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# On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold

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***Dedicated to Professor Masayoshi Nagata on his sixtieth birthday***

## Introduction

Let $X$ be a connected compact Kähler manifold. Let $b_k$ be the Betti numbers of $X$. Then we know that 1) $b_k$ is even if $k$ is odd, and 2) the inequalities $b_k \leqq b_{k+2}$ hold for $0 \leqq k < \dim X$. Indeed, these are immediate consequences of the usual Hodge and the Lefschetz decomposition theorems for the (complex) cohomology group of $X$ respectively, which in turn are obtained via Hodge's theory of harmonic integrals (cf. [27] [28]). On the other hand, Chern in [8] showed that the Lefschetz decomposition theorem is actually a special case of a more general decomposition theorem for harmonic forms which are formulated for a general compact Riemannian manifold $M$ with prescribed holonomy group $G$. Indeed, the case where $G$ is the unitary group U(n) corresponds just to the Kähler case, where $2n = \dim_{\boldsymbol{R}} M$; while the case $G = \mathrm{Sp(n)}$, $4n = \dim_{\boldsymbol{R}} M$, corresponds in a certain sense (cf. below) to a compact Kähler symplectic manifold, that is, a compact Kähler manifold which admits a non-degenerate holomorphic 2-form. The main purpose of this paper is then to obtain certain analogues in this context of the Hodge and the Lefschetz decomposition theorems for a compact Kähler symplectic manifold $Y$ in general (cf. Theorems 3.11 and 3.16). At least they give as corollaries the following: 1) Every odd dimensional Betti number is divisible by four and 2) the inequalities $b_k + 3b_{k-4} \leqq 3b_{k-2} + b_{k-6}$ for Betti numbers hold for $k \leqq (1/2) \dim Y$. Moreover, with some more efforts we also obtain a more specific topological property of $Y$ to the effect that the homogeneous form $g$ of degree $2n$ on $H^2(Y, \boldsymbol{Q})$ defined by the cup product, $g(x) = x^{2n}[Y]$, is an $n$-ple power of a nondegenerate quadratic form up to constant (Theorem 4.7), the result which might have been expected by the results of Beauville [1].

We are also interested in the quaternionic Kähler case, that is, the

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case $G=\mathrm{Sp}(n)\mathrm{Sp}(1)$ (cf. Salamon [22]). In this case we obtain an analogue of the Lefschetz decomposition theorem in the positive scalar curvature case, and as an application, show that for such a manifold the associated quadratic form on the middle dimensional cohomology is always definite, though the proof in this case depends on the results of [22] while the harmonic theory as above serves only as a guide in formulating the results.

Now the outline of each section is as follows. Section 1 is preliminary. In Section 2 we consider a compact Lie subgroup $G$ of SO(d) together with its action on the complexified exterior algebra $\wedge F_C$, where $F=\boldsymbol{R}^d$. Namely we study the canonical decomposition of $\wedge F_C$ as $G$- and $H$-modules, where $H$ is a commutor of $G$ in $\mathrm{GL}_d(\boldsymbol{R})$. Our main interest here is in the following three cases: 1) $G=\mathrm{U(n)}$, $d=2n$, 2) $G=\mathrm{Sp(n)}$, $d=4n$, 3) $G=\mathrm{Sp(n)Sp(1)}$, $d=4n$. Then in Section 3 using the results of Section 2 first we formulate two decomposition theorems for a general compact Riemannian manifold with holonomy group (contained in) $G$, and then, corresponding to the above three cases, specialize successively to the Kähler, hyperkähler, and quaternionic Kähler cases. In the Kähler case they reduce respectively to the Hodge and the Lefschetz decompositions, and in the hyperkähler and quaternionic Kähler cases we obtain the above mentioned results. (Here we note that any hyperkähler manifold has necessarily a structure of a Kähler symplectic manifold, and vice versa in our compact case (cf. [1] [6] [14])). In Section 4 we treat compact Kähler symplectic manifolds and translate the above results for hyperkähler manifolds into those for such manifolds. Also we prove Theorem 4.7 mentioned above. For the proof we use the unobstructedness theorem for the Kuranishi family due to Bogomolov [3], for which we shall give a simple alternative proof at the end of this section by using the Calabi family; the latter is a holomorphic family of compact Kähler symplectic manifolds parametrized by the projective line $\boldsymbol{P}^1$, naturally associated with any hyperkähler manifold (cf. [6] [22]). In Section 5 we study some of its properties in connection with the results obtained in the previous sections.

Finally we note that all the above results including the Bogomolov unobstructedness theorem should be true for any compact symplectic $V$-manifold with Kähler $V$-metric. Indeed, the arguments should work word for word for $V$-manifolds under a suitable interpretation, though we have not checked this in every detail. In view of the existence of natural examples (cf. [10]) such a generalization would also be of some interest.

*Notation.* a) For a rational number $\gamma$, $[\gamma]$ denotes the greatest integer which does not exceed $\gamma$. For integers $a$, $b$, $m$, "$a\equiv b(m)$" means that $m$ divides $a-b$. Then for any nonnegative integer $k$ we set

$$(0.1)\qquad \Delta(k)=\{r;\ 0\leqq r\leqq k,\ k\equiv r\ (2)\}.$$

For an integer $r$ we set $r^+=\max(r, 0)$. b) For a complex vector space $E$, $\boldsymbol{P}(E)$ will denote the projective space of lines of $E$. c) For an algebra $A$ over a field $K$ and an element $a$ of $A$ the $K$-linear transformation of $A$ defined by the left multiplication by $a$ is called simply the $a$-multiplication operator and will often be denoted by $L_a\colon A\to A$. d) $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$: the fields of rational, real, and complex numbers, respectively. $\boldsymbol{Q}^\times=\boldsymbol{Q}-\{0\}$, $\boldsymbol{R}^\times=\boldsymbol{R}-\{0\}$, and $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$.

## § 1. Preliminaries

Let $\boldsymbol{H}$ be the real quaternion division algebra with standard real basis $1, i, j, k$. Let $\boldsymbol{H}^\times=\boldsymbol{H}-0$. We always consider $\boldsymbol{H}^n$ as a right $\boldsymbol{H}$-module with standard quaternion inner product $\langle\ ,\ \rangle$. Denote by $P$ the real subspace of pure quaternions. We have $\boldsymbol{H}=\boldsymbol{R}\oplus P$, and accordingly $\langle\ ,\ \rangle$ is decomposed into $\boldsymbol{R}$- and $P$-parts

$$(1.1)\qquad \langle\ ,\ \rangle=(\ ,\ )+\langle\ ,\ \rangle_P,$$

where $(\ ,\ )$ is the standard inner product on $\boldsymbol{H}^n=\boldsymbol{R}^{4n}$ and $\langle\ ,\ \rangle_P$ is a $P$-valued real alternating form on $\boldsymbol{H}^n=\boldsymbol{R}^{4n}$.

The unitary symplectic group Sp(n) is by definition the group of $\boldsymbol{H}$-linear automorphisms of $\boldsymbol{H}^n$ preserving $\langle\ ,\ \rangle$. Especially Sp(1) is identified with the group of quaternions of norm 1. We set

$$C=\{\lambda\in \mathrm{Sp}(1);\ \lambda^2=-1\}.$$

Then $\boldsymbol{H}^\times$ acts transitively on $C$ by inner automorphisms, with stabilizer at $i\in C$ given by $\boldsymbol{C}^*=\{a+bi\}\subseteqq\boldsymbol{H}^\times$. Therefore we have the natural identifications

$$(1.2)\qquad C=\boldsymbol{H}^\times/\boldsymbol{C}^*=\boldsymbol{P}^1$$

which will be fixed throughout this paper, where $\boldsymbol{P}^1$ is the complex projective line. Note that the naturally induced map $\pi\colon \boldsymbol{H}^\times\to C$ is given by

$$(1.3)\qquad \lambda=\pi(h)=hih^{-1},\quad h\in\boldsymbol{H}^\times.$$

Now we shall review some of the well-known facts on the (finite dimensional) representations of classical groups. We refer to [5] for the following terminology. A (complex or real) representation $\rho$ of a group $G$ is said to be *isotypical* if it is equivalent to a direct sum of one and the same irreducible representation, say $\rho_1$. In this case $\rho$ is also called $\rho_1$-*isotypical*, or $V_1$-*isotypical* if $V_1$ is the $G$-module corresponding to $\rho_1$. Let

$\mathfrak{R}=\mathfrak{R}_G$ be the set of equivalence classes of irreducible complex representations of $G$. Then, if $G$ is a compact Lie group, any $G$-module admits a unique direct sum decomposition

$$V=\bigoplus_{\rho\in\mathfrak{R}} V_\rho$$

of $V$ into "$\rho$-isotypical components" $V_\rho$. The same is also true for real representations. In general for any group such a decomposition, if any, is called the *canonical* ($G$-) *decomposition* of $V$.

*Representations of* U(n) *and* Sp(n) (cf. [29; VII]): The set of equivalence classes of irreducible complex representations of the unitary group U(n) (resp. the unitary symplectic group Sp(n)) is in natural bijective correspondence with the set of non-increasing sequences

$$\underline{f}:=(f_1,\cdots,f_n),\qquad f_1\geqq f_2\geqq\cdots\geqq f_n$$

of integers (resp. nonnegative integers) $f_i$. If $\rho_{\underline{f}}$ is an irreducible representation corresponding to $\underline{f}$, then $\underline{f}$ is called the *signature* of $\rho_{\underline{f}}$.

Let D(n) be the subgroup of diagonal elements of U(n) (resp. Sp(n));

$$\text{(1.4)}\qquad \begin{aligned}\mathrm{D(n)}=\{&\mathrm{diag}\,(\varepsilon_1,\cdots,\varepsilon_n)\ (\text{resp. diag}\,(\varepsilon_1,\cdots,\varepsilon_n,\varepsilon_1^{-1},\cdots,\varepsilon_n^{-1}));\\ &\varepsilon_i\in \boldsymbol{C},\ |\varepsilon_i|=1\}.\end{aligned}$$

Let $\chi_{\underline{f}}$ be the character of $\rho_{\underline{f}}$. Then $\chi_{\underline{f}|D(n)}$ is a Laurent polynomial of $\varepsilon_1,\cdots,\varepsilon_n$, i.e., a finite linear combination of the "Laurent monomials" $\varepsilon_1^{i_1}\cdots\varepsilon_n^{i_n}$, $i_\alpha\in\boldsymbol{Z}$. On the set of such Laurent monomials one puts the lexicographical order with respect to their indices $I=(i_1,\cdots,i_n)$. Then the highest nonzero term which appear in the polynomial expression for $\chi_{\underline{f}}$ is just $\varepsilon_1^{f_1}\cdots\varepsilon_n^{f_n}$. This fact indeed characterizes $\rho_{\underline{f}}$.

*Representations of* Sp(1) *and* $\boldsymbol{H}^\times$ (cf. [5; II.5]): Irreducible complex representations of U(1) (resp. Sp(1)) are up to equivalences indexed by integers (resp. nonnegative integers), say $r$. In the case of $\mathrm{U}(1)=\{\varepsilon\in\boldsymbol{C};|\varepsilon|=1\}$ these are 1-dimensional and given by $\rho_r(\varepsilon)=\varepsilon^r$. In the case of Sp(1) they are described as follows. Let $V_1$ be the underlying complex vector space of the right $\boldsymbol{H}$-module $\boldsymbol{H}$ with respect to the inclusion $\boldsymbol{C}=\{a+bi\}\subseteqq\boldsymbol{H}$. $V_1$ is then naturally a complex Sp(1)-module, and the induced Sp(1)-modules

$$V_r:=\mathrm{S}^r(V),\qquad r>0$$

are all irreducible, where $\mathrm{S}^r$ denote the symmetric products. We let $V_0=\boldsymbol{C}$ be the trivial Sp(1)-module. Then these $V_r$ give the desired representations $\rho_r$. They are all self-conjugate in the sense that $\rho_r$ is equivalent to its complex conjugate $\bar{\rho}_r$ (cf. (1.7) (1.8)).

Consider $\boldsymbol{C}^*$ and $\boldsymbol{H}^\times$ naturally as real algebraic groups Then any rational complex representation of $\boldsymbol{C}^*=\mathrm{U}(1)\times_{\boldsymbol{Z}_2}\boldsymbol{R}^\times$ (resp. $\boldsymbol{H}^\times=\mathrm{Sp}(1)\times_{\boldsymbol{Z}_2}\boldsymbol{R}^\times$) is equivalent to exactly one of the following representations $\rho_{k,r}$, $r$, $k\in\boldsymbol{Z}$, $r\geqq 0$, $k\equiv r$ (2);

$$(1.5)\qquad \rho_{k,r}(g)=t^k\rho_r(\varepsilon),\ g=(\varepsilon, t)\in \mathrm{U}(1)\times_{\boldsymbol{Z}_2}\boldsymbol{R}^\times\ (\text{resp. } \mathrm{Sp}(1)\times_{\boldsymbol{Z}_2}\boldsymbol{R}^\times).$$

We shall denote by $V_{k,r}$ the corresponding $\boldsymbol{H}^\times$-module.

*Real representations of* Sp(1) *and* $\boldsymbol{H}^\times$: i) When $r=2s-1$, $s\geqq 1$, is odd, then the underlying real representation $\rho_r'$ of $\rho_r$ is irreducible. The corresponding real Sp(1)-module, denoted by $V_r^{\boldsymbol{R}}(=V_r)$, has the natural structure of a right $\boldsymbol{H}$-module and the action of Sp(1) is $\boldsymbol{H}$-linear. Correspondingly we have

$$(1.6)\qquad \dim_{\boldsymbol{R}} V_r^{\boldsymbol{R}}=4s$$

and

$$(1.7)\qquad \rho_r'\otimes\boldsymbol{C}\cong\rho_r\oplus\rho_r.$$

ii) Let $r=2s$, $s\geqq 0$, be even. Then the right multiplication by $j$ on $\boldsymbol{H}=V_1$ induces a $\boldsymbol{C}$-antilinear involution $j_r$ on $V_r$, $r\geqq 1$. Further we define $j_0$ as the complex conjugation of $\boldsymbol{C}$. Then the set $V_r^{\boldsymbol{R}}$ of fixed points of $j_r$ is a $2s+1$ dimensional real vector space preserved by Sp(1). The induced real representation $\rho_r'$ of Sp(1) on $V_r^{\boldsymbol{R}}$ is irreducible and

$$(1.8)\qquad \rho_r'\otimes\boldsymbol{C}\cong\rho_r.$$

iii) The above representations $\rho_r'$, $r\geqq 0$, exhaust all the real irreducible representations of Sp(1) up to equivalence.

iv) $V_2^{\boldsymbol{R}}$ is equivalent to the Lie algebra $\mathfrak{sp}(1)$ of Sp(1) with the adjoint representation.

Any real rational irreducible representation $\rho$ of $\boldsymbol{H}^\times$ is up to equivalence of the form $\rho(g)=t^k\rho_r'(\varepsilon)$ for some $k\in\boldsymbol{Z}$ and $r\geqq 0$ in the notation of (1.5). The corresponding real $\boldsymbol{H}^\times$-module will be denoted by $V_{k,r}^{\boldsymbol{R}}$. We note in particular that for any irreducible representation of $\boldsymbol{H}^\times$ its restriction to Sp(1) also is irreducible.

*Hodge decomposition associated to a real representation of* $\boldsymbol{C}^*$ *and* $\boldsymbol{H}^\times$: A real (rational) $\boldsymbol{C}^*$ (resp. $\boldsymbol{H}^\times$) module $W$ is said to be of *weight* $k$ if $t\in\boldsymbol{R}^\times$ acts via the multiplication by $t^k$. In the case of a $\boldsymbol{C}^*$-module, $W$

then has the Hodge structure of weight $k$, i.e., the canonical decomposition of its complexification $W_C$ has the following form;

$$W_C = \bigoplus_{p+q=k} W^{p,q}, \qquad \overline{W}^{p,q} = W^{q,p},$$

where $W^{p,q}$ is the $\rho_{k,p-q}$-isotypical component. We set $h^{p,q} = \dim W^{p,q}$ and call it the Hodge number of $W$. We shall say that $W$ is *regular* if $h^{p,q}=0$ for $p<0$ or $q<0$.

Let $W$ be a real $\boldsymbol{H}^\times$-module of weight $k$. Any $\lambda \in C$ defines an embedding of real algebraic groups;

$$\text{(1.9)} \qquad \lambda_*: \boldsymbol{C}^* \hookrightarrow \boldsymbol{H}^\times, \quad \lambda_*(a+bi) = a+b\lambda, \quad a, b \in \boldsymbol{R}.$$

Via $\lambda_*$, $W$ has the structure of a real $\boldsymbol{C}^*$-module of weight $k$, and hence we have the associated Hodge decomposition with respect to $\lambda$;

$$\text{(1.10)} \qquad W_C = \bigoplus_{p+q=k} W^{p,q}(\lambda), \qquad \overline{W}^{p,q}(\lambda) = W^{q,p}(\lambda).$$

The Hodge numbers $h^{p,q}(W(\lambda))$ are actually independent of $\lambda$ so that we may denote it by $h^{p,q}(W)$.

First we consider the case $W = V^{\boldsymbol{R}}_{k,k}$. As the complexification of $\mathfrak{sp}(1)$ the Lie algebra $\mathfrak{sl}_2(\boldsymbol{C})$ acts $\boldsymbol{C}$-linearly on $W_C$. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

be the standard $\boldsymbol{C}$-basis of $\mathfrak{sl}_2(\boldsymbol{C})$. For any $\lambda \in C$ let

$$H_\lambda = ad(h)H, \quad X_\lambda = ad(h)X, \quad Y_\lambda = ad(h)Y$$

where $h \in \boldsymbol{H}^\times \subseteqq \mathrm{GL}_2(\boldsymbol{C})$ is any element with $\lambda = \pi(h)$ (cf. (1.3)) and ad denotes the adjoint action. Then (1.10) is just the eigenspace decomposition of $W_C$ with respect to the action of $H_\lambda$, where $W^{p,q}(\lambda)$ corresponds to the eigenvalue $p-q$. Further, from the standard facts on the representations of $\mathfrak{sl}_2(\boldsymbol{C})$ (cf. e.g. [5] [28]) we see easily the following: For $0 \leqq p$, $q \leqq k$, $\dim W^{p,q}(\lambda) = 2$ or $1$ according as $k$ is odd or even; otherwise, $W^{p,q}(\lambda) = 0$. In fact, the actions of $X_\lambda$ and $Y_\lambda$ have the property:

$$\text{(1.11)} \qquad \begin{aligned} &X_\lambda\colon W^{p,q}(\lambda) \rightleftarrows W^{p+1,q-1}(\lambda),\ 0 \leqq p < k,\ X_\lambda(W^{k,0}(\lambda)) = 0 \\ &Y_\lambda;\ W^{p,q}(\lambda) \rightleftarrows W^{p-1,q+1}(\lambda),\ 0 < p \leqq k,\ Y_\lambda(W^{0,k}(\lambda)) = 0. \end{aligned}$$

Suppose next that $W$ is isotypical so that $W \cong (V^{\boldsymbol{R}}_{k,r})^{\oplus h_0}$ for unique $k \in \boldsymbol{Z}$ and $r$, $h_0 \geqq 0$. Then by (1.7) and (1.8) $W_C$ is again isotypical as a complex $\boldsymbol{H}^\times$-module with $W_C \cong V^{\oplus h}_{k,r}$, where $h = h_0$ if $k$ is even and $=2h_0$

if $k$ is odd. Then by what we have seen above we get the next lemma easily.

**Lemma 1.1.** *If $W$ is isotypical as above, then in the Hodge decomposition* (1.10) *of $W_C$, $W^{p,q}(\lambda)\neq 0$ if and only if $(k-r)/2\leqq p, q\leqq(k+r)/2$; moreover for any $p$, $q$ in this range we have $h^{p,q}=h$ independently of $(p, q)$, where $h$ is even if $k$ is odd.*

In the general case, we say that an $\boldsymbol{H}^{\times}$-module $W$ is *regular* if it is regular as a real $\boldsymbol{C}^*$-module via some, and hence for any, $\lambda\in \boldsymbol{C}$. For a regular $\boldsymbol{H}^{\times}$-module $W$ its canonical decomposition takes the following form

$$W=\bigoplus_{r\in\Delta(k)} W^{k;r}$$

(cf. (0.1)), where $W^{k;r}$ is the $V^{\boldsymbol{R}}_{k,r}$-isotypical component, and the Hodge decomposition (1.10) of $W_C$ is the direct sum of those of $W^{k;r}_C$. We set

$$h^{k;r}(W):=(1/(r+1))\dim_{\boldsymbol{R}} W^{k;r}.$$

This equals any nonzero Hodge number $h^{p,q}(W^{k;r})$ of $W^{k;r}$. Then in view of Lemma 1.1 together with (1.6) and (1.11) we get the following:

**Lemma 1.2.** *Suppose that $W$ is regular of weight $k$ as above.* 1) *If $k$ is odd, then any Hodge number $h^{p,q}(W)$ is even and $4\,|\dim_{\boldsymbol{R}} W$.* 2) *For any $p\geqq q>0$ with $p+q=k$ we have the inequality $h^{p,q}(W)\geqq h^{p+1,q-1}(W)$; more precisely we have $h^{p,q}(W)-h^{p+1,q-1}(W)=h^{k;p+q-2}(W)$.* 3) *An element $\alpha$ of $W^{p,q}(\lambda)$ is contained in $W^{k;p-q}$ if and only if $X_\lambda\alpha=0$.*

We also note the obvious relations between $h^{p,q}(W)$ and $h^{k;r}(W)$;

$$h^{p,q}(W)=\sum_{r\geqq 2p-k} h^{k;r}(W)$$

and

$$h^{k;r}(W)=h^{p+1,q-1}(W)-h^{p,q}(W),\quad p=(k+r)/2,\quad q=(k-r)/2.$$

Returning again to the irreducible case of $W=W_s:=V_{2s,2s}$ we shall consider the associated period map $p_s$: $C\to\boldsymbol{P}_s:=\boldsymbol{P}(W)$ defined by $p_s(\lambda)=$ the line $W^{2s,0}(\lambda)$ in $W$.

**Lemma 1.3.** *$p_s$ is isomorphic to the Veronese embedding $\boldsymbol{P}^1\to\boldsymbol{P}^{2s}$. Further if we identify the tangent space of $\boldsymbol{P}_s$ at $p_s(\lambda)$ canonically with* $\mathrm{Hom}_C(H^{2s,0}(\lambda), H^{2s-1,1}(\lambda)\oplus\cdots\oplus H^{0,2s}(\lambda))$, *the differential $dp_s$ of $p_s$ at $\lambda$ factors through the subspace* $\mathrm{Hom}_C(H^{2s,0}(\lambda), H^{2s-1,1}(\lambda))$ *and gives an isomor-*

*phism* $T_\lambda \xrightarrow{\sim} \mathrm{Hom}_C(H^{2s,0}(\lambda), H^{2s-1,1}(\lambda))$, *where* $T_\lambda$ *is the tangent space of* $C$ *at* $\lambda$.

*Proof.* Take any $h \in \boldsymbol{H}^\times$ with $\lambda=\pi(h)=hih^{-1}$ (cf. 1.3)). Then we have $\lambda h=hi$; in other words the line $hC$ is the eigenspace with eigenvalue $\sqrt{-1}$ of the $\lambda$-action on $\boldsymbol{H}=V_1$. Hence $H^{2s,0}(\lambda)=\mathrm{S}^{2s}(hC)$ (symmetric product); this implies that $p_s$ is just the Veronese embedding. Let $C_s=p_s(C)$ and $\lambda_s=p_s(\lambda)$. Let $\pi_s$: $W-\{0\}\to\boldsymbol{P}_s$ be the natural projection and $\tilde{C}_s=\pi_s^{-1}(C_s)$. Let $\tilde{\lambda}_s$ be any point of $\pi_s^{-1}(\lambda_s)$. Now consider any element $A$ of $\mathfrak{sl}_2(\boldsymbol{C})$ as a vector field on $W$ with respect to its action on $W$. Note that $A$ is tangent to $\tilde{C}_s$ at any of its point. Then we see readily that the image of $Y_\lambda|_{\tilde{C}_s}$ by $d\pi_s$ at $\tilde{\lambda}_s$ is nonzero and thus generates the image $dp_s(T_\lambda)$ over $\boldsymbol{C}$. The result then follows from (1.11).

*Clebsch-Gordan formula*: Recall finally the Clebsch-Gordan formula (cf. [5]) which gives the isomorphism of Sp(1)-modules

$$V_k\otimes_C V_l\cong\bigoplus_s V_{k+l-2s}, \qquad 0\leqq s\leqq\min(k, l). \tag{1.12}$$

From this we get the irreducible decomposition of the symmetric product $\mathrm{S}^l(V_2)$ of $V_2$ as an Sp(1)-modules;

$$\mathrm{S}^l(V_2)\cong\bigoplus_{s\in\Delta(l)} V_{2s}. \tag{1.13}$$

Let $\Phi_0$ be any element of $\mathrm{S}^2(V_2)$ corresponding to the nonzero element of $V_0=\boldsymbol{C}$ in the above isomorphism $\mathrm{S}^2(V_2)\cong V_4\oplus V_0$. Then the multiplication by $\Phi_0$ gives an embedding of Sp(1)-modules $\mathrm{S}^{l-2}(V_2)\hookrightarrow \mathrm{S}^l(V_2)$ so that we have

$$\mathrm{S}^l(V_2)=\Phi_0\cdot\mathrm{S}^{l-2}(V_2)\oplus V_{2l}. \tag{1.14}$$

Using (1.12) and (1.13) we can also show the following:

**Lemma 1.4.** *Let* $l\equiv\varepsilon(2)$, *where* $\varepsilon=0$ *or* 1. *Let* $r$ *be a nonnegative integer. Then the canonical decomposition of the* Sp(1)-*module* $\mathrm{S}^l(V_2)\otimes_C V_r$ *is given by*

$$\mathrm{S}^l(V_2)\otimes_C V_r\cong\bigoplus_d V_d^{\pi(d)}$$

*where the nonnegative integer* $\pi(d)=\pi(d, l, r)$ *is given by*

$$\pi(d)=(\min((l+\varepsilon)/2, [(r+d+2\varepsilon)/4])+[-(|d-r|+2\varepsilon)/4]+1)^+.$$

*In particular* $\pi(d)\neq 0$ *if and only if* $\max(r-2l, |r_0|)\leqq d\leqq r+2l$, *where* $r_0$ *is defined by* $r+2\varepsilon\equiv r_0(4)$ *and* $-1\leqq r_0\leqq 2$.

## § 2. Representations on complexified exterior algebra

For a positive integer $d$ let $V_{\boldsymbol{R}}:=\boldsymbol{R}^d$ be the Euclidian space with the standard inner product ( , ). Let $V\cong\boldsymbol{C}^d$ be the complexification of $V_{\boldsymbol{R}}$ and $\wedge V=\bigoplus_{k=0}^{d}\wedge^k V$ the exterior algebra of $V$ over $\boldsymbol{C}$.

Let $G$ be a compact Lie subgroup of SO($d$) acting naturally on $V_{\boldsymbol{R}}$. The action extends to an action on $\wedge V$ as $\boldsymbol{C}$-algebra automorphisms. Let $\mathfrak{R}=\mathfrak{R}_G$ be the set of equivalence classes of irreducible complex representations of $G$. (We often identify a representation and its equivalence class.) Let

$$\wedge^k V=\bigoplus_{\rho\in\mathfrak{R}} V_\rho^k \tag{2.1}$$

be the canonical $G$-decomposition of $\wedge^k V$ into $\rho$-isotypical components $V_\rho^k$. For any $\rho\in\mathfrak{R}$ we set $V_\rho^*=\bigoplus_{k\geq 0}V_\rho^k$. Then we have $\wedge V=\bigoplus_{\rho\in\mathfrak{R}}V_\rho^*$.

Let $o\in\mathfrak{R}$ be the class of the trivial representation. Then

$$V_o^*=\bigoplus_k V_o^k\subseteq\wedge V$$

is a graded subalgebra of $\wedge V$ and each $V_\rho^*$ becomes a graded $V_o^*$-module.

Let $H$ be a commutor of $G$ in $\mathrm{GL}_d(\boldsymbol{R})$. Then $H$ acts naturally on $\wedge V$ and we have again the canonical decomposition

$$\wedge^k V=\bigoplus_{\mu\in\mathfrak{R}'} V_\mu^k,\qquad \mathfrak{R}'=\mathfrak{R}_H \tag{2.2}$$

of $\wedge^k V$ into $\mu$-isotypical components $V_\mu^k$. Moreover, the decompositions (2.1) and (2.2) are compatible with each other.

The inner product ( , ) and the standard orientation of $V_{\boldsymbol{R}}$ induce the Hodge $*$-operator

$$*:\wedge^k V\xrightarrow{\sim}\wedge^{d-k}V. \tag{2.3}$$

It is $G$- and $H_1$-equivariant and induces the isomorphisms

$$*_\rho: V_\rho^k\xrightarrow{\sim}V_\rho^{d-k},$$

where $H_1=H\cap\mathrm{SO(d)}$.

We also remark on the real structures of $V_\rho$ and $V_\mu$.

**Lemma 2.1.** 1) *$V_\rho$ is defined over $\boldsymbol{R}$,* i.e., *$V_\rho=\overline{V}_\rho$ if and only if $\rho$ is self-conjugate.* 2) *Suppose that $\mu|_{H_1}$ is irreducible. Then $V_\mu$ is defined over $\boldsymbol{R}$ if and only if $\mu|_{H_1}$ is self-conjugate.*

*Proof.* 1) The necessity is obvious. So suppose that $\rho$ is equivalent to its complex conjugate $\bar{\rho}$. Let $\mathfrak{R}_o$ be the set of equivalence classes of real irreducible representations of $G$. Let $\wedge V_{\boldsymbol{R}}=\bigoplus_{\rho_0\in\mathfrak{R}_0}V_{\rho_0}$ be the (real) canonical decomposition of $\wedge V_{\boldsymbol{R}}$. Then there exists a unique $\rho_0\in\mathfrak{R}_0$ such that $V_\rho\subseteqq V_{\rho_0,\boldsymbol{C}}$. By our assumption either $\rho_{0,\boldsymbol{C}}$ is irreducible or $\rho_{0,\boldsymbol{C}}\cong\rho_1\oplus\rho_1$ for some $\rho_1\in\mathfrak{R}$ (cf. [5; II. § 6]). In either case we have $V_\rho=V_{\rho_0,\boldsymbol{C}}$. The proof of 2) is the same.

Now the purpose of this section is to study more closely the above general construction in the following three special cases: 1) $d=2n$ and $G=\mathrm{U(n)}$, 2) $d=4n$ and $G=\mathrm{Sp(n)}$, and 3) $d=4n$ and $G=\mathrm{Sp(n)Sp(1)}$.

*Case* 1: $d=2n$ and $G=\mathrm{U(n)}$ (cf. Chern [8; § 4]): In this case we consider $V_{\boldsymbol{R}}=\boldsymbol{R}^{2n}$ as the underlying real Euclidian space of the complex Euclidian space $E=\boldsymbol{C}^n$ with the standard Hermitian inner product $\langle\ ,\ \rangle$. Then the natural action of $G$ on $E$ induces the embedding $G\hookrightarrow\mathrm{SO(2n)}$. Then as a $G$-module $V$ is the direct sum $V=E\oplus\bar{E}$, where $\bar{E}$ is the complex conjugate of $E$.

In this case $H=\boldsymbol{C}^*$ and the canonical $H$-decomposition (2.2) is just the usual type decomposition

$$\wedge^k V=\bigoplus_{p+q=k}V^{p,q},\qquad V^{p,q}=\wedge^p E\otimes\wedge^q\bar{E},$$

which is also the Hodge decomposition associated to the real $\boldsymbol{C}^*$-module $\wedge^k V_{\boldsymbol{R}}$ of weight $k$.

Let $\omega\in V_{\boldsymbol{R}}^{1,1}$ be the fundamental 2-form associated to the Hermitian inner product $\langle\ ,\ \rangle$. This is $G$-invariant, and in fact we have

$$V_0^*=\bigoplus_{k\geqq 0}\boldsymbol{C}\omega^k \tag{2.4}$$

in this case (cf. [2; IV. 2]). As usual let $L$: $\wedge V\to\wedge V$ be the $\omega$-multiplication operator and set $\Lambda=*^{-1}L*$. Then an element $v$ of $\wedge V$ is called *effective* (or *primitive*) if $\Lambda v=0$. Denote by $V_e^k$ the subspace of effective elements of $\wedge^k V$, and set $V_e^{p,q}=V_e^k\cap V^{p,q}$. Further for any integer $p,q\geqq 0$ with $p+q\leqq n$ we denote by $\rho(p,q)$ the representation of $G$ of signature $(1^p, 0^{n-(p+q)}, -1^q)$ (cf. § 1). Let $W(p,q)$ be the corresponding $G$-module. We know that (cf. [29; (5.14)])

$$\dim W(p,q)=\binom{n+1}{p}\binom{n+1}{q}(n+1-(p+q))/(n+1). \tag{2.5}$$

Then the following is essentially due to Chern [8; § 4].

**Proposition 2.2.** 1) $V_\rho^*\neq 0$ *if and only if* $\rho=\rho(p,q)$ *for some* $p,q$ *as above.* 2) $V_e^{p,q}$ *is an irreducible G-module isomorphic to* $W(p,q)$ *and we have* $V_\rho=\oplus_r L^r V_e^{p,q}$, $0\leqq r\leqq n-(p+q)$, *for* $\rho=\rho(p,q)$. 3) *The natural map* $\boldsymbol{C}\omega^r\otimes_{\boldsymbol{C}}V_e^{p,q}\to L^r V_e^{p,q}$ *is isomorphic for any* $r$ *as in* 2).

Here are some consequences of the proposition. From 1) and 2) we get the decomposition $V^{p,q}=\oplus_r L^r V_e^{p-r,q-r}$, $r \geqq (p+q-n)^+$, of $V^{p,q}$, or summing these up with respect to $p$, $q$ get the decomposition

$$(2.6) \qquad \wedge^k V = \bigoplus_{p+q=k} \Big( \bigoplus_{r \geqq (k-n)^+} L^r V_e^{p-r,q-r} \Big) = \bigoplus_{r \geqq (k-n)^+} L^r V_e^{k-2r}.$$

The first equality gives the canonical decomposition (2.1) in our special case and the third term gives the Lefschetz decomposition of the exterior algebra (cf. [28; V.3]). 3) then gives the isomorphism

$$(2.7) \qquad L^{n-k}\colon \wedge^k V \to \wedge^{2n-k} V, \qquad k<n$$

(the strong Lefschetz theorem for exterior algebra). The subsequent proof of the proposition thus gives an alternative proof for these facts.

*Proof of Proposition* 2.2. Fix an orthonormal basis $z_1, \cdots, z_n$ of $E$. Let D(n) be the diagonal group of U(n) as in (1.4). Then $z_{i_1} \wedge \cdots \wedge z_{i_p} \wedge \bar{z}_{j_1} \wedge \cdots \wedge \bar{z}_{j_q}$ form a $\boldsymbol{C}$-basis of $\wedge V$ consisting of eigenvectors of the induced D(n)-action with eigenvalues $\varepsilon_{i_1} \cdots \varepsilon_{i_p} \varepsilon_{j_1}^{-1} \cdots \varepsilon_{j_q}^{-1}$ in the notation of (1.4). From this and the characterization of representations by their characters as mentioned in Section 1 one already sees that $V_\rho = 0$ unless $\rho \cong \rho(p, q)$ for some $p$ and $q$.

Now fix $p$, $q \geqq 0$ with $p+q \leqq n$. Let

$$z(p, q) = z_1 \wedge \cdots \wedge z_p \wedge \bar{z}_{n-q+1} \wedge \cdots \wedge \bar{z}_n.$$

Then it is immediate to see that $z(p, q) \in V_e^{p,q}$ (cf. [27; I, n°4]). Moreover, the eigenvalue $\varepsilon_1 \cdots \varepsilon_p \varepsilon_{n-q+1}^{-1} \cdots \varepsilon_n^{-1}$ corresponding to $z(p, q)$ appears as the unique largest term in the character polynomial in $\varepsilon_1, \cdots, \varepsilon_n$ of D(n) on $V^{p,q}$ in the lexicographical order of indices. This implies that there exists a unique irreducible $G$-submodule, say $W_e^{p,q}$, of $V_e^{p,q}$ containing $z(p, q)$ and isomorphic to $W(p, q)$. Recall now that $\omega$ can be written in the form $\omega = (\sqrt{-1}/2) \sum_{i=1}^n z_i \wedge \bar{z}_i$ and hence if we set $k=p+q$,

$$\omega^{n-k} \wedge z(p, q) = (\sqrt{-1}/2)^{n-k} \Big( \prod_{p+1 \leqq i \leqq n-q} z_i \wedge \bar{z}_i \Big) \wedge z(p, q) \neq 0.$$

Since $W(p, q)$ is irreducible and $L$ is $G$-equivariant, this implies that $L^r\colon W_e^{p,q} \to L^r W_e^{p,q}$ is isomorphic for any $r \leqq n-k$. Since $p$, $q$ were arbitrary, we get the inequality

$$\binom{n}{p}\binom{n}{q} = \dim V^{p,q} \geqq \sum_{r=(p+q-n)^+}^{\min(p,q)} \dim W(p-r, q-r).$$

But, using (2.5) we calculate easily that this is actually an equality. Hence

$V^{p,q}$ is the direct sum of $L^r W_e^{p-r,q-r}$. Since $V_e^{p,q}$ is a $G$-submodule of $V^{p,q}$, if it is not irreducible, it contains $L^r W_e^{p-r,q-r}$ for some $r>0$ other than $W_e^{p,q}$, and hence in particular $L^r z(p-r, q-r) \in V_e^{p,q}$. However this is impossible because $\Lambda L^r z(p-r, q-r)\neq 0$ as a simple computation shows. Therefore $V_e^{p,q}$ are always irreducible and we get the direct sum decomposition $V^{p,q}=\bigoplus_{(p+q-n)+\leq r} L^r V_e^{p-r,q-r}$ and natural map $C\omega^r\otimes_C V_e^{p,q}\to L^r V_e^{p,q}$ is injective. From these all the assertions of the proposition follow.

*Case* 2: $d=4n$ and $G=\mathrm{Sp(n)}$. In this case we consider $V_{\boldsymbol{R}}=\boldsymbol{R}^{4n}$ as the underlying real Euclidian space of the right $\boldsymbol{H}$-module $\boldsymbol{H}^n$ with the standard quaternion inner product. Then the natural action of $G$ on $\boldsymbol{H}^n$ induces the natural inclusion $G\hookrightarrow \mathrm{SO(4n)}$ (cf. (1.1)).

The centralizer $H$ of $G$ is identified with $\boldsymbol{H}^\times$ and $H_1:=H\cap \mathrm{SO(4n)}$ with $\mathrm{Sp}(1)$ with the actions induced by the right $\boldsymbol{H}$-module structure of $\boldsymbol{H}^n$ via the inclusions $\mathrm{Sp}(1)\subseteq \boldsymbol{H}^\times\subseteq \boldsymbol{H}$. Any $\lambda\in C$ defines a complex structure $J_\lambda$ on $V_{\boldsymbol{R}}$ via the right multiplication. Associated to $J_\lambda$ we get the usual type decomposition

$$\wedge^k V=\bigoplus_{p+q=k} V^{p,q}(\lambda)$$

of $\wedge^k V$ into $(p, q)$-covectors, and this coincides by definition with the Hodge decomposition of the $\boldsymbol{H}^\times$-module $\wedge^k V$ with respect to $\lambda$ (cf. (1.10)). It follows that $\wedge^k V$ is a regular $\boldsymbol{H}^\times$-module of weight $k$, and the canonical $\boldsymbol{H}^\times$-decomposition of $\wedge^k V$ takes the form

$$\wedge^k V_{\boldsymbol{R}}=\bigoplus_{r\in\Delta(\hat{k})} V^{k;r},\qquad \hat{k}=\min(k, 4n-k), \tag{2.8}$$

where $V^{k;r}$ is the $V_{k,r}^{\boldsymbol{R}}$-isotypical component (cf. (0.1)). Here the inequality $r\leqq 4n-k$ comes from the $*$-isomorphism (2.3).

Now for any $\lambda\in C$ let $\omega_\lambda\in V_{0,\boldsymbol{R}}^2\cap V^{1,1}(\lambda)$ be defined by

$$\omega_\lambda(x, y)=(x, y\lambda),\qquad x, y\in V_{\boldsymbol{R}}=\boldsymbol{H}^n, \tag{2.9}$$

where we have identified $V_{\boldsymbol{R}}$ with its dual via the inner product ( , ). ($\omega_\lambda$ is just the fundamental 2-form associated to ( , ) and the isometric complex structure $J_\lambda$.) Then $V_{0,\boldsymbol{R}}^*$ is generated by $\omega_\lambda$, $\lambda\in C$, as an $\boldsymbol{R}$-algebra (cf. [2; IV. 2]). More precisely we have the following:

**Lemma 2.3.** *$V_0^k=0$ if $k$ is odd. If $k=2l$ is even, then $V_0^k\cong \mathrm{S}^l(V_{2,2})$ as an $\boldsymbol{H}^\times$-module if $k\leqq 2n$ and in general the canonical $\boldsymbol{H}^\times$-decomposition of $V_0^k$ takes the form $V_0^k=\bigoplus_{s\in\Delta(\hat{l})} V_0^{k;2s}$, $\hat{l}=\min(l, 2n-l)$, where $V_0^{k;2s}\cong V_{k,2s}^{\boldsymbol{R}}$. In particular $V_0^2\cong V_{2,2}^{\boldsymbol{R}}$.*

*Proof.* The first assertion is clear. So let $k=2l$. Then we have the natural surjection $\mathrm{S}^l(V_2)\to V_0^k$. For $k\leqq 2n$, in view of (1.13) it thus suffices to show that $\dim V_0^k = \dim S^l(V_2) = (s+1)(s+2)/2$. This in fact will be shown in a more general setting in the proof of Proposition 2.4, 5) below. (So we omit the proof here.) The case $k > 2n$ then follows from this and the $*$-isomorphism (2.3).

We next study the structure of each $V_\rho$ as a $V_0$-module. For any $\lambda \in C$ let $L_\lambda \colon \wedge V \to \wedge V$ be the $\omega_\lambda$-multiplication operator and set $\Lambda_\lambda = *^{-1} L_\lambda *$. Then an element $\alpha \in \wedge^k V$ is said to be *universally effective* if $\Lambda_\lambda \alpha = 0$ for any $\alpha \in C$. We denote by $V_\varepsilon^k$ the subspace of all the universally effective $k$-covectors. Then $V_\varepsilon^k$ is $G$-invariant, each $\lambda$ being $G$-invariant. Moreover, since $g(\omega_\lambda) = \omega_{g^{-1}\lambda g}$ for any $\lambda \in C$ and any $g \in \boldsymbol{H}^\times$, $V_\varepsilon^k$ is even an $\boldsymbol{H}^\times$-module. Therefore if we set $V_\varepsilon^{k;r} = V_\varepsilon^k \cap V_C^{k;r}$ we have the canonical decomposition of $V_\varepsilon^k$ as a complex $\boldsymbol{H}^\times$-module;

$$V_\varepsilon^k = \bigoplus_{r \in \Delta(\hat{k})} V^{k;r}. \tag{2.10}$$

We also set $V_\varepsilon^* = \bigoplus_{k \geq 0} V_\varepsilon^k$.

Define the set $\Psi$ of pairs of integers by

$$\Psi = \{(k, r);\, r \in \Delta(k),\, k + r \leqq 2n\}. \tag{2.11}$$

For any pair of integers $(k, r) \in \Psi$ we denote by $\rho(k; r)$ the irreducible representation of $G$ of signature $(2^{(k-r)/2}, 1^r, 0^{n-(k+r)/2})$ (cf. § 1). Let $W(k; r)$ be the corresponding $G$-module. We denote by

$$\mathrm{Sp(n)Sp(1)} \tag{2.12}$$

the subgroup of SO(4n) generated by $G = \mathrm{Sp(n)}$ and Sp(1).

**Proposition 2.4.** 1) *For any* $\rho \in \Re$, $V_\rho^* \neq 0$ *if and only if* $\rho \cong \rho(k; r)$ *for some* $(k, r) \in \Psi$. $V_\rho^*$ *is always defined over* $\boldsymbol{R}$. 2) $V_\varepsilon^{k;r}$ *if and only if* $(k, r) \in \Psi$. 3) *If, further,* $k \leqq n$, *then* $V_\varepsilon^{k;r}$ *is an irreducible* $\neq 0$ Sp(n)Sp(1)*-module isomorphic to* $W(k; r) \otimes_C V_r$. *In particular* $V_\varepsilon^{k;r} \subseteqq V_{\rho(k;r)}$. 4) $\wedge V$ *is generated by* $V_\varepsilon^*$ *as a* $V_0^*$*-module. Further for any* $l \leqq n$ *and any* $\rho = \rho(k; r)$, $V_\rho^l$ *is generated by* $V_\varepsilon^{k;r}$ *as a* $V_0^*$*-module;* $V_\rho^l = V_0^{l-k} \wedge V_\varepsilon^{k;r}$. 5) *For any triple* $(l, k, r)$ *of integers such that* $(l, r) \in \Psi$, $k \in \Delta(l)$, $r \in \Delta(k)$ (cf. (0.1)) *and* $k \leqq n$ *the natural linear map* $V_0^{l-k} \otimes_C V_\varepsilon^{k;r} \to V_0^{l-k} \wedge V_\varepsilon^{k;r}$ *is isomorphic.*

**Remark 2.5.** It seems that 3) is true for any $(k, r) \in \Psi$. Then the proof below would show that 4) is also true for any $l$ and 5) is true without the assumption that $k \leqq n$ (though the assumption on $(l, r)$ is necessary).

The above proposition completely determines the structure of $\wedge^l V$ for $l \leqq n$ as an Sp(n)Sp(1)-module; namely we have

$$\wedge^l V = \bigoplus_{k \in \Delta(l)} \bigoplus_{r \in \Delta(k)} V^l_{\rho(k;r)}, \qquad l \leqq n+1$$

where for $l \leqq n$

$$V^l_{\rho(k;r)} \cong V_0^{l-k} \otimes_C V_\varepsilon^{k;r} \cong (V_0^{l-k} \otimes_C V_r) \otimes_C W(k; r)$$

as an Sp(n)Sp(1)-module. Further the structure of $V_0^{l-k} \otimes_C V_r$ as an Sp(1)-module can be determined completely by Lemmas 1.4 and 2.3.

Before the proof of Proposition 2.4 we make some preliminary considerations. Choose the orientation of the space $P$ of pure quaternions by requiring $i, j, k$ to be an oriented basis of $P$. Let $\lambda$ be any element of $C$. Take $\lambda_1, \lambda_2 \in C$ in such a way that $\{\lambda, \lambda_1, \lambda_2\}$ is an oriented orthonormal basis of $P$. We set

$$\varphi_\lambda = \omega_{\lambda_1} + \sqrt{-1}\, \omega_{\lambda_2}. \tag{2.13}$$

(Up to multiplicative constants $a$ with $|a|=1$, $\varphi_\lambda$ depends only on $\lambda$ but not on the choice of $\lambda_1, \lambda_2$.) Then we have

$$\varphi_\lambda \in V_0^{2,0}(\lambda), \quad \omega_\lambda \in V_0^{1,1}(\lambda), \quad \bar{\varphi}_\lambda \in V_0^{0,2}(\lambda); \tag{2.14}$$

in particular $\varphi_\lambda, \omega_\lambda, \bar{\varphi}_\lambda$ form a $C$-basis of $V_0^2$, where $\bar{\varphi}_\lambda = \omega_{\lambda_1} - \sqrt{-1}\, \omega_{\lambda_2}$ is the complex conjugate of $\varphi_\lambda$.

In what follows we shall consider only the case $\lambda = i$, and accordingly, we omit the index $\lambda$ in the relevant notation, for instance $V^{p,q} = V^{p,q}(\lambda)$ and (2.14) will be denoted by $\varphi \in V^{2,0}$, $\omega \in V^{1,1}$, $\bar{\varphi} \in V^{0,2}$. We set $E = V^{1,0}$ and $\bar{E} = V^{0,1}$. Let $L_\varphi$ (resp. $L_{\bar{\varphi}}$): $\wedge V \to \wedge V$ be the $\varphi$- (resp. $\bar{\varphi}$-) multiplication operator and set $\Lambda_\varphi = *^{-1} L_{\bar{\varphi}} *$. Then $\Lambda_\varphi$ sends $V^{p,q}$ into $V^{p-2,q}$. Then we have the following:

**Proposition 2.6.** 1) *We have the direct sum decomposition*

$$V^{p,0} = \bigoplus_{r \geqq (n-p)^+} L^r V_{e(\varphi)}^{p-2r}, \qquad r \geqq (n-p)^+ \tag{2.15}$$

*which is orthogonal with respect to the natural Hermitian metric on $V^{p,0}$, where $V_{e(\varphi)}^{s,0} := \{\alpha \in V^{s,0};\ \Lambda_\varphi \alpha = 0\}$ is the space of $\varphi$-effective holomorphic $s$-forms.* 2) *The linear map $L_\varphi^{n-p}$: $V^{p,0} \to V^{2n-p,0}$ is isomorphic for any $p < n$.*

*Proof.* (cf. also [5; p 271 ff]). As in Weil [27; I, n° 3.4] one can readily show that for any $p \geqq 0$

$$[L_\varphi, \Lambda_\varphi] := L_\varphi \Lambda_\varphi - \Lambda_\varphi L_\varphi = 4(p-n)\, id, \qquad \text{on } V^{p,0}$$

where *id* denotes the identity. As usual, this implies the existence of the natural action of the Lie algebra $\mathfrak{sl}_2(C)$ on the vector space $\wedge E$ and this leads to the results 1) and 2) (cf. [28; V]). Note that the orthogonality of (2.15) follows from the fact that $L_\varphi$ and $\Lambda_\varphi$ are adjoint to each other with respect to the metric on $V^{p,0}$.

As in [27; p. 28, Cor.] we obtain the following:

**Corollary 2.7.** 1) $V^{p,0}_{e(\varphi)}\neq 0$ *if and only if* $0<p\leqq n$, *and* 2) *in this range a p-form* $\alpha$ *is* $\varphi$*-effective if and only if* $L_\varphi^{n-p+1}\alpha=0$.

We now fix once and for all a basis

$$z_1, \cdots, z_n, z_1', \cdots, z_n' \tag{2.16}$$

of $E$ such that with respect to this basis $\varphi$ and $\omega$ are written in the form

$$\varphi=\sum_{i=1}^{n} z_i\wedge z_i' \quad \text{and} \quad \omega=(\sqrt{-1}/2)\Big(\sum_{i=1}^{n} z_i\wedge\bar{z}_i+\sum_{j=1}^{n} z_j'\wedge\bar{z}_j'\Big). \tag{2.17}$$

We also need the following lemmas.

**Lemma 2.8.** *Let* $X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\in\mathfrak{sl}_2(\boldsymbol{C})$. *Let* $X_*\colon \bar{E}\to E$ *be the* $\boldsymbol{C}$*-linear isomorphism with respect to the action of* $\mathfrak{sl}_2(\boldsymbol{C})$ *on* $V=E\oplus\bar{E}$ *induced by the action of* $\mathfrak{sp}(1)$ *on* $V_{\boldsymbol{R}}$ *(cf. (1.11)). Then there exists a nonzero constant c such that* $X_*(\bar{z}_i)=cz_i'$ *and* $X_*(\bar{z}_j')=-cz_j$ *for any i and j.*

*Proof.* Let $\varphi_*\colon E\to E^*$ and $\omega_*\colon \bar{E}\to E^*$ be the $\boldsymbol{C}$-linear isomorphisms induced respectively by $\varphi$ and $\omega$, where $\varphi$ (resp. $\omega$) is considered as a non-degenerate alternating (resp. Hermitian) form on $E$ and $E^*$ is the dual of $E$. Since both $X_*$ and $\varphi_*\omega_*^{-1}$ are Sp(n)-isomorphisms and since both $E$ and $\bar{E}$ are Sp(n)-irreducible, by Schur's lemma $X_*=c\varphi_*\omega_*^{-1}$ for some nonzero constant $c$. The lemma then follows from this and (2.17).

**Lemma 2.9.** *Let G be a group acting on a complex vector space F. Let* $\bar{F}$ *be the complex conjugate of F as a G-module. Suppose that there exists a G-invariant Hermitian inner product* $\langle\ ,\ \rangle$ *on F and a decomposition* $F=F_1\oplus\cdots\oplus F_k$ *of F into a direct sum of G-modules. Then if* $\dim_{\boldsymbol{C}}(\bar{F}\otimes_{\boldsymbol{C}}F)^G=k$, *each* $F_i$ *is an irreducible G-module, where* $(\ \ )^G$ *denotes the set of G-invariants.*

*Proof.* $\langle\ ,\ \rangle$ induces the natural $G$-isomorphism of $\bar{F}$ with the dual $F^*$ of $F$. It follows that the condition of the lemma is equivalent to: $\dim_{\boldsymbol{C}}\mathrm{End}_G F=k$, where $\mathrm{End}_G F$ is the vector space of $G$-linear endomorphisms of $F$. Then from

$$\sum_{i=1}^{k} \dim_C \mathrm{End}_G F_i \leqq \dim_C \mathrm{End}_G F$$

we see that $\dim_C \mathrm{End}_G F_i = 1$; hence $F_i$ is irreducible.

**Lemma 2.10.** *Let $G$, $H$ be compact (real) Lie groups. Let $U$, $V$ and $W$ be complex $(G\times H)$-modules. Let $f$: $V\otimes_C W \to U$ be a $(G\times H)$-homomorphism. Suppose that $V$ is $(G\times H)$-irreducible and $W$ is $G$-trivial. Suppose further that there exists a nonzero irreducible $H$-submodule $V'$ of $V$ such that the induced homomorphism $f|_{V'\otimes_C W}$: $V'\otimes_C W \to U$ is injective. Then $f$ itself is injective.*

*Proof.* From the $(G\times H)$-irreducibility of $V$ follows the existence of an irreducible $G$-module $V_1$ and an irreducible $H$-module $V_2$ such that there exists an isomorphism $h$: $V \to V_1 \otimes_C V_2$ of $(G\times H)$-modules. Moreover, we can find a nonzero vector $u \in V_1$ such that $h(V') = \boldsymbol{C}u\otimes V_2$. The kernel $K$ of $f$ is a $(G\times H)$-submodule of $V\otimes_C W \cong V_1\otimes_C (V_2\otimes_C W)$. Hence, since $V_1$ is $H$-irreducible and $W$ is $G$-trivial there exists an $H$-submodule $M$ of $V_2\otimes_C W$ such that $K = V_1\otimes_C M$. Then by our assumption, $\boldsymbol{C}u\otimes_C M = (V_1\otimes_C M)\cap(\boldsymbol{C}u\otimes_C V_2) = 0$. Hence $M = K = 0$.

*Proof of Proposition* 2.4. We use the basis (2.16) of $E$. Then the totality of $z_I\wedge z'_J\wedge z_K\wedge z'_L$, where $I=(i_1, \cdots, i_s)$, $J=(j_1, \cdots, j_{s'})$, $K=(k_1, \cdots, k_t)$, $L=(l_1, \cdots, l_{t'})$, $s$, $s'$, $t$, $t'\geqq 0$, are any increasing sequences of integers from $\{1, \cdots, n\}$, forms a basis of $\wedge V$ consisting of eigenvectors for the action of the diagonal group D(n) of Sp(n) with eigenvalues $\varepsilon_I\varepsilon_J^{-1}\varepsilon_K^{-1}\varepsilon_L$. Here the multi-index notation is used. Now set for any $0\leqq p$, $q\leqq n$

$$\alpha_{p,q} = z_1\wedge\cdots\wedge z_p\wedge \bar{z}'_1\wedge\cdots\wedge\bar{z}'_q. \tag{2.18}$$

If $q\leqq p$, the eigenvalue $\varepsilon_1^2\cdots\varepsilon_q^2\varepsilon_{q+1}\cdots\varepsilon_p$ of $\alpha_{p,q}$ appears as the highest term in the character polynomial in $\varepsilon_1, \cdots, \varepsilon_n$ of the induced action of D(n) on the Sp(n)-module $V^{p,q}$ with respect to the lexicographical order. From these we conclude as in the proof of Proposition 2.2 the first assertion of 1).

The second assertion can be seen as follows. For any $\rho\in\mathfrak{R}$, $V_\rho^k$ is also an Sp(1)-module. Since any complex Sp(1)-module is self-conjugate, by Lemma 1.4 $V_\rho^k$ is defined over $\boldsymbol{R}$.

Next we show 2). For any pair $(k, r)$ of nonnegative integers with $r\in\Delta(k)$ we set $p=(k+r)/2$ and $q=(k-r)/2$. Suppose first that $k+r\leqq 2n$, i.e., $(k, r)\in\Psi$. Then $\alpha=\alpha_{p,q}$ makes sense and we compute readily that $\Lambda_\varphi\alpha = \Lambda_{\bar\varphi}\alpha = \Lambda_\omega\alpha = 0$. Since $\Lambda_\beta$ is linear with respect to $\beta$ and $\{\varphi, \bar\varphi, \omega\}$ is a basis of $V_0^2$, this implies that $\alpha \in V_\varepsilon^{p,q} := V^{p,q} \cap V_\varepsilon^k$. Moreover by Lemma 2.8 we get that $X_*(\alpha)=0$, by noting that $X_*$ operates on $\wedge V$ as a derivation. Hence by 3) of Lemma 1.2, $\alpha \in V_\varepsilon^{k;r}$, and hence $V_\varepsilon^{k;r} \neq 0$. Suppose next that $k+r>2n$, i.e., $p>n$. Then by 2) of Proposition 2.6 $L_\varphi$: $V^{p-2,q} \to V^{p,q}$ is surjective. Hence its Hermitian adjoint $\Lambda_\varphi$: $V^{p,q} \to V^{p-2,q}$ is injective so that the Hodge $(p, q)$-component of $V_\varepsilon^{k;r}$ must vanish. In view of Lemma 1.1 this implies that $V_\varepsilon^{k;r}=0$ as $V_\varepsilon^{k;r}$ is $V_{k,r}^{\boldsymbol{R}}$-isotypical.

*Proof of* 3): For simplicity we consider only the case where $k$ is even. Write $k=2s$. Denote by $\hat{V}$, $\hat{V}_\varepsilon^k$, $\hat{V}_\varepsilon^{k;r}$ the Hodge $(s, s)$-components of $\wedge^k V$, $V_\varepsilon^k$, $V_\varepsilon^{k;r}$ respectively. First we shall show that $\hat{V}_\varepsilon^{k;r}$ is Sp(n)-irreducible. From (2.8) we have the direct sum decomposition $\hat{V}_\varepsilon^k = \bigoplus_{r \in \Delta(k)} \hat{V}_\varepsilon^{k;r}$, where, since $k \leqq n$, each summand $\hat{V}_\varepsilon^{k;r}$ is nonzero by 2) so that the number of summands equals $s+1$. Set

$$B=\hat{V}_\varepsilon^k \otimes_C \hat{V}_\varepsilon^k \quad \text{and} \quad C=\hat{V}^k \otimes_C \hat{V}^k.$$

Then by Lemma 2.9 it suffices to show that $\dim_C B^G \leqq s+1$ for $G=\mathrm{Sp(n)}$. First we note that $B^G \subseteqq C^G$. Set

$$\begin{aligned} &\varphi_1=\varphi\otimes 1 \in V^{2,0}\otimes V^{0,0}, \quad \varphi_2=1\otimes\varphi \in V^{0,0}\otimes V^{2,0}, \quad \omega_1=\omega\otimes 1 \in V^{1,1}\otimes V^{0,0} \\ &\bar{\varphi}_1=\bar{\varphi}\otimes 1 \in V^{0,2}\otimes V^{0,0}, \quad \bar{\varphi}_2=1\otimes\bar{\varphi} \in V^{0,0}\otimes V^{0,2}, \quad \omega_2=1\otimes\omega \in V^{0,0}\otimes V^{1,1}. \end{aligned}$$

We regard $C$ as a subspace of $\wedge E \otimes \wedge \bar{E} \otimes \wedge E \otimes \wedge \bar{E}$ via the natural isomorphism $C \cong \wedge^s E \otimes \wedge^s \bar{E} \otimes \wedge^s E \otimes \wedge^s \bar{E}$. Let $K$ be the subspace of $C$ generated by those elements which are multiples of any of $\varphi_i$, $\bar{\varphi}_j$, $\omega_k \in (\wedge E \otimes \wedge \bar{E} \otimes \wedge E \otimes \bar{E})^G$. By the orthogonality of the Lefschetz decompositions (2.6) and (2.12) we see that $K$ is orthogonal to $B$ with respect to the natural induced metric on $C$. Therefore it suffices to show that $\dim_C C^G/K^G \leqq s+1$. In fact, we can find $s+1$ elements of $C^G$ which generate $C^G/K^G$ as follows. Let $j_{14}$: $\wedge E \otimes \wedge \bar{E} \to \wedge E \otimes \wedge \bar{E} \otimes \wedge E \otimes \wedge \bar{E}$ be the inclusion defined by $j_{14}(x\otimes y)=x\otimes 1\otimes 1\otimes y$, $x \in \wedge E$, $y \in \wedge \bar{E}$. Similarly, we define the inclusions $j_{32}$, $j_{13}$, $j_{24}$. Then we set

$$\omega_{14}=j_{14}(\omega), \quad \omega_{32}=j_{32}(\omega), \quad \varphi_{13}=j_{13}(\varphi), \quad \bar{\varphi}_{24}=j_{24}(\varphi)$$

under the identification

$$\omega \in E\otimes\bar{E} \subseteqq \wedge E \otimes \wedge \bar{E} \quad \text{and} \quad \varphi \in \wedge^2 E \subseteqq E\otimes E \subseteqq \wedge E \otimes \wedge E.$$

Then by using the classical invariant theory [29] one sees readily that the elements

$$\omega_{14}^{s-j}\otimes\omega_{32}^{s-j}\otimes\varphi_{13}^j\otimes\bar{\varphi}_{24}^j, \quad 0\leqq j\leqq s$$

have the required properties. Thus $\hat{V}_\varepsilon^{k;r}$ is Sp(n)-irreducible.

Since any nonzero Hodge component $W_\varepsilon^{s,t}$ of $V_\varepsilon^{k;r}$ are Sp(n)-isomorphic to one another (cf. (1.11)), this implies in particular that $W_\varepsilon^{p,q}$ is Sp(n)-irreducible, where $p=(k+r)/2$ and $q=(k-r)/2$ as before. Then since $\alpha_{p,q}\in W_\varepsilon^{p,q}$, by the first part of the proof we see that $W_\varepsilon^{p,q}\cong W(k;r)$ as an Sp(n)-module (cf. the proof of Proposition 2.2). Finally since $V_\varepsilon^{k;r}\cong W_\varepsilon^{p,q}\otimes_C V_r$ as an Sp(n)Sp(1)-module, 3) follows.

*Proof of* 4). Let $\alpha$ be any element of $\wedge^l V$. Then by using the Lefschetz decomposition (2.6) with respect to $\omega_\lambda$ for $\lambda=i, j, k$, we conclude easily that there exists a (non-commutative) polynomial $\Phi(X, Y, Z; X', Y', Z')$ with rational coefficients such that if we set $\alpha_0=\Phi(L_i, L_j, L_k; \Lambda_i, \Lambda_j, \Lambda_k)\alpha$, then $\alpha_0\in\wedge^k V$, $\Lambda_\lambda\alpha_0=0$ for $\lambda=i, j, k$ and $\alpha$ is written in the form $\alpha=L_i^s L_j^t L_k^u\alpha_0$ for some $s, t, u\geq 0$, where $k=l-2(s+t+u)$ (cf. [27; p. 26, Th. 3], [28; V. Th. 3.12]). Since $\Lambda_\lambda$ is linear with respect to $\lambda$ and $i, j, k$ form a basis of $V_{0,R}^2$ it follows that $\alpha_0\in V_\varepsilon^k$. This shows the first half of 4). Further, in this argument if $\alpha\in V_\rho^l$ for some $\rho$, then we have again $\alpha_0\in V_\rho^k$ since both $L_\lambda$ and $\Lambda_\lambda$ are Sp(n)-invariant. In particular if $\rho=\rho(k;r)$ and $k\leq n$, this implies that $\alpha_0\in V_\rho^k\cap V_\varepsilon^k=V_\varepsilon^{k;r}$ by 3). This proves the second statement.

*Proof of* 5). Let $p=(k+r)/2$ and $q=(k-r)/2$ as before. Let $m=n-p$. Set $u_i=z_{q+i}$, $u_i'=z_{q+i}'$, $1\leq i\leq r$. Further we set $\beta=\prod_{i=1}^q z_i\wedge z_i'$. For any $0\leq u, v\leq r$ with $u+v=r$ we define

$$Z_{uv}=\beta\wedge\sum_I s(I)u_I\wedge\bar{u}_{I'}',$$

where the summation is over all the ordered $u$-tuples $I=(i_1, \cdots, i_u)$ with $1\leq i_1<\cdots<i_u\leq r$, $I'=(j_1, \cdots, j_v)$, $1\leq j_1<\cdots<j_v\leq r$ with $\{1, \cdots, r\}=\{i_1, \cdots, i_u, j_1, \cdots, j_v\}$, $s(I)$ is the signature of the permutation $(I, I')$ of $\{1, \cdots, r\}$, and $u_I=u_{i_1}\wedge\cdots\wedge u_{i_u}$, $u_{I'}'=u_{j_1}'\wedge\cdots\wedge u_{j_v}'$. Let $V'$ be the smallest complex Sp(1)-submodule (or equivalently, $\mathfrak{sl}_2(C)$-submodule) of $V_\varepsilon^{k;r}$ containing $Z_{r0}=\pm\alpha_{p,q}\in V_\varepsilon^{k;r}$ (cf. (2.18)). In view of Lemma 2.8, it then readily follows that $Z_{u,r-u}$, $0\leq u\leq r$, form a $C$-basis of $V'$. Since $(l, r)\in\Psi$ we have

$$(l-k)/2\leq n-(q+r). \tag{2.19}$$

Let $s=(l-k)/2$. For any integers $a, b, c\geq 0$ with $a+b+c=s$, we set $\Omega_{abc}=\varphi^a\bar{\varphi}^b\omega^c$. Let $B$ be the set of all such $\Omega_{abc}$. Then $B$ is a $C$-basis of $V_0^{l-k}$ (cf. the proof below for the case $k=0$ so that $V_\rho^{k;r}=V_0^{0;0}=C$). Therefore $A:=\{\Omega_{abc}\otimes Z_{uv};\ \Omega_{abc}\in B, u+v=r\}$ is a $C$-basis of $V_0^{l-k}\otimes V'$. Now by 3) we can apply Lemma 2.10 to the case where $G=\text{Sp(n)}$, $H=\text{Sp(1)}$, $V=V_\varepsilon^{k;r}$, $W=V_0^{l-k}$, $f$ is the linear map of the proposition and $V'$

is as above; thus it suffices to show the injectivity of the natural map $V_0^{l-k} \otimes V' \to \wedge^l V$. Let

$$\bar{A} = \{\Omega_{abc} \wedge Z_{uv};\ \Omega_{abc} \otimes Z_{uv} \in A\}.$$

We have only to show that the elements of $\bar{A}$ are linearly independent; indeed, among the elements of $\bar{A}$, $\Omega_{abc} \wedge Z_{uv}$ is characterized, as one sees readily from (2.19), by the property that it contains the term

$$\beta \wedge \left(\prod_1^u u_\mu\right) \wedge \left(\prod_{u+1}^r \bar{u}'_\nu\right) \wedge \left(\prod_1^a w_i \wedge w'_i\right) \wedge \left(\prod_{a+1}^{a+b} \bar{w}_j \wedge \bar{w}'_j\right) \wedge \left(\prod_{a+b+1}^m w_k \wedge \bar{w}'_k\right)$$

with nonzero coefficients in the expansion in $z_i$, $z'_i$, where $w_j = z_{p+j}$ and $w'_j = z'_{p+j}$, $1 \leqq j \leqq m$.

*Case* 3: $d = 4n$ and $G = \mathrm{Sp(n)Sp(1)}$ with $V_{\boldsymbol{R}} = \boldsymbol{H}^n$ as in Case 2 (cf. (2.12)). In this case $H = \{\pm 1\}$ and the $H$-structure is uninteresting. The structure of $V_0^*$ is as follows. Let Sp(1) act on $P$ by the inner automorphism. Then $\langle\ ,\ \rangle_P : \boldsymbol{H}^n \times \boldsymbol{H}^n \to P$ is Sp(1)-equivariant (cf. (1.1)). We set

$$\Phi = N(\langle\ ,\ \rangle_P), \tag{2.20}$$

where $N$ is the quaternion norm. If we consider $\langle\ ,\ \rangle_P$ as a $P$-valued 2-form on $V_{\boldsymbol{R}} = \boldsymbol{H}^n$ by identifying $V_{\boldsymbol{R}}$ with its dual via $(\ ,\ )$, then $\Phi$ may be regarded as a $G$-invariant 4-form on $V_{\boldsymbol{R}}$. We call $\Phi$ the Kraines form (cf. [19]). Then it is well-known and easy to verify that

$$V_0^* = (V^{\mathrm{Sp(n)}})^{\mathrm{Sp(1)}} = \bigoplus_{0 \leqq s \leqq n} \boldsymbol{C}\Phi^s. \tag{2.21}$$

Let $L_\Phi$ be the $\Phi$-multiplication operator on $\wedge V$. Let $\Lambda_\Phi = *^{-1} L_\Phi *$. Then $L_\Phi$ and $\Lambda_\Phi$ are adjoint to each other with respect to the natural Hermitian inner product of $\wedge V$. We call an element $v \in \wedge V_{\boldsymbol{R}}$ *$\Phi$-effective* if $\Lambda_\Phi v = 0$. Let $V^k_{e(\Phi)}$ be the subspace of $\Phi$-effective $k$-forms, which is clearly $G$-invariant. We set $V^{k;r}_{e(\Phi)} = V^k_{e(\Phi)} \cap V^{k;r}$, where $V^{k;r}$ is as in (2.8).

**Proposition 2.11.** *For $l < n$ the linear map $L_\Phi$: $\wedge^l V \to \wedge^{l+4} V$ is injective and for $l \leqq n+3$ we have the natural direct sum decomposition*: $\wedge^l V = \bigoplus_{0 \leq s \leq [l/4]} L_\Phi^s V^{l-4s}_{e(\Phi)}$.

This is essentially due to Kraines [19], where she has shown the injectivity for $l \leqq n-3$.

*Proof.* The second assertion follows from the first as in [19; Th. 2.6]. So we shall prove the first assertion. First we note that for any integer $n \leqq t \leqq 2n$ and $i \in \Delta(2n-t)$ if we set $V_0^{2t|2i} = \bigoplus_{s \in \Delta(i)} V_0^{2t;2s}$ then we have

$$V_0^{2t|2i} = L_\Phi^{(t-i)/2} V_0^{2i} \tag{2.22}$$

where $V_0^{2t;2s}$ are as in Lemma 2.3. In fact this follows from the consideration of the natural surjections $u_k$: $S^k(V_2) \to V_0^{2k}$ with the help of (1.14) and Lemma 2.3. Note that $\Phi_0$ in (1.14) is mapped by $u_2$ to $\Phi$ up to multiplicative constants.

We show more precisely the injectivity of $L_\Phi^\alpha$: $V^{l;s} \to V^{l+4\alpha;s}$ for any $s \in \Delta(l)$, where $\alpha = n-(l+s)/2$. This is equivalent to the injectivity of $\Lambda_\Phi^\alpha = *^{-1} L_\Phi^\alpha *$: $V^{4n-l;s} \to V^{4n-l-4\alpha;s}$, $*$ being isomorphic. Since $L_\Phi^\alpha$ and $\Lambda_\Phi^\alpha$ are adjoint to each other, the latter is further equivalent to the injectivity of $L_\Phi^\alpha$: $V^{4n-l-4\alpha;s} \to V^{4n-l;s}$. Hence it suffices to show that for each $\rho \in \mathfrak{R} = \mathfrak{R}_{\mathrm{Sp}(n)}$ the induced linear mapping $L_\Phi^\alpha$: $V_\rho^{4n-l-4\alpha;s} \to V_\rho^{4n-l;s}$ is surjective. By Proposition 2.4 and (2.3) we may assume that $\rho = \rho(k; r)$ for some $(k, r) \in \Psi$ with $k \in \Delta(l)$, so that $V_\rho^{k;r} = V_\varepsilon^{k;r}$. Moreover, by 4) of Proposition 2.4, we have a) $V_\rho^{l;s} \subseteqq V_0^{l-k} \wedge V_\rho^{k;r}$ and b) $V_\rho^{4n-l;s} \subseteqq V_0^{4n-l-k} \wedge V_\rho^{k;r}$. In view of (1.12) from a) we get $k+r \leqq s+l$, and from b) we see that $V_\rho^{4n-l;s} \subseteqq V_0^{4n-l-k|r+s+\varepsilon} \wedge V_\rho^{k;r}$, where $\varepsilon = 0$ or 2 is determined by $k+l+s+r \equiv \varepsilon$ (4). Then we have $\alpha \leqq \beta$ with

$$\beta = n-(k+l+s+r+\varepsilon)/4.$$

Hence by (2.22) we get $V_\rho^{4n-l;s} \subseteqq L_\Phi^\beta V^{r+s+\varepsilon} \wedge V_\rho^{k;r} \subseteqq \operatorname{Im} L_\Phi^\alpha$ as desired.

## § 3. Decomposition theorem for the cohomology group of a Riemannian manifold

Fix a compact connected oriented $C^\infty$ manifold $M$ of dimension $d$. Let $T_{\boldsymbol{R}}^*$ be the cotangent bundle of $M$. Let $G_0$ be a closed subgroup of $\mathrm{GL}_d(\boldsymbol{R})$. The examples of $G_0$ we have in mind are: Case 1, $\mathrm{GL}_n(\boldsymbol{C})$ $(d=2n)$; Case 2, $\mathrm{GL}_n(\boldsymbol{H})$ $(d=4n)$; and Case 3, $\mathrm{GL}_n(\boldsymbol{H}) \cdot \boldsymbol{H}^\times (d=4n)$.

A *$G_0$-structure* is by definition a reduction of the structure group of $T_{\boldsymbol{R}}^*$ in $G_0$. Set $V_{\boldsymbol{R}} = \boldsymbol{R}^d$. Let $A := \mathrm{End}_{G_0} V_{\boldsymbol{R}}$ be the $\boldsymbol{R}$-algebra of $G_0$-linear endomorphisms of $V_{\boldsymbol{R}}$. Then $A$ naturally operates on $T_{\boldsymbol{R}}^*$ by bundle endomorphisms. The structures of $A$ in the above three cases are given respectively by: Case 1, $\boldsymbol{C}$; Case 2, $\boldsymbol{H}$; and Case 3, $\boldsymbol{R}$. Accordingly, in the first two cases we call the corresponding $G_0$-structure *almost complex structure* and *almost quaternionic structure*, respectively. In the thrid case we call the $G_0$-structure a *twisted almost quaternionic structure*.*) In fact,

*) In [22] this is rather called the almost quaternionic structure (when $n>1$). The present usage of "almost quaternionic structure" is in accordance with [25].

in this case the bundle $\mathrm{End}_R T_R^*$ of algebras contains a subbundle of algebras whose fiber is isomorphic to $\boldsymbol{H}$ (cf. [22]). In these cases if, further, the $G_0$-structure admits a torsion-free connection, we shall call it a *complex structure*, *quaternionic structure*, and *twisted quaternionic structure*, respectively. In fact, in Case 1, then the almost complex structure defined by $i \in \boldsymbol{C}$ is known to be integrable. Similarly in Case 2 the almost complex structures defined by $\lambda \in C \subseteqq \boldsymbol{H}$ are all integrable, and hence we obtain a family of compact complex manifolds $\{M_\lambda\}_{\lambda \in C}$ parametrized by $C = \boldsymbol{P}^1$ (cf. [15, IX. 3]). We shall call this family the *Calabi family* associated to the given quaternionic structure (cf. [6]).

Now fix a Riemannian metric $g$ on $M$. Let $(\ ,\ )$ be the standard Euclidian inner product of $V_R$. Let $K$ be the linear holonomy group of the Riemannian manifold associated to the Riemannian connection $\Gamma$ considered as a subgroup of SO(d) (with respect to some reference point $x_0 \in M$ and an isometry $u\colon (T_{x_0}^*, g_{x_0}) \cong (V_R, (\ ,\ ))$ (cf. [15]). Then the structure group $T_R^*$ is naturally reduced to $K$; in particular if $K$ is contained in $G := G_0 \cap \mathrm{SO(d)}$, $M$ has naturally the associated $G_0$-structure with a torsion-free connection (cf. [15]). In the above three cases, $G = \mathrm{U(n)}$, Sp(n) and Sp(n)Sp(1), respectively. In these cases we call the Riemannian manifold $(M, g)$ a *Kähler manifold, a hyperkähler manifold* (cf. [6]) and a *quaternionic Kähler manifold*, respectively, except that in the last case we assume that $n > 1$ for some reason (cf. [22]). In fact in Case 1, $g$ becomes a Kähler metric in respect to the underlying complex structure, and similarly in Case 2, $g$ becomes a Kähler metric on each $M_\lambda$ so that $\{M_\lambda\}$ is now a family of compact Kähler manifolds.

The importance of the above three cases comes from Berger's holonomy classification theorem [2], which tells us that if the identity component $K_0$ of $K$ acts irreducibly on $V_R$, and if $(M, g)$ is not locally symmetric, $K_0$ is equivalent to one of the following subgroups of SO(d): i) SO(d), ii) U(n) or SU(n) ($d=2n$), iii) Sp(n) or Sp(n)Sp(1) ($d=4n$) iv) Spin(7), v) $\mathrm{G}_2$.

Now we shall fix a compact Lie subgroup $G$ of SO(d) and assume that the holonomy group of $(M, g)$ is contained in $G$. Let $V$ be the complexification of $V_R$ with the induced Hermitian inner product. Let $T^* = T_R^* \otimes C$ be the complexified cotangent bundle of $M$. Let $\wedge^k T^*$ be its $k$-th exterior power, which is considered as a $C^\infty$ complex vector bundle with typical fiber $\wedge^k V$ and with structure group $G$. Then for any $G$-invariant subspace $F$ of $\wedge^k V$ we have the naturally associated vector subbundle of $\wedge^k T^*$ with typical fiber $F$ and with structure group $G$, which we shall call the *subbundle associated to F*. In general, we call such a subbundle of $\wedge^k T^*$ *admissible*.

Now let $\Delta_k$ be the Laplacian associated to the metric $g$ operating on the space $A^k(M)$ of $C^\infty$ $k$-forms on $M$. Let $\mathfrak{H}^k(M)$ be the subspace of harmonic $k$-forms with respect to $\Delta_k$. For any subbundle $W$ of $\wedge^k T^*$ a section of $W$ will be called a *W-form* (of degree $k$). We denote by $\mathfrak{H}^k_W(M)$ the space of harmonic $W$-forms. The following theorem of Chern [8] is fundamental for our purposes.

**Theorem** (Chern). *Let $W$ be any admissible subbundle of $\wedge^k T^*$. Then the Laplacian $\Delta_k$ commutes with the orthogonal projection $p_W\colon \wedge^k T^* \to W$ (with respect to the natural Hermitian structure of $\wedge^k T^*$). If $\wedge^k T^*$ is a direct sum of admissible subbundles $W_i$, $1 \leqq i \leqq s$, then we have the direct sum decomposition $\mathfrak{H}^k(M) = \oplus_{1\leqq i\leqq s} \mathfrak{H}^k_{W_i}(M)$.*

Let $W$ be any complex (resp. real) $C^\infty$ subbundle of $\wedge^k T^*$ (resp. $\wedge^k T^*_{\boldsymbol{R}}$). Then we denote by $H^k_W(M, \boldsymbol{K})$ the subspace of $H^k(M, \boldsymbol{K})$ consisting of those classes which are representable by $d$-closed $C^\infty W$-forms, where $\boldsymbol{K} = \boldsymbol{C}$ (resp. $\boldsymbol{R}$). Then the de Rham isomorphism $\mathfrak{H}^k(M) \to H^k(M, \boldsymbol{C})$ induces the natural injection $j_W\colon \mathfrak{H}^k_W(M) \to H^k_W(M, \boldsymbol{C})$. As an immediate consequence of the above theorem we note the following:

**Lemma 3.1.** *If $W$ is admissible, then $j_W$ is isomorphic so that $\mathfrak{H}^k_W(M) \cong H^k_W(M, \boldsymbol{C})$ naturally.*

*Proof.* Let $\alpha$ be any $d$-closed $C^\infty$ $W$-form. Let $H\alpha$ be the harmonic part of $\alpha$. Then by the above theorem $H\alpha$ is again a $W$-form. (Since $\Delta_k$ commutes with $p_W$, the harmonic projection operator $H$ also commutes with $p_W$ (cf. [27; IV]).) This implies the surjectivity of $j_W$.

Now in the two theorems below we shall summarize some of the consequences which we shall draw from the above theorem. First, note that associated to the canonical $G$-decomposition $\wedge^k V = \oplus_{\rho\in\mathfrak{R}} V^k_\rho$ of (2.1) we have the natural decomposition of complex vector bundles

$$\wedge^k T^* = \bigoplus_{\rho\in\mathfrak{R}} W^k_\rho,$$

where $W^k_\rho$ is the subbundle associated to the subspace $V^k_\rho$. Then for any $\rho \in \mathfrak{R}$ we set $H^*_\rho(M, \boldsymbol{C}) = H^*_{W_\rho}(M, \boldsymbol{C})$.

**Theorem 3.2.** *In the above notation and assumptions there exists a natural direct sum decomposition*

$$H^*(M, \boldsymbol{C}) = \bigoplus_{\rho\in\mathfrak{R}} H^*_\rho(M, \boldsymbol{C}). \tag{3.1}$$

*Moreover, if $0 \in \mathfrak{R}$ is the class of the trivial representation, then $H^*_0(M, \boldsymbol{C})$ is a subalgebra defined over $\boldsymbol{R}$ of the cohomology ring $H^*(M, \boldsymbol{C})$ naturally* *isomorphic to* $V_0^*$, *and each* $H_\rho^*(M, \boldsymbol{C})$ *is naturally a graded* $H_0^*(M, \boldsymbol{C})$-*module so that* (3.1) *is a decomposition as an* $H_0^*(M, \boldsymbol{C})$-*module.*

Let $H$ be the commutor of $G$ in $\mathrm{GL}_d(\boldsymbol{R})$ and $G_0$ the commutor of $H$ in $\mathrm{GL}_d(\boldsymbol{R})$. Then $H$ operates naturally on $T_{\boldsymbol{R}}^*$ and this action induces the $\boldsymbol{C}$-linear action on the bundle $\wedge^k T^*$. Then associated to the canonical $H$-decomposition $\wedge^k V = \bigoplus_{\mu \in \mathfrak{R}'} V_\mu^k$ of (2.2) we have the canonical decomposition

$$\wedge^k T^* = \bigoplus_{\mu \in \mathfrak{R}'} W_\mu^k \tag{3.2}$$

of $C^\infty$ vector bundles, where $W_\mu^k$ is the subbundle associated to the subspace $V_\mu^k$. Then we set $H_\mu^k(M, \boldsymbol{C}) = H_{W_\mu}^k(M, \boldsymbol{C})$.

**Theorem 3.3.** *In the above notation and assumptions* $H^*(M, \boldsymbol{C})$ *has the natural structure of an H-module defined over* $\boldsymbol{R}$ *such that* $H_\rho^*(M, \boldsymbol{C})$ *and* $H_\mu^*(M, \boldsymbol{C})$ *are all H-submodules for any* $\rho \in \mathfrak{R}$ *and* $\mu \in \mathfrak{R}'$. *Further, the canonical H-decomposition of* $H^*(M, \boldsymbol{C})$ *is of the form*

$$H^*(M, \boldsymbol{C}) = \bigoplus_{\mu \in \mathfrak{R}'} H_\mu^*(M, \boldsymbol{C}), \tag{3.3}$$

*where* $H_\mu^*(M, \boldsymbol{C})$ *is the* $\mu$-*isotypical component. Moreover,* (3.3) *depends only on the underlying* $G_0$-*structure, but not on the particular choice of the metric* $g$.

**Remark 3.4.** The decompositions (3.1) and (3.3) are compatible; for instance if we set $H_{\mu,\rho}^*(M, \boldsymbol{C}) = H_\mu^*(M, \boldsymbol{C}) \cap H_\rho^*(M, \boldsymbol{C})$, then we have

$$H_\mu^*(M, \boldsymbol{C}) = \bigoplus_{\rho \in \mathfrak{R}} H_{\mu,\rho}^*(M, \boldsymbol{C}). \tag{3.4}$$

The following result due to Lichnerowicz is useful for our argument.

**Lemma 3.5.** 1) *Let* $\alpha$ *be any parallel k-form and* $\beta$ *a* $C^\infty$ *l-form. Then* $\Delta(\alpha \wedge \beta) = \alpha \wedge \Delta\beta$, *where* $\Delta$ *is the Laplacian. In particular,* $\alpha$ *is harmonic.* 2) *Let h be a* $C^\infty$ *endomorphism of the bundle* $\wedge^k T^*$. *Suppose that h is parallel as an element of the bundle* End $\wedge^k T^*$. *Then for any* $C^\infty$ *k-form* $\alpha$ *we have* $\Delta(h\alpha) = h\Delta\alpha$.

For the proof see Lichnerowicz [20] (cf. also [21; (4.3.88)]). We write $\mathfrak{H}_\rho^s(M) = \mathfrak{H}_{W_\rho}^s(M)$. Using the above lemma we show the following:

**Lemma 3.6.** 1) *A* $C^\infty$ *k-form* $\alpha$ *is in* $\mathfrak{H}_0^k(M)$ *if and only if it is parallel.* 2) *For any* $\rho \in \mathfrak{R}$, *any* $\alpha \in \mathfrak{H}_0^k(M)$ *and* $\beta \in \mathfrak{H}_\rho^l(M)$ *we have* $\alpha \wedge \beta \in \mathfrak{H}_\rho^{k+l}(M)$.

*Proof.* 1) If $\alpha$ is parallel, then $\alpha$ is a $W_0^k$-form and further is harmonic by Lemma 3.5. Thus $\alpha \in \mathfrak{H}_0^k(M)$. Conversely, suppose that $\alpha$ is in $\mathfrak{H}_0^k(M)$. We use the argument of Bochner-Yano for which we refer to Goldberg [11; p. 88]. Let

$$\alpha=(1/k!)\sum \alpha_{i_1\cdots i_k}du_{i_1}\wedge\cdots\wedge du_{i_k}$$

be the local expression of $\alpha$ with respect to local coordinates $u_1, \cdots, u_d$, where $\alpha_{i_1\cdots i_k}$ are skew-symmetric in their indices. Let $R_{ijkl}$, $R_{ij}$, $D_j$ be the local expressions with respect to $u_i$ of the Riemannian curvature tensor, the Ricci curvature tensor, and the covariant differentiation associated to the Riemannian connection $\Gamma$, respectively. Then the following is true (cf. [11; (3.2.9)]);

$$\int_M (F(\alpha)+D_j\alpha_{i_1\cdots i_k}D^j\alpha^{i_1\cdots i_k})*1=0, \tag{3.5}$$

where

$$F(\alpha)=R_{ij}\alpha^{ii_2\cdots i_k}\alpha^j{}_{i_2\cdots i_k}+((k-1)/2)R_{ijkl}\alpha^{iji_3\cdots i_k}\alpha^{kl}{}_{i_3\cdots i_k}.$$

(If $k=1$, the second term is assumed to be zero.) However, the argument there certainly shows that if, further, $\alpha$ is parallel, then even the integrand of (3.5), which reduces to $F(\alpha)$ then, vanishes indentically on $M$. Since every element of each fiber of the bundle $W_0^k$ is obtained as the restriction of some parallel $k$-forms, this in turn implies that even for any $C^\infty$ $W_0^k$-form $\beta$, $F(\beta)$ vanishes identically on $M$. Then we apply this fact to an arbitrary harmonic $W_0^k$-form $\alpha$ and then use (3.5) to conclude that $D_j\alpha_{i_1\cdots i_k}=0$, i.e., that $\alpha$ is parallel.

2) By 1) $\alpha$ is parallel. Hence the result follows from Lemma 3.5.

**Corollary 3.7.** *Let $\alpha \in \mathfrak{H}_0^k(M)$ and $\bar{\alpha}$ corresponding cohomology class. Let $L_\alpha$ (resp. $L_{\bar{\alpha}}$) be the $\alpha$- (resp. $\bar{\alpha}$-) multiplication operator on $\mathfrak{H}^*(M)$ (resp. $H^*(M, \boldsymbol{R})$). Then the de Rham isomorphism $\mathfrak{H}^*(M)\to H^*(M, \boldsymbol{R})$ induces the isomorphism of* Ker $L_\alpha$ *and* Ker $L_{\bar{\alpha}}$, *where* Ker *denotes the kernel.*

Let $P^*(M)$ be the space of parallel forms. By Lemma 2.6 we obtain the canonical isomorpnisms

$$H_0^*(M, \boldsymbol{C})\xleftarrow[\sim]{d}\mathfrak{H}_0^*(M)=P^*(M)\xrightarrow[\sim]{r}V_0^*, \tag{3.6}$$

where $d$ is the de Rham-Hodge isomorphism, the middle equality is just the content of 1) of Lemma 3.6 and $r$ is obtained by restriction to the fiber $(\wedge^*T_{x_0})$ over $x_0$ identified with $\wedge^*V$ by $\wedge^*u$. Now the proof of Theorem 3.2 is immediate.

*Proof of Theorem* 3.2. (3.1) is an immediate consequence of the theorem of Chern and Lemma 3.1. The second assertion follows from (3.6), since $P^*(M)$ is a $C$-subalgebra of $A^*(M)$ and $r$ is an algebra isomorphism. Finally, the last assertion follows from Lemmas 3.1 and 3.6, 2).

*Proof of Theorem* 3.3. The definition of $H^*_\mu(M, C)$ depends only on the underlying $G_0$-structure (which determines $H$ as the commutor of $G_0$); hence the same is true for (3.3) if such a decomposition exists. For the remaining assertions, by identifying $\mathfrak{H}^*_\mu(M):=\mathfrak{H}^*_{W_\mu}(M)$ with $H^*_\mu(M, C)$ and $\mathfrak{H}^*_\rho(M)$ with $H^*_\rho(M, C)$ by Lemma 3.1, we have only to prove the corresponding assertions for the corresponding space of harmonic forms. First, we note that each $h \in H$, regarded as a section of the bundle End $\wedge^k T$ of endomorphisms of $\wedge^k T^*$, is clearly parallel with respect to $\Gamma$. Then by Lemma 3.5 the action of $H$ preserves $\mathfrak{H}^k(M)$ and is defined over $\boldsymbol{R}$. On the other hand, the action of $H$ clearly preserves each subbundles $W^k_\rho$ and $W^k_\mu$ and hence the spaces $\mathfrak{H}^k_\mu(M)$ and $\mathfrak{H}^k_\rho(M)$ also. Further, as an $H$-module $\mathfrak{H}^k_\mu(M)$ is clearly $\mu$-isotypical. Hence the direct sum decomposition $\mathfrak{H}^k(M)=\bigoplus_{\mu\in\mathfrak{N}'}\mathfrak{H}^k_\mu(M)$ associated to (3.2) by the theorem of Chern is just the canonical decomposition of $\mathfrak{H}^k(M)$.

**Remark 3.8.** $H^*_\rho(M, C)$ is defined $\boldsymbol{R}$ over if and only if $\rho$ is self-conjugate. Similarly, if $\mu|_{H_1}$ is irreducible, $H^*_\mu(M, C)$ is defined over $\boldsymbol{R}$ if and only if $\mu|_{H_1}$ is self-conjugate. These facts follow from the above proofs and Lemma 2.1. In these cases we can speak of the real cohomology groups $H^*_\rho(M, \boldsymbol{R})$ and $H^*_\mu(M, \boldsymbol{R})$.

Let $V_i$, $i=1, 2$, be $G$-invariant subspaces of $\wedge^{k_i}V$ with $k_1<k_2$ such that the exterior product induces the linear map $\gamma$: $V^s_0\otimes V_1\to V_2$, where $s=k_1-k_2$. Let $W_i$ be the associated subbundles of $\wedge^{k_i}V$ with fiber $V_i$. Then $\gamma$ induces a bundle map $W^s_0\otimes W_1\to W_2$, and then by Lemma 3.6 it further induces linear map $\gamma_{\mathfrak{H}}$: $\mathfrak{H}^s_0(M)\otimes\mathfrak{H}^{k_1}_{W_1}(M)\to\mathfrak{H}^{k_2}_{W_2}(M)$ of the corresponding spaces of harmonic forms. Let $\gamma_H$: $H^s_0(M, C)\otimes H^{k_1}_{W_1}(M, C)\to$ $H^{k_2}_{W_2}(M, C)$ be the corresponding map of cohomology groups (cf. Lemma 3.1).

**Lemma 3.9.** *If $\gamma$ is injective, then $\gamma_H$ also is injective. If $\gamma$ is isomorphic, then $\gamma_H$ also is isomorphic.*

*Proof.* The first assertion is clear. We have only to show that $\gamma_{\mathfrak{H}}$ is surjective assuming that $\gamma$ is isomorphic. Let $\beta$ be any element of $\mathfrak{H}^{k_2}(M)$. Fix a basis $\{\alpha_1, \cdots, \alpha_b\}$ of $\mathfrak{H}^s_0(M)$. Since the restriction induces the isomorphism of $\mathfrak{H}^s_0(M)$ with each fiber of $W^s_0\to M$, the assumption that $\gamma$

is isomorphic implies that we can write $\beta=\sum_{i=1}^{b}\alpha_i\wedge\beta_i$ for some $C^\infty$ $W_1$-forms $\beta_1,\cdots,\beta_b$. Applying the Laplacian $\Delta$ to this equality and using Lemmas 3.5 and 3.6, we obtain $0=\Delta\beta=\sum_{i=1}^{b}\alpha_i\wedge\Delta\beta_i$. Since $\Delta\beta_i$ is again a $W_1$-form by the theorem of Chern, by the injectivity of $\gamma$ we get that $\Delta\beta_j=0$ for any $j$. This implies that $\beta_j$ are harmonic as desired.

Note that the lemma is still true if we replace $V_0^s$ by any of its subspaces and $H_0^s(M, \boldsymbol{C})$ by the subspace correponding to it via (3.6). This is clear from the above proof.

Let $H_1=H\cap \mathrm{SO(d)}$. Then the Hodge $*$-operator $*: \wedge^d T\rightleftarrows\wedge^{d-k}T$ is both $G$- and $H_1$-equivariant. Therefore for any $\rho\in\mathfrak{R}$ it induces successively $H_1$-isomorphisms $\mathfrak{H}_\rho^k(M)\rightleftarrows\mathfrak{H}_\rho^{d-k}(M)$ and

$$*: H_\rho^k(M, \boldsymbol{C})\rightleftarrows H_\rho^{d-k}(M, \boldsymbol{C}). \tag{3.7}$$

This reduces in principle the study of the structure of $H^k(M, \boldsymbol{C})$ to the case $k\leqq d/2$. We note, however, that the definition depends explicitly on the choice of the metric.

It is interesting to ask if the following statement is true in general: Any element $h$ of $H$ acts on $H^*(M, \boldsymbol{R})$ as an algebra automorphism. If this is the case, then the Poincaré duality makes $H^k(M, \boldsymbol{R})$ and $H^{d-k}(M, \boldsymbol{R})$ into dual $H_1$-modules.

We now specialize the previous considerations to the three cases mentioned at the beginning of this section. We start with:

*Kähler case* (cf. Chern [8; § 4]): In this case $d=2n$, $G=\mathrm{U(n)}$, $H=\boldsymbol{C}^*$ and $G_0=\mathrm{GL}_n(\boldsymbol{C})$. Further $(M, g)$ has naturally the structure of a compact Kähler manifold, which we denote by $(X, \tilde{g})$. In this case Theorems 3.3 and 3.2 reduce to the usual Hodge and the Lefschetz decomposition theorems for $X$, respectively. We shall give these reductions briefly.

Let $H^{p,q}(X)$ be the subspace of $H^{p+q}(X, \boldsymbol{C})$ of those classes representable by $d$-closed $(p, q)$-forms. Then we have the Hodge decomposition

$$H^k(X, \boldsymbol{C})=\bigoplus_{p+q=k} H^{p,q}(X), \qquad \overline{H}^{p,q}(X)=H^{q,p}(X),$$

which depends only on the underlying complex structures. This is just the canonical decomposition (3.3) of the $\boldsymbol{C}^*$-module $H^*(M, \boldsymbol{C})$ defined over $\boldsymbol{R}$.

Let $\omega$ be the Kähler class corresponding to $\tilde{g}$. Then we have $H_0^*(X, \boldsymbol{R})=\bigoplus_{k\geqq 0}\boldsymbol{R}\omega^k$ by (2.4) and (3.6). Let $L$ be the $\omega$-multiplication operator as usual on $H^*(M, \boldsymbol{R})$. For $k\leqq n$ we set

$$H_e^k(X, \boldsymbol{R})=\{\alpha\in H^k(X, \boldsymbol{R});\ L^{n-k+1}\alpha=0\}$$

(the set of $\omega$-effective (or primitive) classes) and set $H_e^{p,q}(X)=H_e^k(X, \boldsymbol{C})\cap H^{p,q}(X)$. Then for the representation $\rho=\rho(p, q), p+q\leqq n$, we have $H_\rho^*(X, \boldsymbol{C})=\bigoplus_{r\geqq 0} L^r H_e^{p,q}(X)$ and $H^*(X, \boldsymbol{C})$ is a direct sum of such submodules; in particular $H_\rho^*(X, \boldsymbol{C})=H_0^*(X, \boldsymbol{C})\cdot H_e^{p,q}(X)$. This follows from Proposition 2.2, Theorem 3.2 and Lemma 3.8. Moreover, adding with respect to $(p, q)$ and taking the real part, we get the Lefschetz decomposition

$$H^k(X, \boldsymbol{R})=\bigoplus_{(k-n)^+\leqq r} L^r H_e^{k-2r}(X, \boldsymbol{R}).$$

Moreover by (2.7) and Lemma 3.9, we have the isomorphism

$$L^{n-k}\colon H^k(X, \boldsymbol{R})\cong H^{2n-k}(X, \boldsymbol{R}) \quad \text{(the strong Lefschetz theorem).}$$

These depend only on the Kähler class but not on the metric itself.

Recall also two corollaries of the above decompositions for comparison with the hyperkähler case. Let $b_k$ be the $k$-th Betti number of $X$. Then

$$\text{(3.8)} \qquad b_k \text{ is even if } k \text{ is odd}$$

and

$$\text{(3.9)} \qquad b_k-b_{k-2}=b_{k,e}\geqq 0$$

where $b_{k,e}=\dim H_e^k(X, \boldsymbol{R})$.

We also quote the following fact (cf. [27; p. 74, Th. 4]):

**Lemma 3.10.** *$H=\boldsymbol{C}^*$ acts as algebra automorphisms on the cohomology ring $H^*(M, \boldsymbol{R})$.*

*Hyperkähler case*: In this case $d=4n$, $G=\mathrm{Sp(n)}$, $H=\boldsymbol{H}^\times$ and $G_0=\mathrm{GL}_n(\boldsymbol{H})$. $(M, g)$ is a hyperkähler manifold of dimension 4n and admits a family of compact Kähler manifolds, $\{M_\lambda\}_{\lambda\in C}$, the Calabi family associated to $(M, g)$. In this case our results will be summarized in the two theorems (Theorems 3.11 and 3.16) below, which correspond to Theorems 3.3 and 3.2, respectively. The decomposition (3.2) in this case takes the form

$$\text{(3.10)} \qquad \wedge^k T_{\boldsymbol{R}}^*=\bigoplus_{r\in\Delta(k)} W^{k;r}$$

corresponding to the canonical $\boldsymbol{H}^\times$-decomposition (2.8) of $\wedge^k V_{\boldsymbol{R}}$. We set $H^{k;r}(M, \boldsymbol{R})=H_{W^{k;r}}^k(M, \boldsymbol{R})$.

**Theorem 3.11.** *Let $(M, g)$ be a compact hyperkähler manifold of dimension $4n$. Then the real cohomology ring $H^*(M, \boldsymbol{R})$ has the natural structure of an $\boldsymbol{H}^\times$-algebra, where the elements of $\boldsymbol{H}^\times$ act as algebra automorphisms. With respect to this action the canonical $\boldsymbol{H}^\times$-decomposition of $H^k(M, \boldsymbol{R})$ is given by*

$$(3.11) \qquad H^k(M, \boldsymbol{R}) = \bigoplus_{r \in \Delta(\hat{k})} H^{k;r}(M, \boldsymbol{R}), \qquad \hat{k} = \min(k, 4n-k),$$

*where $H^{k;r}(M, \boldsymbol{R})$ is the $V^{\boldsymbol{R}}_{k,r}$-isotypical component. Moreover, the above structure depends only on the underlying quaternionic structure but not on the particular metric $g$.*

**Remark 3.12.** It is known that $g$ is necessarily an Einstein metric which becomes a Kähler Einstein metric on each $M_\lambda$. Therefore the apparent freedom of $g$ asserted in the last part of the theorem would actually be very little, unlike the Kähler case.

By Lemma 1.2 and Theorem 3.11, corresponding to (3.8) in the Kähler case we get:

**Corollary 3.13.** *Every odd dimensional Betti number of $M$ is divisible by* 4.

Theorem 3.11 follows immediately from Theorem 3.3 and Remark 3.8 except possibly the following:

**Lemma 3.14.** *Any element of $\boldsymbol{H}^\times$ acts as an algebra automorphism of $H^*(M, \boldsymbol{R})$.*

*Proof.* Let $h$ be any element of $\boldsymbol{H}^\times$. We may write $h = a + b\lambda$ for unique $a, b \in \boldsymbol{R}$ and $\lambda \in C$. On the other hand, $\lambda$ defines a natural almost complex structure $J_\lambda$ on $M$ and the triple $(M, g, J_\lambda)$ with $g$ as above becomes a compact Kähler manifold as we have already remarked. Then by Lemma 3.10 $h$ acts as an algebra automorphism of $H^*(M, \boldsymbol{R})$.

For any $\lambda \in C$ let $\tilde{\omega}_\lambda$ be the parallel 2-form on $M$ corresponding to the fundamental form $\omega_\lambda \in V^2_{0,\boldsymbol{R}}$ (cf. (2.9)) by the isomorphism of (3.6). We shall denote again by the same letter $\omega_\lambda$ the de Rham class of $\tilde{\omega}_\lambda$ in $H^2(M, \boldsymbol{R})$. Then $\tilde{\omega}_\lambda$ is just the Kähler form, and hence $\omega_\lambda$ the Kähler class, associated to the Kähler metric defined by $g$ on $M_\lambda$ (cf. [6] [15; IX. 4]).

**Definition.** We call the real 3-dimensional subspace $F$ of $H^2(M, \boldsymbol{R})$ generated by $\omega_\lambda$ for $\lambda \in C$ the *hyperkähler* (*HK*) 3-*space* associated to $(M, g)$.

Note that $F$ is an $\boldsymbol{H}^{\times}$-submodule of $H^2(M, \boldsymbol{R})$ isomorphic to $V^{\boldsymbol{R}}_{2;2}$; further, $H^*_0(M, \boldsymbol{R})$ is an $\boldsymbol{H}^{\times}$-submodule by Theorem 3.3. From Lemma 2.3 and the isomorphism (3.6) we get the following:

**Proposition 3.15.** *$H^2_0(M, \boldsymbol{R})=F$, and $H^*_0(M, \boldsymbol{R})$ is generated by $F$ as an $\boldsymbol{R}$-algebra. If $k=2l$, $H^k_0(M, \boldsymbol{R})$ is isomorphic as an $\boldsymbol{H}^{\times}$-module to $S^l(F)$ if $k\leqq 2n$ and in general $H^k_0(M, \boldsymbol{R})\cong\bigoplus_{s\in\Delta(\hat{l})} V^{\boldsymbol{R}}_{k,2s}$, $\hat{l}=\min(l, 2n-l)$.*

For any $\lambda \in C$ let $L_\lambda$ be the $\omega_\lambda$-multiplication operator on $H^*(M, \boldsymbol{R})$. For any $k\leqq 2n$ we set

$$H^k_\varepsilon(M, \boldsymbol{R})=\{\alpha \in H^k(M, \boldsymbol{R});\, L_\lambda^{2n-k+1}\alpha=0,\ \lambda \in C\}.$$

An element of $H^k_\varepsilon(M, \boldsymbol{R})$ is said to be *universally effective*. Since $F$ is $\boldsymbol{H}^{\times}$-invariant, $H^*_\varepsilon(M, \boldsymbol{R})$ is an $\boldsymbol{H}^{\times}$-submodule of $H^*(M, \boldsymbol{R})$. Hence if we set $H^{k;r}_\varepsilon(M, \boldsymbol{R})=H^k_\varepsilon(M, \boldsymbol{R})\cap H^{k;r}(M, \boldsymbol{R})$, the canonical $\boldsymbol{H}^{\times}$-decomposition of $H^k_\varepsilon(M, \boldsymbol{R})$ is clearly given by

$$H^k_\varepsilon(M, \boldsymbol{R})=\bigoplus_{r\in\Delta(k)} H^{k;r}_\varepsilon(M, \boldsymbol{R}). \tag{3.11}$$

For simplicity we set $N^*=H^*_0(M, \boldsymbol{R})$.

**Theorem 3.16.** *Let $(M, g)$ be a compact hyperkähler manifold of dimension $4n$ as above. Then we have the following: 1) As an $N^*$-module $H^*(M, \boldsymbol{R})$ is generated by the graded submodule $H^*_\varepsilon(M, \boldsymbol{R})$. 2) If $l\leqq n$, we have a natural direct sum decomposition of $\boldsymbol{H}^{\times}$-modules;*

$$H^l(M, \boldsymbol{R})=\bigoplus_{k\in\Delta(l)} N^{l-k}H^k_\varepsilon(M, \boldsymbol{R}). \tag{3.12}$$

*Moreover in this case, the natural map*

$$N^{l-k}\otimes_{\boldsymbol{R}}H^k_\varepsilon(M, \boldsymbol{R})\to N^{l-k}H^k_\varepsilon(M, \boldsymbol{R}) \tag{3.13}$$

*is an isomorphism of $\boldsymbol{H}^{\times}$-modules. 3) $H^{k;r}_\varepsilon(M, \boldsymbol{R})=0$ unless $(k, r)\in\Psi$. 4) All the above structures depend only on the hyperkähler 3-space $F$ associated to $(M, g)$ but not on the particular metric (cf. Remark 3.12, however).*

**Remark 3.17.** 1) (3.12) and (3.13) correspond, respectively, to (a weak version of) the Lefschetz decomposition and the strong Lefschetz theorems in the Kähler case. (3.12) is expected to be true in general, while (3.13) is in general not true for $l>n$; the problem would then to determine the kernel of the map exactly. 2) If $l\leqq n$, any element of $H^l(M, \boldsymbol{R})$ is written uniquely in the form $\alpha=\sum_{s,t,u\geqq 0}\omega_i^s\omega_j^t\omega_k^u\alpha_{stu}$ by 3), where $\alpha_{stu}$ are universally effective; moreover $\omega_i^s\omega_j^t\alpha_{stu}$ is $\omega_k$-effective etc. as follows from the uniqueness of the expression.

*Proof of Theorem* 3.16. Let $\mathfrak{H}^k_\varepsilon(M)$ be the space of real harmonic $k$-forms $\alpha$ with $L^{2n-k+1}_\lambda \alpha=0$ for any $\lambda \in C$. Then by Corollary 3.7 the natural injection $j$: $\mathfrak{H}^k_\varepsilon(M) \to H^k_\varepsilon(M, \boldsymbol{R})$ of $\boldsymbol{H}^\times$-modules is isomorphic. Further, by considering the isotypical components of both sides, we see that $j$ induces an isomorphism $\mathfrak{H}^{k;r}_\varepsilon(M) \rightleftarrows H^{k;r}_\varepsilon(M, \boldsymbol{R})$, where $\mathfrak{H}^{k;r}_\varepsilon(M)=\mathfrak{H}^k_\varepsilon(M) \cap \mathfrak{H}^k_{W^{k;r}}(M)$. 3) then follows from Proposition 2.4.

For 1) we repeat formally the proof of Proposition 2.4, 2) with the $l$-covector $\alpha$ there replaced by a harmonic $k$-form in $\mathfrak{H}^k(M)$ and with $\omega_\lambda$ considered as the corresponding Kähler class on $M_\lambda$. Then we have $\mathfrak{H}^l(M)=N^{l-k} \wedge \mathfrak{H}^k_\varepsilon(M)$, where $N^*$ is identified with $\mathfrak{H}^*_0(M)$ via (3.6). In view of the above isomorphism $\mathfrak{H}^k_\varepsilon(M) \cong H^k_\varepsilon(M, \boldsymbol{R})$, this gives 1).

The proof in fact shows more; namely, we have $\mathfrak{H}^l_\rho(M)=N^{l-k} \wedge \mathfrak{H}^k_{\rho,\varepsilon}(M)$ for any $\rho \in \mathfrak{R}$, where $\mathfrak{H}^k_{\rho,\varepsilon}(M)=\mathfrak{H}^k_\rho(M) \cap \mathfrak{H}^k_\varepsilon(M)$. On the other hand, by Proposition 2.4, if $l \leqq n$, then $\mathfrak{H}^k_{\rho,\varepsilon}(M)=0$ unless $\rho=\rho(k; r)$ for some $r \in \Delta(k)$, and in this case $V^k_{\rho,\boldsymbol{R}}=V^{k;r}_\varepsilon$. Hence $\mathfrak{H}^k_{\rho,\varepsilon}(M)$ coincides with $\mathfrak{H}^{k;r}_\varepsilon(M)$. In particular, we have obtained: $\mathfrak{H}^l_{\rho(k;r)}(M)=N^{l-k} \wedge \mathfrak{H}^{k;r}_\varepsilon(M)$, or $H^l_{\rho(k;r)}(M, \boldsymbol{C})=N^{l-k} H^{k;r}_\varepsilon(M, \boldsymbol{R})$. Hence by Theorem 3.2, Remark 3.8 together with Proposition 2.4 we have the direct sum decomposition

$$H^l(M, \boldsymbol{R})=\bigoplus_{k \in \Delta(l)} \bigoplus_{r \in \Delta(k)} H^l_{\rho(k;r)}(M, \boldsymbol{R}).$$

Hence by (3.11)

$$H^l(M, \boldsymbol{R})=\bigoplus_{k \in \Delta(l)} N^{l-k}\Big(\bigoplus_{r \in \Delta(k)} H^{k;r}_\varepsilon(M, \boldsymbol{R})\Big)=\bigoplus_{k \in \Delta(l)} N^{l-k} H^k_\varepsilon(M, \boldsymbol{R}).$$

This shows (3.12). Finally by Lemma 3.9 the injectivity of

$$N^{l-k} \otimes_{\boldsymbol{R}} H^k_\varepsilon(M, \boldsymbol{R}) \to N^{l-k} H^k_\varepsilon(M, \boldsymbol{R})$$

would follow from the injectivity of $V^{l-k}_0 \otimes_{\boldsymbol{R}} V^{k;r}_\varepsilon \to \wedge^l V$, and hence of $V^{l-k}_0 \otimes_{\boldsymbol{R}} V^k_\varepsilon \to \wedge^l V$, which is in fact true by Proposition 2.4. 2) is proved. Finally 4) is clear from the definitions.

By the above theorem and (3.11), as an $\boldsymbol{H}^\times$-module

$$H^l(M, \boldsymbol{R}) \cong \bigoplus_{k \in \Delta(l)} \bigoplus_{r \in \Delta(k)} N^{l-k} \otimes_{\boldsymbol{R}} H^{k;r}(M, \boldsymbol{R}), \quad l \leqq n.$$

Therefore if $h^{k;r}_\varepsilon := \dim H^{k;r}_\varepsilon(M, \boldsymbol{R})$ are given, the structure of $H^k(M, \boldsymbol{R})$ as a real $\boldsymbol{H}^\times$-module, namely the dimension $h^{k;r}=\dim H^{k;r}(M, \boldsymbol{R})$ of its isotypical components are determined by Lemma 1.4:

$$h^{k;r}(M)=\sum_{k \in \Delta(l)} \sum_{r \in \Delta(k)} \sum_{d} \pi(d, (l-k)/2, r) h^{k;r}_\varepsilon. \tag{3.14}$$

(See Lemma 1.4 for the definition of $\pi$.)

In addition to Corollary 3.13, the theorem imposes some more topological restrictions on the underlying $C^\infty$ manifold of a hyperkähler manifold. Let $b_k$ be the betti number of $M$ and we set $b_{k,\varepsilon}=\dim_{\boldsymbol{R}} H^k_\varepsilon(M,\boldsymbol{R})$. Then from Proposition 3.15 together with (3.12) (3.13) we get the following:

**Corollary 3.18.** *Suppose that $l \leqq n$. Then we have*

$$b_l = \sum_{k\in\Delta(l)} (1/8)(l-k+2)(l-k+4)b_{k,\varepsilon}. \tag{3.15}$$

Taking the first, second and third differences of the sequence $\{b_l\}$ with unit difference 2, we successively get the formulas

$$b_l - b_{l-2} = \sum_{k\in\Delta(l)} (1/2)(l-k+2)b_{k,\varepsilon}$$
$$b_l - 2b_{l-2} + b_{l-4} = \sum_{k\in\Delta(l)} b_{k,\varepsilon}$$
$$b_l - 3b_{l-2} + 3b_{l-4} - b_{l-6} = b_{l,\varepsilon},$$

where $b_k=0$ if $k<0$. In particular, corresponding to (3.9) in the Kähler case we get:

**Corollary 3.19.** *If $l \leqq n$, we have the inequality*

$$b_l + 3b_{l-4} \geqq 3b_{l-2} + b_{l-6}.$$

Also from (3.15) we get for instance:

$$b_{2s} \geqq (s+1)((b_2-1)s+2)/2 \geqq (s+1)(s+2)/2, \quad 2s \leqq n,$$
$$b_{2s+1} \geqq s(s+1)b_3/2 \quad \text{if } b_1=0, \qquad 2s+1 \leqq n.$$

The above results show the rapidly increasing character of the Betti numbers of a hyperkähler manifold.

For any $\lambda \in C$ we have the Hodge decomposition

$$H^k(M, \boldsymbol{C}) = \bigoplus_{p+q=k} H^{p,q}(M)_\lambda \tag{3.16}$$

with respect to $\lambda$ associated to the regular $\boldsymbol{H}^\times$-module $H^k(M,\boldsymbol{R})$ (cf. (1.10)). The geometric interpretation of this decomposition is as follows.

**Proposition 3.20.** *Let $\{M_\lambda\}_{\lambda\in C}$ be the Calabi family associated to $(M,g)$. Then $H^{p,q}(M)_\lambda$ coincides with the Hodge $(p,q)$-component $H^{p,q}(M_\lambda)$ of the Kähler manifold $M_\lambda$; $H^{p,q}(M)_\lambda = H^{p,q}(M_\lambda)$. In particular* (3.16) *is just the Hodge decomposition of $M_\lambda$.*

*Proof.* This follows immediately from the definition of the $H^{\times}$-module structure of $H^k(M, \boldsymbol{R})$ and from the definition of the complex structure on $M_\lambda$ as that induced by the multiplication by $\lambda$ on $T^*_{\boldsymbol{R}}$.

From Lemma 1.2 we get the following:

**Corollary 3.21.** *The Hodge number $h^{p,q}(M_\lambda)$ is even if $p+q$ is odd. For any $p\geqq q\geqq 0$ we have the inequalities $h^{p,q}(M_\lambda)\geqq h^{p+1,q-1}(M_\lambda)$.*

*Quaternionic Kähler case*: In this case $d=4n$, $G=\mathrm{Sp}(n)\mathrm{Sp}(1)$, $H=\boldsymbol{R}^{\times}$, and $G_0=\mathrm{GL}_{4n}(\boldsymbol{R})$. $(M, g)$ is a quaternionic Kähler manifold. Such a manifold has recently been investigated by Salamon [22] extensively. In order to give the reader an idea what the manifold is like, and also for later reference we shall recall some of its properties from [22] first: 1) $(M, g)$ is an Einstein manifold so that in particular its scalar curvature $t$ is constant. If $t>0$, then $M$ is simply connected and every odd dimensional Betti number of $M$ vanishes. 2) To $(M, g)$ there is naturally associated a compact complex manifold $Z$ with a $C^\infty$-submersion $\pi$: $Z\to M$ with each fiber a complex projective line $\boldsymbol{P}^1$ (called the twistor space of $(M, g)$). If $t>0$, $Z$ is a (projective) Fano manifold (i.e., $c_1(Z)$ is positive) with holomorphic contact structure. Further, the Hodge numbers satisfy

$$h^{p,q}(Z)=0 \quad \text{if } p\neq q, \tag{3.17}$$

the fact which holds for any Fano manifold with holomorphic contact structure. 3) When $t>0$, certain compact Riemannian symmetric spaces constructed by Wolf are the only known examples.

Now since $H=\boldsymbol{R}^{\times}$, Theorem 3.3 is uninteresting in this case. We shall see certain consequences of Theorem 3.2. Denote by the same letter $\Phi$ the parallel 4-form corresponding to the Kraines form $\Phi\in V^4_{0,\boldsymbol{R}}$ (cf. (2.20)). Let $[\Phi]$ be the de Rham cohomology class of $\Phi$ in $H^4(M, \boldsymbol{R})$. We call $[\Phi]$ the Kraines class of $(M, g)$ (cf. [19]). Let $L_\Phi$ be the $[\Phi]$-multiplication operator on $H^*(M, \boldsymbol{R})$. Set $\Lambda_\Phi=*^{-1}L_\Phi *$, where $*$ is defined by the isomorphism $H^*(M)\cong H^*(M, \boldsymbol{R})$. Then a class $\alpha$ of $H^*(M, \boldsymbol{R})$ is called *$\Phi$-effective* if $\Lambda_\Phi\alpha=0$. Let $H^*_{e(\Phi)}(M, \boldsymbol{R})$ be the space of $\Phi$-effective classes. Then the following is true:

**Theorem 3.22.** *Let $(M, g)$ be a quaternionic Kähler manifold of dimension $4n$. Then* 1) *for any $k<n$, the linear map $L_\Phi$: $H^k(M, \boldsymbol{R})\to H^{k+4}(M, \boldsymbol{R})$ is injective. In particular we have $b_k\leqq b_{k+4}$ in this range, where $b_k$ denotes the Betti number.* 2) *We have the direct sum decomposition $H^k(M, \boldsymbol{R})=\bigoplus_{0\leq s\leq[k/4]} L^s_\Phi H^{k-4s}_{e(\Phi)}(M, \boldsymbol{R})$, $k\leqq n+3$.*

*Proof.* By (2.21) and (3.6) we have $H^*_0(M, \boldsymbol{R})=\bigoplus_{k\geqq 0}\boldsymbol{R}[\Phi]^k$. The theorem then follows easily from Proposition 2.11, Theorem 3.2 and Lemma 3.9.

Note that 1) of Theorem 3.19 for $k \leqq n-3$ is already in Kraines [19], where the theorem of Chern is already used. Note also that since the decomposition (2.8) is Sp(n)Sp(1)-invariant, by Theorem 3.2 the direct sum decomposition $H^k(M, \boldsymbol{R}) = \bigoplus_{r \in \Delta(\hat{k})} H^{k;r}(M, \boldsymbol{R})$ of Theorem 3.11 still holds though there is no natural action of $H^\times$ in this case. On the other hand, 2) of the above theorem has the obvious disadvantage in that the definition of $H^k_{e(\Phi)}(M, \boldsymbol{R})$ depends explicitly on the metric and is not expressed purely in cohomological terms. In this respect it is interesting to consider the following variation of the definition of $\Phi$-effectivity: We set

$$H^k_{\varepsilon(\Phi)}(M, \boldsymbol{R}) = \{\alpha \in H^k(M, \boldsymbol{R});\ L_\Phi^{n-k+1}\alpha = 0\}.$$

The definition clearly depends only on the Kraines class $[\Phi]$ and not on the particular choice of the metric. In any case, we can get the "strong Kraines" and "Kraines decomposition" in either definition of $\Phi$-effectivity when the scalar curvature $t$ is positive, by using directly the results of Salamon mentioned earlier.

**Theorem 3.23.** *Let $M$ be a quaternionic Kähler manifold of dimension $4n$ with positive scalar curvature. Then*: 1) *for any $0 \leqq k < n$, $L_\Phi^{n-k}$ induces an isomorphism $L_\Phi^{n-k}\colon H^{2k}(M, \boldsymbol{R}) \to H^{4n-2k}(M, \boldsymbol{R})$, and* 2) *for any $k \geqq 0$ we have the direct sum decompositions*

$$H^{2k}(M, \boldsymbol{R}) = \bigoplus_r L_\Phi^r H^{2k-4r}_{\varepsilon(\Phi)}(M, \boldsymbol{R}), \quad (k-n)^+ \leqq r \leqq [k/2].$$

**Remark 3.24.** As mentioned above, 2) also is true with $H^{2k-4r}_{\varepsilon(\Phi)}(M, \boldsymbol{R})$ replaced by $H^{2k-4r}_{e(\Phi)}(M, \boldsymbol{R})$. Indeed, this can be proved in the same way as in [19; Th. 2.6] by using 1) and Corollary 3.7. It is expected that $H^{2k}(M, \boldsymbol{R}) = H^{2k;0}(M, \boldsymbol{R})$ for any $k \geqq 0$ in this case.

*Proof.* We shall write $H^k(X)$ for $H^k(X, \boldsymbol{R})$, $X = M, Z$, where $Z$ is the twistor space of $M$. Let $\omega = c_1(Z)$. Then $\omega$ is a Kähler class on $Z$. As usual let $L$ be the $\omega$-multiplication operator on $H^*(Z)$. Since $\pi^*\colon H^*(M) \to H^*(Z)$ is injective by Leray-Hirsch, we consider $H^*(M)$ as a subring of $H^*(Z)$. Then we have the canonical direct sum decomposition

$$H^i(Z) = H^i(M) \oplus L H^{i-2}(M) \tag{3.18}$$

and $\omega^2 = a[\Phi]$ for some positive number $a > 0$ so that

$$L^2 = a L_\Phi \quad \text{on} \quad H^*(Z) \tag{3.19}$$

(cf. [22; (2.6) and p. 151]). The strong Lefschetz theorem on $Z$ gives us the isomorphism

$$L^{(2n+1)-i}\colon H^i(Z) \xrightarrow{\sim} H^{2(2n+1)-i}(Z), \quad i \leqq 2n-1, \tag{3.20}$$

In particular, we have the inclusion

$$L^{2n-i}\colon H^i(Z) \hookrightarrow H^{4n-i}(Z). \tag{3.21}$$

Now we consider the case $i=2k$. Then by (3.18) (3.19) and (3.21) we get the inclusion

$$L_\Phi^{n-k}\colon H^{2k}(M) \hookrightarrow H^{4n-2k}(M).$$

By the Poincaré duality this is actually isomorphic. This proves 1).

Next we show 2). We consider the Lefschetz decomposition on $Z$;

$$H^{2k}(Z) = \oplus_r L^r H_e^{2k-2r}(Z), \quad (2k-2m-1)^+ \leqq r \leqq k. \tag{3.22}$$

*Claim*: For any $0 \leqq l \leqq n$ we have the inclusion $H_e^{2l}(Z) \subseteqq H^{2l}(M)$ with respect to the decomposition (3.18).

*Proof.* First we note that for any $2l \leqq 2n$ and $u \in H^{2l}(Z)$, $u$ is an element of $H_e^{2l}(Z)$ if and only if $L^{2(n-l+1)}u = L^{(2n+1)-2l+1}u = 0$. On the other hand, by (3.19) we have $L^{2(n-l+1)} = a^{n-l+1} L_\Phi^{n-l+1}$. Hence $L^{2(n-l+1)}$ preserves each factor of (3.18). Therefore it suffices to show that $L^{2(n-l+1)}|_{LH^{2l-2}}(M)$ is injective. Indeed, this follows from the strong Lefschetz theorem on $Z$:

$$L^{2(n-l+1)+1} = L^{(2n+1)-(2l-2)}\colon H^{2l-2}(Z) \xrightarrow{\sim} H^{(4n+2)-(2l-2)}(Z)$$

and the inclusion $H^{2l-2}(M) \subseteqq H^{2l-2}(Z)$.

Now for any $\alpha \in H^{2k}(M)$ write

$$\alpha = \sum_r L^r \alpha_r, \quad \alpha_r \in H_e^{2k-2r}(Z)$$

according to (3.22). Then by Claim above $\alpha_r \in H^{2k-2r}(M)$. Hence if we set $\alpha' = \sum_{r:\text{even}} L^r \alpha_r$ and $\alpha'' = \sum_{r:\text{odd}} L^{r-1} \alpha_r$, then $\alpha' \in H^{2k}(M)$ and $\alpha'' \in H^{2k-2}(M)$ in view of (3.19) and $\alpha = \alpha' + L\alpha''$. From (3.18) we get $L\alpha'' = 0$. Therefore $\alpha = \alpha' = \sum_{r:\text{even}} L_\Phi^{r/2} \hat{\alpha}_r$, $\hat{\alpha}_r = \alpha_r / a^{r/2}$ uniquely. Moreover from (3.18) and (3.19) we get $\hat{\alpha}_r \in H_{e(\Phi)}^{2k-2r}(M)$ with $r$ even. This completes the proof.

**Remark 3.25.** The method of the above proof also gives another proof of the fact that $b_i(M) = 0$ for any odd $i$. Indeed, from (3.20) we get the isomorphism $L_\Phi^{n-k}\colon H^{2k+1}(M) \xrightarrow{\sim} H^{2(2n+1)-(2k+1)}(M) = H^{4n-2k+1}(M)$,

$1 \leqq i = 2k+1 \leqq 2n-1$. Hence by the Poincaré duality $b_{2k+1}(M) = b_{4n-2k+1}(M) = b_{2k-1}(M)$ for $0 \leqq k \leqq n-1$. Thus $b_{2k+1}(M) = b_{-1}(M) = 0$.

Note also that the following fact is proved in the course of the proof of the theorem:

$$(3.23) \qquad H^{2k}_{e}(Z) = H^{2k}_{\varepsilon(\Phi)}(M).$$

Now we shall prove a theorem concerning the index of $M$ as an application of the above theorem, just in the same way as the Hodge index theorem is proved by means of the Lefschetz decomposition (cf. [12] [27]). Recall first that $M$ is naturally oriented so that the signature $\tau = \tau(M)$ of $M$ can be defined as the signature of the quadratic form $h$: $H^{2n}(M, \boldsymbol{R}) \times H^{2n}(M, \boldsymbol{R}) \to H^{4n}(M, \boldsymbol{R}) \cong \boldsymbol{R}$.

**Theorem 3.26.** *Let $b_{2n}(M)$ be the $2n$-th Betti number of $M$. Then we have $b_{2n}(M) = (-1)^n \tau(M)$. In other words, $h$ is positive or negative definite according as $n$ is even or odd.*

The theorem is due to Salamon when $n=2$ ([22; Th. 7.3]). See also [13] for the corresponding result when $n=1$. Now for any integer $k \geqq 0$ define a symmetric bilinear form $A$ on $H^{2k}(M, \boldsymbol{R})$ by

$$A(a, b) = L^{n-k}_{\Phi} ab[M] \in \boldsymbol{R},$$

where $[M]$ denotes the evaluation on the fundamental cycle of $M$. $A$ reduces to $h$ in the case $k=n$. Hence Theorem 3.26 is a special case of the following:

**Theorem 3.27.** *The symmetric bilinear form $(-1)^k A(a, b)$ is positive definite on $H^{2k}(M, \boldsymbol{R})$.*

*Proof.* We write $H^i_\varepsilon$ for $H^i_{\varepsilon(\Phi)}(M, \boldsymbol{R})$. The theorem clearly would follow if we show the following two facts: 1) With respect to $A$, $L^r_\Phi H^{2k-4r}_\varepsilon$ and $L^{r'}_\Phi H^{2k-4r'}_\varepsilon$ are orthogonal to each other if $r \neq r'$. 2) $(-1)^k A$ is positive definite on $L^r_\Phi H^{2k-4r}_\varepsilon$ for any $r$.

*Proof of* 1). Let $a \in H^{2k-4r}_\varepsilon$ and $b \in H^{2k-4r'}_\varepsilon$ be arbitrary elements. We may assume that $r' < r$. Then $n-k+r+r' > n-k+2r'+1$, which implies that $L^{n-k+r+r'}_\Phi H^{2k-4r'}_\varepsilon = 0$. Hence $L^{n-k}_\Phi L^r_\Phi a L^{r'}_\Phi b = a L^{n-k+r+r'}_\Phi b = 0$.

*Proof of* 2). Let $a$ be as above. Let $C$ be the Weil operator on $Z$, i.e., the action of $i \in \boldsymbol{C}^*$ on $H^*(M, \boldsymbol{C})$. Since $a$ is real of type $(k-2r, k-2r)$ by (3.17), we have $C\bar{a} = a$. Moreover by (3.23), $a$ is $\omega$-effective. Therefore by [27; p. 77, Cor] if $a \neq 0$ we have

$$(-1)^k L^{2n+1-(2k-4r)}_\Phi a^2[Z] = (-1)^k L^{2n+1-(2k-4r)}_\Phi a C\bar{a}[Z] > 0.$$

On the other hand, if we put $R=\int_{\pi^{-1}(x)}\omega>0$ for $x\in M$ (which is independent of $x$) we have by (3.19)

$$A(L_{\Phi}^{r}a,\, L_{\Phi}^{r}a)=L_{\Phi}^{2n-(2k-4r)}a^{2}[X]=(1/Ra^{n-k+2r})L^{2n+1-(2k-4r)}a^{2}[Z].$$

Thus 2) follows.

$A(a, b)$ is defined for any quaternionic Kähler and hyperkähler manifolds. It is interesting to study the properties of $A(a, b)$ in the general case.

## § 4. Compact Kähler symplectic manifolds

Let $X$ be a compact Kähler manifold of even dimension $2n$. Then $X$ is said to be *symplectic* if there exists a holomorphic 2-form $\varphi$ on $X$ which is nondegenerate, i.e., its $n$-th exterior power $\varphi^{n}$ is nowhere vanishing. In partiuclar, in this case the first Chern class $c_1(X)$ of $X$ vanishes. On the structure of such a manifold the *Bogomolov splitting theorem* is fundamental (cf. [4] [1] [14]).

**Theorem.** *Let $X$ be a compact Kähler symplectic manifold. Then there exists a finite unramified Galois covering $\tilde{X}\to X$ such that $\tilde{X}$ is isomorphic to a product $T\times Y_1\times\cdots\times Y_r$, where $T$ is a complex torus and $Y_i$, $1\leqq i\leqq r$, are simply connected Kähler symplectic manifolds with $h^{2k,0}(Y_i)=1$ and $h^{2k-1,0}(Y_i)=0$ for any $k\geqq 1$. Here the direct factors are uniquely determined by $\tilde{X}$ up to permutation.*

**Remark 4.1.** If $\dim T=0$ in the above theorem, then $X$ itself is already simply connected so that it is isomorphic to $Y:=Y_1\times\cdots\times Y_r$; this is the case, for instance, if the fundamental group is finite, or if $h^{2,0}(X)=1$ and $\dim X>2$. This can be deduced from the following two facts: 1) If Aut denotes in general the group of automorphisms, then Aut $Y$ is naturally a semidirect product of the product of Aut $Y_i$, $1\leqq i\leqq r$, by the finite group of "permutations" of those factors which are isomorphic to each other. 2) For any $i$, Aut $Y_i$ contains no finite symplectic automorphism groups acting freely on $Y_i$.

If $(M, g)$ is a hyperkähler manifold, then in the associated Calabi family $\{M_\lambda\}_{\lambda\in C}$ each $M_\lambda$ is a Kähler symplectic manifold; indeed, if $\varphi_\lambda$ is element of $V_0^{2,0}(\lambda)$ defined by (2.13), then the parallel 2-form on $M$ which corresponds to $\varphi_\lambda$ with respect to the isomorphism in (3.6) is a nondegenerate holomorphic 2-form on $M_\lambda$. We shall denote this form still by the same letter $\varphi_\lambda$ and call it the canonical holomorphic 2-form on $M_\lambda$.

Conversely, any compact Kähler symplectic manifold $X$ determines naturally a family of hyperkähler structures on the underlying oriented $C^\infty$ manifold $M$, as was shown by Beauville [1], Kobayashi [14] and others by using the result of Yau [30]. To be more precise, let $\Omega$ be the open cone of $H^{1,1}(X)_{\boldsymbol{R}}$ consisting of Kähler classes. For any class $\omega \in \Omega$ there exists a unique Kähler metric $\tilde{g}$ with the vanishing Ricci curvature by Yau. Then the underlying Riemannian manifold of $(X, \tilde{g})$ is a hyperkähler manifold and gives rise to a quaternionic structure on $M$, which is unique exactly when $X$ is simply connected (cf. Remark 4.1); moreover in this case the quaternionic structure depends only on the corresponding point $\bar{\omega}$ of the projectified cone $\Omega/\boldsymbol{R}^+$. We also note that the Calabi family $\{M_\lambda\}$ associated to any quaternionic structure on $M$ as above contains $(X, \tilde{g})$ as one of the members. Now combining the above facts with those of Section 3 we get the following:

**Theorem 4.2.** *Let $X$ be a compact Kähler symplectic manifold. Then associated to any quaternionic structure on the underlying $C^\infty$ manifold $M$ of $X$ constructed as above we have the natural decompositions of $H^k(X, \boldsymbol{C})$ and $H^k(X, \boldsymbol{R})$ as stated respectively in Theorem* 3.11 *and Theorem* 3.16. *They are compatible with the Hodge decomposition of $H^k(X, \boldsymbol{C})$.*

**Remark 4.3.** Let $\psi$ be the canonical holomorphic 2-form on $X = M_\lambda$ for some $\lambda \in \Lambda$. Let $\psi_0, \psi_1$ be the real and imaginary parts of $\psi$ respectively. Then the space $H^k_\varepsilon(X, \boldsymbol{R})$ of universally effective classes in Theorem 3.16 can be described in terms of the holomorphic structure of $X$ by $H^k_\varepsilon(X, \boldsymbol{R}) = \{\alpha \in H^k(X, \boldsymbol{R});\ L^d_\omega\alpha = L^d_{\psi_0}\alpha = L^d_{\psi_1}\alpha = 0\}$, where $d = 2n - k + 1$.

We put together some consequences of the above theorem which is independent of the choice of quaternionic structures.

**Proposition 4.4.** *Let $X$ be a compact Kähler symplectic manifold of dimension $2n$. Let $b_i$ be the Betti numbers of $X$. Then the following are true:* 1) *For any $l \leqq n$, $b_l + 3b_{l-2} \geqq 3b_{l-4} + b_{l-6}$ and hence also $b_l + b_{l-4} \geqq 2b_{l-2}$. We have $b_{2s} \geqq (s+1)((b_2-1)s+2)/2$ for $2s \leqq n$ and $b_{2s+1} \geqq (s+1)(s+2)b_3/2$ for $2s+1 \leqq n$, if $b_1 = 0$.* 2) *$b_i$ is divisible by* 4, *if $i$ is odd. A Hodge number $h^{p,q}$ of $X$ is even, if $p+q$ is odd.* 3) *For any $p \geqq q \geqq 0$, we have the inequality $h^{p,q} \geqq h^{p+1,q-1}$.*

We shall next give another result on the Hodge cohomology group $H^q(X, \Omega^p_X)$ of $X$. Let $\varphi$ be a nondegenerate holomorphic 2-form with class $\psi \in H^{2,0}(X)$. Let $L_\psi$: $H^q(X, \Omega^p_X) \to H^q(X, \Omega^{p+2}_X)$ (resp. $L_{\bar{\psi}}$: $H^q(X, \Omega^p_X) \to H^{q+2}(X, \Omega^p_X)$) be the $\psi$- (resp. $\bar{\psi}$-) multiplication operator on the graded ring $H^*(X, \Omega^*_X)$. We then define for $\gamma = \psi, \bar{\psi}$ the space of $\gamma$-effective Hodge classes by

$$H^q_{e(\gamma)}(X, \Omega^p_X)=\{\alpha \in H^q(X, \Omega^p_X);\ L^{n-s+1}_\gamma\alpha=0\},$$

where $s=p$ (resp. $q$), if $\gamma=\psi$ (resp, $\bar{\psi}$).

**Theorem 4.5.** 1) *The linear maps* $L^{n-p}_\psi$: $H^q(X, \Omega^p_X)\to H^q(X, \Omega^{2n-p}_X)$ *for* $p<n$, *and* $L^{n-q}_{\bar{\psi}}$: $H^q(X, \Omega^p_X)\to H^{2n-q}(X, \Omega^p_X)$ *for* $q<n$, *are both isomorphic.* 2) *We have for any* $p, q\geqq 0$ *the direct sum decompositions*

$$H^q(X, \Omega^p_X)=\bigoplus_{r\geqq(n-p)+} L^r H^q_{e(\psi)}(X, \Omega^{p-2r}_X), \quad \textit{and}$$
$$H^q(X, \Omega^p_X)=\bigoplus_{r\geqq(n-q)+} L^r H^{q-2r}_{e(\bar{\psi})}(X, \Omega^p_X).$$

*Proof.* Let $L_\varphi$: $\Omega^p_X\to\Omega^{p+2}_X$ be defined by the exterior multiplication by $\varphi$. Then by Proposition 2.6 together with its corollary, we see that $L^{n-p}_\varphi$ induces the isomorphism of $O_X$-modules $L^{n-p}_\varphi$: $\Omega^p_X\xrightarrow{\sim}\Omega^{2n-p}_X$, $p<n$, and moreover, that if $\Omega^p_{e(\varphi)}$ is the kernel of $L^{n-p+1}_\varphi$, we get the direct sum decomposition of $O_X$-modules

$$\Omega^p_X=\bigoplus_{r\geqq(n-p)+} L^r_\varphi\Omega^{p-2r}_{e(\varphi)}. \tag{4.1}$$

From the first assertion, 1) for $L_\psi$ follows at once. As for 2), we first note that from (4.1) we have the direct sum decomposition

$$H^q(X, \Omega^p_X)=\bigoplus_{r\geqq(n-p)+} H^q(X, L^r_\varphi\Omega^{p-2r}_{e(\varphi)}).$$

Hence it suffices to show that

$$H^q(X, L^r_\varphi\Omega_{e(\varphi)})\cong L^r_\psi H^q_{e(\psi)}(X, \Omega^p_X) \tag{4.2}$$

for general $p$. First we show that the natural inclusion (cf. (4.1)) $H^q(X, \Omega^p_{e(\varphi)})\hookrightarrow H^q_{e(\psi)}(X, \Omega^p_X)$ is isomorphic. For this purpose we fix any hyperkähler structure $(M, g)$ on $X$ such that the associated $HK$ 3-space contains $\psi$. Take any element $\bar{\alpha}$ of $H^q_{e(\psi)}(X, \Omega^p_X)$ and represent it by a harmonic $(p, q)$-form $\alpha$. Let $\wedge^{p,0}T^*$ and $\wedge^{0,q}T^*$ be the bundle of $(p, 0)$- and $(0, q)$-forms. Let $\wedge^{p,0}_{e(\varphi)}T^*$ be the subbundle of $\wedge^{p,0}T^*$ corresponding to the subsheaf $\Omega^p_{e(\varphi)}$ of $\Omega^p_X$. Let $W^{p,q}=\wedge^{p,0}_{e(\varphi)}T^*\otimes\wedge^{0,q}T^*$. Then by Corollary 3.7 applied to the parallel form $\varphi^{n-p+1}$ we see that $L^{n-p+1}_\varphi\alpha=0$ on $M$ and hence $\alpha$ is a harmonic $W^{p,q}_\varphi$-form. This implies that $\bar{\alpha}\in H^q(X, \Omega^p_{e(\varphi)})$ as desired. (4. 2) then follows from Lemma 3.9 and the ensuing remark. Finally the results for $\bar{\psi}$ is obtained, if we take the complex conjugation of the results for $\psi$.

**Corollary 4.6.** 1) $h^{p,q}(X)=h^{2n-p,q}(X)=h^{p,2n-q}(X)$. 2) *For any* $p<n$ *and any* $q\geqq 0$, $h^{p,q}(X)\leqq h^{p+2,q}(X)$; *for any* $p\geqq 0$ *and any* $q<n$, $h^{p,q}(X)\leqq h^{p,q+2}(X)$.

The corollary implies that among the Hodge numbers $h^{p,q}(X)$ of $X$ those with $0 \leqq p,\ q \leqq n$ are really essential. In view of 1) above the inequlaities in 3) of Proposition 4.4 for $p > n$ follows from the relation $h^{s,t} \leqq h^{s+1,t+1}$, $s,\ t \leqq 2n$, which holds for any compact Kähler manifold.

Next, using the fact that $\boldsymbol{H}^{\times}$ acts on $H^*(X, \boldsymbol{R})$ as algebra automorphisms, we shall prove a remarkable topological property of a compact Kähler symplectic manifold. Namely we show the following:

**Theorem 4.7.** *Let $X$ be a compact Kähler symplectic manifold with $h^{2,0}(X)=1$. Let $g(x)$ be the homogeneous form of degree $2n$ on $H^2(X, \boldsymbol{Q})$ defined by $g(x)=x^{2n}[X]$, $x \in H^2(X, \boldsymbol{Q})$, where $[X]$ denotes the evaluation on the fundamental class of $X$. Then there exist a constant $c \in \boldsymbol{Q}^+$ and a nondegenerate quadratic form $f$ of singnature $(3, b-3)$ on $H^2(X, \boldsymbol{Q})$ such that $g=cf^n$, where $b=b_2(X)$.*

We remark that by Beauville [1] it has already been known that $g$ is divisible by a quadratic form $f$ as above. Further, the theorem is known for two typical series of known examples of compact Kähler symplectic manifolds (cf. [1]).

For the proof we use the Bogomolov unobstructedness theorem [3]. Let $X$ be a compact complex manifold. We say that $X$ is *unobstructed* (under deformation) if the parameter space $S$ of the Kuraniashi family $(f\colon X \rightarrow S,\ X_o=X,\ o \in S)$ of $X$ is (reduced and) nonsingular. Then the theorem is as follows.

**Theorem.** *Any compact Kähler symplectic manifold is unobstructed in the sense defined above.*

Using the Calabi family we shall give a simple alternative proof of this theorem at the end of this section. We shall also recall some basic facts on the period map of a symplectic compact Kähler manifold $X$ with $h^{2,0}(X)=1$ (cf. [1] [10; § 4]).

Let $E=H^2(X, \boldsymbol{C})$. Let $\boldsymbol{P}=\boldsymbol{P}(E)$ be the associated projective space. Let $q \in \boldsymbol{P}$ be the line $H^{2,0}=H^{2,0}(X)$ of $E$. Then the (holomorphic) tangent space $T_q$ of $\boldsymbol{P}$ at $q$ is naturally identified with $\mathrm{Hom}_{\boldsymbol{C}}(H^{2,0}, H^{1,1} \oplus H^{0,2})$. Now let $(f\colon X \rightarrow S,\ X_o=X,\ o \in S)$ be a deformation of $X$. Let $T$ be the Zariski tangent space of $S$ at $o$. Let $\rho\colon T \rightarrow H^1(X, \Theta_X)$ be the Kodaira-Spencer map [17]. Let $p\colon (S, o) \rightarrow (\boldsymbol{P}, q)$ be the period map associated to 2-forms for $f$. Then $dp$ factors through the subspace $\mathrm{Hom}_{\boldsymbol{C}}(H^{2,0}, H^{1,1})$ of $T_q$;

$$dp\colon T \rightarrow \mathrm{Hom}_{\boldsymbol{C}}(H^{2,0}, H^{1,1}). \tag{4.3}$$

The cup product $H^1(X, \Theta_X)\times H^0(X, \Omega_X^2)\to H^1(X, \Omega_X^1)$ defines a natural isomorphism $\alpha\colon H^1(X, \Theta_X)\to \mathrm{Hom}_C(H^{2,0}, H^{1,1})$. Moreover we have the commutativity $dp=\alpha\rho$. If, further, $f$ is the Kuranishi family of $X$, then $\rho$ is isomorphic; hence (4.3) is isomorphic and $p$ is an embedding. Note also that the Kuranishi family is universal for any compact Kähler symplectic manifold (cf. [10]).

On the other hand, let $g$ be the homogeneous form of degree $2n$ on $E$ defined in Theorem 4.7. Let $N$ be the (reduced) hypersurface of $\boldsymbol{P}$ defined by $g=0$. Then $q\in N$ and the image of $\boldsymbol{P}$ is contained in $N$;

$$p\colon (S, o)\to (N, q). \tag{4.4}$$

By considering the dimensions we see that (4.4) is isomorphic if $S$ is nonsingular; in particular $N$ also is nonsingular at $q$.

Now we shall fix once and for all a Kähler class $\omega$ on $X$ and a Kähler form $\tilde{\omega}$ which represents it. Let $(M, g)$ be the hyperkähler manifold associated to $\omega$ by Theorem 3.1, where $M$ is the underlying $C^\infty$ manifold of $X$. Let $F\subseteqq H^2(M, \boldsymbol{R})$ be the corresponding $HK$ 3-space. Let $\{M_\lambda\}_{\lambda\in C}$ be the Calabi family associated to $(M, g)$. Let $\omega_\lambda$ be the canonical Kähler class on $M_\lambda$ and $\varphi_\lambda$ the canonical holomorphic 2-form on $M_\lambda$ with class $\psi_\lambda\in H^{2,0}(M_\lambda)$. Then in the canonical $\boldsymbol{H}^\times$-decompostion $H^2(M, \boldsymbol{R})=H^{2;2}(M, \boldsymbol{R})\oplus H^{2;0}(M, \boldsymbol{R})$ of $H^2(M, \boldsymbol{R})$ (cf. (3.11)), we have $H^{2;2}(M, \boldsymbol{R})=F$ and

$$F=(\boldsymbol{C}\psi_\lambda\oplus\boldsymbol{C}\bar{\psi}_\lambda)_{\boldsymbol{R}}\oplus\boldsymbol{R}\omega_\lambda \tag{4.5}$$

for any $\lambda\in C$; in particular the Hodge decomposition of $H^2(M_\lambda, \boldsymbol{C})$ is of the form

$$H^2(M_\lambda, \boldsymbol{C})=\boldsymbol{C}\psi_\lambda\oplus(\boldsymbol{C}\omega_\lambda\oplus H^{2;0}(M, \boldsymbol{C}))\oplus\boldsymbol{C}\bar{\psi}_\lambda.$$

Now $\{M_\lambda\}$ forms a $C^\infty$ trivial holomorphic family $f_C\colon Z_C\to C$, $M_\lambda=f^{-1}(\lambda)$, by Salamon [22] (cf. Sect. 5 below). So we get a globally defined period map $p_C\colon C\to\boldsymbol{P}$, which factors through $\boldsymbol{P}(F_C)\subseteqq\boldsymbol{P}$ by (4.5). Then the resulting map $p'_C\colon C\to\boldsymbol{P}':=\boldsymbol{P}(F_C)$ may naturally be identified with the period map associated to the Hodge decomposition of $F_C\cong V_{2,2}$ considered in Lemma 1.3. In particular, by that lemma the differential $dp_C$ of $p_C$ at $\lambda\in C$ gives an isomorphism

$$\begin{aligned} dp_C\colon T_\lambda &\cong \mathrm{Hom}\,(H^{2,0}(M_\lambda), \boldsymbol{C}\omega_\lambda) \\ &\subseteqq \mathrm{Hom}\,(H^{2,0}(M_\lambda), H^{1,1}(M_\lambda)\oplus H^{0,2}(M_\lambda)), \end{aligned} \tag{4.6}$$

where $T_\lambda$ is the tangent space of $C$ at $\lambda$.

We shall fix some more notation. Let $E_{\boldsymbol{R}}=H^2(M, \boldsymbol{R})$. Let $\mathrm{Gr}_{\boldsymbol{R}}$

(resp. Gr) be the Grassmann manifold of 3-dimensional real (resp. complex) subspaces of $E_R$ (resp. $E$). We identify an element of $\mathrm{Gr}_R$ or Gr with the corresponding subspaces of $E_R$ with $E$, respectively. Further, by identifying $F \in \mathrm{Gr}_R$ with $F_C \in \mathrm{Gr}$ we consider $\mathrm{Gr}_R$ as a real submanifold of Gr. Indeed, Gr may be considered as a complexification of $\mathrm{Gr}_R$ along $\mathrm{Gr}_R$ in the sense that for any $x \in \mathrm{Gr}_R$ we have an isomorphism

$$(4.7) \qquad (\mathrm{Gr}, \mathrm{Gr}_R, x) \cong (C^N, R^N, 0), \quad N = 3(b-3),$$

as a germ of real analytic manifolds.

On the other hand, consider the triple

$$(4.8) \qquad (Y, \omega_Y, u)$$

consisting of a compact Kähler symplectic manifold $Y$ with $h^{2,0}(Y) = 1$, a Kähler class $\omega_Y$ on $Y$ and a diffeomorphism $u: M \xrightarrow{\sim} Y$. Let $\tilde{g}_Y$ be the Kähler-Einstein metric corresponding to the class $\omega_Y$. Then by Theorem 4.1 if we set $g_Y = u^* \tilde{g}_Y$, $(M, g_Y)$ is again a hyperkähler manifold. In this case $(M, g)$, the corresponding *HK* 3-space, the induced Sp(1)-action on $H^*(M, C)$ etc. are all said to be *associated* to $(Y, \omega_Y, u)$. If $Y = X$, $\omega_Y = \omega$ and $u$ = identity, we denote by $F(X)$ the associated *HK* 3-space. Further, we call a 3-space $F \in \mathrm{Gr}_R$ *HK* if it is the *HK* 3-space associated to some triple $(Y, \omega_Y, u)$ as in (4.8).

Now we shall first show the following assertion (A) weaker than Theorem 4.7:

(A) *There exists a quadratic form h on E such that* $g = h^n$.

For this purpose it clearly suffices to prove the following three lemmas. (We identify $g$ with its extensions to $E_R$ and to $E$ as above.)

**Lemma 4.8.** *If* $F \in \mathrm{Gr}_R$ *is HK, then there exists a definite quadratic form* $h_F$ *on F such that* $g|_F = \pm h_F^n$.

**Lemma 4.9.** *Let* $B = \{F \in \mathrm{Gr}_R;\ F \text{ is } HK\}$. *Then B is open in* $\mathrm{Gr}_R$ *so that in particular B is Zariski dense in* Gr.

**Lemma 4.10.** *Let B be any Zariski dense subset of* Gr. *Suppose that for any* $F \in B$ *the conclusion of Lemma* 4.8 *is true. Then there exists a quadratic form h on E such that* $g = h^n$.

*Proof of Lemma* 4.8. Let $F$ be any *HK* 3-space associated to the triple $(Y, \omega_Y, u)$ as in (4.8). Then with respect to the associated Sp(1)-module structure, $F$ is isomorphic to $\mathfrak{sp}(1)$ with the adjoint representation (cf. Proposition 3.15). On the other hand, $g$ is Sp(1)-invariant by Lemma

3.14, and hence, $g|_F$ is an Sp(1)-invariant homogeneous form of degree $2n$ on $F$. Since the invariant polynomial ring of $\mathfrak{sp}(1)$ is generated over $\boldsymbol{R}$ by the Killing form of $\mathfrak{sp}(1)$ (cf. [15; XII]) it follows that we may write $g|_F = \pm h_F^n$ for some quadratic form $h_F$ on $F$.

*Proof of Lemma* 4.9. Let $F$ be the $HK$ 3-space associated to a triple $(Y, \omega_Y, u)$ as in (4.8). It suffices to show that for some neighborhood $U$ of $F$ in $\mathrm{Gr}_{\boldsymbol{R}}$, any element $F' \in U$ is again an $HK$ 3-space. (The last assertion would then follow from (4.7)). Without loss of generality we may assume that $Y = X$, $\omega_Y = \omega$, and $u$=identity, so that by (4.5) $F_{\boldsymbol{C}} = (H^{2,0}(X) \oplus H^{0,2}(X)) \oplus \boldsymbol{C}\omega$. Let now $(f\colon X \to S, X_o = X, o \in S)$ be the Kuranishi family of $X$. By the Bogomolov unobstructedness theorem $S$ is smooth. Let $(S, o) \to (N, q)$ be the associated period map (4.4). Let $\mathrm{Gr}_2$ be the Grassmann manifold of 2-planes in $E_{\boldsymbol{R}}$. Let $L_0 = (H^{2,0}(X) \oplus H^{0,2}(X))_{\boldsymbol{R}} \in \mathrm{Gr}_2$. Then the map $l \mapsto e(l) := (l \oplus \bar{l})_{\boldsymbol{R}}$ is easily seen to define isomorphism $e\colon (N, q) \rightleftarrows (\mathrm{Gr}_2, L_0)$ of germs of real analytic manifolds.

On the other hand, let $\mu\colon H \to S$ be a real $C^\infty$ vector subbundle of the trivial bundle $E_{\boldsymbol{R}} \times S \to S$ such that $H_0 = H^{1,1}(M)_{\boldsymbol{R}} \subseteqq E_{\boldsymbol{R}}$. Let $\sigma$ be a $C^\infty$ section of $H$ such that $\sigma(0) = \omega \in H_0$. Then in view of the direct sum decomposition $E_{\boldsymbol{R}} = L_0 \oplus H^{1,1}(M)_{\boldsymbol{R}}$, by considering the differential at $u$, one sees readily that the map $v\colon (H, \omega) \to (\mathrm{Gr}_{\boldsymbol{R}}, F)$ of real analytic manifolds defined by $v(\beta) = ep(\mu(\beta)) \oplus \boldsymbol{R}\beta \subseteqq E_{\boldsymbol{R}}$ is a submersion, where $\beta \in H_s \subseteqq E \times s = E$.

Now we shall actually construct a vector subbundle $H \to S$ and a $C^\infty$ section $\sigma$ as above. Fix a $C^\infty$-family $\{u_s\}_{s \in S}$ of diffeomorphisms $u_s\colon X \to X_s := f^{-1}(s)$ with $u_0$=identity. This induces the trivialization of the local system:

$$(4.9) \qquad R^2 f_* \boldsymbol{R} \rightleftarrows E_{\boldsymbol{R}} \times S.$$

Let $\{\tilde{\omega}_s\}_{s \in S}$ be a $C^\infty$ family of Kähler forms $\tilde{\omega}_s$ on $X_s$ with $\tilde{\omega}_0 = \tilde{\omega}$. Let $\mathfrak{H}_s$ be the space of harmonic forms of type (1, 1) on $X_s$ with respect to the Kähler metric associated to $\tilde{\omega}_s$. Then $H := \bigcup_{s \in S} \mathfrak{H}_s \to S$ has the natural structure of a $C^\infty$-vector bundle on $S$, and $\sigma(s) := \tilde{\omega}_s \in \mathfrak{H}_s$ defines a $C^\infty$-section $\sigma$ of $H$. Identifying $\mathfrak{H}_s$ with $H^{1,1}(X_s)_{\boldsymbol{R}}$ naturally, and using (4.9), we may view $H$ as a vector subbundle of $E_{\boldsymbol{R}} \times S \to S$. Then it suffices to show that for the submersion $v\colon (H, \omega) \to (\mathrm{Gr}, F_0)$ associated to $H$ and $\sigma$, the 3-spaces in the image of $v$ are all $HK$. In fact, if we take a neighborhood $U$ of $\sigma(S)$ in $H$ in such a way that any $\tilde{\beta} \in U_s := U \cap f^{-1}(s)$ is a Kähler form on $X_s$, by our construction of $v$ above it follows readily that for any $\tilde{\beta} \in U_s$, $v(\beta)$ is just the $HK$ 3-space determined by the triple $(X_s, \beta, u_s)$, where $\beta$ is the de Rham class of $\tilde{\beta}$.

*Proof of Lemma* 4.10. For any complex space $Y$ let $Y_{\text{red}}$ denote the underlying reduced subspace. Let $\boldsymbol{P}=\boldsymbol{P}(E)$ as above. Let $Z$ be the hypersurface of degree $2n$ in $\boldsymbol{P}$ defined by $g$ so that $Z_{\text{red}}=N$. For any $P \in \mathrm{Gr}$ we set $Z_P=Z\cap P$. Then by our assumption, for any $F \in B$ we have

$$Z_{F_{\boldsymbol{C}}}=nZ_{F_{\boldsymbol{C}},\text{red}} \qquad \text{with } Z_{F_{\boldsymbol{C}}} \text{ smooth.} \tag{4.10}$$

Suppose first that $N$ is reducible. Then there exists a nonempty Zariski open subset $U$ of Gr such that for any $P \in U$, we have $P \not\subseteq Z$ and $N_{P,\text{red}}$ is reducible, where $N_P=N\cap P$. By (4.10) we must have $B\subseteq \mathrm{Gr}-U$, which contradicts the Zariski density of $B$. Hence $N$ is irreducible. In particular, we may write $Z=kN$ for some $k>0$. Now take any Zariski open subset $V$ of Gr such that for any $P \in V$, we have $P\not\subseteq N$ and $N_P$ is reduced and irreducible. Then again by the Zariski density of $B$ we have $B\cap U \neq \varnothing$. Take any $P=F_{\boldsymbol{C}} \in B\cap V$. Then by (4.10) we get $Z_P=n(Z_P)_{\text{red}}=kN_P$; hence $k=n$ and $Z=nN$. Thus we have $g=h^n$ for a suitable quadratic form on $E$ which defines $N$.

Next we show the following assertion (B):

(B) *There exists a quadratic form $f$ on $E_{\boldsymbol{Q}}:=H^2(M, \boldsymbol{Q})$ and a constant $c \in \boldsymbol{Q}^{\times}$ such that $g=cf^n$.**)

We first prove a lemma. Let $c_2=c_2(X) \in H^4(M, \boldsymbol{Q})$ be the (rational) second Chern class of $X$. Then define a homogeneous form $g_0$ of degree $2n-2$ on $E_{\boldsymbol{Q}}$ by

$$g_0(x)=x^{2n-2}c_2[M], \qquad x \in E_{\boldsymbol{Q}}=H^2(M, \boldsymbol{Q}).$$

**Lemma 4.11.** *Let $h$ be the quadratic form on $E$ obtained in* (A). *Then there exists a nonzero constant $d \in \boldsymbol{C}$ such that $g_0=dh^{n-1}$.*

*Proof.* First we note that $g_0$ is nonzero, since $g_0(\beta)\neq 0$ for any Kähler class $\beta$ on $X$ by Chen-Ogiue [8]. Let $F$ be an $HK$ 3-space associated to some triple $(Y, \omega_Y, u)$ as in (4.8). Then since $c_1(Y)=0$, $u^*c_2(Y)=-p_1(M)/2$, where $p_1(M)$ is the first rational Pontrjagin class of $M$ (cf. [12]). Since $c_2$ is always of type (2, 2), from this (cf. Lemma 5.4 below) we see that $c_2$ is, and hence by Lemma 3.14 $g_0$ also is, invariant under the Sp(1)-action associated to $(Y, \omega_Y, u)$. Hence if $h_F$ is as in Lemma 4.8 we can show that

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*) Professor Kazuya Kato kindly informed me a purely algebraic proof of (B) (based on (A)). The proof here nevertheless seems to be of some geometric interest (cf. Remark 4.12 below).

$$(4.11) \qquad g_0|_F = rh_F^{n-1}$$

for some nonzero constant $r$, by the same argument as in Lemma 4.8. Hence if $Z_0$ is the hypersurface of $\boldsymbol{P}$ of degree $2n-2$ defined by $g_0$, then by the same proof as in Lemma 4.10 we see that we may write $Z_0=(n-1)\bar{N}$ for $\bar{N}:=Z_{0,\mathrm{red}}$. On the other hand, by (4.11) for any $F \in B$ we have $\bar{N}_{F_C}=N_{F_C}$ in the notation of the proof of Lemma 4.10. Then again by using the Zariski density of $B$ we see that $\bar{N}=N$. Hence $Z_0=(n-1)N$. The lemma follows from this at once.

*Proof of* (B). By Lemmas 4.10 and 4.11 we have $g=h^n$ and $g_0=dh^{n-1}$ in the complex polynomial ring $\boldsymbol{C}[E]$. Since $g$ and $g_0 \in \boldsymbol{Q}[E_{\boldsymbol{Q}}]$, $f:=g/g_0=h/d \in \boldsymbol{Q}[E_{\boldsymbol{Q}}]$, where $E_{\boldsymbol{Q}}=H^2(X, \boldsymbol{Q})$. Then we have $g=cf^n$ with $c=d^n \in \boldsymbol{Q}^\times$ as desired.

**Remark 4.12.** Let $C^*(X)$ be the subring of $H^*(X, \boldsymbol{Q})$ generated by the rational Chern classes $c_i(X) \in H^{2i}(X, \boldsymbol{Q})$ of $X$. Note that since the structure group of $X$ is reducible to $\mathrm{Sp}(n, \boldsymbol{C})$, $c_i(X)=0$ for any odd $i$ (cf. [15; XII]). Thus we may write $C^*(X)=\oplus_k C^{4k}(X)$. Note further that since $c_1(X)=0$, each $c_{2k}(X)$ is expressible in terms of universal rational polynomials of the Pontrjagin classes of $M$ (cf. [12]). Thus $C^*(X)$ actually depends only on $M$, but not on the particular Kähler symplectic structure $X$ on $M$. So denote it by $C^*(M)$. Now any homogeneous element $\alpha$ of $C^{4k}(M)$ defines a homogeneous form $g_\alpha$ of degree $2(n-k)$ on $H^2(M, \boldsymbol{Q})$ by $g_\alpha(x)=\alpha x^{2(n-k)}[M]$. Then by the same argument as above we see that $g_\alpha=c_\alpha f^{n-k}$ for some $c_\alpha \in \boldsymbol{Q}$ and for the quadratic form $f$ as in (B) which is independent of $\alpha$. This defines a rational linear form $\alpha \to c_\alpha$ on the vector space $C^{4k}(M)$ naturally associated to $M$.

In view of (B), Theorem 4.7 now follows from the next assertion (C). (This is essentially contained in Beauville [1], but for completeness we shall give a proof in our context.)

(C) *Let $f$ be as in* (B). *Then $f$ is nondegenerate and its signature can be made into* $(3, b-3)$ *if we replace $f$ by $-f$ if necessary.*

*Proof.* Let $W$ be the subspace of $\omega$-effective classes of type (1, 1) in $E_{\boldsymbol{R}}$. Let $F=F(X)$ be the $HK$ 3-space associated to $(X, \omega, \mathrm{id}_X)$. Then we have the direct sum decomposition

$$(4.12) \qquad E_{\boldsymbol{R}}=F \oplus W,$$

which coincides with the canonical $\boldsymbol{H}^\times$-decomposition

$$E_{\boldsymbol{R}}=H^{2;2}(M, \boldsymbol{R}) \oplus H^{2;0}(M, \boldsymbol{R})$$

associated to $(X, \omega, \mathrm{id}_X)$. (Note that $V^{2;0}=V^{1;1}_{e;\boldsymbol{R}}$ in the notation of Section 2.) In particular, (4.12) is orthogonal with respect to $f$, since $f$ is Sp(1)-invariant as well as $g$. We know already that (cf. Lemma 4.8) $f$ is definite on $F$. Hence by replacing $f$ by $-f$ if necessary, we may assume that $f$ is positive definite on $F$. (C) then follows from:

**Lemma 4.13.** *$f$ is negative definite on $W$.*

*Proof.* Let $E(\omega)=\{a \in E;\ \omega^{2n-1}a[X]=0\}$ be the hyperplane of $E$ consisting of $\omega$-effective classes and $\boldsymbol{P}(\omega):=\boldsymbol{P}(E(\omega))$ the associated hyperplane in $\boldsymbol{P}$. Let $N$ be the quadric in $\boldsymbol{P}$ defined by $f=0$. Set $N(\omega)=\boldsymbol{P}(\omega)\cap N \subseteqq \boldsymbol{P}(\omega)$. Let $(f\colon X\to S, X_o=X, o \in S)$ be the Kuranishi family of $X$ and $p\colon (S, o)\to(N, q)$ the period map, which is isomorphic since $S$ is nonsingular by the unobstructedness theorem.

Let $\{u_s\}$ be as in the proof of Lemma 4.9. Let $\omega_s=(u_s^{-1})^*\omega \in H^2(X_s, \boldsymbol{R})$. Let $S(\omega)=\{s \in S;\ \omega_s$ is of type $(1, 1)$ on $X_s\}$. Then $o \in S(\omega)$ and $S(\omega)$ is a smooth hypersurface in $S$ (cf. [10; Prop. 4.2], [1]). Then $p$ maps $S(\omega)$ into $\boldsymbol{P}(\omega)$ and hence into $N(\omega)$ because $\omega_s^{2n-1}\psi_s[X_s]=0$ for any element $\psi_s$ of $H^{2,0}(X_s)$. Since $N(\omega)$ is also a hypesurface, we see that $p$ induces an isomorphism of germs

$$(S(\omega), o)\cong(N(\omega), q); \tag{4.13}$$

in partiuclar, $N(\omega)$ is smooth at $q$.

Now we define another quadratic form $f_\omega$ on $E$ by $f_\omega(x)=\omega^{2n-2}x^2[X]$. Since $f_\omega(\omega)>0, f_\omega\not\equiv 0$. Let $N^\omega=\{f_\omega=0\}$ be the quadric in $\boldsymbol{P}$ defined by $f_\omega$. Let $N^\omega(\omega)=N^\omega\cap \boldsymbol{P}(\omega)$. Since $N^\omega$ is irreducible as one sees by restricting $f_\omega$ to $F$, $N^\omega(\omega)$ is again a quadric in $\boldsymbol{P}(\omega)$. Now for any $s \in S(\omega)$, $p(s) \in N^\omega$ because $\psi_s$ is of type $(2, 0)$. Therefore for the same reason as above $p$ induces an isomorphism

$$p\colon (S(\omega), o)\xrightarrow{\sim}(N^\omega(\omega), q).$$

It follows from this and (4.13) that $N^\omega(\omega)=N(\omega)$ because both are irreducible. Hence $f=c_0 f_\omega$ on $E(\omega)_{\boldsymbol{R}}$ for some nonzero constant $c_0 \in \boldsymbol{R}$. On the other hand, we know that $E(\omega)_{\boldsymbol{R}}=(H^{2,0}(X)\oplus H^{0,2}(X))_{\boldsymbol{R}}\oplus W$ and $f_\omega$ is negative definite on $W$ and positive definite on $F$ (cf. [27; p. 78, Cor.]). Therefore $c_0>0$ and the lemma, and hence Theorem 4.7 also, is proved.

Finally we shall prove the Bogomolov unobstructedness theorem stated after Theorem 4.7. We use the following lemmas.

**Lemma 4.14.** 1) *Let $\pi\colon \tilde{X}\to X$ be a finite unramified Galois covering of compact Kähler symplectic manifolds. If $\tilde{X}$ is unobstructed, then so is $X$.*

2) *Let $Y$ and $Z$ be compact complex manifolds. Let $X=Y\times Z$. Suppose that $H^1(Z, O_Z)=H^0(Z, \Theta_Z)=0$. Then if both $Y$ and $Z$ are unobstructed, $X$ also is unobstructed.*

*Proof.* By Kodaira-Spencer [18] for the unobstructedness of $X$ it suffices to construct a deformation of $X$ whose Kodaira-Spencer map is isomorphic. 1) Let $G$ be the Galois group of $\pi$. By the universality of the Kuranishi family $\tilde{f}: \tilde{X}\to\tilde{S}$, $\tilde{X}_o=X$, $o\in\tilde{S}$, $G$ acts naturally on $\tilde{S}$ (cf. [10; § 3]). Let $S:=\tilde{S}^G$ be the set of fixed points, which is nonsingular since $\tilde{S}$ is. Let $X=\tilde{X}_S/G$ and let $f: X\to S$ be the natural projection. Then as a deformation of $X=X_o$ the Kodaira-Spencer map of $f$ is isomorphic in view of the natural isomorphism $H^1(\tilde{X}, \Theta_{\tilde{X}})^G\cong H^1(X, \Theta_X)$. 2) follows from the fact that the Kodaira-Spencer map of the product of the Kuranishi families of $Y$ and of $Z$ as a deformation of $X$ is isomorphic in view of the natural isomorphism $H^1(X, \Theta_X)\cong H^1(Y, \Theta_Y)\oplus H^1(Z, \Theta_Z)$, which in turn comes from the Künneth formula and the assumption.

Let $S=(S, o)$ be a germ of complex spaces. Let $T$ be the Zariski tangent space of $S$ at $o$. An element $\theta\in T$ is said to be *unobstructed* if there exists a smooth analytic subspace $A$ of $S$ such that its tangent space $T_A$ contains $\theta$ with respect to the natural inclusion $T_A\subseteqq T$.

**Lemma 4.15.** *Let $U$ be the set of unobstructed elements of $T$. Suppose that $T$ has a real structure in the sense that it admits an antilinear involution and that $U$ contains an open subset $V$ of the real part $T_{\boldsymbol{R}}$. Then $S$ is smooth.*

*Proof.* Choose an embedding of $S$ into $T$ and consider $S$ as a subspace of $T$. Let $C$ be the tangent cone of $S$ (in $T$). We may naturally consider $C$ as a subspace of $T$. Then it suffices to show that $C=T$. For any smooth analytic subspace $A$ of $S$ we have the natural inclusion $T_A\subseteqq C$. Hence $C$ contains $V$ so that any homogeneous polynomial $f$ in the ideal of $C$ vanishes on $V$. Since $V$ is open in the real part $T_{\boldsymbol{R}}$, $f$ must vanish identically even in a neighborhood of $V$ in $T$. This forces $f$ to be identically zero on the whole $T$. Thus $C=T$ as desired.

*Proof of the unobstructedness theorem.* By the Bogomolov splitting theorem together with Lemma 4.14 we see that it suffices to show the theorem only for those $X$ with $h^{2,0}(X)=1$. Let $\Omega=\{\omega\in H^{1,1}(X)_{\boldsymbol{R}};\ \omega$ is a Kähler class$\}$, which is open in $H^{1,1}(X)_{\boldsymbol{R}}$. Take any element $\omega$ of $\Omega$. Let $f_C: Z\to C$ be the Calabi family associated to $(X, \omega)$ and $\lambda\in C$ the point corresponding to $X$, i.e., $X=M_\lambda$ (cf. Theorem 4.1). Let $(f: X\to S$, $X_o=X$, $o\in S)$ be the Kuranishi family of $X$ and $\tau: (C, \lambda)\to(S, o)$ the universal morphism associated to $f_C$. Let $T_\lambda$ and $T$ be the (Zariski) tangent spaces of $(C, \lambda)$ and $(S, o)$, respectively. Let $p$ and $p_C$ be the period maps associated to $f$ and $f_C$, respectively. Then we have $p_C = p\tau$ with $p_C$ an embedding. Hence $\tau$ also is an embedding, and in the natural identification of $T$ with $\mathrm{Hom}_C(H^{2,0}(X), H^{1,1}(X))$, $d\tau(T_\lambda)$ is identified with the subspace $\mathrm{Hom}(H^{2,0}(X), \boldsymbol{C}\omega)$ (cf. (4.3) (4.6)). Now fix once and for all a nonzero element $\psi$ of $H^{2,0}(X)$. Then since $h^{2,0}(X)=1$, we may further identify $T$ with $H^{1,1}(X)$ and $d\tau(T_\lambda)$ with $\boldsymbol{C}\omega$. Since $\omega$ was arbitrary, with this identification we have shown that any element of $\Omega \subseteqq T$ is unobstructed in the sense defined before Lemma 4.15. Since $\Omega$ is open in the real part $H^{1,1}(X)_{\boldsymbol{R}}$ of $H^{1,1}(X)$ with its natural real structure, $S$ is smooth by Lemma 4.15 as desired.

## § 5. Calabi family associated to a hyperkähler manifold

Let $M$ be an oriented $C^\infty$ manifold of dimension $4n$. Then a quaternionic structure on $M$ gives rise to a family of complex structures on $M$, $\{M_\lambda\}_{\lambda \in C}$, parametrized by $C$, which we have called the Calabi family associated to the structure. Salamon [22] further showed that there exists a natural structure of a complex manifold on the union $Z := \bigcup_\lambda M_\lambda$ such that the natural projection $f: Z \to C$ is holomorphic. He also showed some remarkable properties of the complex manifold $Z$ as in the case of the Penrose twistor construction when $n=1$. Especially we see that associated to any quaternionic structure on $M$ we get canonically a triple

$$(f: Z \to \boldsymbol{P}^1, \pi, \tau)$$

consisting of 1) a smooth morphism $f: Z \to \boldsymbol{P}^1$ of complex manifolds, 2) a $C^\infty$ submersion $\pi: Z \to M$ such that $\mu := \pi \times f: Z \to M \times \boldsymbol{P}^1$ is a diffeomorphism, and 3) an antiholomorphic involution $\tau$ of $Z$ such that $\mu\tau\mu^{-1} = \mathrm{id}_M \times \tau_0$, where $\tau_0$ is the antiholomorphic fixed point free involution on $\boldsymbol{P}^1$ induced by the right multiplication by $j$ on $\boldsymbol{H}$ with respect to the identification (1.2); further, $f$ and $\pi$ are required to have the following property: Let $Z_m = \pi^{-1}(m)$ be the fiber over $m \in M$. Then 1) $Z_m$ is a complex submanifold of $Z$ (which is mapped isomorphically to $\boldsymbol{P}^1$ by $f$) and 2) the holomorphic normal bundle $N_m$ of $Z_m$ in $Z$ is isomorphic to the direct sum $H^{\oplus 2n}$ of the hyperplane bundle $H$ on $\boldsymbol{P}^1$;

$$N_m \cong H^{\oplus 2n}. \tag{5.1}$$

Here, if $(f': Z' \to \boldsymbol{P}^1, \pi', \tau')$ is another such triple, then we say that $(f, \pi, \tau)$ and $(f', \pi', \tau')$ are isomorphic if there exists a biholomorphic map $g: Z \to Z'$ such that $f = f'g$, $\pi = \pi'g$ and $g\tau = \tau'g$. In the next theorem we shall show that the above correspondence is invertible. When $n=1$, this is a part of Penrose twistor programme (cf. [13] [24]) and the idea is essentially the same in the general case.

**Theorem 5.1.** *There exists a natural bijective correspondence between the set of isomorphism classes of quaternionic structures on $M$ and the set of isomorphism classes of the triples as above.*

Let $V_1=\boldsymbol{H}$, considered as a complex vector space as in Section 1. Let $j_1$ be the antilinear isomorphism on $V_1$ induced by the right multiplication by $j$. The commutor $K_1$ of $j_1$ in $\mathrm{GL}_2(V_1)$ is naturally identified with $\boldsymbol{H}^\times$;

$$K_1=\boldsymbol{H}^\times. \tag{5.2}$$

We may naturally identify the dual $V_1^*$ of $V_1$ with $H^0(\boldsymbol{P}^1, H)$ as an Sp(1)-module. For any $\lambda \in C=\boldsymbol{P}^1$ we get an exact sequence of complex vector spaces

$$0 \longrightarrow K_\lambda \longrightarrow H^0(\boldsymbol{P}^1, H) \xrightarrow{r_\lambda} H_\lambda \longrightarrow 0$$

where $r_\lambda$ is the restriction map to the fiber $H_\lambda$ over $\lambda$ and $K_\lambda$ is the kernel of $r_\lambda$. Since $\lambda \in \mathrm{Sp}(1)$ fixes $\lambda \in C$, $\lambda$ acts on this sequence.

**Lemma 5.2.** *$K_\lambda$ is just the eigenspace of $\lambda$ with eigenvalue $-\sqrt{-1}$ in $H^0(\boldsymbol{P}^1, H)$.*

*Proof.* The tautological line $L_\lambda \subseteq E$ is just the eigenspace of $\lambda$ with eigenvalue $\sqrt{-1}$ as we have seen in the proof of Lemma 1.3. On the other hand, $K_\lambda$ is just the annihilator of $L_\lambda$ with respect to the above identification $V_1^*=H^0(\boldsymbol{P}^1, H)$. From this the lemma follows immediately.

*Proof of Theorem* 5.1. First we recall how to construct $Z$ from a given quaternionic structure on $M$ [22]. We first define an almost complex structure $J$ on $M \times C$ as follows. Let $(m, \lambda)$ be an arbitrary point of $M \times C$. Then $J_{(m,\lambda)}$ is the direct sum $J_{(m,\lambda)}=J_{\lambda,m} \oplus \hat{J}_\lambda$, where $J_\lambda$ is the almost complex structure of $M$ defined by $\lambda \in C$ and $\hat{J}$ is that induced on $C$ by the complex structure of $\boldsymbol{P}^1$. Then by Salamon [22; Th. 4.1] $J$ is integrable; further, if we define $f: Z \to C$ and $\pi: Z \to M$ by the natural projections and set $\tau=\mathrm{id}_M \times \tau_0$, then we see that $(f, \pi, \tau)$ is a triple with the desired property (cf. [22; 4.2]) for (5.1)).

Suppose conversely that we are given a triple $\mathscr{T}=(f: Z \to \boldsymbol{P}^1, \pi, \tau)$ as above. We shall construct a quaternionic structure on $M$ naturally associated to $\mathscr{T}$.

*a*) Let $D_Z$ be the Douady space of $Z$ which parametrizes the universal family of compact complex subspaces of $Z$. Since $M$ is naturally a parameter space of the $C^\infty$ family $\{Z_m\}_{m\in M}$ of compact submanifolds of $Z$, $M$ is naturally considered as a subset of $D_Z$. By (5.1) for any $m$ we have

$$H^1(Z_m, N_m)=0 \quad \text{and} \quad \dim H^0(Z_m, N_m)=4n. \tag{5.3}$$

Hence $D_Z$ is smooth of pure dimension $4n$ along $M$ by Kodaira [16]. Let $U$ be a sufficiently small smooth neighborhood of $M$ in $D_Z$. Let $(\rho\colon B\to U,\ B\subseteqq U\times Z)$ be the universal family restricted to $U$:

$$\begin{tikzcd} B\subseteqq U\times Z \arrow[d, "\rho"'] \arrow[d, "\rho_1", shift left=4] \\ U \end{tikzcd} \qquad \rho_1\text{: first projection.} \tag{5.4}$$

We set $Z_u=\rho^{-1}(u)$ for $u\in U$ consistent with the previous notation $Z_m$.

*b*) Let $N_U$ be the normal bundle of $B$ in $U\times Z$ and $\mathcal{N}_U$ the $O_B$-module of holomorphic sections of $N_U$. Then by (5.3) $\rho_*\mathcal{N}_U$ is locally free and if $E$ is the corresponding holomorphic vector bundle, then for any $u$ the fiber $E_u$ is naturally identified with $H^0(Z_u, N_u)$, where $N_u$ is the normal bundle of $Z_u$ in $Z$. Further, there exists a natural isomorphism $\varphi\colon \Theta_U\to\rho_*\mathcal{N}_U$ of $O_U$-modules, where $\Theta_U$ is the sheaf of holomorphic vector fields on $U$ (cf. [17; (12.2)]). We denote by $\phi\colon T_U\to E$ the corresponding isomorphism of vector bundles. Here $T_U$ denotes the holomorphic tangent bundle of $U$.

*c*) Since $\tau$ is antiholomorphic, it induces a natural antiholomorphic involution of the universal family (5.4), and hence in particular, of $B$ and $D_Z$, denoted respectively by $\tau_B$ and $\tau_D$. Then $\tau_D$ and $\tau_B$ lift to antiholomorphic involutions $\tau_D$ and $\tau_N$ on the bundles $T_U\to U$ and $N_U\to B$, respectively, which are antilinear on the fibers $(\tau_T(v)=\overline{\tau_{D*}(v)})$, $v\in T_U$ etc.). On the other hand, since $\tau$ preserves each $Z_m$, $\tau_B$ fixes any point of $M\subseteqq D_Z$, and in fact one sees easily that $M$ is precisely the fixed point submanifold of $\tau_D$ on $U$ (by restricting $U$ if necessary). Let $T_M$ be the (real) tangent bundle of $M$. Then $T_M$ is identified with the fixed point bundle of $\tau_T$ on $T_U|_M$, namely, $T_M$ is the real part of the real structure on $T_U|_M$ induced by $\tau_T$.

*d*) Identify $\rho^{-1}(M)\subseteqq B$ with $Z$ by the second projection in (5.4). Let $N$ be the restriction of $N_U$ to $Z$. Then $\tau_B$ preserves $Z$ and $\tau_N$ acts as an antilinear involution on $N\to Z$. Let $E_M$ be the restriction of $E$ to $M$. $E_M$ is a $C^\infty$ complex vector bundle on $M$ such that for any open subset $U$ of $M$ a $C^\infty$ section of $E$ on $U$ is naturally identified with a $C^\infty$ section of the bundle $N$ on $\rho^{-1}(U)$ which is holomorphic when restricted to each fiber $Z_m$. It follows then that $\tau_N$ induces naturally an antilinear involution $\tau_E$ of the bundle $E$; further, from the naturality of the isomorphism $\phi$ we infer readily that $\phi$ is $(\tau_T, \tau_E)$-equivariant, i.e., preserves the real structures. Thus we have obtained the natural identification of $T_M$ and the real part $E'_M$ with respect to $\tau_E$.

*e*) For each $u \in U$, let $G_u$ be the group of bundle automorphisms of the vector bundle $N_u \to B_u$. Then $G_u$ can be put together to form a complex Lie group over $U$, $G \to U$. By restricting $U$ if necessary, we may assume by (5.1) that $(B_u, N_u) \cong (\boldsymbol{P}^1, H^{\oplus 2n})$ for any $u$ so that in particular there exists a natural smooth morphism $\alpha_u$: $\boldsymbol{P}(N_u) \to \boldsymbol{P}^{2n-1}$ defined up to automorphisms of $\boldsymbol{P}^{2n-1}$. Therefore we get naturally a projective bundle $\boldsymbol{P}_0 \to U$ with fiber $\boldsymbol{P}^{2n-1}$ and a smooth morphism $\alpha$: $\boldsymbol{P}(N_U) \to \boldsymbol{P}_0$ over $U$ inducing $\alpha_u$ on $\boldsymbol{P}(N_u) = \boldsymbol{P}(N_U)_u$. Let $\psi_u$: $N_u - \{0\} \to \boldsymbol{P}(N_u) \to \boldsymbol{P}^{2n-1}$ be the composite map, where $\{0\}$ denotes the zero section. Let $K_u$ be the subgroup of elements of $G_u$ which induce the identity on $\boldsymbol{P}^{2n-1}$ via $\psi_u$. The existence of $\alpha$ shows that $K_u$ can be put together to form a complex Lie subgroup $K \to U$ of $G \to U$.

*f*) The involution $\tau_N$ on $N_U$ induces naturally an antiholomorphic involution on $G \to U$ (cf. (5.5) below). Further, if we let $\tau$ act on $\boldsymbol{P}_0$ trivially, $\alpha$ is $\tau$-equivariant; it follows that $\tau$ preserves $K$ under this action. Let $K_0$ be the fixed point set of $\tau$ on $K$. Then $K_0$ is a real Lie subgroup over $M$ of the restriction of $K \to U$ to $M$. By our construction $G$ operates on $E \to U$ naturally and the induced action of $K_0$ on $E_M \to M$ preserves the real part $E'_M$ by the definition of $K_0$. Thus via the identification $T_M = E'_M$ in *d*), $K_0 \to M$ becomes naturally a bundle of automorphisms of the tangent bundle $T_M$.

*g*) By restricting what we have obtained over each fiber $Z_m$ we see that $K_0 \to M$ is a $C^\infty$ fiber bundle with typical fiber $\boldsymbol{H}^\times$ and with structure group $\boldsymbol{H}^\times$ acting by adjoint action on itself. Indeed, we have by (5.2)

$$G_u \cong \mathrm{GL}_2(\boldsymbol{C}) \times_{\boldsymbol{C}^*} \mathrm{GL}_{2n}(\boldsymbol{C}), \qquad K_u \cong \mathrm{GL}_2(\boldsymbol{C}),$$
$$K_{0,u} = \boldsymbol{H}^\times = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \in \mathrm{GL}_2(\boldsymbol{C});\ x, y \in \boldsymbol{C} \right\}. \tag{5.5}$$

Since the structure group of $K_0 \to M$ factors through $\mathrm{PSp}(1) \cong \boldsymbol{H}^\times / \boldsymbol{R}^\times$ and the corresponding principal PSp(1)-bundle is associated to the $C^\infty$ $\boldsymbol{P}^1$-bundle $\pi$: $Z \to M$ which is trivial by our assumption, $K_0 \to M$ is actually trivial. Thus we get a global action of $\boldsymbol{H}^\times$ on $T_M$.

*h*) For any $m \in M$, as a complex $\boldsymbol{H}^\times$-module we have

$$T_{U,m} \cong E_m \cong H^0(Z_m, N_m) \cong H^0(\boldsymbol{P}^1, H)^{\oplus 2n} \cong V_{1,1}^{\oplus 2n}$$

and hence as a real $\boldsymbol{H}^\times$-module $T_{M,m} \cong (V_{1,1}^{\boldsymbol{R}})^{\oplus 2n}$. Therefore the action actually comes from an $\boldsymbol{H}$-module structure of $T_M$, i.e., $M$ has a natural almost quaternionic structure. We denote by $J_\lambda$ the associated almost complex structures on $T_M$ for any $\lambda \in C \subseteqq \boldsymbol{H}$.

*i*) Let $\mathscr{J}$ be the almost complex structure on $M \times \boldsymbol{P}^1$ induced from $Z$ by $\mu$. For each $\lambda \in C = \boldsymbol{P}^1$, $\mathscr{J}$ induces an almost complex structure $\mathscr{J}_\lambda$ on $M = M \times \lambda$. We shall show that under a natural identification,

$$J_\lambda = \mathscr{J}_\lambda. \tag{5.6}$$

Let $m$ be any point of $M$. Let $T_m$ be the tangent space of $M$ at $m$, and $N_m$ the holomorphic normal bundle of $Z_m$ in $Z$. Note that $N_m = N_{U|Z_m}$. Now $\mu$ induces the real linear isomorphism $N_{m,\lambda} \rightrightarrows T_m$ and the resulting complex structure on $T_m$ is just the one given by $\mathscr{J}_\lambda$, where $N_{m,\lambda}$ is the fiber of $N_m$ over $(m, \lambda)$. On the other hand, the complex structure on $T_m$ induced by $J_\lambda$ is induced via the isomorphisms

$$T_m \cong E'_m \cong H^0(Z_m, N_m)_{\boldsymbol{R}} \quad (:= H^0(Z_m, N_m)^{\tau_N})$$

and the natural action of $\lambda \in \boldsymbol{H}^\times$ on $H^0(Z_m, N_m)$ with $\lambda^2 = -1$.

*j*) By the definition of $E$ we have a natural homomorphism $\pi^*E \to N$ of $C^\infty$ complex vector bundles on $Z$. On the fiber over $(m, \lambda)$ this reduces to the $\lambda$-linear surjection

$$\nu \colon H^0(Z_m, N_m) \to N_{m,\lambda}.$$

(Since $\lambda \in C$ is fixed by $\lambda \in \boldsymbol{H}^\times$, $\lambda$ acts on $N_{m,\lambda}$.) In view of the description of $J_\lambda$ and $\mathscr{J}_\lambda$ in *i*), it suffices for (5.6) to show that $\nu$ induces by restriction an isomorphism

$$(H^0(Z_m, N_m)_{\boldsymbol{R}}, \lambda) \rightrightarrows N_{m,\lambda} \tag{5.7}$$

of complex vector spaces. Let $V_\lambda^\pm$ be the eigenspace with eigenvalue $\pm\sqrt{-1}$ for the action of $\lambda$ on $H^0(Z_m, N_m)$. Then we claim that the kernel of $\nu$ coincides with $V_\lambda^-$. In fact, taking a suitable isomorphism in (5.1) we see that $\nu$ is $\lambda$-isomorphic to the direct sum of $2n$-copies of the surjection $r_\lambda \colon H^0(\boldsymbol{P}^1, H) \to H_\lambda$; and then our claim follows from Lemma 5.2. Then $\nu$ induces a $\boldsymbol{C}$-linear isomorphism $V_\lambda^+ \rightrightarrows N_{m,\lambda}$ and (5.7) then follows from the natural isomorphism $(H^0(Z_m, N_m)_{\boldsymbol{R}}, \lambda) \rightrightarrows V_\lambda^+$.

*k*) (5.6) in particular shows that each $J_\lambda$ is integrable. Then it follows from Proposition 5.3 just below that our almost quaternionic structure is actually a quaternionic structure. Thus we have associated to any triple $(f, \pi, \tau)$ naturally a quaternionic structure on $M$. Finally in view of (5.6) and the construction at the beginning of the proof we see that the above correspondences "quaternionic structures" $\leftrightarrow$ "triples" are inverses to each other, thus completing the proof of the theorem.

**Proposition 5.3.** *Let $M$ be an oriented $C^\infty$ manifold with an almost quaternionic structure. Let $\{J_\lambda\}$ be the associated family of almost complex structures on $M$. Then the almost quaternionic structure admits a torsion-free connection if and only if $J_\lambda$ are all integrable.*

For the proof we need some preliminaries, which will be given in a more general form than is needed for the proof of the proposition. Let $A_q^k(M)$, $0 \leqq q \leqq [k/2]$, be the space of $C^\infty$ sections of the subbundle $W^{k;k-2q} \otimes_{\boldsymbol{R}} \boldsymbol{C} \subseteqq \wedge^k T^*$ (cf. (3.10)). Then we have a direct sum decomposition $A^k(M) = \oplus_q A_q^k(M)$, where $A^k(M)$ is the space of $C^\infty$ $k$-forms on $M$. Define a decreasing filtration $\{E^\cdot\}$ of $A^k(M)$ by the complex vector subspaces

$$E^q(A^k(M)) = A_q^k(M) \oplus A_{q+1}^k(M) \oplus \cdots \oplus A_{[k/2]}^k(M), \quad 0 \leqq q \leqq [k/2].$$

We set $E^q(A^k(M)) = 0$ for $q > k/2$. Then clearly we have

$$\mathrm{Gr}_E^q(A^k(M)) := E^q(A^k(M))/E^{q+1}(A^k(M)) \cong A_q^k(M). \tag{5.8}$$

Then $\{E^\cdot\}$ is clearly defined over $\boldsymbol{R}$, since each $A_q^k(M)$ is.

**Lemma 5.4.** *A $k$-form $\alpha \in A^k(M)$ is in $E^q(A^k(M))$ if and only if for any $\lambda \in C$ its Hodge $(k-t, t)$ component $\alpha^{k-t,t}(\lambda)$ with respect to $\lambda$ vanishes whenever $t < q$.*

*Proof.* The necessity is obvious from the definition of $E^q$. Restricting ourselves to fibers over each point of $M$, the sufficiency is easily seen to reduce to proving the following assertion: If $x$ is an element of $V_{k,r}$ for any $(k, r)$ with respect to $r \in \Delta(k)$ (cf. (0, 1)) then its Hodge $(s, t)$-component $x^{s,t}(\lambda) \neq 0$ for some $\lambda$ if $(k-r)/2 \leqq s$, $t \leqq (k+r)/2$. Indeed, the subspace $V(s, t) := \{x \in V_{k,\boldsymbol{R}};\ x^{s,t}(\lambda) = 0,\ \lambda \in C\}$ is an $\boldsymbol{H}^\times$-submodule of the irreducible $V_{k,r}$ so that $V(s, t) = 0$.

From the above lemma we deduce the following:

**Lemma 5.5.** *Suppose that $J_\lambda$ are integrable. For any integer $q \geqq 0$, $E^q(A^k(M))$ is a subcomplex of the de Rham complex $A^\cdot(M)$.*

*Proof.* Let $\alpha$ be an arbitrary element of $E^q(A^k(M))$. Then for any $t < q$ and any $\lambda \in C$ we have

$$(d\alpha)^{k+1-t,t}(\lambda) = \bar{\partial}_\lambda(\alpha^{k+1-t,t-1}(\lambda)) + \partial_\lambda(\alpha^{k-t,t}(\lambda)) = 0$$

by the above lemma and the integrability of $J_\lambda$, where $\partial_\lambda$ and $\bar{\partial}_\lambda$ are the (1, 0)- and (0, 1)-components of $d$ for $\lambda$, respectively; hence $d\alpha \in E^q A^{k+1}(M)$ again by the above lemma.

Therefore $(A^{\bullet}(M), E^{\bullet})$ becomes a filtered complex and the associated complex induced on each graded module $\mathrm{Gr}_E^q(A^{\bullet}(M))$ is isomorphic to

$$(5.9) \qquad (A_q^{\bullet}(M), d_q),$$

where $d_q$ is the restriction of $d$ to $A_q^{\bullet}(M)$ followed by the projection $E^q(A^{\bullet}(M)) \to A_q^{\bullet}(M)$.

*Proof of Proposition* 5.3. By Salamon [23; Th. 2.2] the almost quaternionic structure in question admits a torsion-free connection if and only if $(A_k^{\bullet}(M), d_k)$ is a complex. By the above argument the condition is certainly satisfied if $J_\lambda$ are all integrable. See e.g. [15; IX, § 3] for the converse.

Note in passing that the filtered complex $(A^{\bullet}(M), E^{\bullet})$ gives rise to a spectral sequence

$$(5.10) \qquad E_1^{p,q}(M) := H^q(A_p^{\bullet}(M)) \Rightarrow H^k(A^{\bullet}(M)), \quad k = p+q$$

with

$$E_\infty^{p,q}(M) \cong \mathrm{Gr}_E^p(H^k(A^{\bullet}(M))),$$

where we denote by the same letter $E$ the induced filtration on $H^{\bullet}(M, \boldsymbol{C}) \cong H^{\bullet}(A^{\bullet}(M))$. Note also that the above objects are all defined over $\boldsymbol{R}$ with respect to the standard complex conjugation, since each $A_q^k(M)$ is.

Now let $(M, g)$ be a compact hyperkähler manifold of dimension $4n$. Let $(f\colon Z \to C, \pi, \tau)$ be the triple associated to $(M, g)$ via Theorem 5.1. Theorems 3.1 and 5.1 suggest that the decomposition

$$(3.11) \qquad H^k(M, \boldsymbol{R}) = \bigoplus_{r \in \Delta(k)} H^{k;r}(M, \boldsymbol{R})$$

of $H^k(M, \boldsymbol{R})$ should be described in terms of the associated triple $(f, \pi, \tau)$. Let $\Omega_{Z/C}^{\bullet}$ be the relative Poincaré complex for the smooth morphism $f\colon Z \to C$. We have the associated spectral sequence of hypercohomology

$$(5.11) \qquad E_1^{p,q} := H^q(Z, \Omega_{Z/C}^p) \Rightarrow \boldsymbol{H}^{p+q}(Z, \Omega_{Z/C}^{\bullet}).$$

Let $F^{\bullet}$ be the associated decreasing filtration on the abutment $\boldsymbol{H}^{\bullet}(Z, \Omega_{Z/C}^{\bullet})$. Then concerning the above question the following holds.

**Proposition 5.6.** 1) *There exists a natural* $\boldsymbol{C}$*-linear isomorphism* $u\colon H^k(M, \boldsymbol{C}) \to \boldsymbol{H}^k(Z, \Omega_{Z/C}^{\bullet})$ *for any* $k \geqq 0$. 2) *For any* $r \in \Delta(\hat{k})$ *we set* $q = (k-r)/2$. *Then we have an inclusion* $u(H^{k;r}(M, \boldsymbol{C})) \subseteq F^q(H^k(M, \boldsymbol{C}))$ *which induces an isomorphism* $u^{k;r}\colon H^{k;r}(M, \boldsymbol{C}) \to \mathrm{Gr}_F^q(\boldsymbol{H}^k(Z, \Omega_{Z/C}^{\bullet}))$ *by* *passing to the associated graded module.* 3) *There exists a natural real structure on* $\boldsymbol{H}^k(Z, \Omega^{\cdot}_{Z/C})$ *such that the filtration* $F^{\cdot}$, *and* $u$ *and* $u^{k;r}$ *are all defined over* $\boldsymbol{R}$.

*Proof of* 1). Let $f^{\cdot}O_C$ be the sheaf-theoretic inverse image of the structure sheaf $O_C$ of $C$. Then $\Omega^{\cdot}_{Z/C}$ is a resolution of $f^{\cdot}O_C$. Hence we have a natural isomorphism

$$(5.12) \qquad \boldsymbol{H}^k(Z, \Omega^{\cdot}_{Z/C}) \cong H^k(Z, f^{\cdot}O_C).$$

On the other hand, since $R^q f_*(f^{\cdot}O_C) \cong R^q f_* \boldsymbol{C} \otimes_C O_C$ for any $q$ and since $R^q f_* \boldsymbol{C}$ is naturally isomorphic to the constant sheaf associated to the trivial local system $H^q(M, \boldsymbol{C}) \times C \to C$, we see that the Leray spectral sequence $E_2^{p,q} := H^p(C, R^q f_*(f^{\cdot}O_C)) \Rightarrow H^{p+q}(Z, f^{\cdot}O_C)$ degenerates and gives natural isomorphisms

$$H^k(Z, f^{\cdot}O_C) \cong H^0(C, R^k f_* \boldsymbol{C} \otimes_C O_C) \cong H^0(S, O_C) \otimes_C H^k(M, \boldsymbol{C}) \cong H^k(M, \boldsymbol{C}).$$

Then 1) follows from this and (5.12).

Before proving 2) and 3) we make some preliminary considerations. We first recall how the spectral sequence (5.8) is realized. Let $\mathscr{A}^q$ (resp. $\bar{\mathscr{A}}^q$) be the sheaf of germs of $C^\infty$ (0, $q$)- (resp. ($q$, 0)-) forms on $Z$. We consider the double complex $\mathscr{K}^{\cdot\cdot} := \Omega^{\cdot}_{Z/C} \otimes_{O_Z} \mathscr{A}^{\cdot}$ which gives the Dolbeault resolution of the relative Poincaré complex $\Omega^{\cdot}_{Z/C}$. Let $K^{\cdot\cdot} = \Gamma(Z, \mathscr{K}^{\cdot\cdot})$ and let $K^{\cdot}$ be the associated simple complex. Define as usual the Hodge filtration $\{F^p\}$ of $K^{\cdot}$ by $F^p(K^{\cdot}) = \bigoplus_{p' \geq p, q' \geq 0} K^{p',q'}$. Then (5.11) is nothing but the spectral sequence associated to the filtered complex $(K^{\cdot}, F^{\cdot})$.

Let $\bar{\Omega}^{\cdot}_{Z/C}$ be the antiholomorphic relative Poincaré complex. Then using the antiholomorphic involution $\tau$ of $Z$ we define an antilinear involution $\sigma_{p,q}$ of $K^{p,q}$ by

$$\sigma_{p,q}(\alpha) = \overline{\tau^*(\alpha)}, \quad \alpha \in K^{p,q},$$

where $\tau^*(\alpha)$ is the pull-back of $\alpha$ by $\tau$ considered as an element of $\bar{K}^{p,q} := \Gamma(Z, \bar{\Omega}^p_{Z/C} \otimes_{O_Z} \bar{\mathscr{A}}^q)$ and then $\overline{\tau^*(\alpha)}$ is its image under the complex conjugation considered as an antilinear isomorphism $\bar{K}^{p,q} \to K^{p,q}$. Then $\sigma_{p,q}$ altogether induce a natural antilinear involution $\sigma$ of the filtered complex $(K^{\cdot}, F^{\cdot})$, which then defines the natural real structures on various spaces naturally involved in the spectral sequence (5.11) such as $H^q(Z, \Omega^p_{Z/C})$, $\boldsymbol{H}^{\cdot}(Z, \Omega^{\cdot}_{Z/C})$, $\mathrm{Gr}^p \boldsymbol{H}^{\cdot}(Z, \Omega^{\cdot}_{Z/C})$.

Now the pull-back by $\pi$ induces a homomorphism of the filtered complexes

$$\pi^*: (A^{\cdot}(M), E^{\cdot}) \to (K^{\cdot}, F^{\cdot}).$$

In fact for any element $\alpha$ of $E^q(A^\cdot(M))$, $\pi^*\alpha$, restricted to the fiber $Z_\lambda$, has the vanishing Hodge $(p', q')$-component for any $q'<q$ by Lemma 5.4. Further, since the elements of $\pi^*A(M)$ are $\tau$-invariant and $\pi^*A(M)$ is preserved by the complex conjugation, $\pi^*$ is compatible with the natural real structures. It follows that $\pi^*$ induces a homomorphism of the spectral sequences compatible with the real structures

$$\begin{array}{ccc} E_1^{q,k-q}(M) & & E_1^{q,k-q} \\ \| & & \| \\ H^k(A_q^\cdot(M)) & \xrightarrow{\pi^*} & H^{k-q}(Z, \Omega^q_{Z/C}) \\ \Downarrow & & \Downarrow \\ H^k(M, C) & \xrightarrow{\pi^*} & H^k(Z, \Omega^\cdot_{Z/C}). \end{array} \tag{5.13}$$

The map $\pi^*$ on the bottom line is factored as

$$H^k(M, C) \to H^k(Z, C) \to H^k(Z, f^\cdot O_C) \overset{\sim}{\to} H^k(Z, \Omega^\cdot_{Z/C})$$

and from this one sees easily that this isomorphism $\pi^*$ coincides with the one obtained in the proof of 1). Now we use the hyperkähler structure.

**Lemma 5.7.** *For any $(k, r)$ with $r \in \Delta(\hat{k})$ set $q=(k-r)/2$. Then there exists a natural inclusion $j_M$: $H^{k;r}(M) \to E_\infty^{q,k-q}(M)$, where $H^{k;r}(M)$ is the space of harmonic $W^{k;r}$-forms. Moreover, if we set $j_Z=\pi_\infty^* j_M$, where $\pi_\infty^*$: $E_\infty^{q,k-q}(M) \to E_\infty^{q,k-q}(Z)$, then $j_Z$ also is injective.*

*Proof.* Recall that

$$E_\infty^{q,k-q}=\{\alpha \in E^q(A^k(M));\ d\alpha=0\}/\{\beta+d\gamma;\ \beta \in E^{q+1}(A^k(M)) \text{ with } d\beta=0,\ \gamma \in A^{k-1}(M) \text{ with } d\gamma \in E^q\}. \tag{5.14}$$

It is then clear that we have a natural map $j_M$: $H^{k;r}(M) \to E_\infty^{q,k-q}(M)$. It now suffices to show that $j_Z$ defined as above is injective. Let $\alpha$ be a nonzero element of $H^{k;r}(M)$ such that $j_Z(\alpha)=0$. By the formula for $E_\infty^{p,q}(Z)$ similar to (5.14) we see that

$$\alpha=\beta+d\gamma \tag{5.15}$$

for some $\beta \in F^{q+1}(K^k(Z))$ and $\gamma \in K^{k-1}(Z)$ such that $d\gamma \in F^q$. Then by Lemma 5.4 there exists a $\lambda \in C$ such that the Hodge $(q, k-q)$-component $\alpha^{q,k-q}(\lambda)$ of $\alpha$ with respect to $\lambda$ is nonvanishing;

$$\alpha^{q,k-q}(\lambda) \neq 0. \tag{5.16}$$

Let $\alpha_\lambda$ and $\beta_\lambda$ be the restrictions of $\pi^*\alpha$ and $\pi^*\beta$ to $Z_\lambda$, respectively. Then from (5.15) $\alpha_\lambda$ and $\beta_\lambda$ are cohomologous. Then since $\alpha_\lambda$ is harmonic, $H_\lambda\beta_\lambda=\alpha_\lambda$ and hence

$$(H_\lambda\beta_\lambda)^{q,k-q}=\alpha_\lambda^{q,k-q}=\alpha^{q,k-q}(\lambda),$$

where $H_\lambda\beta_\lambda$ is the harmonic part of $\beta_\lambda$. Since the harmonic projection $H_\lambda$ preserves the Hodge components and $\beta_\lambda^{q,k-q}=0$ as $\beta\in F^{q+1}(K^k(Z))$, we have $(H_\lambda\beta_\lambda)^{q,k-q}=0$. This contradicts (5.16).

Now we can complete the proof of Proposition 5.6.

*Proof of* 2), 3). By Lemma 5.7 above we have an inequality

$$\dim H^{k;r}(M)\leq\dim E_\infty^{q,k-q}(X) \quad \text{for } X=M,\ Z. \tag{5.17}$$

On the other hand, from (3.11) we have

$$b_k(M)=\sum_r \dim H^{k;r}(M), \tag{5.18}$$

while from 1) we have

$$b_k(M)=\sum_q \dim E_\infty^{q,k-q}(X) \quad \text{for } X=M,\ Z. \tag{5.19}$$

From (5.17), (5.18) and (5.19) we get that the inclusion $j_M, j_Z$ of Lemma 5.7 are actually isomorphic. Since $j_M$ factors through $E^q(A^\cdot(M))$, and hence $j_Z$ through $F^q(K^\cdot(Z))$, by identifying $H^{k;r}(M)$ with $H^{k;r}(M, \boldsymbol{C})$ we obtain 2). 3) is then clear since (5.13) and $j_M$ are defined over $\boldsymbol{R}$.

**Remark 5.8.** The above proof shows that

$$E_\infty^{q,k-q}(X)=0 \quad \text{for any } q>[\hat{k}/2] \text{ for } X=M,\ Z.$$

It is not clear whether the map $\pi^*$ for the $E_1^{p,q}$-terms $\pi^*\colon H^k(A_q^\cdot(M))\to H^q(Z, \Omega_{Z/C}^{k-q})$ are isomorphic or not. We only note that we have again inclusions $j\colon H^{k;r}(M)\to H^k(A_q^\cdot(M))$ and $j'\colon H^{k;r}(M)\to H^q(Z, \Omega_{Z/C}^{k-q})$ with $j'=\pi^*j$. When $q=k$, above is isomorphic by Salamon [23] and by using this and the result of the next proposition we can conclude that both $j$ and $j'$ are isomorphic in this case. In particular, the elements of $H^k(A_k^\cdot(M))$ are represented by harmonic forms; this is reminiscent of the Kähler case, where the Dolbeault cohomology classes are representable by harmonic forms.

Next we shall calculate the $E_1^{p,q}$-term of the spectral sequence (5.11).

**Proposition 5.9.** *Let $f\colon Z \to C$ be the Calabi family associated to a hyperkähler manifold $(M, g)$. Let $h^{p,q} := h^{p,q}(M_\lambda)$ for any $\lambda \in C$ (which is independent of $\lambda$). Then $R^q f_* \Omega^p_{Z/C}$ is isomorphic to the direct sum of $h^{p,q}$ copies of $O_C(H^{q-p})$; $R^q f_* \Omega^p_{Z/C} \cong O_C(H^{q-p})^{\oplus h^{p,q}}$. Further, $h^{p,q}(Z/C) := \dim H^q(Z, \Omega^p_{Z/C})$ is given by $h^{p,q}(Z/C) = (q-p+1)h^{p,q}$ if $q \geqq p$ and $= (p-q)h^{p,q}$ if $q < p$.*

The fact $H^0(C, f_* \Omega^p_{Z/C}) = 0$ has been shown in [25] and [13]. The above result is a generalization of this. However, the Hodge numbers $h^{p,q}(Z)$ of $Z$ are yet to be determined. The proof of the proposition will be given after a lemma.

For any $\lambda \in C$ consider $V_k$ as a complex $S^1$-module via the embedding $\lambda_*\colon S^1 \to \mathrm{Sp}(1)$ of (1.9). Let $V_k = \bigoplus_{r \equiv k(2), |r| \leqq k} V_{k;r}(\lambda)$ with $\dim V_{k;r}(\lambda) = 1$ be the canonical decomposition as an $S^1$-module, where $V_{k;r}(\lambda)$ is the eigenspace with eigenvalue $\varepsilon^r$, $\varepsilon \in S^1$. Define a decreasing filtration $\{F^\cdot_\lambda\}$ on $V_k$ by

$$F^p_\lambda(V_k) = \bigoplus_{r \geqq 2p-k} V_{k;r}(\lambda).$$

Denote by $\mathrm{Gr}^p_\lambda(V_k)$ the associated graded modules. Now we fix an $\mathrm{Sp}(1)$-invariant nondegenerate symmetric bilinear form $f$ on $V_k$. Then for any $\lambda \in C$ and nonzero $u \in V_{k;r}(\lambda)$ and $v \in V_{k;s}(\lambda)$ we have

(5.20) $\qquad f(u, v) = 0$ if $r+s \neq 0$, and $\neq 0$ if $r+s = 0$.

**Lemma 5.10.** *$\mathrm{Gr}^p_\lambda(V_k)$ depends holomorphically on $\lambda$ so that it defines a holomorphic line bundle on $C$, denoted also by the same letter $\mathrm{Gr}^p_\lambda(V_k)$. As a line bundle $\mathrm{Gr}^p_\lambda(V_k)$ is isomorphic to $H^{k-2p}$.*

*Proof.* Under the identification $C = \boldsymbol{P}^1 = \boldsymbol{H}^\times/\boldsymbol{C}^*$ of (1.3) if $\pi(h) = \lambda$, then $V_{1;1}(\lambda) = h\boldsymbol{C}$ by the proof of Lemma 1.3. This implies that $\{V_{1;1}(\lambda)\}_{\lambda \in C}$ defines a holomorphic line bundle isomorphic to $H^{-1}$. Then for any $p > 0$, $\mathrm{S}^p(V_{1;1}(\lambda))$, and hence $\mathrm{S}^p(V_{1;1}(\lambda)) \otimes_{\boldsymbol{C}} V_{k-p}$ also, depends holomorphically on $\lambda$ and $\mathrm{S}^p(V_{1;1}(\lambda)) \cong H^{-p}$ as a holomorphic line bundle, where $\mathrm{S}^p(\ )$ denotes the symmetric product. On the other hand, one sees readily that $F^p(V_k)$ coincides with the natural injective image of

$$\mathrm{S}^p(V_{1;1}(\lambda)) \otimes_{\boldsymbol{C}} V_{k-p} = \mathrm{S}^p(V_{1;1}(\lambda)) \otimes_{\boldsymbol{C}} \mathrm{S}^{k-p}(V_1) \quad \text{in } \mathrm{S}^k(V_1) = V_k$$

and therefore that

$$\mathrm{Gr}^p_\lambda(V_k) \cong \mathrm{S}^p(V_{1;1}(\lambda)) \otimes \mathrm{Gr}^0_\lambda(V_{k-p}) \cong H^{-p} \otimes \mathrm{Gr}^0_\lambda(V_{k-p}).$$

Finally by (5.20), the quadratic form $f$ induces in general a duality between the line bundles $\mathrm{Gr}^p_\lambda(V_k)$ and $\mathrm{Gr}^{k-p}_\lambda(V_k)$. In particular, $\mathrm{Gr}^0_\lambda(V_{k-p}) \cong \mathrm{Gr}^{k-p}_\lambda(V_{k-p})^{-1} \cong H^{k-p}$. The lemma follows.

*Proof of Proposition* 5.9. Let $E^{p,q}$ be the vector bundle on $C$ corresponding to the locally free sheaf $R^q f_* \Omega^p_{Z/C}$. Set $W_k = H^k(M, \boldsymbol{C})$, considered as a regular $\boldsymbol{H}^\times$-module of weight $k$. Then $\{F^{\cdot}_\lambda\}$ with $F^p_\lambda(W) = \bigoplus_{p \leq p'} H^{p',q'}(M_\lambda)$ defines a decreasing filtration on $W_k$ and in view of Lemma 5.10 the associated graded module $\mathrm{Gr}^p_\lambda(W)$ vary holomorphically with $\lambda$ and isomorphic to $E^{p,q}$; indeed if $W_k = \bigoplus_{r \in \Delta(k)} W_{k;r}$ is the canonical $\boldsymbol{H}^\times$-decomposition of $W_k$, then $\mathrm{Gr}^p_\lambda(W) \cong \bigoplus_r \mathrm{Gr}^p_\lambda(W_{k;r})$, while, because of shifts in weights, $\mathrm{Gr}^p_\lambda(W_{k;r})$ is isomorphic to a direct sum of copies of $\mathrm{Gr}^{p-(k+r)/2}_\lambda(V_r)$ which is isomorphic to $H^{k-2p}$ by Lemma 5.10. The first part of the lemma thus follows. The second part then follows from the exact sequence

$$0 \to H^1(C, R^{q-1} f_* \Omega^p_{Z/C}) \to H^q(Z, \Omega^p_{Z/C}) \to \Gamma(C, R^q f_* \Omega^p_{Z/C}) \to 0$$

coming from the Leray spectral sequence for $f$.

Finally we shall study the structure of the general members of the Calabi family. In general for a complex manifold $X$ and for an analytic subset $A$ of pure codimension $k$ of $X$ we denote by $c(A)$ the associated cohomology class in $H^{2k}(X, \boldsymbol{Q})$. A class $\beta \in H^{2k}(X, \boldsymbol{Q})$ is said to be *analytic* if $\beta = c(A)$ for some $A$ as above. Our purpose is to show the following:

**Proposition 5.11.** *Let $(M, g)$ be a compact hyperkähler manifold with the associated Calabi family $\{M_\lambda\}_{\lambda \in C}$. Let $\beta$ be any element of $H^{2k}(M, \boldsymbol{Q})$. Let $C(\beta) = \{\lambda \in C;\ \beta$ is analytic on $M_\lambda\}$. Then* 1) *if $k$ is even, $C(\beta) = C$, $\{\pm\lambda\}$, or $= \emptyset$ and* 2) *if $k$ is odd, then $C(\beta) = \{\lambda\}$ or $= \emptyset$, where $\lambda = \lambda(\beta)$ is some point of $C$.*

**Corollary 5.12.** *Except for countable numbers of $\lambda$, $M_\lambda$ contains no analytic subvariety of odd dimension; in particular, it contains no curves and hypersurfaces, and hence the algebraic dimension $a(M_\lambda) = 0$ and $f: Z \to C$ is the algebraic reduction of $Z$. If, further, $h^{2,0}(M_\lambda) = 1$, then $M_\lambda$ is even simple (cf. below for the definition).*

**Remark 5.13.** 1) The proposition is inspired by Calabi [6; Th. 4.1], where the case of $K3$ surfaces is treated. 2) For an even $k$ the case $C(\beta) = C$ can actually occur even if $h^{2,0}(M_\lambda) = 1$. Roughly speaking any hyperkähler submanifold $(M', g')$ defines a family $\{M'_\lambda\}$ of complex submanifolds $M'_\lambda$ of $M_\lambda$. 3) By [10; § 4] there exists a countably many $\lambda$ such that $M_\lambda$ is projective. However, a simple example shows that for some intermediate value $0 < k < \dim_{\boldsymbol{C}} M_\lambda$ it can happen that $a(M_\lambda) \neq k$ for any $\lambda$.

First we prove an algebraic lemma. Let $x$ be any element of $V_{2s}^{\boldsymbol{R}}$. For any $\lambda \in C$ we set $l_\lambda(x)=\min\{d;\, x \in \bigoplus_{p,q\leq d} V^{p,q}(\lambda)\}$, where $V^{p,q}(\lambda)$ is the Hodge $(p, q)$-component with respect to $\lambda$ of $V_{2s}=(V_{2s}^{\boldsymbol{R}})_{\boldsymbol{C}}$. By definition, $s\leqq l_\lambda(x)\leqq 2s$. For any $d$ let $C_d(x)=\{\lambda \in C;\, l_\lambda(x)=d\}$, and $n_d(x)=\sharp C_d(x)$. Since $V^{p,q}(-\lambda)=\bar{V}^{p,q}(\lambda)$, "$\lambda \in C_d(x)$" implies "$-\lambda \in C_d(x)$". Hence $n_d(x)$ is always even.

**Lemma 5.14.** *With the above notation we have* $n_{2s-1}(x)+2n_{2s-2}(x)+\cdots+sn_s(x)=2s$. *In particular* $n_s(x)=0$ *or* 2.

*Proof.* We argue in the framework of Lemma 1.3 and its proof, so we freely use the notation there. Let $Q(x)=\bigcup_{s\leqq d\leqq 2s-1} C_d(x)$. Then by (5.20) we have $Q(x)=\{\lambda;\, f(x, \tilde{\lambda}_s)=0\}$, where $\lambda_s=p_s(\lambda) \in \boldsymbol{P}_s$ and $\tilde{\lambda}_s$ is any point of $\pi_s^{-1}(\lambda_s)$. Let $\boldsymbol{P}_x$ be the hyperplane of $\boldsymbol{P}_s$ defined by $f_x(\lambda):=f(x, \lambda)=0$. For any $\lambda \in C$ let $m_\lambda$ be the intersection multiplicity of $C_s$ and $\boldsymbol{P}_x$ at $\lambda_s$; $m_\lambda=(C_s\cdot\boldsymbol{P}_x)_{\lambda_s}$. We claim that $m_\lambda=2s-d$ whenever $\lambda \in C_d(x)$. Since $\deg C=2s$, this would prove the lemma.

Note first that $m_\lambda$ is just the order of zeroes of $f_x$ restricted to $C_s$. Since $Y_\lambda$ is tangent to $\tilde{C}_s$ at $\tilde{\lambda}_s$ and $d\pi_{s*}(Y_\lambda)\neq 0$ at $\lambda_s$, we have $m_\lambda=\min\{b;\, (Y_\lambda^b f_{x|\tilde{C}_s})(\tilde{\lambda}_s)=0\}$. Now by (1.11) $Y_\lambda^b f_x(\tilde{\lambda}_s)=f_x(Y_\lambda^b\tilde{\lambda}_s)=f(Y_\lambda^b\tilde{\lambda}_s, x)$, $\tilde{\lambda}_s \in H^{2s,0}(\lambda)$; further by Lemma 5.4, the $(2s-d, d)$-component (with respect to $\lambda$) of $x$ is nonzero because $x$ is real. Therefore by (5.20) we see that $Y^b f_x(\lambda)=0$ if $b<2s-d$ and $\neq 0$ if $b=2s-d$. Hence $m_\lambda=2s-d$ as desired.

**Lemma 5.15.** *Let $\beta$ be any element of $H^{2k}(M, \boldsymbol{R})$. Let $C_0(\beta)=\{\lambda \in C;\, \beta$ is of type $(k, k)$ on $M_\lambda\}$. Then one of the following three cases can occur: $C_0(\beta)=C$, $=\{\pm\lambda\}$ for some $\lambda \in C$, or $=\emptyset$.*

*Proof.* Let $\bigoplus_\gamma E_\gamma$ be a decomposition of $H^{2k}(M, \boldsymbol{C})$ into Sp(1)-irreducible components. In particular, $E_\gamma\cong V_{2l}$ as a complex Sp(1)-module for some $0\leqq l\leqq k$. Let $\beta=\sum_\gamma\beta_\gamma$ be the corresponding decomposition of $\beta$. Then we have $C_0(\beta)=\bigcap_\gamma C_0(\beta_\gamma)$; therefore it suffices to prove the lemma for each $\beta_\gamma$, while since $E_\gamma\cong V_{2l}$, the lemma for $C_0(\beta_\gamma)$ follows immediately from Lemma 5.14.

*Proof of Proposition* 5.11. By the above lemma we have only to show that if $k$ is even (resp. odd) and $\beta$ is analytic on $M_\lambda$, then it is also (resp. never) analytic on $M_{-\lambda}$. Let $A_\lambda$ be an analytic subset on $M_\lambda$ with $c(A_\lambda)=\beta$. Then $A_{-\lambda}:=\pi_{\lambda}^{-1}\pi(A_\lambda)$ is an analytic subset of $M_\lambda$, with original (resp. reversed) orientation. Hence $c(A_{-\lambda})=\beta$ (resp. $-\beta$). This already takes care of the even case. In the odd case, $\beta=-(-\beta)$ cannot be analytic on $M_{-\lambda}$, since $M_{-\lambda}$ is a Kähler manifold.

As for the corollary, only the last assertion needs a proof. First we recall the following definition (cf. [9]). A compact complex manifold $X$ is said to be *simple* if there is no analytic family $\{A_t\}_{t\in T}$ of compact complex subspaces of $X$ with $0<\dim A_t<\dim X$ which covers the whole $X$ in the sense that $X=\bigcup_{t\in T}A_t$. It clearly suffices to prove the following:

**Proposition 5.16.** *Let $X$ be a symplectic compact Kähler manifold with $h^{2,0}(X)=1$ and containing no divisor. Then $X$ is simple.*

*Proof.* Let $\dim X=2n$. If $n=1$, the lemma is well-known. So we assume that $n>1$. Then as follows easily from the Bogomolov splitting theorem (cf. § 4), $X$ has the finite fundamental group, i.e., in that theorem the torus factor does not appear in the decomposition of $\tilde{X}$. Therefore by replacing $X$ by its universal covering we may assume from the beginning that $X$ is simply connected. By our assumption, $a(X)=0$. Suppose that $X$ is not simple. Then by [9; Lemma 9] we can find surjective morphisms $\pi$: $Z\to X$ and $\rho$: $Z\to T$ of compact complex manifolds such that $\dim Z=\dim X$ and $\dim X>\dim T>0$. Then since $X$ is simply connected and contains no divisors, by the purity of branch loci $\pi^*$ must actually be bimeromorphic. Then, since $h^{2,0}(T)\leqq h^{2,0}(X)=1$ and $X$ is symplectic, $h^{2,0}(T)$ must vanish. Then by a theorem of Kodaira $T$ must be a Moishezon manifold since by [26] $T$ is bimeromorphic to a compact Kähler manifold. Then we have a contradiction: $0=a(X)\geqq a(T)=\dim T>0$. Hence $X$ is simple.

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