducible sextics with n nodes. There is a natural map Cr: $\boldsymbol{M}_n \to \boldsymbol{M}_n^{\mathrm{Cr}}$ which associates the Cremona equivalence class of C to the projective equivalence class of C.

The following theorem provides a positive answer to the question asked at the end of Eection 6 in this case.

Theorem (7.1). *Let $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ be two polarized K3 surfaces in $\boldsymbol{M}_n$ $(n \geq 2)$. Let μ: $S \to X$ and μ': $S' \to X'$ be the minimal desingularizations and let C and C' be the associated irreducible sextics with n nodes. Assume that the Picard numbers of S and S' are both equal to $n+1$. Then $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ have the same period in $D_n^\circ/\mathrm{Aut}_{\boldsymbol{Z}}(D_n)$ if and only if C and C' are Cremona equivalent.*

Let $\boldsymbol{M}_n^\circ$ denote the subset of $\boldsymbol{M}_n$ which consists of polarized $K3$ surfaces whose minimal desingularizations have Picard numbers $n+1$ and let $\boldsymbol{M}_n^{\circ \mathrm{Cr}}$

[5] Here we implicitly assume that C is a "general" sextic with n nodes. The meaning of "general sextic with n nodes" is as follows: the line $\overline{p_ip_j}$ $(i \neq j)$ through the nodes p_i and p_j of C intersects C transversely at the two residual points in $C \cap \overline{p_ip_j}$ for all pairs (i, j). If the minimal resolution S of the associated $K3$ surface X in $\boldsymbol{M}_n$ has Picard number n+1, the curve C is general.

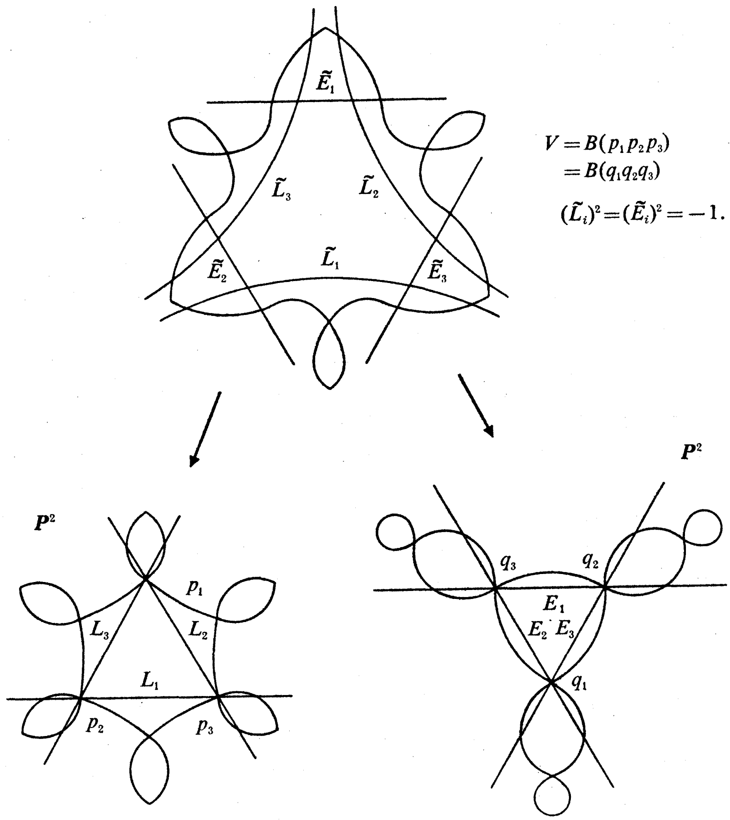


Figure 1.

denote the image of M_n° under the map Cr. Then M_n° (resp. $M_n^{\circ\mathrm{Cr}}$) is an open dense subset in M_n (resp. M_n^{Cr}). Similarly, we can define an open dense subset $D_n^\circ{}^*$ of the period space D_n° such that $D_n^\circ{}^*/\Gamma_n$ is isomorphic to M_n°.

Then from Theorem (7.1), we get:

Corollary (7.2). *The set* $M_n^{\circ\mathrm{Cr}}$ *is isomorphic to* $D_n^\circ{}^*/\mathrm{Aut}_{\boldsymbol{Z}}(D_n)$ *and the period map* $p_n: M_n^\circ \to D_n^\circ{}^*/\mathrm{Aut}_{\boldsymbol{Z}}(D_n)$ *can be naturally identified with the map* $\mathrm{Cr}: M_n^\circ \to M_n^{\circ\mathrm{Cr}}$.

Proof of Theorem (7.1). If $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ have the same period, by Theorem (6.1) there exists an isomorphism $\Phi: S \xrightarrow{\sim} S'$. Since we assume that the Picard numbers of S and S' are both equal to $n+1$, we have $\mathrm{NS}(S)=M(X)$ and $\mathrm{NS}(S')=M(X')$; in this case, Φ induces isomorphisms

$\Phi^*: M(X') \simeq M(X)$ and $\Phi^*: N(X') \simeq N(X)$. Let $p_1, \cdots, p_n$ (resp. $q_1, \cdots, q_n$) be the nodes of C (resp. C') and let $\lambda: V = B(p_1, \cdots, p_n) \to \boldsymbol{P}^2$ (resp. $\lambda': V' = B(q_1, \cdots, q_n) \to \boldsymbol{P}^2$) be the blow up of $\boldsymbol{P}^2$ at $p_1, \cdots, p_n$ (resp. $q_1, \cdots, q_n$). We then get finite maps $g: S \to V$ and $g': S' \to V'$ of degree 2; let $\iota: S \to S$ and $\iota': S' \to S'$ be the corresponding involutions. Since $g^* H^2(V, \boldsymbol{Q}) = M(X) \otimes \boldsymbol{Q}$, $M(X)$ is the invariant part of the cohomology $H^2(S, \boldsymbol{Z})$ under the action of ι^*; a similar statement holds for S'. Thus, Φ^* maps the invariant part of $H^2(S', \boldsymbol{Z})$ to the invariant part of $H^2(S, \boldsymbol{Z})$, so that the isomorphism Φ descends to an isomorphism $\bar{\Phi}: V \simeq V'$, which maps the proper transform of C to the proper transform of C'. Hence C and C' are Cremona equivalent. The converse statement in the theorem is obvious.

Q.E.D.

Remark (7.3). A $K3$ surface X of Todorov type (0, 9) is the double cover of $\boldsymbol{P}^2$ branched along two cubics D_1 and D_2 intersecting transversely (when X is general in moduli). Under the Cremona transformations whose fundamental points are contained in $D_1 \cap D_2$, D_1 and D_2 are mapped to cubics D_1' and D_2'. Let $\boldsymbol{M}_{0,9}^{\mathrm{Cr}}$ denote the Cremona equivalence classes of pairs or cubics. Then as in Theorem (7.1) and Corollary (7.2), the natural morphism $\mathrm{Cr}: \boldsymbol{M}_{0,9}^{\circ} \to \boldsymbol{M}_{0,9}^{\mathrm{Cr}}$ can be identified with the period map $p_{0,9}$: $\boldsymbol{M}_{0,9}^{\circ} \to \boldsymbol{D}_{0,9}/\mathrm{Aut}_{\boldsymbol{Z}}(D_{0,9})$. Here we denote by $\boldsymbol{M}_{0,9}^{\circ}$ the moduli space of $K3$ surfaces of Todorov type (0, 9) whose minimal desingularizations have Picard numbers 10.

We now explore the relationship between the degrees of the period maps and "Coble's numbers". Let P_n^2 denote the set of projective equivalence classes of unordered n-tuples of points in $\boldsymbol{P}^2$. The Cremona group Cr_2 acts on P_n^2 in a natural manner (cf. [3], [8]). Let $p_1, \cdots, p_n$ denote a set of n points in $\boldsymbol{P}^2$. We let $\mathrm{Cob}(p_1, \cdots, p_n)$ denote the number of projective equivalence classes which are Cremona equivalent to the given set $(p_1, \cdots, p_n)$ and call this the *Coble number* of the set of points.

The following theorem is due to Coble [3], [5].

Theorem (7.4). (1) *For* $5 \leq n \leq 10$, *let* $p_1, \cdots, p_n$ *be the* n *nodes of a general irreducible sextic curve* C *with* n *nodes in* $\boldsymbol{P}^2$. *Then the Coble number* $\mathrm{Cob} = \mathrm{Cob}(p_1, \cdots, p_n)$ *is given as follows*:

n	5	6	7	8	9	10
Cob	1	72	288	8640	$2^8 \cdot 960$	$2^{13} \cdot 31 \cdot 51$

(2) *If* $p_1, \cdots, p_9$ *are the base points of a pencil of cubics, then the Coble number* $\mathrm{Cob}(p_1, \cdots, p_9)$ *is equal to* 960.

Note that for $n \leq 8$, there exists a sextic curve C with nodes at any specified set of n points $p_1, \cdots, p_n$; for $n=9$ and 10 the sets of points we are considering are rather special.

Comparing Table 6 and the Coble numbers, one would expect to find some relations between the degrees of the period maps p_n and $p_{0,9}$ and the Coble numbers. In fact, we can reprove this theorem of Coble by using our computations of the degrees of these period maps.

Let $\psi_n: \boldsymbol{M}_n \rightarrow \boldsymbol{P}_n^2$ denote the natural map which associates to the projective equivalence class of the nodal sextic C the projective equivalence class of its set of nodes. If n is less than or equal to 8, the map ψ_n is generically surjective, and if n is greater than 8, the image of ψ_n is a proper subset of $\boldsymbol{P}_n^2$. Let $\boldsymbol{P}_n^{2,\mathrm{Cr}}$ denote the set of Cremona equivalence classes of unordered n-tuples of points in $\boldsymbol{P}^2$ and let $W_n: \boldsymbol{P}_n^2 \rightarrow \boldsymbol{P}_n^{2,\mathrm{Cr}}$ be the natural map. We have the commutative diagram

$$\begin{array}{ccc} \boldsymbol{M}_n & \xrightarrow{\psi_n} & \boldsymbol{P}_n^2 \\ {\scriptstyle \mathbf{Cr}}\downarrow & & \downarrow{\scriptstyle W_n} \\ \boldsymbol{M}_n^{\mathrm{Cr}} & \longrightarrow & \boldsymbol{P}_2^{2,\mathrm{Cr}}. \end{array}$$

Let $\lambda: V \rightarrow \boldsymbol{P}^2$ be the blow up of n points $p_1, \cdots, p_n$ which are either the n nodes of an irreducible sextic curve C or the 9 base points of a pencil of cubic curves $|D|$. (We refer to the latter situation as the "(0, 9) case".) Consider the linear system $|-2K_V|$, which contains the proper transform of C on V (or the proper transform of D_1+D_2 for $D_i \in |D|$ in the (0, 9) case). One can easily see the following lemmas (cf. [8]):

Lemma (7.5). *Let* Aut(V) *denote the automorphism group of* V. *If* C *(or* $|D|$*) is generic, we have the following*:

(1) *If* $n=5$, Aut$(V) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^4$ *and all non-trivial elements in* Aut(V) *act on* $|-2K_V|$ *non-trivially.*

(2) *If* $n=6$, Aut$(V)=\{1\}$.

(3) *If* $n=7$, Aut$(V)=\boldsymbol{Z}/2\boldsymbol{Z}$ *and it acts non-trivially on* $|-2K_V|$.

(4) *If* $n=8$, 9, or 10, *all elements of* Aut(V) *act trivially on* $|-2K_V|$.

(5) *In the* (0, 9) *case, all elements of* Aut(V) *act trivially on* $|-2K_V|$.

Proof. A proof of this lemma for $n \leq 8$ can be found in Dolgachev [8]; we will treat the (0, 9) case and the cases $n=9$ and $n=10$.

In the (0, 9) case, the pencil of cubics $|D|$ lifts to an elliptic pencil $|-K_V|$ on V which has a section. Moreover, $\mathrm{Sym}^2 H^0(-K_V) \cong H^0(-2K_V)$, so it suffices to show that Aut(V) acts trivially on $|-K_V| \cong \boldsymbol{P}^1$. If $\gamma \in$ Aut(V) acts nontrivially, by composing with a translation along the fibers we may assume that γ preserves some specified section of the pencil.

Suppose that $|D|$ is generic. It must then be a Lefschetz pencil so that the elliptic pencil $|-K_V|$ together with a choice of section gives V the structure of a (rational) Weierstrass fibration over $\boldsymbol{P}^1$, and this structure is preserved by the action of γ. A moduli space for such fibrations has been constructed by Miranda [16] (using geometric invariant theory) from their description in terms of a pair of sections $g_4 \in \mathrm{Sym}^4 W$, $g_6 \in \mathrm{Sym}^6 W$, where $W = H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(1))$). To construct the moduli space, one must form the quotient of the space $U = \mathrm{Sym}^4 W \oplus \mathrm{Sym}^6 W$ by the group $G = \mathrm{SL}(W) \times \boldsymbol{C}^*$, which acts via the natural actions of $\mathrm{SL}(W)$ in the first factor, and the action $\lambda \mapsto (\lambda^4, \lambda^6)$ in the second factor; Miranda shows that the set of stable points for this action is nonempty. In particular, the automorphisms of the generic rational Weierstrass fibration are given by the elements of G which act trivially on U. But from the standard representation theory of SL_2 it follows that the only elements of G acting trivially on U are those of the form $(\pm 1) \times (\pm 1)$; since these elements act trivially on $\boldsymbol{P}^1$ as well, we conclude that the generic rational Weierstrass fibration has no automorphism acting nontrivially on $\boldsymbol{P}^1$. This means that γ must act trivially on $\boldsymbol{P}^1$ when $|D|$ is generic, proving (5).

If $n=9$, $|-2K_V|$ is an elliptic pencil with one double fiber; by a result of Dolgachev [7] (cf. also [23; pp. 22–24]), V is the logarithmic transform of a Weierstrass fibration $f\colon W \to \boldsymbol{P}^1$. Conversely, by [23; Proposition 1.5] the logarithmic transform of order 2 along a single smooth fiber of a generic Weierstrass fibration has a minimal model isomorphic to $\boldsymbol{P}^2$, and so must coincide with the blow up of the nodes of some sextic curve in $\boldsymbol{P}^2$. Our results in the (0, 9) case now imply that when W is generic, the set of singular fibers of f is not invariant under any nontrivial automorphism of $\boldsymbol{P}^1$. Since the set of non-multiple singular fibers is preserved under logarithmic transform, V has the same property; this implies that every automorphism of V acts trivially on $|-2K_V| \cong \boldsymbol{P}^1$.

Finally, if $n=10$ and V is generic, the linear system $|-2K_V|$ has (projective) dimension zero, so any element of $\mathrm{Aut}(V)$ acts trivially on $|-2K_V|$.
Q.E.D.

For an automorphism g of V, we define a Cremona transformation $h_g = \lambda \circ g \circ \lambda^{-1}$ which makes the following diagram commutative:

$$\begin{array}{ccc} V & \xrightarrow{\;g\;} & V \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\lambda} \\ \boldsymbol{P}^2 & \overset{h_g}{\dashrightarrow} & \boldsymbol{P}^2 \end{array}$$

Let $C \in \boldsymbol{M}_n$ denote an irreducible sextic curve with n nodes at the given points $p_1, \cdots, p_n$ and let $h_g(C)$ be the image of C under the Cremona transformation h_g. Then we have the following:

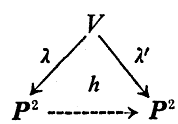
Lemma (7.6). *Let n be an integer such that $4 \leq n \leq 10$.*

(1) *For any element g in* Aut(V), *the curve $h_g(C)$ is an irreducible sextic with n nodes at the same set of points $p_1, \cdots, p_n$. In particular, we have* $\mathrm{Cr}(C) = \mathrm{Cr}(h_g(C))$ *and* $\psi_n(C) = \psi_n(h_g(C))$.

(2) *Assume that all non-trivial elements in* Aut(V) *act on* $|-2K_V|$ *nontrivially and C is general. Then for a non-trivial element g in* Aut(V), *the curve $h_g(C)$ is not projectively equivalent to the original curve C.*

(3) *Conversely, let C' be an irreducible sextic with n nodes at the same set of points $p_1, \cdots, p_n$ which is Cremona equivalent to C. Then there exists an element g in* Aut(V) *such that the Cremona transformation $h_g = \lambda \circ g \circ \lambda^{-1}$ maps C to C'.*

Proof. The assertion (1) follows directly from the definition of the Cremona transformation h_g. To prove (2), let $\bar{C}$ be the proper transform of C by λ. Then we have $\bar{C} \in |-2K_V|$. The assumption means that for a non-trivial element g in Aut(V), $g(\bar{C}) \neq \bar{C}$. This also implies that $h_g(C) \neq C$. If we take the set of n points $p_1, \cdots, p_n$ in general position (this is possible since C is general), there are no non-trivial elements in Aut$(\boldsymbol{P}^2)$ which stabilize the set of n nodes $p_1, \cdots, p_n$ if $n \geq 4$. Since the set of n nodes of $h_g(C)$ coincides with that of C, (i.e., $p_1, \cdots, p_n$), this implies that $h_g(C)$ is not projectively equivalent to C. (3) By assumption, there exists a Cremona transformation $h: \boldsymbol{P}^2 \dashrightarrow \boldsymbol{P}^2$ with base points $p_1, \cdots, p_n$ which maps C to C'. Then there exists a birational morphism $\lambda': V \rightarrow \boldsymbol{P}^2$ which makes the following diagram commutative:



Since h maps C to C', h also maps the set of n points $p_1, \cdots, p_n$ to the same set. Hence λ' is also the blow up of n points $p_1, \cdots, p_n$. By the universality of the blow up, there exists an automorphism $g: V \rightarrow V$ such that $\lambda' = \lambda \circ g$. Thus we have $h = \lambda \circ g \circ \lambda^{-1}$ as desired. Q.E.D.

From these lemmas and the commutative diagram above, we have the following

Proposition (7.7). *Let $W_n: P_n^2 \rightarrow P_n^{2,\mathrm{Cr}}$ be the natural map so that the degree of W_n is equal to the Coble number.*

(1) *If* $n=5$, $\deg W_n=(\deg p_n)/16$.
(2) *If* $n=6$, $\deg W_n=\deg p_n$.
(3) *If* $n=7$, $\deg W_n=(\deg p_n)/2$.
(4) *If* $n=8$, $\deg W_n=\deg p_n$.
(5) *If* $n=9$ *or* 10, *on the image of* ψ_n, $\deg W_n=\deg p_n$.
(6) *In the* (0, 9) *case, on the image of the natural map* $\psi_{0,9}: M_{0,9}\to P_9^2$, $\deg W_9=\deg p_{0,9}$.

Proposition (7.7) and Tables 6 and 8 now reprove Coble's Theorem (7.4).

§ 8. The Cayley Symmetroid

In this section, we study $K3$ surfaces of Todorov type (0,10): these are related to quartic $K3$ surfaces with 10 nodes in $\boldsymbol{P}^3$. By using the period map of these $K3$ surfaces, we will find an interpretation of Coble's classical results [4] on the Cayley symmetroids. Coble's results have recently been given another modern interpretation by Cossec and Dolgachev [6]; our treatment of this topic depends heavily on their work, and will be relatively brief.

Let $(X, \mathscr{L}, \Sigma)$ be a (polarized) $K3$ surface of Todorov type (0, 10). We call $(X, \mathscr{L}, \Sigma)$ *simple* if the linear system $|\mathscr{L}|$ is not hyperelliptic, and if there are exactly 10 irreducible rational curves $C_1, \cdots, C_{10}$ on S which are orthogonal to $\mu^*\mathscr{L}$. (The generic $K3$ surface of Todorov type (0, 10) is simple by [20; Corollary 7.6]). For a simple surface, the distinguished partial desingularization Σ of X actually coincides with X, so that the pair $(X, \mathscr{L})$ contains all the information we need to discuss the moduli. X is embedded by $|\mathscr{L}|$ as a quartic hypersurface with exactly 10 ordinary double points; such a quartic hypersurface is called a *Cayley symmetroid.*

The original definition of a Cayley symmetroid was as follows (cf. [4]): take a 4×4 symmetric matrix A whose entries are linear forms on $\boldsymbol{P}^3$. The quartic hypersurface X defined by the equation $\det(A)=0$ is then a Cayley symmetroid. But one easily checks that on the minimal desingularization $\mu: S\to X$, $(1/2)(c_1(\mu^*(\mathscr{L}))+\sum c(C_i)) \in H^2(S, \boldsymbol{Z})$. Hence, $(X, \mathscr{L}, X)$ is a $K3$ surface of Todorov type (0, 10) as well.

Lemma (8.1). *If X is a Cayley symmetroid whose minimal desingularization has Picard number* 11, *then no four of the nodes of X are coplanar.*

Proof. Assume that four nodes p_1, p_2, p_3, p_4 of X are coplanar. Projecting from the point p_1 to a complementary $\boldsymbol{P}^2$, we get a morphism ψ: $B_{p_1}(X)\to \boldsymbol{P}^2$ of degree two, where $B_{p_1}(X)$ denotes the blow up of X at p_1. The branch divisor of this morphism is a plane sextic curve with 9 nodes.

Since the four nodes on X are coplanar, three of the nodes in the branch curve must be collinear. If l denotes a line containing three nodes, then l meets the branch curve *only* in those three nodes so that the inverse image of l in $B_{p_1}(X)$ has two components l_1 and l_2. l_1-l_2 then lifts to a divisor class on the minimal desingularization $\mu\colon S\to X$ whose first Chern class is not in the linear span of $c_1(\mu^*\mathscr{L})$ and the exceptional classes $c_1(C_i)$, so that S has Picard number greater than 11, contrary to hypothesis. Q.E.D.

Let X be a Cayley symmetroid with ten nodes $p_1, \cdots, p_{10}$ and let μ: $S\to X$ be the minimal desingularization; suppose that the Picard number of S is 11. Let $\mathscr{L}=\mu^*\mathcal{O}_{\boldsymbol{P}^3}(1)$ and $C_i=\mu^{-1}(p_i)$. Choose four of the points p_1, p_2, p_3, p_4; since these are not coplanar, we can define the *standard Cremona transformation* $T_{p_1p_2p_3p_4}$: $\boldsymbol{P}^3\to\boldsymbol{P}^3$ with fundamental points p_1, p_2, p_3, p_4 (as in Section 6). It is easy to see that the image $X'\subset\boldsymbol{P}^3$ of X under $T_{p_1p_2p_3p_4}$ is also a symmetroid and that there is a morphism $\mu'\colon S\to X'$. Let $q_1, \cdots, q_{10}$ denote the nodes of X' and put $\mathscr{L}'=\mu'^*(\mathcal{O}_{\boldsymbol{P}^3}(1))$ and $C_i'=(\mu')^{-1}(q_i)$. For a suitable numbering of the q_i's, we then have the following relations in the Picard group of S:

$$\mathscr{L}'=\mathscr{L}^{\otimes 3}\otimes\mathcal{O}_S(-2C_1-2C_2-2C_3-2C_4)$$
$$\mathcal{O}_S(C_i')=\mathscr{L}\otimes\mathcal{O}_S(-C_1-C_2-C_3-C_4+C_i)\qquad\text{for } 1\leq i\leq 4$$
$$\mathcal{O}_S(C_i')=\mathcal{O}_S(C_i)\qquad\text{for } 5\leq i\leq 10.$$

We shall now relate the action of regular Cremona transformations in $\mathrm{Cr}_3^{\mathrm{reg}}$ on the Todorov lattice $M_{0,10}$ to the action of the *Weyl group* $W_{2,4,6}$ associated to the Dynkin diagram $T_{2,4,6}$ (cf. [8], [6]). Fix an isomorphism $\alpha\colon\{1, \cdots, 10\}\to\mathrm{Sing}\,\Sigma_{0,10}$, and let e_i denote $e_{\alpha(i)}$. Define the "roots" β_i in $M_{0,10}$ as follows:

$$\beta_0=\lambda-e_1-e_2-e_3-e_4$$
$$\beta_i=e_i-e_{i+1}\qquad\text{for } 1\leq i\leq 9.$$

If we define a new bilinear form on $M_{0,10}$ by $(x, y)=-(x\cdot y)/2$, then $(\beta_i, \beta_i)=2$; the Dynkin diagram of the (generalized) root system spanned by $\beta_0, \beta_1, \cdots, \beta_9$ (with respect to the bilinear form $(\ ,\ )$) is given by the diagram $T_{2,4,6}$ in Figure 2.

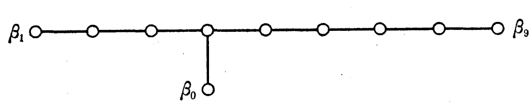


Figure 2.

Define the simple reflection s_j, $0 \leq j \leq 9$ on $M_{0,10}$ as follows:

$$s_j(x) = x - (\beta_j, x)\beta_j.$$

It is easy to check that s_j defines an isometry of $M_{0,10}$; the *Weyl group* $W_{2,4,6}$ is the group of isometries of $M_{0,10}$ generated by these simple reflections s_j, $j = 0, \cdots, 9$.

The simple reflection s_j, $j = 1, \cdots, 9$ acts on $M_{0,10}$ by leaving e_i fixed for $i \neq j, j+1$, and interchanging e_j and e_{j+1}. The simple reflection s_0 acts on $M_{0,10}$ by the following formula:

$$\begin{aligned} &s_0(\lambda) = 3\lambda - 2(e_1 + e_2 + e_3 + e_4) \\ &s_0(e_i) = \lambda - (e_1 + e_2 + e_3 + e_4) + e_i \qquad \text{for } 1 \leq i \leq 4 \\ &s_0(e_i) = e_i \qquad \text{for } 5 \leq i \leq 10. \end{aligned}$$

Let $\alpha_j = \alpha \circ (j, j+1) \circ \alpha^{-1} \in \mathrm{Aut}(\mathrm{Sing}\, \Sigma_{0,10})$, where $(j, j+1)$ denotes a simple transposition.

Proposition (8.2). *Let* $(X, \mathscr{L})$ *be a* (*simple*) K3 *surface of Todorov type* (0, 10), *let* $\mu: S \to X$ *be the minimal desingularization, let* $\phi: H^2(S, \boldsymbol{Z}) \to \Lambda$, $\psi: \mathrm{Sing}\, X \to \mathrm{Sing}\, \Sigma_{0,10}$ *be a special marking of X, and suppose that S has Picard number* 11.

(1) *For* $1 \leq j \leq 9$, $(s_j \circ \phi, \alpha_j \circ \psi)$ *is a special marking of X.*

(2) *Let X′ be the image of X under the standard Cremona transformation* $T_{p_1p_2p_3p_4}$. *Then* $(s_0 \circ \phi, \psi)$ *is a special marking of X′.*

(3) *If* $w \in W_{2,4,6}$ *there exists a regular Cremona transformation T whose fundamental points are contained in* Sing X *and a permutation* $\sigma \in \mathfrak{S}_{0,10}$ *such that* $(w \circ \phi, \sigma \circ \psi)$ *is a special marking of the image X′ of X under T.*

(4) *If T is a regular Cremona transformation whose fundamental points are contained in* Sing X, *and X′ is the image of X under T then there exist* $w \in W_{2,4,6}$ *and* $\sigma \in \mathfrak{S}_{0,10}$ *such that* $(w \circ \phi, \sigma \circ \psi)$ *is a special marking of X′.*

Proof. (1) follows directly from the definitions of special marking. (2) is a consequence of the formulas we have given above for the actions of s_0 on $M_{0,10}$ and of $T_{p_1p_2p_3p_4}$ on the Picard group of S. (3) follows from (1) and (2), by the definition of $W_{2,4,6}$. For (4), it is enough to prove the statement when T is a standard Cremona transformation whose fundamental points are contained in Sing X. But any such transformation can be expressed as the composite of a permutation of Sing X and the transformation $T_{p_1p_2p_3p_4}$; the statement now follows from (1) and (2). Q.E.D.

The reflections s_j all lie in $O_-(M_{0,10})$, so there is a natural homomorphism $W_{2,4,6} \to O_-(M_{0,10})$. We let $f: W_{2,4,6} \to O(G_{0,10})/(\pm 1)$ be the composite of this homomorphism with the canonical homomorphism $O_-(M_{0,10})\to O(G_{0,10})/(\pm 1)$. The following lemma is originally due to Coble [4] (a recent modern proof has been given by Cossec and Dolgachev [6]):

Lemma (8.3). *$W_{2,4,6}$ contains three distinguished elements K, B, and G (called the Kantor, dilated Bertini, and dilated Geiser involutions) such that the smallest normal subgroup $\overline{W}(2)$ of $W_{2,4,6}$ containing K, B, and G satisfies $W_{2,4,6}/\overline{W}(2)\cong Sp(8, 2)$.*

We use Lemma (8.3) to deduce

Lemma (8.4). *The homomorphism f: $W_{2,4,6}\to O(G_{0,10})/\pm 1$ is surjective.*

Proof. By using the explicit generators of $G_{0,10}$ given in [20; Proposition 6.2], one can check that the homomorphism f is non-trivial but that K, B, and G (the involutions from Lemma (8.3)) all lie in Ker f; by Lemma (8.3), $\overline{W}(2)$ is contained in Ker f. Since $W_{2,4,6}/\overline{W}(2)\cong Sp(8, 2)$ is a simple group, we have $\overline{W}(2)=\operatorname{Ker} f$ and $f(W_{2,4,6})\cong Sp(8, 2)$. On the other hand, by Corollary (2.4), there is an exact sequence

$$1\longrightarrow \boldsymbol{Z}/2\boldsymbol{Z}\longrightarrow O(G_{0,10})\longrightarrow Sp(8,2)\longrightarrow 1$$

and it is easy to see that the $\boldsymbol{Z}/2\boldsymbol{Z}$ subgroup (which comes from the $w^3_{2,2}$ summand in the normal form decomposition of $G_{0,10}$) is generated by -1. Thus, $O(G_{0,10})/(\pm 1)\cong Sp(8, 2)$, so that f must be surjective. Q.E.D.

We can now give an answer to the question posed at the end of Section 6 in the Todorov case (0, 10).

Theorem (8.5). *Let $(X_1, \mathscr{L}_1)$ and $(X_2, \mathscr{L}_2)$ be two $K3$ surfaces of Todorov type* (0, 10), *regarded as Cayley symmetroids in $\boldsymbol{P}^3$. Suppose that the minimal desingularizations of X_1 and X_2 have Picard number* 11. *Then $(X_1, \mathscr{L}_1)$ and $(X_2, \mathscr{L}_2)$ have the same period in $D_{0,10}/\operatorname{Aut}_{\boldsymbol{Z}}(D_{0,10})$ if and only if there exists a regular Cremona transformation $T\in \mathrm{Cr}_3^{\mathrm{reg}}$ such that $T(X_1, \mathscr{L}_1)=(X_2, \mathscr{L}_2)$.*

Proof. Suppose that $(X_1, \mathscr{L}_1)$ and $(X_2, \mathscr{L}_2)$ have the same period. By Theorem (6.1), the minimal desingularizations of X_1 and X_2 are isomorphic; we use the isomorphism to identify them with a single smooth $K3$ surface S, with $\mu_i\colon S\to X_i$ the desingularization maps for $i=1, 2$.

Let $\phi_i\colon H^2(S, \boldsymbol{Z})\to \Lambda$, $\psi_i\colon \operatorname{Sing} X\to \operatorname{Sing} \Sigma_{0,10}$ be special markings of X_i for $i=1, 2$. By Lemma (8.4), there is some $w\in W_{2,4,6}$ which has the same image in $O(G_{0,10})/(\pm 1)$ as does $(\phi_2\circ\phi_1^{-1})|_{M_{0,10}}$. By Proposition (8.2) (3), there is a regular Cremona transformation T and a permutation $\sigma\in\mathfrak{S}_{0,10}$

such that if $(X_3, \mathscr{L}_3)=T(X_1, \mathscr{L}_1)$ then $(w \circ \phi_1, \sigma \circ \psi_1)$ is a special marking of X_3.

By construction, $(\phi_2 \circ (w \circ \phi_1)^{-1})|_{M_{0,10}}$ maps to $(-1)^k$ in $O(G_{0,10})$ for some k; let $\tau=(1_{M_{0,10}})\oplus(-1)^k(\phi_2 \circ (w \circ \phi_1)^{-1})|_{N_{0,10}}$. Then $\tau \in O_-(\Lambda)$, so that if we define $\phi_3=\tau \circ w \circ \phi_1$ and $\psi_3=\sigma \circ \psi_1$, then (ϕ_3, ψ_3) is also a special marking of X_3. But since $(\phi_3 \circ \phi_2^{-1})|_{N_{0,10}}=(-1_{N_{0,10}})^k$, ϕ_2 and ϕ_3 correspond to the same point of $D_{0,10}/\Gamma_{0,10}$; hence $(X_3, \mathscr{L}_3)=T(X_1, \mathscr{L}_1)$ is isomorphic to $(X_2, \mathscr{L}_2)$.

The converse statement is an obvious consequence of Proposition (8.2) (4). Q.E.D.

Let $\boldsymbol{M}_{0,10}^{\mathrm{Cr}}$ denote the moduli of *regular* Cremona equivalence classes of Cayley symmetroids $X \subset \boldsymbol{P}^3$, and let Cr: $\boldsymbol{M}_{0,10} \to \boldsymbol{M}_{0,10}^{\mathrm{Cr}}$ be the natural map (assigning the Cremona equivalence class to the projective equivalence class). From Theorem (8.5) and Table 8 in Section 5, we have the following corollaries which explain the relation between Cremona transformations and the period map for $K3$ surfaces of Todorov type (0, 10).

Corollary (8.6). *Let $\boldsymbol{M}_{0,10}^{\circ}$ denote the moduli of the Cayley symmetroids whose minimal resolution have Picard number* 11, *and let*

$$p_{0,10}\colon \boldsymbol{M}_{0,10} \longrightarrow D_{0,10}/\mathrm{Aut}_{\boldsymbol{Z}}(D_{0,10})$$

be the period map. Then the image of $\boldsymbol{M}_{0,10}^{\circ}$ under $p_{0,10}$ and under the map Cr *are isomorphic, and the period map can be identified with the natural map* Cr. *In particular, the degree of* Cr *is equal to* $2^8 \cdot 51$.

Corollary (8.7) (Coble [4]). *The number of distinct projective equivalence classes of Cayley symmetroids which are regular-Cremona-equivalent to a fixed general Cayley symmetroid is equal to* $2^8 \cdot 51$.

The reader will notice that the key step in the proof of Theorem (8.5) was Lemma (8.3). This lemma was implicitly used by Coble [4]. Cossec and Dolgachev [6] have recently clarified the relationship between the Weyl group $W_{2,4,6}$ and the Reye congruences, and they proved that the automorphism group of the generic "nodal" Enriques surface is isomorphic to the group $\overline{W}(2)$ which was described in Lemma (8.3). Note that the covering $K3$ surface of such a "nodal" Enriques surface is the minimal desingularization S of a Cayley symmetroid X.

Coble [3] also proves that under regular Cremona transformations of $\boldsymbol{P}^3$, a set of eight points which are the base points of a net of quadrics in $\boldsymbol{P}^3$ are transformed to 36 distinct projective equivalence classes of such sets of eight points. This fact is related to the period map $p_{1,10}$ for $K3$ surfaces of Todorov type (1, 10), which has degree 36.

We close this paper by posing a problem. For $K3$ surfaces of Todorov type (α, k), let $N=k-7$ so that the linear system $|\mathscr{L}|$ maps X to $\boldsymbol{P}^N$. Does there exist any explanation of the degree of the period map $p_{\alpha,k}$ in terms of regular Cremona transformations of $\boldsymbol{P}^N$, when N is greater than 3?

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