Advanced Studies in Pure Mathematics 10, 1987
Algebraic Geometry, Sendai, 1985
pp. 361–398

Subadditivity of the Kodaira Dimension: Fibers of General Type

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I. Introduction

The aim of classification theory of algebraic varieties is to exhibit the order in the behavior of algebraic varieties. From this point of view, curves were well understood already in the nineteenth century—surfaces turn out to be much more complicated, and it is only recently that their theory achieved a satisfactory level of completeness. The outlines of an order among higher dimensional varieties has only started to emerge.

One possible approach is to define good numerical invariants of varieties and then use these invariants to relate varieties. One possible candidate is $h^0(\omega_X)$ where ω_X is the dualizing sheaf of a smooth projective variety X. For curves this invariant contains all the "discrete" information about the variety, but already surfaces with $h^0(\omega_X)=0$ have very little in common. It seems better to consider the asymptotic behavior of the numbers $P_m(X)=h^0(\omega_X^m)$. There is a largest number k such that $0<\limsup P_m(X)m^{-k}<\infty$; this k is called the Kodaira dimension of X (denoted by $\kappa(X)$). We put $\kappa(X)=-\infty$ if $P_m(X)=0$ for every m. $\kappa(X)$ is an integer and can take the values $-\infty, 0, 1, \cdots, n$ for n-dimensional varieties. Varieties satisfying $\kappa(X)=\dim X$ are said to be of general type.

One of the basic problems about the behavior of the Kodaira dimension was formulated by Iitaka [I].

Conjecture. *Let $f: X\to Y$ be a surjective map between smooth projective varieties and let F be the generic fiber of f. Then*

$$\kappa(X)\geqslant\kappa(F)+\kappa(Y);$$

i.e., the Kodaira dimension is subadditive for algebraic fiber spaces.

So far various special cases of this conjecture have been proved. The most important cases when an affirmative answer was obtained are the following:

Received November 11, 1985.

(i) $\dim F \leqslant 2$

(ii) $\kappa(Y) \geqslant \dim Y - 1$

The main contributors were Fujita, Kawamata, Ueno and Viehweg. The survey articles of Esnault [E] and Mori [Mo] contain an overview of the work done in this area and references.

The starting point of the investigations is the following observation. One would like to find sections of ω_X^m. Now $h^0(\omega_X^m) = h^0(f_*\omega_X^m)$ and $f_*\omega_X^m = f_*\omega_{X/Y}^m \otimes \omega_Y^m$. $f_*\omega_{X/Y}^m$ is a torsion free sheaf whose rank is $h^0(\omega_F^m)$. If for instance $f_*\omega_{X/Y}^m$ is generated by global sections, then we find approximately $h^0(\omega_F^m) \cdot h^0(\omega_Y^m)$ sections of $f_*\omega_X^m$ and this proves the required inequality. In general any result that asserts that $f_*\omega_{X/Y}^m$ is very close to having sections has some consequences for algebraic fiber spaces. Therefore the main focus of attention was the investigation of the sheaves $f_*\omega_{X/Y}^m$. The results can be collected around three main directions:

(i) The study of $f_*\omega_{X/Y}$. This sheaf is closely related to the topological sheaf $R^{n-k}f_*C_X$, and the theory of variations of Hodge structures provides a powerful method to attack. The main result, due to Fujita [F] and Kawamata [Ka1] is that $f_*\omega_{X/Y}$ is semipositive (cf. also Zucker [Z2] and [Ko]). Ultimately every proof found so far hinges upon the properties of these sheaves.

(ii) The study of $f_*\omega_{X/Y}^m$, $m \geqslant 2$. These sheaves are much harder to control than $f_*\omega_{X/Y}$. On the other hand they seem to satisfy certain stability properties that are not true for $m=1$. In fact if $f_*\omega_{X/Y}$ is a little bit positive then $f_*\omega_{X/Y}^m$ is very positive. These results are due to Viehweg [V2] [V3] and they merit further investigation.

(iii) Base change and covering tricks, the most profound ones being introduced by Viehweg [V2] [V3].

The difference between the results of types (i) and (ii) can be well illustrated in the case Y is an elliptic curve. Then $f_*\omega_{X/Y}^m$ is a vector bundle, and it can be written as a sum of indecomposables $\sum E_i^m$. One would like to find sections of $\sum E_i^m$. A general result that always holds is that $\deg E_i^m \geqslant 0$. Since $h^0(E_i^m) \geqslant \deg E_i^m$ this nearly solves the problem. If $\deg E_i^m = 0$ then one can say more:

(i) If $\deg E_i^1 = 0$ then E_i^1 is a line bundle and some tensor power of it is trivial. Therefore one can find a section in some $(f_*\omega_{X/Y})^{\otimes k}$. Note that this result is very unstable, i.e. will not hold for small deformations of E_i^1.

(ii) If $m \geqslant 2$ then the above results are unknown. Instead one has the following: if $\deg E_i^m > 0$ for one i then $\deg E_i^m > 0$ for every i. This can be formulated as follows: if $m \geqslant 2$ then either $\deg f_*\omega_{X/Y}^m = 0$ or $f_*\omega_{X/Y}^m$ is ample. A similar assertion is definitely false for $m=1$.

In order to understand the higher dimensional analog of this situation, a simple observation is needed. If all the fibers of f are isomorphic then X

is essentially the direct product of F and Y, hence $f_*\omega^m_{X/Y}$ should be trivial. Similarly, if Y is covered by a family of curves such that the fibers of f are isomorphic along the curves, then $f_*\omega^m_{X/Y}$ should be trivial along these curves.

This situation can be analyzed by introducing the notion $\mathrm{Var}\, f=$ (number of effective parameters of the birational equivalence classes of fibers). Using this Viehweg [V3] formulated the following:

Conjecture. *Let f: $X\to Y$ be a surjective map between smooth projective varieties and assume that* $\mathrm{Var}\, f=\dim Y$. *Then for some $m>0$ $f_*\omega^m_{X/Y}$ is big (big is an appropriate technical version of ampleness).*

He proved that this conjecture is stronger than the Iitaka conjecture. Its advantage is that it is more amenable to various reductions that change Y and X. Using these the main result of this article can be formulated as follows.

Theorem. *Let f: $X\to Y$ be a surjective map between smooth projective varieties of characteristic zero and assume that the generic fibre F is of general type. Then Viehweg's conjecture, and consequently Iitaka's conjecture are true.*

Viehweg himself proved this statement under the additional assumption that some multiple of the canonical class of F is base point free ([V2], [V3]). This solves the problem completely if $\dim F\leqslant 2$. The same line of argument works if the canonical ring of F happens to be finitely generated; certain technical problems make this case much harder (see Kawamata [Ka3]).

It seems to be worthwhile to compare Viehweg's argument with the one in this paper.

The first problem is to make sense of the nebulous definition of $\mathrm{Var}\, f$ given above. In his case this is obtained for free, during the proof. However in general a preliminary analysis is required; this is done in Chapter II.

The second step is to reduce the problem to the special case when $h^0(\omega_F)$ is "large". This is done in Chapter III, closely following the original argument of Viehweg.

For the next step he uses an argument developed by Fujita [F] and Kawamata [Ka1] [Ka2]. Let f: $X^n\to Y^k$ be the given map, $Y^0\subset Y$ an open subset above which f is smooth, and let f^0: $X^0\to Y^0$ be the induced map. Let F_y be the fiber above $y\in Y^0$. Then via Hodge theory $H^0(F_y, \omega_{F_y})\subset H^{n-k}(F_y, C)$, and this defines an inclusion $f^0_*\omega_{X^0/Y^0}\hookrightarrow R^{n-k}f^0_*C\otimes\mathcal{O}_{Y^0}$. $R^{n-k}f^0_*C$ carries a variation of Hodge structures and this induces a metric on the vector bundle $V^0=f^0_*\omega_{X^0/Y^0}$. The curvature of V^0 is always non negative, and in fact positive if the local Torelli problem holds for the fibers, i.e. F can be reconstructed (at least locally) from the datum $H^0(F, \omega_F) \subset H^{n-k}(F, C)$. This step is a compromise: the original problem is about birational properties of the fibers, and the Torelli problem is a biregular question, which depends on the birational model chosen. Therefore this method works only if a good pick of the birational model of the fibers is possible; this accounts for the extra assumptions in [V3] and [Ka3].

Under suitable additional assumptions $V=f_*\omega_{X/Y}$ is a vector bundle, and one would like to have information about V instead of V^0. Although the Hodge metric degenerates near $Y-Y^0$, it acquires only mild singularities, and therefore one can hope that the expected integrals in the curvature of V^0 compute the Chern classes of V. If dim $Y=1$ this is relatively easy. A proof of the general case is claimed in [Ka2, Theorem 3]. Unfortunately, the proof is incorrect. The problem is in the last two lines of the proof on p. 6. Indeed, if

$$g=\log[(-\log|z_1|)^{2m}+(-\log|z_2|)^{2m}-2(-\log|z_1|)^m(-\log|z_2|)^m +(-\log|z_1|)+(-\log|z_2|)],$$

then the coefficient of $dz_1\wedge dz_2\wedge d\bar{z}_2$ in $\partial g\wedge\bar{\partial}\partial g$ is $(z_1z_2\bar{z}_2)^{-1}\cdot f(z_1, z_2)$ for some f and direct computation shows that $f(z, z)$ is asymptotically $(m^2/4)(-\log|z|)^{2m-4}$ as $z\to 0$, and for $m\geqslant 2$ its limit is not zero.

Fortunately recent results of Cattani-Kaplan-Schmid [C-K-S, 5.30] about the asymptotic behavior of Hodge metrics furnish the required result. This was observed independently by Kawamata as well. A similar result, needed for the present proof, is discussed in Chapter V.

The approach via Torelli is replaced by a different one here. The idea is particularly simple if dim $Y=1$. First consider $V=f_*\omega_{X/Y}$; this is a vector bundle over Y. If deg $V>0$ then the former methods work well. If deg $V=0$, then V inherits from the above-mentioned variation of Hodge structures a Hermitian metric which is flat. Now consider im$[V^{\otimes m}\to f_*\omega^m_{X/Y}]$. If this has positive degree, then again the former methods work. It cannot have negative degree, and if its degree is zero, then $K\subset V^{\otimes m}$, the kernel of this map, is a flat subbundle. If $y\in Y$ is general, then $V_y=H^0(F_y, \omega_{F_y})$ are the linear functions on a projective space where the canonical image of F_y lies. Then K_y is the set of degree m equations satisfied by the canonical image. If K is flat, then this implies that the degree m equations of the canonical images do not depend on $y\in Y$. If one assumes that the canonical maps of the fibers are birational and then one chooses m to be large, this implies that the fibers are birational, which proves the theorem in this case.

For higher dimensional Y some more information is needed about the local properties of variations of Hodge structures; this is obtained in Chapter IV. The proof is finally completed in Chapter VI.

Acknowledgment. A large part of these results were obtained while I stayed at Nagoya University with the financial support of the Japan Association for Mathematical Sciences. I am grateful to the numerous Japanese mathematicians who made my stay useful and enjoyable.

During the preparation of this article I received partial support from the NSF.

Notation.

(i) With the exception of Chapter II the base field will always be assumed to be $\boldsymbol{C}$.

(ii) If X is a smooth projective variety the map given by $|mK_X|$ is called the m-canonical map, its image the m-canonical image. These are denoted by ϕ_{mX} and $\phi_m(X)$ respectively. For $m=1$ we usually omit 1 (i.e. $\phi(X)$).

(iii) If m is sufficiently large and divisible, then up to birational equivalence $\phi_m(X)$ and ϕ_m are independent of m. Any representative of them is called the Iitaka variety (denoted by $I(X)$) and the stable canonical map (or Iitaka fibration).

(iv) An algebraic fiber space is a surjective map between varieties with connected generic fiber.

(v) The symbol $\approx$ will denote birational isomorphism. The symbol $\cong$ will denote isomorphism.

(vi) A sheaf $\mathscr{F}$ over a scheme X is said to be generically generated by global sections if the natural map $\mathcal{O}_X \otimes H^0(X, \mathscr{F}) \to \mathscr{F}$ is generically surjective.

(vii) A sheaf $\mathscr{F}$ on a variety X is said to be weakly positive (w.p. for short) if for every ample line bundle H and every $a>0$, there exists $b>0$ such that $(S^b(H \otimes S^a(\mathscr{F})))^{**}$ is generically generated by global sections. (Here $**$ denotes double dual.) This definition is slightly different from [V2, 1.2] since $\mathscr{F}$ is not assumed to be torsion-free. Clearly $\mathscr{F}$ is w.p. iff $\mathscr{F}$/torsion is w.p.

(viii) A sheaf $\mathscr{F}$ on a variety X is called big if for every ample H there is an $a>0$ such that $S^a(\mathscr{F}) \otimes H^{-1}$ is w.p. The basic properties of w.p. sheaves are listed in [V2, 1.4]. The same properties hold for big sheaves too.

(ix) The end (or lack) of a proof is denoted by $\square$.

II. Generic moduli

Let $f: X \to Y$ be a map whose fibers are reduced and irreducible. Then one can introduce an equivalence relation on the closed points of Y by setting $p, q \in Y$ equivalent iff $f^{-1}(p)$ and $f^{-1}(q)$ are birationally isomorphic. This equivalence relation is given by a subset $\mathrm{Br}\, E \subset Y \times Y$ such that $(p, q) \in \mathrm{Br}\, E$ iff p and q are equivalent. The aim of this section is to analyze this equivalence relation. We are especially interested in endowing the set $Y/\mathrm{Br}\, E$ with a "natural" scheme structure. In general this is a very difficult problem. However, it turns out, that by considering a suitable open set $Y_0 \subset Y$ the quotient $Y_0/\mathrm{Br}\, E$ behaves much better and this approach will be pursued in the sequel.

For the rest of this section the ground field will be assumed to be uncountable. This assumption is needed bacause we use the naive definition of equivalence relation instead of the scheme theoretic one. Another possibility would be to use Weil's universal domain, which is the approach adopted by Matsusaka [Ma1]. The reader should be able to translate the arguments to either approach without difficulty.

The techniques and ideas of this section are taken from various works of Matsusaka, especially [Ma1] and [Ma2]. The cases he considers overlap very little with the ones that will be studied here; therefore I will give complete proofs.

Definition 2.1. (i) A subset $E \subset Y \times Y$ is called an algebraic equivalence relation if E is an equivalence relation on the closed points of Y and E is a locally closed subscheme of $Y \times Y$.

(ii) $E \subset Y \times Y$ is called a pro-algebraic equivalence relation if it is an equivalence relation on the closed points of Y and E is a countable union of locally closed subschemes of $Y \times Y$.

(iii) a (pro)-algebraic equivalence relation E is called closed if E is a (countable union of) closed subscheme(s) of $Y \times Y$.

Definition 2.2. Let $f: X \to Y$ be a map between algebraic varieties whose fibers are reduced and irreducible. Let $p_i: Y \times Y \to Y$ be the coordinate projections. Define $\mathrm{Br}\, E(f, X, Y)$ (or $\mathrm{Br}\, E$ if no confusion is likely) to be the set of points $z \in Y \times Y$ s.t. $z \times_{p_1} X$ and $z \times_{p_2} X$ are birational (over $k(z)$).

Proposition 2.3. (i) *Let $f: X \to Y$ be as above and assume that f is projective. Then* $\mathrm{Br}\, E(f, X, Y)$ *is a pro-algebraic equivalence relation. On the closed points it agrees with the definition given at the beginning of the section.*

(ii) *If furthermore f is smooth and none of its fibers are birationally ruled, then* $\mathrm{Br}\, E(f, X, Y)$ *is closed.*

Proof. (i) The only part which is not clear is that $\mathrm{Br}\, E$ is pro-algebraic.

If $g\colon U\to V$ is a birational isomorphism of n-dimensional projective varieties then consider its closed graph $\Gamma(g)\subset U\times V$. This $\Gamma(g)$ is an n-dimensional cycle, the intersection numbers $\Gamma(g)\cdot p_U^{-1}(pt)$ and $\Gamma(g)\cdot p_V^{-1}(pt)$ are 1, and $\Gamma(g)$ is irreducible. Conversely, any such cycle is the graph of a birational map.

Now consider $f\times f\colon X\times X\to Y\times Y$ and let $H(f, X, Y)\subset \mathrm{Hilb}(X\times X/Y\times Y)$ be the subscheme of relative ($\dim X-\dim Y$)-cycles that satisfy the above conditions fiberwise. The conditions are algebraic; therefore $H(f, X, Y)$ is a countable union of algebraic varieties. One has a natural map $H(f, X, Y)\to Y\times Y$ and clearly $\mathrm{Br}\,E$ is the set-theoretic image of this map. If $H_i\subset H(f, X, Y)$ is an irreducible component, then the image of H_i in $Y\times Y$ is constructible, hence a union of locally closed subschemes. This proves that $\mathrm{Br}\,E$ is pro-algebraic.

(ii) Let $\mathrm{Br}\,E=\bigcup E_i$. We have to prove that $\bar{E}_i\subset \mathrm{Br}\,E$ for every i. It is sufficient to check this condition after a base change $g\colon T\to Y$ where T is a spectrum of a DVR. This case is nothing else but [M-M, Theorem 1]. □

Proposition 2.4. *Let $E\subset Y\times Y$ be an algebraic equivalence relation. Then there exist an open $Y_0\subset Y$ and a map $g\colon Y_0\to Z$ onto an algebraic variety such that (at least set theoretically) the fibers of g are exactly the equivalence classes of $E_0=E\cap(Y_0\times Y_0)$.*

Proof. Let $\bar{Y}$ be a compactification of Y and let $\bar{E}\subset \bar{Y}\times Y$ be the closure of E and $p\colon \bar{E}\to Y$ the projection map onto the second factor. Let $Y'\subset Y$ be the open set above which p is flat. Let $g'\colon Y'\to \mathrm{Hilb}(\bar{Y})$ be the map that sends $y\in Y'$ to the Hilbert point of $p^{-1}(y)$, and let Z' be the image of Y'. Each fiber of g' contains an open dense subset which is an equivalence class of $E\cap(Y'\times Y')$. To get Y_0 first we pick a $Y''\subset Y'$ such that $g'|Y''$ is equidimensional. Now we have to get rid of the equivalence classes of smaller than expected dimension. These form a locally closed subset D of Y'' and let $Y_0=Y''-\bar{D}$, $g=g'|Y_0$, $Z=g(Y_0)$. These choices clearly satisfy the statement of the proposition. □

Theorem 2.5 (Generic moduli for varieties of general type). *Let $f\colon X\to Y$ be a smooth projective map whose fibers are varieties of general type. Then there exist an open subset $Y_0\subset Y$ and a map $g\colon Y_0\to Z$ such that X_u and X_v for $u, v\in Y_0$ are birational iff $g(u)=g(v)$.*

Proof. Let F be the generic fiber of f and choose k so that the k-canonical map given by $|kK_F|$ is birational. Let $Y'\subset Y$ be the open set such that $\dim|kK_{X_u}|=\dim|kK_F|=N$ holds for $u\in Y'$. The k-canonical images of X_u ($u\in Y'$) are closed subvarieties of $\boldsymbol{P}^N$, and for some $Y''\subset Y'$

they all have the same Hilbert-polynomial if $u \in Y''$. Let $S \subset \mathrm{Hilb}(\boldsymbol{P}^N)$ be the set of Hilbert points of the k-canonical images of X_u ($u \in Y''$). S is a locally closed subscheme and $G = \mathrm{PGL}(N+1)$ operates on S and this operation defines an algebraic equivalence relation $E \subset S \times S$. Let q: $S_0 \to Z$ be the quotient map of Proposition 2.4, $Y_0 = \{u \in Y''$: some k-canonical image of X_u is in $S_0\}$, and g: $Y_0 \to Z$ be the induced map. Let $u, v \in Y_0$. Then X_u and X_v are birational iff their k-canonical images are projectively equivalent. By the definition of q and g, this is the case iff $g(u) = g(v)$. This proves the theorem. □

The next result describes the generic structure of pro-algebraic equivalence relations. It is formulated for closed ones for simplicity. A similar statement holds in general.

Theorem 2.6. *Let $E \subset Y \times Y$ be a closed pro-algebraic equivalence relation. Then there exist an open set $Y_0 \subset Y$ and a surjective map g: $Y_0 \to Z$ with connected fibers such that for $E_0 = E \cap (Y_0 \times Y_0)$ the following statements hold*:

(i) *any equivalence class of E_0 is a union of fibers of g*;

(ii) *there are countably many proper closed subvarieties $Z_i \subset Z$ such that if $u \in Y_0$ and $g(u) \notin \bigcup Z_i$ then the equivalence class of E_0 that contains u is a countable union of fibers of g.*

Furthermore, g, viewed as a rational map of Y, is unique.

Proof. The equivalence class containing $u \in Y$ will be denoted by $E(u)$. Let $E = \bigcup E_i$ where the E_i are irreducible reduced closed subvarieties of $Y \times Y$. Then $E(u) = \bigcup E_i(u)$ where $E_i(u) = p_1(E_i \cap p_2^{-1}(u))$. A point $v \in E(u)$ will be called 1-fold if $v \in E_i(u) \cap E_j(u)$ implies that $E_i(u)$ contains $E_j(u)$ or vice versa. The notion of a 1-fold point depends only on the set $E(u)$. A property P is said to hold for very general points if it holds outside a countable union of closed proper subschemes.

One can assume that $E_i \subset E_j$ implies $i = j$. Then $V_{ij} = \{u \mid E_i(u) \subset E_j(u)\}$ is a proper closed subset of Y for $i \neq j$. Let $u \in Y - \bigcup_{i \neq j} Y_{ij}$. Then $E_i(u) \not\subset E_j(u)$ if $i \neq j$. Let $E_1(u) \ni u$. Then $E_1(u) \not\subset V_{ij}$; hence there is a 1-fold point $v \in E_1(u)$ such that $v \notin V_{ij}$ for any $i \neq j$. $E(v) = E(u)$ and $v \in E(v)$ is a 1-fold point. Since $v \notin V_{ij}$ for $i \neq j$, v is contained in exactly one of the $E_i(v)$'s.

Since E contains the diagonal of $Y \times Y$ it has a component, say E_1, that contains the diagonal. By the above remark only one component can contain (v, v), hence E_1 is the unique component containing the diagonal.

Let $\bar{Y}$ be a compactification of Y, $\bar{E}_1 \subset \bar{Y} \times Y$ the closure of E_1, and p: $\bar{E}_1 \to Y$ the second projection. $\bar{E}_1$ is irreducible and p has a section (the diagonal). Therefore the general fiber of p is irreducible. By passing to an open subset Y' of Y, we may assume that all fibers of p are irreducible and of the same dimension. Let $E_1'=E_1\cap(Y'\times Y')$.

Claim 2.7. With the above notation $E_1'\subset Y'\times Y'$ defines an algebraic equivalence relation.

Proof. E_1' is clearly an algebraic set which contains the diagonal. If t denotes the operation of interchanging the two copies of Y' then tE_1' is again a component of E'. It contains the diagonal, and therefore $tE_1'=E_1'$; i.e., E_1' is symmetric.

To see that E_1' is transitive, let $u\in Y-\bigcup_{i\neq j}V_{ij}$. Then $E_1'(u)\not\subset V_{ij}$ for $i\neq j$. Let $v\in E_1'(u)$ be a 1-fold point. Then $E_1'(v)\subset E_1'(u)$. By the choice of Y' they are both closed, irreducible and of the same dimension; hence $E_1'(v)=E_1'(u)$ for every $v\in E_1'(u)$. Now let $u_0\in Y$ and $v_0\in E_1'(u_0)$ be arbitrary. Pick a general one-parameter family $(u_t)\subset Y$, and $(v_t)\subset Y$ such that $v_t\in E_1'(u_t)$. For very general t, $E_1'(v_t)=E_1'(u_t)$ as we saw above. Hence the same holds for each t, and therefore $E_1'(v_0)=E_1'(u_0)$.

Now the transitivity is easy. Assume that $(u, t), (t, v)\in E_1'$. Then $(t, u)\in E_1'$ by symmetry, and hence $t\in E_1'(u)$ and $t\in E_1'(v)$. Therefore $E_1'(u)=E_1'(t)=E_1'(v)$. So $u\in E_1'(v)$. This proves the claim. □

Now one can apply Proposition 2.4 for $E_1'\subset Y'\times Y'$ to obtain $Y_0\subset Y$, $E_{10}=E_1\cap(Y_0\times Y_0)$ and $g\colon Y_0\to Z$ such that the fibers of g are exactly the E_{10} equivalence classes of Y_0.

Let $E_0=E\cap(Y_0\times Y_0)$. If $v\in E_0(u)$, then $E_{10}(v)\subset E_0(u)$ and therefore $E_0(u)$ is the union of certain fibers of g; this proves (i).

(ii) follows once we establish that for very general u, $\dim E_i'(u)\leqslant \dim E_1'(u)$. Then, since $E'(u)$ has countably many components, it is a union of fibers of g and of dimension at most the dimension of the fibers. Therefore each component is a fiber, and $E'(u)$ is a union of countably many fibers.

Assume that $\dim E_i'(u)>\dim E_1'(u)$ for some i and general u. If $v\in E_i'(u)$, then $u\in tE_i'(v)$. Since $\dim tE_i'=\dim E_i'>\dim E_1'$, we have that $\dim tE_i'(v)>\dim E_1'(u)$. This contradicts the fact that for very general u, $E_1'(u)$ is the only component of $E'(u)$ containing u, and therefore (ii) is proved.

The uniqueness of g is clear, and this completes the proof. □

Definition 2.8. Let $f\colon X\to Y$ be a smooth projective map and assume for simplicity that none of the fibers are ruled. Let $E=\mathrm{Br}\,E(f, X, Y)$ and let g and Z be as in 2.6. Then

(i) $\overline{k(Z)}$, or more precisely $g^*\overline{k(Z)}\subset\overline{k(Y)}$ is called the minimal closed field of definition of X/Y (or of its generic fiber). This notion goes back to Matsusaka [Ma2] and Shimura [Sh].

(ii) dim Z will be called the variation of f and it will be denoted by Var f. The equivalence of this definition and that of Viehweg [V2] will be established shortly.

Corollary 2.9. *Let $f: X \to Y$ as in 2.8 be defined over* $\mathbf{C}$ *and assume that* Var $f = \dim Y$. *If $u \in Y$ is a very general point and $g: \Delta \to Y$ is an analytic arc through u then not all fibers of f over $g(\Delta)$ are birational.*

Proof. By 2.6 there are only countably many $v \in Y$ such that X_v is birational to X_u. □

Remark 2.10. I do not know any example where Br E is actually non-algebraic. No such example seems to exist for dim $X/Y \leqslant 2$.

In general, unfortunately, the family X/Y will not descend to a family over Z. This is however nearly true, as shown by the following:

Theorem 2.11. *Let $f: X \to Y$ be as in* 2.8. *Then there exist a smooth projective map $q: V \to U$ such that* $\dim U = \mathrm{Var}\, q = \dim Z$, *a variety R, a generically finite and surjective map $b: R \to Y$ and a surjective map $c: R \to U$ such that $R \times_Y X$ and $R \times_U V$ are birationally isomorphic over R (i.e., the birational isomorphism respects the projections onto R).*

Proof. One can pick an $i: U \to Y$ such that $g \cdot i: U \to Z$ is generically finite and surjective. Let $V = X \times_Y U$, $q: V \to U$ the projection. $q: V \to U$ clearly satisfies the requirements.

Let $p_i: E_1 \to Y$ be the projections and let $W = E_1 \times_Y U$ where we consider E_1/Y via the second projection. The natural map $p_1: W \to Y$ is generically finite and surjective. Over W we get two families: $W \times_Y X$ and $W \times_U V$ where $r: W \to U$ is the natural map. By construction, for general $w \in W$ the fibers of these two families over w are birationally isomorphic.

Let $H \subset \mathrm{Hilb}((W \times_Y X) \times_W (W \times_U V)/W)$ be the subset of the relative Hilbert scheme parametrizing birational isomorphisms between fibers of $W \times_Y X/W$ and $W \times_U V/W$. As in the proof of 2.3 one can see that H is a countable union of subschemes and by the above observations it has a component H_1 which maps generically surjectively onto W. Choose $s: R \to H_1$ such that the resulting map $R \to W$ is generically finite and surjective. Let $b: R \to W \to Y$ and $c: R \to W \to U$ be the composite maps. Then (id, s): $R \to R \times_W H_1$ gives a section of $\mathrm{Hilb}((R \times_Y X) \times_R (R \times_U V)/R)$ and the corresponding cycle is a graph of a birational isomorphism between $R \times_Y X$ and $R \times_U V$. This proves the theorem. □

III. The hard covering trick

The aim of this section is to prove the Hard Covering Trick which will be used in the final proof to reduce the problem to the study of another fiber space whose fibers are much better behaved. This trick was first introduced by Viehweg [V3]. Unfortunately his statement and proof are buried in Section 4 of his paper, and he does not treat it in the generality that is needed in the subsequent applications. Therefore a complete proof will be presented here.

Definition 3.1. [V1] (i) Let L be a line bundle and $L^N \cong \mathcal{O}(\sum v_j E_j)$ for some $N>0$. Define

$$L^{(i)} = L^i \otimes \mathcal{O}\left(-\sum \left[\frac{iv_j}{N}\right] E_j\right)$$

where $[x]$ denotes the integral part of a real number x. It is important to note that $L^{(i)}$ depends not only on L and i but also on the section of L^N given by $1 \in \Gamma(\mathcal{O}(\sum v_j E_j))$. However this will not lead to any confusion.

(ii) Let L be a line bundle on Y and let $s: \mathcal{O} \to L^N$ be a section. This defines an algebra structure on $A = \sum_0^{N-1} L^{-i}$ via $s^{-1}: L^{-N} \to \mathcal{O}$. We have a natural map $\tau': Y' = \mathrm{Spec}_Y A \to Y$. Y' is called the (cyclic) covering obtained by extracting the N-th root of s.

Lemma 3.2 (Viehweg, [V1, 1.4]). *Using the notation of* 3.1 *assume furthermore that Y is smooth and that* $\mathrm{div}(s) = \sum v_j E_j$ *is a normal crossing divisor. Let $\tau: Y'' \to Y$ be the normalization of Y' and $d: V \to Y''$ a resolution of singularities. Then*

(i) $(\tau \circ d)_* \omega_V = \tau_* \omega_{Y''} = \Sigma_0^{N-1} \omega_Y \otimes L^{(i)}$; *and*

(ii) $(\tau \circ d)_* \mathcal{O}_V = \tau_* \mathcal{O}_{Y''} = \Sigma_0^{N-1} (L^{(i)})^{-1}$;

(iii) [V3, 1.1] *There is a natural inclusion*

$$(\tau \circ d)_* \omega_V^k \to \Sigma_0^{N-1} \omega_Y^k \otimes L^{k \cdot N - k - N + 1 + i}$$

which is an isomorphism outside the singular locus of $\sum v_j E_j$. □

We record the following simple statement for future reference.

Lemma 3.3. *Let $f: X \to Y$ be projective, X smooth and let $g: X' \to X$ be a proper birational map, X' again smooth. Then* $(fg)_* \omega_{X'}^k = f_* \omega_X^k$. □

The following is the basic set-up to be considered in this section:

Construction 3.4 (Viehweg, [V3, §4]). Let X, Y be smooth quasiprojective varieties, $f: X \to Y$ a projective map. Let $V \subset f_*(\omega_{X/Y}^N)$ be a weakly positive locally free subsheaf.

Let $p\colon P=\boldsymbol{P}_Y(\mathrm{Sym}\ V^*)\to Y$ be the associated projective bundle, $\mathcal{O}(1)$ the tautological bundle (in particular $p_*\mathcal{O}(1)=V^*$). Let $X_1=X\times_Y P$ and $f_1\colon X_1\to P$ be the second projection.

Note 1. $f_{1*}(\omega^N_{X_1/P})\otimes\mathcal{O}(1)$ has a distinguished section.

Proof. Indeed, taking p_* we get $f_*(\omega^N_{X/Y})\otimes V^*=\mathcal{H}om(V, f_*(\omega^N_{X/Y}))$ and the inclusion gives a global section. □

Note 2. There is a map $q\colon S\to P$ finite and surjective such that $q^*\mathcal{O}(1)\cong K^N$ for some K. Moreover, we can assume that S/P is a cyclic covering with a smooth branch locus in general position.

Proof. Let H be a line bundle on P such that $H^N\otimes\mathcal{O}(1)$ is very ample. Let s be a general section and let $q\colon S\to P$ be the extraction of N-th root of s. □

Now we continue the construction. Let $X_2=X_1\times_P S$, $f_2\colon X_2\to S$ the natural map. With a suitable choice of S, X_2 will be smooth. Assume that such a choice of S was made. By flat base change we have $q^*(f_{1*}\omega^k_{X_1/P})\cong f_{2*}(\omega^k_{X_2/S})$. The distinguished section of Note 1 therefore gives a section of $f_{2*}(\omega^N_{X_2/S})\otimes K^N$ and therefore a global section of $(\omega_{X_2/S}\otimes K)^N$.

Applying embedded resolution to this section one obtains an $X_3\to X_2$ and a distinguished section s in $(\omega_{X_3/S}\otimes K)^N$ such that $\operatorname{div}(s)$ is a normal crossing divisor (use 3.3 to do the proper identifications).

Now we extract an N-th root of this section and let X_4 be a resolution of the resulting cyclic cover. Let $f_4\colon X_4\to S$ and $d\colon X_4\to X_3$ be the natural maps.

This can be summarized in the following diagram:

$$\begin{array}{ccccccccc}
X_4 & \xrightarrow{d} & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X \\
\downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
S & = & S & = & S & \xrightarrow{q} & P & \xrightarrow{p} & Y
\end{array}$$

The following is the main result of this section:

Theorem 3.5. *Notation as in* 3.4. *If* $f_{4*}\omega^k_{X_4/Y}$ *is big for some* $k>0$ *then* $\det f_*\omega^m_{X/Y}$ *is big for some* $m>0$.

Remark 3.6. There is an unfortunate asymmetry in the statement of the theorem. This has a purely technical reason—namely that $\det f_*{}^k_{X/Y}$ can be big by "accident" (cf. [V2, 5.2]). In the actual applications this will not cause any problem.

Proof. By 3.2 (iii) there is a natural inclusion

$$d_*\omega^k_{X_4/S}\longrightarrow\sum_0^{N-1}\omega^k_{X_3/S}\otimes(\omega_{X_3/S}\otimes f_3^*K)^{kN-k-N+1+i}$$

which is generically an isomorphism. Taking f_{3*} we get

$$\begin{aligned}f_{4*}\omega^k_{X_4/S}&\longrightarrow\sum_0^{N-1}K^{kN-k-N+1+i}\otimes f_{3*}(\omega^{kN-N+1+i}_{X_3/S})\\&=\sum_0^{N-1}K^{kN-k-N+1+i}\otimes f_{2*}(\omega^{kN-N+1+i}_{X_2/S}),\end{aligned}$$

(the latter equality by 3.3). This map is injective. Fix an i such that the image of $f_{4*}\omega^k_{X_4/S}$ has a nonzero projection into the i-th summand. Let $c=kN-k-N+1+i$, and let F be the image of $f_{4*}\omega^k_{X_4/S}$ in $K^c\otimes f_{2*}(\omega^{c+k}_{X_2/S})$. This gives an exact sequence

$$0\longrightarrow F\otimes K^{-c}\longrightarrow f_{2*}(\omega^{c+k}_{X_2/S})\longrightarrow Q\longrightarrow 0.$$

We can take the first Chern class, and obtain

$$\det f_{2*}(\omega^{c+k}_{X_2/S})=\det F\otimes\det(Q/\text{torsion})\otimes K^{-c\cdot\mathrm{rk}F}\otimes\mathcal{O}(c_1(\text{torsion})).$$

This makes sense over the open set where the occuring sheaves are locally free. The complement of this set has codimension at least 2.

F is big since it is a quotient of the big sheaf $f_{4*}\omega^k_{X_4/S}$; Q/torsion is weakly positive since it is a quotient of $f_{2*}\omega^{k+c}_{X_2/S}$; $\mathcal{O}(c_1(\text{torsion}))$ is weakly positive since c_1 (torsion) is effective. Therefore we get $K^{c\cdot\mathrm{rk}F}\otimes\det f_{2*}\omega^{c+k}_{X_2/S}=M$ (outside a codimension 2 set) where M is a big rank one sheaf over S.

Let H_1 be an ample line bundle on Y, and let $H=q^*p^*H_1$. For some $e>0$ we have that $M^e\otimes H^{-1}$ is big, and one can even assume that N divides e. Setting $g\cdot N=c\cdot\mathrm{rk}\,F\cdot e$ we obtain that

$$(q^*\mathcal{O}(1))^g\otimes(\det f_{2*}\omega^{c+k}_{X_2/S})^e\otimes H^{-1}$$

is big. This being the pull-back of

$$\mathcal{O}(g)\otimes(\det f_{1*}\omega^{c+k}_{X_1/P})^e\otimes(p^*H_1)^{-1},$$

one can conclude that this latter sheaf is big as well. In particular, for some $s>0$ the double dual of

$$\mathcal{O}(sg)\otimes(\det f_{1*}\omega^{c+k}_{X_1/P})^{es}\otimes(p^*H_1)^{-s}$$

has a section.

Applying p_* to this sheaf we get that

$$\begin{aligned}&S^{sg}(V^*)\otimes((\det f_*\omega^{c+k}_{X/Y})^{es})^{**}\otimes H^{-s}\\&\qquad=\mathcal{H}om\,(S^{sg}(V)\otimes H^s,\,((\det f_*\omega^{c+k}_{X/Y})^{es})^{**})\end{aligned}$$

has a section too. Therefore one has a non-trivial map

$$S^{sg}(V)\otimes H^{s}\longrightarrow((\det f_{*}\omega_{X/Y}^{c+k})^{es})^{**}.$$

By assumption V is weakly positive, and so is $S^{sg}(V)$; hence $S^{sg}(V)\otimes H^{s}$ is big. This implies that $\det f_{*}\omega_{X/Y}^{c+k}$ is also big. This is what we wanted to prove. □

For Theorem 3.5 to be of some use the fiber space X_4/S should be easier to handle than X/Y. This is indeed the case and the key to this phenomenon is the study of the fibers of X_4/S. This will be done in the rest of this section.

Proposition 3.7. *Notation as in* 3.4. *There is an open set* $S_0\subset S$ *such that the fibers of* f_4 *over* S_0 *are obtained as follows*:

Let $s\in S_0$, $t=q(s)$ *and* $y=p(t)$. *Then* t *corresponds to a* 1-*dimensional vector subspace of* V_y, *and therefore to an element of the linear system* $|N K_{X_y}|$. *Extracting an N-th root of this section of* $\omega_{X_y}^{N}$, *one obtains a variety which is birational to* $f_4^{-1}(s)$.

Proof. Straightforward from the construction. □

Theorem 3.8. *Let* X' *be a smooth projective variety,* $h\in H^0(X',\omega_{X'}^{N})$ *a section and let* Y' *be a smooth model of the covering obtained by extracting N-th root of h. Then, there exists n such that for every* $N\gg 0$ *divisible by n and for every sufficiently general h the following conditions hold*:

(i) $H^0(Y',\omega_{Y'})$ *gives the stable canonical map*;

(ii) $I(Y')$ *is of general type.*

Proof. All this really makes sense only if $\kappa(X')\geqslant 0$, so we shall assume this. We pick a j such that the j-canonical map of X' is stable and let ϕ_j: $X'\to Z'\subset \boldsymbol{P}^{?}$ be the closed image. We may assume that ϕ_j is in fact a morphism.

For a given N and h we consider the following diagram (note that $\dim Z'=\dim I(Y')$, cf. [U2, 1.8]):

$$\begin{array}{ccc} X' & \longrightarrow & Z' \\ \uparrow & & \uparrow \\ Y' & \longrightarrow & I(Y') \end{array}$$

First we blow up $\sigma\colon Z\to Z'$ such that Z is smooth and the branching divisor of the field extension $k(I(Y'))\supset k(Z)$ is a divisor with normal crossing on Z. Let L be the pull-back of $\mathcal{O}(1)$ to Z. Now choose a blow-up X of X' such that X is smooth, it dominates Z and the branching divisor of the field extension $k(Y')\supset k(X)$ is a divisor with normal crossings. Let Y be the normalization of X in $k(Y')$ and U be the normalization of Z in $k(I(Y'))$. Both Y and U have rational singularities. The morphisms are named according to the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle p}\uparrow & & \uparrow{\scriptstyle q} \\ Y & \xrightarrow{g} & U \end{array}$$

We remark that there is a natural map $f^*L\to\omega_X^j$ (since X dominates Z') and that Y is the normalization of the covering of X obtained from $h\in H^0(X', \omega_{X'}^N)=H^0(X, \omega_X^N)$. Since g has connected fibers, $\mathcal{O}_U=g_*\mathcal{O}_Y$.

First we prove that U is of general type.

$$\begin{aligned} q_*\omega_U &= \mathcal{H}om(q_*\mathcal{O}_U, \omega_Z)=\mathcal{H}om(f_*p_*\mathcal{O}_U, \omega_Z) \\ &= \mathcal{H}om(f_*\textstyle\sum_0^{N-1}(\omega_X^{(i)})^{-1}, \omega_Z) \\ &= \textstyle\sum_0^{N-1}\mathcal{H}om(f_*(\omega_X^{(i)})^{-1}, \omega_Z). \end{aligned}$$

(We used 3.2, (ii) for the covering $p\colon Y\to X$).

Claim 3.9. If $js|N$ then the natural map $f^*L^s\to\omega_X^{js}$ factors through $f^*L^s\to\omega_X^{(js)}\to\omega_X^{js}$.

Proof. Since f^*L is generated by global sections this follows once we prove that $H^0(\omega_X^{(j)})=H^0(\omega_X^j)$. Let $N=kj$.

Let $A+\sum b_iE_i$ be a generic divisor in $|jK_X|$, $\sum b_iE_i$ the fixed part. Then $kA+\sum kb_iE_i\in|NK_X|$. If $\operatorname{div}(h)=M+\sum a_iE_i$, then $a_i\leqslant k\cdot b_i$ since h is general. Therefore

$$\omega_X^{(j)}=\omega_X^j\otimes\mathcal{O}\left(-\sum\left[\frac{ja_i}{N}\right]E_i\right)\supset\omega_X^j\otimes\mathcal{O}(-\sum b_iE_i)=\mathcal{O}(A).$$

This proves the claim for $s=1$. The proof for $s>1$ is the same. □

The inclusion $f^*L^s\to\omega_X^{(j\cdot s)}$ gives $(\omega_X^{(j\cdot s)})^{-1}\to f^*L^{-s}$, hence a map $f_*(\omega_X^{(js)})^{-1}\to L^{-s}$. Therefore we have an injection $\omega_Z\otimes L^s\to\mathcal{H}om(f_*(\omega_X^{js})^{-1}, \omega_Z)$.

$H^0(Z, \omega_Z\otimes L^s)=H^0(Z', \mathcal{O}(s)\otimes\sigma_*\omega_Z)$ and $\sigma_*\omega_Z$ is independent of the Z chosen. There exists an s_0, depending on Z' only, such that sections of $\mathcal{O}(s)\otimes\sigma_*\omega_Z$ separate points over an open set. These sections lift to sections of ω_U so U is of general type.

In order to prove (i) we have to look at the situation in more detail.

Lemma 3.10. *Let V be a smooth projective variety with $\kappa(V)=0$. Let $m=m(V)=\text{g.c.d}\{s:|sK_V|\neq\phi\}$. Then $|mK_V|\neq\phi$. Furthermore the normalized cyclic cover defined by (the unique divisor of) $|s\cdot mK_V|$ is the disjoint union of s copies of the normalized cyclic cover defined by $|mK_V|$.*

Proof. Let $s_1K_V\sim\sum a_iA_i$ and $s_2K_V\sim\sum b_jB_j$. Then $s_2\sum a_iA_i=s_1\sum b_jB_j$ since $|s_1s_2K_V|$ is zero-dimensional. If $s_1>s_2$ then this gives that $(s_1-s_2)K_V\sim\sum a_iA_i-\sum b_jB_j$ and the r.h.s. is effective. This proves the first claim.

The second claim follows from the easy general fact: if $h\in H^0(L^N)$, then the normalized kN-th root obtained from $h^k\in H^0(L^{kN})$ is the disjoint union of k-copies of the normalized N-th root obtained from h. □

Claim 3.11. Let V be a general fiber of f, $m=m(V)$. Then the degree of $g\colon U\to Z$ is Nm^{-1}.

Proof. If N is not divisible by m then $H^0(\omega_X^N)=0$, hence Nm^{-1} is an integer in all meaningful cases.

Let $z\in Z$, $V=f^{-1}(z)$, $W=p^{-1}V$. Then W is the normalized covering obtained from $h|V\in H^0(V,\omega_V^N)$. By 3.10, W has Nm^{-1} connected components, and these are exactly the fibers of g over the points in $q^{-1}(z)$. This proves the claim. □

Now consider the sections of ω_Y. $p_*\omega_Y=\sum_0^{N-1}\omega_X\otimes\omega_X^{(i)}$ and as in 3.9 one can easily obtain factorizations

$$f^*L^s\to\omega_X\otimes\omega_X^{(sj-1)}\to\omega_X^{sj}\qquad\text{if } sj \text{ divides } N.$$

In particular we have an inclusion $L\to f_*p_*\omega_X$, and therefore S, the image of the canonical map of Y, dominates Z'. Since we have rational maps $U\to S\to Z$ in order to prove that S and U are birational it is sufficient to prove that $\deg S/Z=Nm^{-1}$.

On Y we have a $\boldsymbol{Z}_N$ action coming from the cyclic covering structure. Let g be a generator of this action. Let $y_1\in Y$ and $y_t=g^ty_1$. If $f_i\in H^0(\omega_X^{(a(i))}\otimes\omega_X)$, then f_i can be viewed as a section of ω_Y, and one can compare f_1/f_2 at y_1 and y_t. It is easy to see that there is a primitive N-th root of unity ε such that $(f_1/f_2)(y_t)=\varepsilon^{t(a(1)-a(2))}\cdot(f_1/f_2)(y_1)$.

Let $x\in X$ be a general point and let $p^{-1}(x)=\{y_1,\cdots,y_N\}$. If $\phi\colon Y\to S$ is the canonical map then $\phi(y_1),\cdots,\phi(y_N)\in S$ all have the same image in Z, namely $f(x)$. Therefore we are done if we can prove that there are at least Nm^{-1} different points among the $\phi(y_i)$'s.

f_1/f_2 is a coordinate function of S and $(f_1/f_2)(y_1)\neq 0$ for general x if f_1 and f_2 are not zero. If $a(1)-a(2)=m$, then among the numbers $\varepsilon^{t(a(1)-a(2))}\cdot(f_1/f_2)(y_1)$ there are exactly Nm^{-1} different ones for $t=1,\cdots,N$, and therefore there are at least Nm^{-1} different ones among the points $\phi(y_t)$.

Now we pick $a(1)=sj+m-1$, $a(2)=sj-1$. As we saw we have an inclusion $L^s \to f_*(\omega_X \otimes \omega_X^{(a(2))})$ and so we can find f_2 for any $s>0$.

As in the proof of 3.9, we get an isomorphism $H^0(X, \omega_X \otimes \omega_X^{(a(1))}) \cong H^0(X, \omega_X^{sj+m})$ if $sj+m-1$ divides N. Since $H^0(X, \omega_X^k)=H^0(X', \omega_{X'}^k)$ the non-vanishing of this plurigenus does not depend on the particular birational model X.

We have an inclusion $L^s \otimes f_*\omega_X^m \to f_*\omega_X^{sj+m}$. The fiber of $f_*\omega_X^m$ at z is $H^0(V, \omega_V^m)=\boldsymbol{C}$ by 3.10 and therefore $f_*\omega_X^m$ is not zero. Thus for large $s=s_1$, $H^0(L^{s_1} \otimes f_*\omega_X^m)$ and hence $H^0(X, \omega_X \otimes \omega_X^{(a(1))})$ are not zero. We emphasize again that the choice of s_1 depends only on X' and not on N or on $h \in H^0(X', \omega_{X'}^N)$.

Summarizing the results: with this choice of s_0, s_1 and j if $n=ms_0s_1 \cdot (s_1 j+m-1)$ divides N and h is sufficiently general, then Y' satisfies the conditions (i) and (ii). This was to be proved. □

IV. Local study of $f_*\omega_{X/Y}$

During the past twenty years it was gradually understood that many important properties of the sheaves $f_*\omega_{X/Y}$ can be derived using the connection between these sheaves and certain variations of Hodge structures. It seems to be more convenient to consider the problems in the context of an arbitrary variation of Hodge structure (VHS for short). The aim of this chapter is to analyze certain properties of VHS's. The geometric applications will be left to subsequent sections.

Definition 4.1. The definition of a VHS will not be reproduced here. Chapters 2 and 7 of [Sch] contain a good summary of the necessary results. We recall the notation.

Let X be a complex manifold and let H be a local system of $\boldsymbol{R}$-vector spaces. Let $\mathscr{H}=H \otimes \mathcal{O}_X$, a vector bundle, $\mathscr{H}=\mathscr{F}^0 \supset \mathscr{F}^1 \supset \cdots \supset \mathscr{F}^n \supset 0$ the Hodge filtration with vector subbundles. n will be called the weight of the VHS. $\mathscr{F}^n$ will be denoted $\mathscr{F}^b$ (b for bottom) if we do not want to specify the weight. $\mathscr{H}^{p,n-p}=\mathscr{F}^p \cap \overline{\mathscr{F}^{n-p}}$ are the Hodge components, which are real analytic vector bundles over X. A polarization of H is a bilinear form, symmetric for n even, skew for n odd, such that $S(\mathscr{H}^{p,q}, \mathscr{H}^{v,s})=0$ unless $p=s$, $q=v$ and $i^{p-q}S(v, \bar{v})>0$ if $0 \neq v \in \mathscr{H}^{p,q}$. We introduce the Weil operator C: $\mathscr{H} \to \mathscr{H}$ given by $Cv=i^{p-q}v$ if $v \in \mathscr{H}^{p,q}$. This is not holomorphic over X. Let $h(\cdot, \cdot)=S(C\cdot, \bar{\cdot})$ be the Hodge metric, it is a positive definite Hermitian form on $\mathscr{H}$.

Definition 4.2. The flat structure $H \subset \mathscr{H}$ defines an integrable connection $d: \mathscr{H} \to \mathscr{H} \otimes \Omega^1_X$. $\mathscr{F}^b$ is a subbundle of $\mathscr{H}$ and d induces a connection ∇ on $\mathscr{F}^b$ by

$$\nabla : \mathscr{F}^b \longrightarrow \mathscr{H} \xrightarrow{d} \mathscr{H} \otimes \Omega^1 \longrightarrow \mathscr{F}^b \otimes \Omega^1,$$

where the last map is orthogonal projection onto $\mathscr{F}^b$ using h.

The composite map

$$\rho : \mathscr{F}^b \longrightarrow \mathscr{H} \xrightarrow{d} \mathscr{H} \otimes \Omega^1 \longrightarrow (\mathscr{H}/\mathscr{F}^b) \otimes \Omega^1$$

is called the second fundamental form of $\mathscr{F}^b$ in $\mathscr{H}$. The curvature form of the connection ∇ is $\theta_{\mathscr{F}^b} = -\rho^* \wedge \rho$ [Sch, 7.18], where $*$ is the adjoint of ρ with respect to h, and therefore it is positive semidefinite.

Definition 4.3. Let $j: S \to \mathscr{F}^b$ be a subbundle and $q: \mathscr{F}^b \to Q$ the corresponding quotient bundle, $s: Q \to \mathscr{F}^b$ the orthogonal splitting of q (not holomorphic in general).

$$\sigma : S \longrightarrow \mathscr{F}^b \xrightarrow{\nabla} \mathscr{F}^b \otimes \Omega^1 \longrightarrow Q \otimes \Omega^1$$

is the second fundamental form of S in $\mathscr{F}^b$. Via s, h induces a Hermitian metric on Q and the curvature of Q in this metric is given by

$$\Theta_Q = q\Theta_{\mathscr{F}^n} s + \sigma \wedge \sigma^*.$$

Since $\Theta_{\mathscr{F}^n}$ is positive semidefinite, and $\sigma \wedge \sigma^*$ is positive semidefinite too, Θ_Q is again positive semidefinite.

In the applications it will be crucial to understand when Θ_Q will be positive definite. This question will be studied next. The starting point is a simple but very useful theorem. I am grateful to L. Lempert for pointing out to me that it was known earlier to various people.

Theorem 4.4 (Sommer [So], Bedford-Kalka [B-K]). *Let L be a line bundle over X and h a C^∞ metric on L with nonnegative curvatue. Then there exist an open dense $U \subset X$ and a C^∞ foliation of U with complex analytic leaves such that the curvature is zero exactly along the leaves.*

For the reader's convenience we sketch the proof. Let e be a local holomorphic section of L and $f = \log h(e, e)$. Then the curvature is $\partial\bar{\partial} f$, and its nonnegativity means that $i\partial\bar{\partial} f(\bar{v}, v) \geq 0$ for any holomorphic tangent vector v.

If $x \in X$ then in a suitable coordinate system $z_1, \cdots, z_n$ at x we have $\partial\bar{\partial} f = dz_1 \wedge d\bar{z}_1 + \cdots + dz_i \wedge d\bar{z}_i$ and we see that $i\partial\bar{\partial} f(\bar{v}, v) = 0 \Leftrightarrow i\partial\bar{\partial} f(\cdot, v) \equiv 0 \Leftrightarrow v$ is in the space $\langle \partial/\partial z_{i+1}, \cdots, \partial/\partial z_n \rangle$. The condition $i\partial\bar{\partial} f(\cdot, v) \equiv 0$

defines a vector subbundle $V \subset T_U$ over some open dense set $U \subset X$. We can view $\partial\bar{\partial}f$ as a 2-form on the real manifold U, and $\partial\bar{\partial}f(\cdot, w) \equiv 0$ defines a subbundle W of the real tangent bundle. For $x \in U$ in the above coordinate system $W_x = \langle \partial/\partial x_{i+1}, \partial/\partial y_{i+1}, \cdots, \partial/\partial x_n, \partial/\partial y_n \rangle$.

The form $\partial\bar{\partial}f$ is d-closed and hence by the Frobenius theorem W is integrable (cf. [N, 2,11]). This gives a C^∞ foliation of U. The leaves are holomorphic since at each point their tangent space is the same as the tangent space of a holomorphic subvariety ($z_1 = \cdots = z_i = 0$ in the above coordinates) hence the Cauchy-Riemann equations are satisfied. This proves the theorem. □

We remark that in general the foliation will not be holomorphic.

Since the curvature of L is zero along the leaves, the above theorem reduces the problem of understanding the case of non-strictly-positive curvature to the analysis of the zero curvature case on smaller dimensional manifolds.

Proposition 4.5. *Notation as in* 4.3. *The second fundamental form* σ: $S \to Q \otimes \Omega^1$ *is identically zero iff* $sQ \subset \mathscr{F}^b$ *is a holomorphic subbundle.*

Proof. Let $f \in \Gamma(S)$ be a holomorphic section and $g \in \Gamma(sQ)$ be a C^∞ section. Since S and sQ are orthogonal we have $i^n S(g, \bar{f}) = h(g, f) = 0$. Apply $\bar{\partial}$ to obtain $i^n S(\bar{\partial}g, \bar{f}) + i^n S(g, \overline{\partial f}) = 0$.

$i^n S(g, \overline{\partial f}) = h(g, \partial f)$. Since ∇f is the projection of $df = \partial f$ to $\mathscr{F}^b$ and $g \in \Gamma(\mathscr{F}^b)$ we have that $h(g, \partial f) = h(g, \nabla f)$. By assumption σ is identically zero hence $\nabla S \subset S \otimes \Omega^1$, and therefore $h(g, \nabla f) = 0$. This yields that

$$h(\bar{\partial}g, f) = i^n S(\bar{\partial}g, \bar{f}) = -i^n S(g, \overline{\partial f}) = h(g, \nabla f) = 0.$$

This holds for an arbitrary f and therefore sQ is $\bar{\partial}$ stable. Thus sQ is holomorphic as shown by the following well known:

Lemma 4.6. (i) *Let V be a holomorphic vector bundle over $U \subset \mathbf{C}^n$ and let $W \subset V$ be a complex C^∞ subbundle. Then W is holomorphic iff it is $\partial/\partial\bar{z}_k$ stable for every k.*

(ii) *Let S be a $\mathbf{C}^m$ local system over $U \subset \mathbf{C}^n$, and $T \subset S \otimes \mathcal{O}_U$ a holomorphic subbundle. Then T is flat iff it is $\partial/\partial z_k$ stable for every k.*

Proof. We prove only (ii), the other part being similar. We can pick suitable local bases $e_1, \cdots, e_m$ of S and $f_1, \cdots, f_t$ of T such that $f_i = e_i + \Sigma_{j>t} f_{ij} e_j$. Applying $\partial/\partial z_k$ we get $\partial f_i/\partial z_k = \Sigma_{j>t} \partial f_{ij}/\partial z_k e_j$. The right hand side is in the span of f_i's, and this implies that $\partial f_{ij}/\partial z_k = 0$ for every i, j, k, hence all the f_{ij} are constant. The converse is clear. □

This completes the proof of the only if part of 4.5. The other implication is clear. □

Theorem 4.7. *Notation as in* 4.3.

(i) *Let* $A \subset \mathscr{F}^b$ *be a flat subbundle. Then its orthogonal complement is holomorphic.*

(ii) *Let* $P^0 \subset \mathscr{F}^b$ *be the maximal flat subbundle and* P^+ *its orthogonal complememt. If* $q\colon \mathscr{F}^b \to Q$ *is a quotient of* $\mathscr{F}^b$ *and* Θ_Q *is identically zero, then* $P^+ \subset \ker q$, $P^0 \cap \ker q$ *is flat and* $sQ \subset P^0$ *is flat.* P^0 *is the maximal zero curvature quotient of* $\mathscr{F}^b$.

Proof. Note that $\mathscr{F}^b \subset H \otimes \mathcal{O}_X$ and therefore it makes sense to talk about flat subbundles of $\mathscr{F}^b$. Let B be the orthogonal complement of A and let $f \in \Gamma(B)$ be a C^∞ section, $g \in \Gamma(A)$ a flat section. $h(g,f) = i^n S(g, \bar{f}) = 0$ by assumption. Using $\partial g = 0$ we get

$$0 = \partial h(g,f) = i^n S(\partial g, \bar{f}) + i^n S(g, \partial \bar{f}) = h(g, \bar{\partial} f).$$

Therefore B is $\bar{\partial}$ stable, hence holomorphic. This proves (i).

Now let Q be a quotient of $\mathscr{F}^b$ whose curvature is identically zero. This implies that the second fundamental form is zero, and therefore $sQ \subset \mathscr{F}^b$ is holomorphic by 4.5. Let $f \in \Gamma(sQ)$, $g \in \Gamma(S)$ be holomorphic sections. Then

$$0 = \partial h(f, g) = i^n S(\partial f, \bar{g}) + i^n S(f, \partial \bar{g}) = i^n S(\partial f, \bar{g}).$$

By assumption $\Theta_Q = q \Theta_{\mathscr{F}^b} s = 0$ and $\Theta_{\mathscr{F}^b} = -\rho^* \wedge \rho$. Therefore ρ, the second fundamental form of $\mathscr{F}^b$ in $\mathscr{H}$, is zero on sQ, which means that $\partial f \in \mathscr{F}^b \otimes \Omega^1$. Therefore $i^n S(\partial f, \bar{g}) = h(\partial f, g) = 0$. This implies that sQ is ∂ stable, hence flat by 4.6 (ii).

Since P^0 is the maximal flat subbundle this means that $sQ \subset P^0$ and therefore S—the orthogonal complement of sQ—contains P^+.

If f and g are flat sections of P^0 then $h(f, g) = i^n S(f, \bar{g})$ is constant, and therefore the curvature of P^0 is identically zero. This completes the proof of the theorem. □

Proposition 4.8. *Let* H_i *be* VHS's *over* X. *Then*

$$P^0(\mathscr{F}_1^b \otimes \mathscr{F}_2^b) = P^0(\mathscr{F}_1^b) \otimes P^0(\mathscr{F}_2^b).$$

Proof. Let e_s (resp. d_t) be a flat basis of H_1 (resp. H_2), and $f_i = \sum f_{is} e_s$ (resp. $g_j = \sum g_{jt} d_t$) be a basis of $\mathscr{F}_1^b$ (resp. $\mathscr{F}_2^b$). Let $\sum \lambda_{ij} f_i \otimes g_j$ be a flat section of $\mathscr{F}_1^b \otimes \mathscr{F}_2^b$. Then

$$\sum \lambda_{ij} f_i \otimes g_j = \sum \lambda_{ij} f_i g_{js} d_s = \sum_s d_s \sum_i (\sum_j \lambda_{ij} g_{jk}) f_i.$$

This is flat iff $h_k = \sum_i (\sum_j \lambda_{ij} g_{jk}) f_i$ is flat for all k and these are sections of $\mathcal{F}_1^b$. Let $h_1, \cdots, h_r$ be the maximal linearly independent set among the h_k's. Then we can write $\sum \lambda_{ij} f_i \otimes g_j = \sum_i^r h_p \otimes e_p$, where the e_p are certain flat sections of H_2. This sum is in $\mathcal{F}_1^b \otimes \mathcal{F}_2^b$. Therefore $e_p \in \mathcal{F}_2^b$, and this proves the proposition. □

Remark 4.9. Theorem 4.7 can be viewed as the local analog of [Z1, 10.1]. As a global application we derive the following result although it will not be needed in the sequel.

Proposition 4.10. *Let C be a smooth projective curve, $U \subset C$ open. Let H be a* VHS *over U with unipotent monodromies around $C-U$. Let $'\mathcal{F}$ be the canonical extension of $\mathcal{F}^b$. Then $'\mathcal{F}$ can be written as $'\mathcal{F} = A + B$ where A is ample and B is flat. A, as a subbundle of $'\mathcal{F}$, is unique.*

Proof. Let $P^0 \subset \mathcal{F}^b$ be the maximal flat subbundle. Then $P^0 + \overline{P^0} \subset \mathcal{H}$ defines a sub VHS, and so does its orthogonal complement with respect to $S(\ ,\ )$. Let $\mathcal{H} = (P^0 + \overline{P^0}) + \mathcal{H}'$. $\mathcal{F}^b(\mathcal{H}') \subset \mathcal{F}^b$ is a complement to P^0 and it extends to a subbundle $'\mathcal{F}^b(\mathcal{H}') \subset {'\mathcal{H}'} \subset {'\mathcal{H}}$, which we call A. $h(\ ,\ )$ gives a flat unitary metric on P^0, so the monodromy of P^0 around $C-U$ is unitary. On the other hand it must be unipotent, hence trivial. Therefore P^0 extends over the punctures to a vector bundle B keeping its flat structure.

By [H1, 2.4] A is ample if every quotient of A has positive degree. Let Q be a quotient of A. Outside finitely many points of C, A has a natural positive semidefinite metric coming from $h(\ ,\ )$. This induces a positive semidefinite metric on Q (outside finitely many points). If Θ is the curvature of Q then the integral $(-1/2\pi i)\int_C \operatorname{tr}\Theta$ converges and represents $c_1(Q)$ (cf. [F]). If $\deg Q \leqslant 0$ then this implies that $\Theta \equiv 0$ and therefore A has a flat quotient, contradicting 4.7. Therefore A is ample.

Since A is ample, $\operatorname{Hom}(A, B) = 0$, hence A is unique. □

Remark 4.11. For higher dimensional base a similar but weaker statement holds. One still obtains a decomposition $'\mathcal{F} = A + B$ where B is flat. A is however not ample in general; it satisfies only the following weaker property: A has no flat quotients, even after generically finite pull-backs. This result can be easily obtained from 4.10 by restriction to a general curve section.

V. Estimates of degenerating Hodge metrics

The aim of this section is to prove the following technical:

Theorem 5.1. *Let X be a smooth n-dimensional variety, $X \subset \bar{X}$ a smooth compactification such that $\bar{X} - X = D$ is a normal crossing divisor. Let H be a* VHS *over X with unipotent monodromies around D and let $'\mathcal{F}^b$ be the canonical extension of the lowest piece of the Hodge filtration. Finally let $'\mathcal{F}^b \to Q$ be a vector bundle quotient of $'\mathcal{F}^b$. The Hodge metric of $'\mathcal{F}^b$ induces a metric on Q, let Θ be its curvature form. Then*

$$\left(\frac{-1}{2\pi i}\right)^n \int_X (tr\Theta)^n = c_1(Q)^n.$$

Remark 5.2. Aside from the fact that the Hodge metric has singularities at D, this is the well-known formula for Chern classes. For metrics that acquire singularities essentially the only problem is to establish the convergence of certain integrals (cf. [Mu, 1.1]). Therefore the main part of the theorem is the implicit claim that the above integral converges. This is a local question around D and so we may forget about X and consider an arbitrary VHS over $(\Delta^*)^n$. Instead of a quotient bundle, we shall consider a line subbundle of $'\mathcal{H}$, the passage to Q will then be relatively straightforward.

Cattani-Kaplan-Schmid [C-K-S] gave a detailed analysis of the behavior of the Hodge metric of $\mathcal{H}$ near the singularities. Their results are the starting point of our computations. Next we recall some of their results and their notation.

Definition 5.3. (i) Let Δ be the complex disc of radius e^{-1}, $\Delta^* \subset \Delta$ the punctured disc. On Δ^n we fix coordinates $s_1, \cdots, s_n$ such that $(\Delta^*)^n = (\prod s_i \neq 0)$. For simplicity of notation we introduce $s_{n+1} = e^{-1}$. By M we shall denote the region $\{s \in (\Delta^*)^n \mid |s_i| \leqslant |s_{i+1}|\ i = 1, \cdots, n\}$. For $l_1, \cdots, l_n$ integers we define

$$e(l_1, \cdots, l_n)(s) = \prod_j \left(\frac{-\log|s_j|}{-\log|s_{j+1}|}\right)^{l_j/2}.$$

It is clear from the definitions that if $k_i \leqslant l_i$ and $s \in M$, then $e(\underline{k})(s) \leqslant e(\underline{l})(s)$.

(ii) Let H be a VHS over $(\Delta^*)^n$. The monodromy of H around $s_i = 0$ is B_i. We assume that all the B_i's are unipotent and let $N_i = \log B_i$. The N_i are nilpotent and commute.

(iii) If N is a nilpotent endomorphism of a vector space V then N defines a so-called weight filtration of V; it is an increasing filtration $\cdots \subset W_i \subset W_{i+1} \subset \cdots$. For our purpose it is sufficient to know that the W_i can be built up from the subspaces $\ker N^r$ and $\operatorname{im} N^s$, and therefore we have the following: if M and N commute then $MW_i \subset W_i$. (See [G, p. 255] for the precise definition and for the result.)

(iv) In the situation of (ii), if N is a linear combination of the N_i's then N defines a weight filtration $W_{\bullet}$ on any H_s. This turns out to be a flat filtration of H. Of special interest are the special cases $W^j_{\bullet}=W_{\bullet}(N_1+\cdots+N_j)$ for $j=1, \cdots, n$. One can choose a multivalued flat multigrading $H=\Sigma_{l_1,\dots,l_n}H_{l_1,\dots,l_n}$ such that

$$W^1_{l_1}\cap\cdots\cap W^n_{l_n}=\Sigma_{k_i\leqslant l_i}H_{k_1,\dots,k_n}.$$

We define the bundle map $e(s)$ by the rule: $e(s)$ acts on $H_{l_1,\dots,l_n}$ as multiplication by $e(l_1, \cdots, l_n)(s)$. (To be precise we should restrict to a region of M where H is single-valued, but this technical problem will be unimportant.)

(v) In each $H_{\bullet\bullet}$ we choose a flat multivalued basis, and all these together give a flat basis $(v_{\bullet})$ of H. The fomula

$$(\tilde{v}_{\bullet})(s)=\exp\left(-\frac{1}{2\pi i}\Sigma N_j\log s_j\right)(v_{\bullet})(s)$$

gives a single-valued basis of $\mathscr{H}=H\otimes\mathcal{O}_{\Delta^n}$ which extends to a basis of $'\mathscr{H}$.

We order the basis $(v_{\bullet})$ somehow to get $(v_1, \cdots)$ and define $e_i(s)=e(l_1, \cdots, l_n)(s)$ if $v_i\in H_{l_1,\dots,l_n}$. Then $e(s)$ acts on H by $v_i\mapsto e_i(s)\cdot v_i$.

(vi) On $(\Delta^*)^n$ one defines the Poincaré metric by declaring the coframe

$$\left\{\frac{ds_i}{s_i\log|s_i|},\ \frac{d\bar{s}_i}{\bar{s}_i\log|s_i|}\right\}$$

to be unitary. This defines a frame of every Ω^k which we shall refer to as the Poincaré frame.

(vii) A function f on Δ^n will be called nearly bounded if for a suitable choice of local coordinates $(s_{\bullet})$ there are $C, k>0$ and $\varepsilon>0$ such that for every ordering of the coordinate functions $s_1, \cdots, s_n$ at least one of the following conditions is satisfied for every $s\in M=\{|s_1|\leqslant\cdot\leqslant|s_n|\}$.

(a): $|f|\leqslant C$,

(b_m): $|s_1|\leqslant\exp(-|s_m|^{-\varepsilon})$ and $|f|\leqslant C(-\log|s_m|)^k$ (for $m=2, \cdots, n$).

A form η on Δ^n will be called nearly bounded if for a suitable choice of coordinates $(s_{\bullet})$, if we write η in terms of the Poincaré frame then the coefficient functions will be nearly bounded (with the given choice of the coordinates). If η_1 and η_2 are nearly bounded with the same choice of coordinates, then $\eta_1\wedge\eta_2$ is nearly bounded.

A form η on Δ^n will be called almost bounded if there is a proper bimeromorphic map p: $W\to\Delta^n$ such that W is nonsingular and every $w\in W$ has a neighborhood where $p^*\eta$ is nearly bounded.

A form η on a compact manifold will be called almost bounded if every point has a neighborhood where η is almost bounded.

(viii) Let $h(\ ,\ )$ be the Hodge metric on $\mathscr{H}$. By a slight abuse of notation we denote by h the matrix function $h_{ij}=h(\tilde{v}_i, \tilde{v}_j)$ as well. Let $\tilde{h}$ be the matrix function $\tilde{h}_{ij}=e_i^{-1}h_{ij}e_j^{-1}$; so $\tilde{h}=e^{-1}he^{-1}$. The important result of [C-K-S] that we shall use is a good description of $\tilde{h}$ on M. To formulate this we need a final definition.

(ix) Let $L=\{$Laurent polynomials in the variables $(-\log|s_j|)^{1/2}\}$. Let $C^\omega(\Delta^n)$ denote the real analytic functions on Δ^n, and let $C^\omega(\Delta^n)\otimes L$ be the finite tensor product. Let $BM=\{f\in C^\omega(\Delta^n)\otimes L \mid f$ bounded on $M\}$. A not quite trivial but important property of this function space is that the operators $s_j\log|s_j|\partial/\partial s_j$ and $\bar{s}_j\log|s_j|\partial/\partial\bar{s}_j$ map BM into itself (cf. [C-K-S]).

Proposition 5.4 (Cattani-Kaplan-Schmid, [C-K-S, 5.19]). *With the above notation* $\tilde{h}$ *and* $(\det\tilde{h})^{-1}$ *have entries in BM. In particular they are bounded on M.* □

For technical reason we shall need a basis of $'\mathscr{H}$ that is slightly different from $(\tilde{v}_i)$.

Proposition 5.5. *Let $N(s)$ be a C^ω matrix function defined in some neighborhood of the origin. Assume that $N(s)$ commutes with the N_i's for every s. Let $v'_{\bullet}$ be the basis of $'\mathscr{H}$ given by $(v'_{\bullet})(s)=\exp(N(s))\cdot(\tilde{v}_{\bullet})(s)$. Let h' be the matrix $h'_{ij}=e_i^{-1}h(v'_i, v'_j)e_j^{-1}$. Then h' and $(\det h')^{-1}$ have entries in BM. In particular they are bounded on M.*

Proof. The matrix of h in the basis $(v'_{\bullet})$ is given by

$$(h(v'_i, v'_j))=\exp(N)(h(v_i, v_j))^t(\exp(N))$$

where ^{t} denotes transpose. Therefore

$$h'=(e^{-1}\exp(N)e)\tilde{h}^t(e^{-1}\exp(N)e).$$

Since $\det(e^{-1}\exp(N)e)=\det\exp(N)$ is an invertible function near the origin, all we have to show is that $e^{-1}\exp(N)e$ has entries in BM. By assumption $N(s)$ commutes with the N_j's, and the same is true for $\exp(N(s))$. As we remarked in 5.3, (iii), this implies that $\exp(N(s))\,W_q^p\subset W_q^p$ for every p, q. Let $\exp(N(s))=(a_{ij})$. Then $e^{-1}\exp(N(s))e=(e_i^{-1}a_{ij}e_j)$. Assume that $a_{ij}\neq 0$ and let $v_i\in H_{l_1,\dots,l_n}$ and $v_j\in H_{k_1,\dots,k_n}$. Then $\exp(N(s))v_i\in W_{l_1}^1\cap\cdots\cap W_{l_n}^n$. One of its components is $a_{ij}v_j$, and this implies that $k_s\leqslant l_s$ for every s. By 5.3, (i) this means that $e_j(s)\leqslant e_i(s)$ and hence $e_i^{-1}e_j$ is bounded on M. Since a_{ij} is a C^ω function this implies that $e_i^{-1}a_{ij}e_j\in BM$. This completes the proof. □

For convenience of reference we record the following easy:

Lemma 5.6. *Let $G=(g_{ij})$ be a positive definite Hermitian form. Assume that for some constant b we have $|g_{ij}|\leqslant b$, $|\det G|^{-1}\leqslant b$. Then there exists a $c>0$ depending only on b and* rank G *such that for any vector $w=(w_k)$ we have ${}^t\bar{w}Gw\geqslant c\sum|w_iw_j|$.* □

Proposition 5.7. *Let $L\subset{}'\mathscr{H}$ be a line subbundle, $u(s)$ a local generator of L at 0. In the basis $(v'_\cdot)$ of 5.5, $u(s)$ can be written as $u(s)=\sum f_i(s)v'_i$, $f_i(s)$ holomorphic at 0, and $f_i(0)\neq 0$ for at least one i. Assume that $f_i(s)=\prod s_j^{b_{ij}}g_i$ where $g_i(0)\neq 0$ for every i.*

Let $f=h(u, u)$,

$$\theta=\frac{\partial f}{f}\quad \textit{and}\quad \Theta=\frac{\bar{\partial}\partial f}{f}-\frac{\partial f}{f}\wedge\frac{\bar{\partial} f}{f}$$

be the connection and curvature of L.

Then θ and Θ are nearly bounded near 0 *(with the given choice of coordinates).*

Proof. We have $f(s)=\sum h'_{ij}e_ie_jf_i\bar{f}_j$. Only the case of θ will be worked out in detail since the case of the curvature is similar. Since we use the Poincaré frame, the coefficient functions are $s_k\log|s_k|\partial_kf(s)\cdot f(s)^{-1}$. It is clearly sufficient to estimate each $s_k\log|s_k|\partial_k(h'_{ij}e_ie_jf_i\bar{f}_j)f(s)^{-1}$ individually. By 5.5 and 5.6, $|f(s)|\geqslant c\sum|e_ie_jf_i\bar{f}_j|$, and in particular $|f(s)|\geqslant c|e_ie_jf_i\bar{f}_j|$. Differentiating the product and using this estimate we get the following terms:

(i) $c^{-1}\cdot s_k\log|s_k|\partial_kh'_{ij}$ is bounded by 5.3, (ix).

(ii) $c^{-1}\cdot h'_{ij}s_k\log|s_k|\partial_k(e_ie_j)\cdot(e_ie_j)^{-1}=c^{-1}\cdot h'_{ij}\cdot$(constant) as an explicit computation yields;

(iii) we write $f_i=\prod s_j^{b_{ij}}g_i$ and we have two terms

$$c^{-1}\cdot h'_{ij}s_k\log|s_k|\partial_k(\prod s_j^{b_{ij}})\cdot\prod s_j^{-b_{ij}}=c^{-1}\cdot h'_{ij}\cdot\log|s_k|\cdot b_{ik},$$

which is bounded by $(-\log|s_k|)$; and

$$c^{-1}\cdot h'_{ij}\partial_k(g_i)\cdot g_i^{-1}$$

which is bounded;

(iv) $\partial_k\bar{f}_j=0$ so the last term vanishes.

Therefore we have to pay close attention only to the first case of (iii). This is

$$h'_{ij}\cdot s_k\log|s_k|e_ie_j\partial_k(\prod s_q^{b_{iq}})g_i\bar{f}_j\cdot f(s)^{-1}.$$

By assumption $f_p(0)\neq 0$ for some p; therefore $f(s)\geqslant ce_p^2$. Using this, the above expression is bounded by $c's_k(-\log|s_k|)e_ie_je_p^{-2}$ which is of the form

$$c'\cdot s_k \prod(-\log|s_j|)^{a_j}\leqslant c's_k(-\log|s_1|)^a.$$

If $\varepsilon=a^{-1}$ then this is bounded in the region $|s_1|\geqslant\exp(-|s_k|^{-\varepsilon})$. If $|s_1|\geqslant\exp(-|s_k|^{-\varepsilon})$, then we can use the original estimate $c''(-\log|s_k|)$. □

Remark 5.8. The condition that we imposed on the f_i is very strong and artificial. For general f_i the above proof will not work. The crucial point is to estimate $s_k\cdot\partial_k f_i\cdot f_i^{-1}$ in the region $|s_1|\leqslant\exp(-|s_k|^{-\varepsilon})$ for $k\geqslant 2$. If $n=2$ then this expression is bounded near 0, but for $n\geqslant 3$ it is unbounded for certain f_i's, though it is bounded for "most" points of the region.

If we start with any function f_i, then after some blowing up its divisor of zeros will be a normal crossing divisor. To be able to use this we have to relate the degeneration of Hodge structure on the blow-up to the original degeneration. This is what we do next.

5.9. Let Y be a smooth variety and $D\subset Y$ a divisor with normal crossing. We shall consider blow-ups with center $Z\subset Y$ satisfying the following property: for every $z\in Z$ one can choose local coordinates (y_i) at z such that D is a union of some of the $(y_i=0)$ and Z is the intersection of some of the $(y_i=0)$ near z. If BY denotes the blow-up, $\bar{D}$ the total transform of D and E the exceptional divisor, then $\bar{D}\cup E$ is a divisor with normal crossings. Such blow-ups will be called permissible.

Now assume that H is a VHS over $Y-D$ with unipotent monodromies around D, and let N_i be the monodromy logarithms around the components D_i of D. Assume that Z is irreducible and is contained in $D_1,\cdots,D_j$ but not in the others. Since $B_ZY-\bar{D}-E=Y-D$, H gives a VHS on $B_ZY-\bar{D}-E$. This again has unipotent monodromies and the monodromy logarithm around E is $N_1+\cdots+N_j$. Repeating this process we get the following:

Lemma 5.10. *Let H be a* VHS *over $(\Delta^*)^n$ with unipotent monodromies, N_i be the monodromy logarithms. Let $p\colon W\to\Delta^n$ be a sequence of permissible blow-ups with closed centers. If E is any exceptional divisor of p, then the monodromy logarithm N_E of H around E is of the form $N_E=\Sigma a_iN_i$ for certain integers $a_i\geqslant 0$.* □

5.11. Now let $w\in W$ be a point, and we pick coordinates (s_i) at w such that p^*H is a VHS outside $\prod s_i=0$. Let M_i be the monodromy logarithm around $s_i=0$. As in 5.3, (iv) we define the weight filtration W^j using the M_i's. Using 5.10 we obtain that W^j is the weight filtration of some $N=\sum b_i N_i$, $b_i \geqslant 0$. By a result of Cattani-Kaplan [C-K, 3.3], the weight filtration of N depends only on the set $\{i|b_i \neq 0\}$. This gives that for any choice of W, w, s_i, we have to consider only finitely many different weight filtration; even the possible sequences $W^1_\bullet, \cdots, W^n_\bullet$ are finite in number.

For each of these possible sequences we choose a multivalued basis $(v^j_\bullet)$ as in 5.3, (v). We have finitely many bases $(v^1_\bullet), \cdots, (v^j_\bullet), \cdots$; we fix them for the sequel.

If L is a line subbundle of $'\mathscr{H}$ then for each of these bases $(v^j_\bullet)$ we can write a local generator $u(s)$ of L as $u(s)=\sum_i f^j_i \tilde{v}^j_i$, where f^j_i are holomorphic near the origin. One can choose a sequence of permissible blow-ups p: $W \to \Delta^n$ such that the divisor $F=p^{-1}(\prod s_i=0) \cup_{ij} p^{-1}(f^j_i=0)$ is a normal crossing-divisor. $p^* {'\mathscr{H}}$ is the canonical extension of $p^*\mathscr{H}$ to W, p^*L is a line subbundle of $p^* {'\mathscr{H}}$ and we have $p^*u(w)=\sum p^* f^j_i p^* \tilde{v}^j_i$.

For any $w \in W$ we choose coordinates $s_1, \cdots, s_n$ such that F is contained in $\prod s_i=0$ near w. We may assume that $(v^1_\bullet)$ is the basis that is constructed from the weight filtrations that we get from H, w, s_i as in 5.3, (iv). We have $p^*u(w)=\sum p^* f^1_i p^* \tilde{v}^1_i$, and $p^* f^1_i = \prod s_i^{b_{ij}} g_i$ for some $b_{ij} \geqslant 0$ and $g_i(w) \neq 0$. We are seemingly in the situation of 5.4. Unfortunately p^* and $\tilde{}$ do not commute (i.e. $p^*\tilde{v}^1_\bullet \neq \widetilde{p^* v^1_\bullet}$) and therefore 5.4 does not apply.

5.12. To compare $p^*\tilde{v}_\bullet$ and $\widetilde{p^* v_\bullet}$ we can work with one blow-up at a time. For simplicity we compute the case of blowing up a closed point, the general case being similar. Let H be a VHS over $(\Delta^*)^n$, with coordinates s_i and monodromy logarithms N_i. If we blow up the origin, let w a point in the exceptional divisor with local coordinates $s'_i=(s_i/s_1)-a_i$, $i \geqslant 2$ and $s'_1=s_1$. The monodromy logarithms at w are $\sum N_i$ around $s'_1=0$, N_i around $s'_i=0$ if $a_i=0$ and 0 around $s'_i=0$ if $a_i \neq 0$. $\tilde{v}$ is given by

$$\tilde{v}_\bullet = \exp\Big(-\frac{1}{2\pi i}\sum N_i \log s_i\Big) v_\bullet$$

$$= \exp\Big(-\frac{1}{2\pi i}\sum N_i \,(\log s'_1 + \log(s'_i + a'_i))\Big) v_\bullet$$

$$= \exp\Big(-\frac{1}{2\pi i}\sum_{a_i \neq 0} N_i \log(s'_i + a_i)\Big)$$

$$\cdot \exp\Big(-\frac{1}{2\pi i}\Big(\big(\sum N_i\big)\log s'_1 + \sum_{a_i=0} N_i \log s'_i\Big)\Big) v_\bullet$$

This shows that

$$p^*\tilde{v}_{\cdot}=\exp\left(-\frac{1}{2\pi i}\sum_{a_i\neq 0} N_i \log(s_i'+a_i)\right)\cdot\widetilde{p^*v}_{\cdot}.$$

If we take $N(s')=-(1/2\pi i)\sum_{a_i\neq 0} N_i \log(s_i'+a_i)$, then $N(s')$ is holomorphic near w and commutes with each of the N_i's. This shows the following:

Proposition 5.13. *If $p\colon W\to\Delta^n$ is a composite of permissible blow-ups and $w\in W$ an arbitrary point, then there exists a choice of local coordinates s_i' near w and a holomorphic matrix function $N(s')$ such that*

(i) *$N(s')$ commutes with the N_i's;*

(ii) *$p^*\tilde{v}=\exp(N(s'))\cdot\widetilde{p^*v}$, where $\widetilde{p^*v}$ is computed in the coordinates s_i' at w.* □

5.14. This is nearly what we want. Let M_i be the monodromy around $s_i'=0$. We may assume that $M_{j+1}=\cdots=M_n=0$; the rest are nonzero. This means that the VHS extends across the hyperplanes $s_{j+1}'=0, \cdots, s_n'=0$. Let us look at F (given in 5.11) near w. Since F contains the degeneracy locus of H, $s_1'=0, \cdots, s_j'=0$ are certainly components of F. The other components are given by some $s_{j+1}''=0, \cdots, s_m''=0$ and we extend this by $s_{m+1}'', \cdots, s_n''$ to get a local coordinate system.

$\widetilde{p^*v}$ in the coordinate system $(s_1', \cdots, s_n')$ is given by $\exp(-(1/2\pi i)\cdot\sum_1^j M_i \log s_i')p^*v$, and in the coordinate system $(s_1', \cdots, s_j', s_{j+1}', \cdots, s_n'')$ by

$$\exp\left(-\frac{1}{2\pi i}\left(\textstyle\sum_1^j M_i \log s_i'+\sum_{j+1}^n M_i \log s_i''\right)\right)p^*v.$$

$M_i=0$ for $i>j$ so these two expressions coincide.

Putting all these results together we obtain the following:

Proposition 5.15. *Let H be a* VHS *over $(\Delta^*)^n$ and let L be a line subbundle of $'\mathscr{H}$, u a local generator of L. Let θ (resp. Θ) be the connection (resp. curvature) of the induced Hodge metric of L computed in the frame $u(s)$ (for Θ this does not matter). Then both θ and Θ are almost bounded in a neighborhood of the origin.* □

Before finishing the proof we have to compute some integrals.

Proposition 5.16. (i) *Let η be a nearly bounded $2n$-form on Δ^n with compact support. Then $\int\eta<\infty$.*

(ii) *Let η be a nearly bounded $(2n-1)$-form on Δ^n with compact support, and let U be the set $\prod|s_i|=\varepsilon$. Then $\lim_{\varepsilon\to 0}\int_{U_\varepsilon}\eta=0$.*

Proof. (i) Let $d\mu$ denote the Poincaré measure $|z|^{-2}(-\log|z|)^{-2}dz\wedge d\bar{z}$ on Δ^n. Direct computation shows that $\int_{|z|\leq c} d\mu=4\pi(-\log c)^{-1}$, and therefore the measure of any compact subset of Δ^n is finite. To prove that $\int\eta$ converges one has to check it for the bounding functions that we used in 5.3, (viii).

Case (a) is the above mentioned result. Case (b_m) gives the integral

$$\int_{|s_2|<\cdots<|s_n|}(-\log|s_m|)^k\int_{|s_1|<\exp(-|s_m|-\varepsilon)}d\mu\wedge d\mu^{n-1}$$
$$=\int_{|s_2|<\cdots<|s_n|}4\pi(-\log|s_m|)^k\,|s_m|^\varepsilon d\mu^{n-1},$$

which is convergent since $|s_m|^\varepsilon(\log|s_m|)^k$ is bounded. This proves (i).

In order to prove (ii) we can proceed in two ways. One can use a direct computation as above. In this approach it is more convenient to use the set $U'_\varepsilon=\{s|\min|s_i|=\varepsilon\}$.

Another more general approach is the following. Consider the 1-form

$$\omega=((-\log|s_i|)^2)^{1/2}\sum\left(\frac{ds_i}{s_i}+\frac{d\bar{s}_i}{\bar{s}_i}\right).$$

This is orthogonal to the foliation U_ε, it has length one everywhere and written in terms of the Poincaré frame it has bounded coefficients. This implies that $\omega\wedge\eta$ is nearly bounded.

The dual of ω determines a flow v_t on $(\Delta^*)^n$. If we fix an ε, then we can parametrize $\{s|\prod|s_i|<\varepsilon\}$ by $[0,\infty)\times U_\varepsilon$, the map given by $(t,s)\to v_t(s)$. The flow goes to infinity since the Poincaré metric is complete. We know that $\int_{\Delta^n}\omega\wedge\eta<\infty$, so $\int_{[0,\infty)\times U_\varepsilon}\omega\wedge\eta<\infty$. The image of $\{t\}\times U_\varepsilon$ in Δ^n is some $U_{\varepsilon'}$ and

$$\int_{U_{\varepsilon'}}\frac{\omega\wedge\eta}{dt}=\int_{U_{\varepsilon'}}\eta$$

since ω is orthogonal to $U_{\varepsilon'}$ and unitary. Therefore the above integral transforms to some

$$\int_{[0,\infty)}\left(\int_{U_{\varepsilon(t)}}\eta\right)dt<\infty.$$

This can happen only if $\int_{U_{\varepsilon(t)}}\eta\to 0$ for some sequence $t\to\infty$. This will be enough for the applications. □

Corollary 5.17. (i) *Let η be an almost bounded 2n-form on $X \subset \bar{X}$. Then $\int_X \eta < \infty$.*

(ii) *Let η be an almost bounded $(2n-1)$-form on $X \subset \bar{X}$. Then $\int_X d\eta = 0$.*

Proof. Let ϕ_i be some partition of unity on $\bar{X}$. Then each $\phi_i \eta$ is almost bounded and the statements for them imply the claim for η. This shows that the problems are both local. If $\Delta^n \subset X$ and $p: W \to \Delta^n$ is a proper bimeromorphic map and if η has support in Δ^n, then it is sufficient to prove the claims for $p^*\eta$ on W. Here the question is again local so we are reduced to the case where η is nearly bounded in a Δ^n. Now (i) follows from 5.16, (i).

If V_ε is the set $\{s \mid \prod |s_i| > \varepsilon\}$, then

$$\int_{\Delta^n} d\eta = \lim_{\varepsilon \to 0} \int_{V_\varepsilon} d\eta = \lim_{\varepsilon \to 0} \int_{U_\varepsilon} \eta = 0$$

by 5.16, (ii). □

5.18. Now we are ready to prove Theorem 5.1. Let Q be the quotient of $'\mathscr{F}^b$ and let K be the corresponding subbundle. Let the curvatures be denoted by $\Theta_Q, \Theta_F, \Theta_K$. Then $\operatorname{tr} \Theta_Q = \operatorname{tr} \Theta_F - \operatorname{tr} \Theta_K$. Using this the theorem reduces to proving that

$$\left(-\frac{1}{2\pi i}\right)^n \int_X (\operatorname{tr} \Theta_K)^k (\operatorname{tr} \Theta_F)^{n-k} = c_1(K)^k c_1('\mathscr{F}^b)^{n-k}.$$

Let $q = \operatorname{rk} K$. Then $L = \Lambda^q K$ is a line subbundle of the VHS $\Lambda^{q\prime}\mathscr{H}$. Since $\operatorname{tr} \Theta_K = \Theta_L$ (=the curvature of L), we can prove instead the formula

$$\int_X \Theta_L^k (\operatorname{tr} \Theta_F)^{n-k} = (-2\pi i)^n c_1(L)^k c_1('\mathscr{F}^b)^{n-k}.$$

By 5.15, Θ_L is almost bounded on X, and $\operatorname{tr} \Theta_F$ has bounded coefficients in the Poincaré frame by [C-K-S, 5.23]. Therefore $\Theta_L^k (\operatorname{tr} \Theta_F)^{n-k}$ is almost bounded and the integral is convergent.

Pick a C^∞ metric of L over $\bar{X}$, and let the connection (resp. curvature) of L in this metric be θ'_L (resp. Θ'_L). Although θ_L and θ'_L depend on the choice of a frame, $\theta_L - \theta'_L$ is a well-defined 1-form on X. Let $\eta = (\theta_L - \theta'_L) \cdot \Theta_L^{k-1} (\operatorname{tr} \Theta_F)^{n-k}$. Then η is almost bounded, and hence by 5.17, (ii) $\int_X d\eta = 0$. This shows that

$$\int_X \Theta_L^k (\operatorname{tr} \Theta_F)^{n-k} = \int \Theta_L^{k-1} \Theta'_L (\operatorname{tr} \Theta_F)^{n-k}.$$

Repeating this procedure we get that

$$\int_X \Theta_L^k (\operatorname{tr} \Theta_F)^{n-k} = \int_X (\Theta'_L)^k (\operatorname{tr} \Theta_F)^{n-k}.$$

Similarly we can pick a C^∞ metric on $'\mathscr{F}^b$ with curvature Θ'_F and obtain that

$$\int_X \Theta_L^k (\operatorname{tr} \Theta_F)^{n-k} = \int_X (\Theta'_L)^k (\operatorname{tr} \Theta'_F)^{n-k}.$$

By the usual relationship between the curvature and Chern classes this gives us

$$\int_X \Theta_L^k (\operatorname{tr} \Theta_F)^{n-k} = (-2\pi i)^n c_1(L)^k c_1('\mathscr{F}^b)^{n-k},$$

and this completes the proof of 5.1. □

Remark 5.19. Let L_i be vector subbundles of $'\mathscr{H}$. The above proof shows that any given homogeneous polynomial in the first Chern forms of the L_i's is nearly bounded on $\bar{X}$. Therefore it defines a closed current on $\bar{X}$, hence a cohomology class. This class is the same as the cohomology class obtained by evaluating the polynomial at the first Chern classes of the L_i's. Using this, one can derive the following generalization of [C-K-S, 5.23].

Theorem 5.20. *Notation as in* 5.1. *Let A be a vector subbundle of $'\mathscr{H}$ and $q: A \to B$ a quotient bundle. Then the Chern forms of the induced Hodge metric are nearly bounded and closed, hence define a cohomology class. This class is the same as the corresponding Chern class of B.*

Proof. Assume that we have a filtration $A = A_0 \supset A_1 \supset \cdots \supset A_m = 0$ such that A_i/A_{i+1} is a line bundle for every i, and that $\ker q = A_j$ for some j. Then the Chern forms of B are polynomials in the first Chern forms of the A_i's.

Locally such a filtration can always be found and this proves that the Chern forms of B are nearly bounded. If a filtration exists globally then one can imitate 5.18 to complete the proof.

There exists a generically finite and surjective map $p: \bar{Y} \to \bar{X}$ such that p^*A has such a filtration; thus our two cohomology classes are equal if pulled back to $\bar{Y}$. Therefore the classes are already equal on $\bar{X}$. This proves the theorem. □

VI. Proof of the Main Theorem

In this chapter the results of the previous ones will be put together to prove the main theorem. This will be done in two steps. The first is a reduction step. We show that it is sufficient to prove the theorem for certain fiber spaces satisfying several additional properties. To such fiber spaces will the results of Chapters IV and V then be applied. For clarity and for convenience of later reference, the first step will be done in a general setting. For this we introduce some conditions:

Condition 6.1. Let W be a class of smooth projective varieties. Consider the following conditions:

(i) If $X \in W$, X and X' are birational then $X' \in W$.

(ii) If $f: X \to Y$ is a surjective map between smooth projective varieties such that the generic fiber is in W, then there exist countably many proper closed subsets $U_i \subset Y$ such that if $y \notin \cup U_i$ then $f^{-1}(y) \in W$.

(iii) The converse of (ii).

Definition 6.2. A fiber space $f: X \to Y$ will be called a W-fiber space if the generic fiber is in W.

Condition 6.3. For a W satisfying 6.1 consider the following conditions:

(1) If $X \in W$, $D \in |mK_X|$, $n|m$ then any irreducible component of the variety X' obtained by taking the n-th root of D is again in W.

(ii) Let $f: X \to C$ be a W-fiber space, $\dim C=1$, not necessarily projective. Let $D \in |mK_{X/C}|$ and let $f': X' \to C$ be the fiber space obtained by taking the n-th root of D. Assume that $X' \approx C \times F'$ as a fiber space for some F'. Then $X \approx C \times F$ for some F and D is of the form $\pi_2^* D' +$ (components of fibers) + (components whose multiplicity is divisible by n) for some $D' \in |mK_F|$.

Proposition 6.4. *Let W be the class of all varieties of general type. Then conditions* 6.1 *and* 6.3 *are satisfied.*

Proof. 6.1, (i) holds by definition, while (ii) and (iii) are easy in this case, see e.g. [L-S]. Any cover of a variety of general type is of general type again, giving 6.3, (i). By a result of Maehara [M, Appendix], if $X' \approx C \times F'$ then $X \approx C \times F$ and the natural map $X' \to X$ is given by $\mathrm{id}_C \times (F' \to F)$. The branch locus of $X' \to X$ is given by those components of D whose multiplicity is not divisible by n; this gives the last part. $\square$

Condition 6.5. For a fiber space $f: X \to Y$ consider the following conditions:

(i) X and Y are smooth and projective.

(ii) There is a normal crossing divisor $D=\sum D_i \subset Y$ such that f is smooth above $Y^0=Y-D$.

(iii) The monodromies of f around D are unipotent. (This implies that $N=f_*\omega_{X/Y}$ is locally free [Ka1, Theorem 5]).

(iv) $M_k=\operatorname{im}[N^{\otimes k}\to f_*\omega^k_{X/Y}]$ is locally free for every $k>0$.

(v) $\operatorname{Var} f=\dim Y$.

(vi) If F is the generic fiber of f then $|K_F|$ gives the stable canonical map and $I(F)$ is of general type.

Theorem 6.6. *Let W be a class of varieties satisfying* 6.1 *and* 6.3. *The following statements are equivalent.*

(i) *For every W-fiber space f: $X\to Y$ such that* $\operatorname{Var} f=\dim Y$, *we have $f_*\omega^k_{X/Y}$ is big for some $k>0$.*

(ii) *For every W-fiber space f: $X\to Y$ satisfying conditions* 6.5, *we have* $\det f_*\omega^k_{X/Y}$ *is big for some k.*

Proof. Obviously (i)$\Rightarrow$(ii). The other implication requires more work. By [V3, 3.5] it is sufficient to prove that $\det f_*\omega^k_{X/Y}$ is big for some $k>0$ for W-fiber spaces as in (i). Fix one fiber space f: $X\to Y$, and let F be the generic fiber. By 3.8 there is an N such that for generic $s\in H^0(\omega^N_F)$ the variety F' obtained by extracting the N-th root of s satisfies 6.5, (vi). Consider $f_*\omega^N_{X/Y}$. By leaving out a codimension two set of Y we may assume that $f_*\omega^N_{X/Y}$ is locally free and leaving out a codimension two set does not affect the bigness of $\det f_*\omega^k_{X/Y}$. $f_*\omega^N_{X/Y}$ is weakly positive by [V2, III] and therefore we can apply the hard covering trick 3.4 to obtain f_4: $X_4\to S$. By 3.5 it is sufficient to prove that $f_{4*}\omega^k_{X_4/S}$ is big for some $k>0$. As we remarked, f_4 satisfies 6.5 (vi) and it also satisfies (v) by 6.3, (ii).

If $U\to S$ is generically finite and surjective, $V=X_4\times_S U$, g: $V\to U$, then g is again a W-fiber space satisfying 6.5, (v) and (vi). Using [V3, 3.5] again we are reduced to showing that $\det g_*\omega^k_{V/U}$ is big for some $k>0$ for W-fiber spaces satisfying 6.5, (v) and (vi). We may obviously assume 6.5, (i) too.

If τ: $U'\to U$ is generically finite and surjective, let $U^f\subset U$ be the flat locus of this map. $V^f=g^{-1}(U^f)$. Since $U-U^f$ has codimension at least two, $\det g_*\omega^k_{V/U}$ is big iff $\det g_*\omega^k_{V^f/U^f}$ is big. Let $V'=U'\times_U V$, g': $V'\to U'$. By [V2, 3.5] we have an injection $g'_*\omega^k_{V'/U'}|\tau^{-1}U^f \hookrightarrow \tau^* g_*\omega^k_{V^f/U^f}$. Therefore to prove that $\det g_*\omega^k_{V/U}$ is big, it is sufficient to prove this for some g': $V'\to U'$ obtained as above. By [Ka1, 18] one can choose g': $V'\to U'$ so that 6.5, (i) (ii) (iii) (v) and (vi) are all satisfied. We remark that by 6.5, (iii) $N=g'_*\omega_{V'/U'}$ will commute with any further base change. Before considering condition (iv) it is convenient to make some definitions.

Definition 6.7. Recall that for a variety F we have the canonical map ϕ: $F \to \mathrm{Proj}\,(H^0(\omega_F))$. The closure of the image is called the canonical image and is denoted by $\phi(F)$. If f: $X \to Y$ is a fiber space such that $f_*\omega_{X/Y}$ is locally free, let $P = \mathrm{Proj}\,(f_*\omega_{X/Y})$ be a $\boldsymbol{P}^?$-bundle over Y. For general $y \in Y$ we have $P_y = \mathrm{Proj}\,(H^0(\omega_{F_y}))$. Let $\phi(X/Y)$ be the closure of the unions of $\phi(F_y) \subset P_y$. For general $y \in Y$ we have $\phi(X/Y)_y = \phi(F_y)$. By $\deg \phi(X/Y)$ we mean $\deg \phi(F_y) \subset P_y$ for general y. Let $\mathcal{O}(1)$ be the tautological line bundle on P, p: $P \to Y$. Then $p_*\mathcal{O}(k) = S^k(f_*\omega_{X/Y})$.

6.8. We return to the proof of 6.6. Let $N' = g'_*\omega_{V'/U'}$; it is locally free. Let $P' = \mathrm{Proj}\,(N')$, $Z' = \phi(V'/U') \subset P'$ the relative canonical image. By Hironaka's flattening theorem we can choose ρ: $T \to U'$ a birational map such that for $\bar{Z} = T \times_{U'} Z' \subset \rho^* P' = : P$ the irreducible component $Z \subset \bar{Z}$ dominating Z' is flat over T. Let $R = T \times_{U'} V'$, t: $T \to R$. By further blowing up R we may assume that t: $T \to R$ satisfies all conditions of 6.5 except possibly (iv) and that $\phi(T/R) \subset P$ is flat over T. I claim that this implies that $M_k = \mathrm{im}\,[(t_*\omega_{T/R})^{\otimes k} \to t_*\omega^k_{T/R}]$ is locally free for k large. Let $N = t_*\omega_{T/R}$. Then clearly $M_k = \mathrm{im}\,[S^k N \to t_*\omega^k_{T/R}]$. Let K_k be the kernel of this map.

For general $r \in R$, let F_r be the fiber of t above r. Then $N_r = H^0(\omega_{F_r})$ and the above map at r is the natural map $S^k H^0(\omega_{F_r}) \to H^0(\omega^k_{F_r})$. Elements of $S^k H^0(\omega_{F_r})$ can be thought of as sections of $\mathcal{O}_{P_r}(k)$ and the image of a section is zero in $H^0(\omega^k_{F_r})$ iff this section vanishes along $\phi(F_r)$. Let I be the ideal sheaf of $\phi(T/R) \subset P$. Then the above considerations show that $K_r \subset p_* I \otimes \mathcal{O}_P(k)$ and they are equal over an open subset of R.

Since $\phi(T/R)$ is flat over R, $p_* I \otimes \mathcal{O}_P(k)$ has constant rank for $k \gg 0$ and commutes with base change. Therefore $M'_k = S^k N / (p_* I \otimes \mathcal{O}_P(k))$ is locally free for k large. We have a natural surjective map $M_k \to M'_k$ which is generically an isomorphism. M_k is torsion-free since $t_*\omega^k_{T/R}$ is. Hence $M_k = M_{k'}$ and so M_k is locally free for k large.

This result is weaker than (iv) of 6.5. but will be enough for our purpose. To obtain the general result one can use the notion of very flat families along the same lines. See [H1, III. Exercise 9.5].

This completes the proof of the theorem. □

Corollary 6.9. *The statement* 6.6, (i) *is implied by the following*: (iii) *For every W-fiber space* f: $X \to Y$ *satisfying conditions* 6.5, *we have* det M_k *is big for some* k.

Proof. Let Q_k be the quotient $f_*\omega^k_{X/Y}/M_k$. Then $\det f_*\omega^k_{X/Y} = \det M_k \otimes c_1(Q_k)$. $c_1(Q_k)$ is weakly positive by [V2, III]. If $\det M_k$ is big then $\det f_*\omega^k_{X/Y}$ is big as well. Now 6.6 implies the result. □

Corollary 6.10. *Condition* (iii) *of* 6.9 *implies the following: Strong subadditivity of the Kodaira dimension: If $f\colon X \to Y$ is a W-fiber space with generic fiber F then*

$$\kappa(X) \geqslant \kappa(F) + \kappa(Y).$$

Furthermore if $\kappa(Y) \geqslant 0$ then

$$\kappa(X) \geqslant \kappa(F) + \mathrm{Var}\, f.$$

Proof. Follows from 6.6 and [V2, II]. □

Proposition 6.11. *Let $f\colon X \to Y$ be a smooth projective map, Y a complex manifold. Let $M_k = \mathrm{im}\,[(f_*\omega_{X/Y})^{\otimes k} \to f_*(\omega_{X/Y}^k)]$. Assume that M_k is a vector bundle and that it has zero curvature in the induced Hodge metric. Then $f_*\omega_{X/Y}$ has zero curvature too, so it is a flat subbundle of $\mathcal{O}_Y \otimes R^{n-k} f_* \boldsymbol{C}$* ($n = \dim X$, $k = \dim Y$).

Proof. Let $f_*\omega_{X/Y} = P^+ + P^0$ be the decomposition given by 4.7. We have to prove that $P^+ = 0$. Let $(f_*\omega_{X/Y})^{\otimes k} = P_k^+ + P_k^0$ be the analogous decomposition. By 4.8, $(P^+)^{\otimes k} \subset P_k^+$. Let K_k be the kernel of the map m_k: $(f_*\omega_{X/Y})^{\otimes k} \to f_*(\omega_{X/Y}^k)$. If M_k is flat in the induced metric then by 4.7, (ii), $(P^+)^{\otimes k} \subset P_k^+ \subset K_k$. Assume that P^+ is not zero and for general $y \in Y$ let $v \in P_y^+ \subset H^0(\omega_{F_y})$. The map m_k at y is the natutal multiplication $H^0(\omega_{F_y})^{\otimes k} \to H^0(\omega_{F_y}^k)$. If $v \in \Gamma(\omega_{F_y})$ is not zero then $v^{\otimes k} \in \Gamma(\omega_{F_y}^k)$ is again not zero. Therefore $m_k(v^{\otimes k}) \neq 0$, a contradiction to $(P^+)^{\otimes k} \subset K_k$. This proves the proposition. □

Remark 6.12. In general it is of interest to study multiplication maps

$$\otimes f_*(\mathscr{F}^{\otimes \nu_i}) \to f_*(\mathscr{F}^{\otimes \Sigma \nu_i})$$

for any torsion-free sheaf $\mathscr{F}$. These maps all have the special property that pure tensors never map to zero. This property should imply various statements about "preservation of positivity"; 6.11 is a simple useful example.

Corollary 6.13. *Assume in addition to* 6.11 *that $k \geqslant \deg \phi(X/Y)$ and that Y is simply connected. There is a natural isomorphism $P = Y \times \boldsymbol{P}^?$ and via this isomorphism $\phi(X/Y) = Y \times V$ for some $V \subset \boldsymbol{P}^?$. In particular the canonical images of the fibers of f are all isomorphic for general y.*

Proof. By 6.11 $f_*\omega_{X/Y}$ is flat, therefore isomorphic to $\mathcal{O}_Y^r$ for some r since Y is simply connected, the isomorphism being unique up to an element of GL $(r, \boldsymbol{C})$ (and not GL $(r, \mathcal{O}_Y)$!). This defines the isomorphism $P = Y \times \boldsymbol{P}^{r-1}$.

If $y \in Y$ is general then K_y is the kernel of $H^0(\omega_{F_y})^{\otimes k} \to H^0(\omega_{F_y}^k)$. The elements of this kernel are exactly the degree k equations of $\phi(F_y)$. By 4.7, (ii) K is a flat subbundle of $(f_*\omega_{X/Y})^{\otimes k} = \mathcal{O}_Y^{rk}$, and therefore the degree k equations of $\phi(F_y)$ are unchanged as y varies. If $k \geqslant \deg(X/Y)$ this implies that $\phi(F_y)$ is unchanged as a subvariety of the given $\boldsymbol{P}^{r-1}$. This proves the corollary. □

Theorem 6.14. *Let $f: X \to Y$ be a fiber space such that the generic fiber is of general type. Assume that conditions* 6.5 *are satisfied. Then* det M_k *is big for* $k \geqslant \deg \phi(X/Y)$.

Proof. $N = f_*\omega_{X/Y}$ is semipositive by [Ka1, 5] hence M_k and det M_k are semipositive. If L is a line bundle which is semipositive (=numerically effective) then L is big iff $c_1(L)^n > 0$ for $n = \dim Y$ ([V1, 3.2]). Therefore we have to prove that $c_1(\det M_k)^n = c_1(M_k)^n > 0$.

The Hodge metric on $N^{\otimes k}$ induces a metric M_k; let Θ be its curvature form. Θ and tr Θ are semipositive, so $(-(1/2\pi i)\operatorname{tr}\Theta)^n = f\cdot$(volume form) for some nonnegative function f. By 5.1 we have

$$\int_Y f\cdot(\text{volume form}) = c_1(M_k)^n.$$

Assume that $c_1(M_k)^n = 0$. Then f is identically zero, hence tr Θ is nowhere positive definite. tr Θ is the curvature form of det M_k. Thus by 4.4, for any sufficiently general $y \in Y$, there is small disc $y \in \Delta \subset Y$ such that tr Θ is zero along Δ. Therefore Θ is zero along Δ. Now 6.13 gives that the canonical images of the fibers of f over Δ are all isomorphic. We assumed that the canonical map of the fibers is birational, and therefore the fibers of f over Δ are all birational. By 2.9 this contradicts $\operatorname{Var} f = \dim Y$. The assumption $c_1(M_k)^n = 0$ leads to a contradiction. Therefore $c_1(M_k)^n > 0$, which proves the theorem. □

6.15. The theorem of the introduction now follows from 6.9, 6.10 and 6.14. This completes the proof. □

6.16. Mazur asked the following interesting question: Let X be a smooth projective variety, $\kappa(X) < \dim X$. Is it true that X contains either a rational curve or a subvariety birational to an abelian variety? This is true if $\dim X \leqslant 2$. It is easy to see that the minimal dimensional counterexample has $\kappa(X) \leqslant 0$. The result of the present article shows that it also satisfies $q(X) = 0$.

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