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Uniqueness of Einstein Kähler Metrics Modulo Connected Group Actions

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Dedicated to Professor Kunihiko Kodaira on his seventieth birthday

§0. Introduction

Throughout this paper, we fix an arbitrary n-dimensional compact complex manifold X *with positive first Chern class* $c_1(X)_{\boldsymbol{R}}>0$. We then put

$\mathscr{K}$: the set of all Kähler forms on X representing $2\pi c_1(X)_{\boldsymbol{R}}$,
$\mathscr{K}^+:=\{\omega \in \mathscr{K} \mid \omega$ has positive definite Ricci tensor$\}$,
$\mathscr{E}:=\{\omega \in \mathscr{K} \mid \omega$ is an Einstein form$\}$,
$C^\infty(X)_{\boldsymbol{R}}$: the space of real-valued C^∞-functions on X,
$\mathrm{Aut}(X)$: the group of holomorphic automorphisms of X,
$G:=\mathrm{Aut}^0(X)$: the identity component of $\mathrm{Aut}(X)$.

Furthermore, $\mathrm{Aut}(X)$ is always assumed to act from the *right* on $\mathscr{K}$ by $(\omega, g) \in \mathscr{K}\times\mathrm{Aut}(X)\mapsto g^*\omega \in \mathscr{K}$.

The main purpose of this paper is to prove the uniqueness of Einstein Kähler metrics, if any, on X up to G-action. Such uniqueness was known only for i) Kähler C-spaces (cf. Matsushima [12]) and ii) some non-homogeneous Einstein manifolds recently discovered by Sakane [13]. Now, the correct statement we obtain has the following stronger form as announced earlier in [9]:

Theorem A. *Fix an element ω_1 of $\mathscr{K}$. Let $\mu^+:\mathscr{K}^+\to\boldsymbol{R}$ be the restriction to $\mathscr{K}^+$ of the $\mathscr{K}$-energy map $\omega \in \mathscr{K}\mapsto M(\omega_1, \omega) \in \boldsymbol{R}$ of the Kähler manifold (X, ω_1) (see Section 1, also [9]). Assume that $\mathscr{E}\neq\phi$. Then*

(i) *μ^+ is bounded from below and takes its absolute minimum exactly on $\mathscr{E}$.*

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(ii) *$\mathscr{E}$ consists of a single G-orbit.*

We now suppose that $\mathscr{E} \neq \phi$, and let K be a maximal compact subgroup of G. By the well-known theorem of Matsushima [11], there exists an element θ of $\mathscr{E}$ such that the isotropy subgroup of G at θ coincides with K. Hence $\mathscr{E}$ is G-equivariantly diffeomorphic to G/K. Note that G/K has a structure of a Riemannian symmetric space, though the choice of its metric is not unique (even up to constant multiple) if the symmetric space G/K is reducible. We now endow $\mathscr{E}$ with the natural Riemannian metric defined in [10] (see also Section 9 of the present paper). Then Theorem A allows us to sharpen a result in [10] and one can determine the structure of $\mathscr{E}$ as follows:

Theorem B. *If $\mathscr{E} \neq \phi$, then $\mathscr{E}$ is G-equivariantly isometric to the Riemannian symmetric space G/K endowed with a suitable metric, and furthermore,* Aut (X) *acts isometrically on $\mathscr{E}$.*

As a straightforward consequence of Theorem B, we obtain:

Theorem C. *Let H be an arbitrary (possibly non-connected) compact subgroup of* Aut (X). *If in addition $\mathscr{E} \neq \phi$, then there always exists an H-invariant Einstein Kähler metric on X.*

We now briefly explain how the proof of Theorem A is carried out. Let ω_0 be an arbitrary element of $\mathscr{K}$ and $R(\omega_0)$ be the corresponding Ricci form (cf. (1.1)). (Later in this introduction, we set $\omega_0 := R(\tilde{\omega})$ for some element $\tilde{\omega}$ of $\mathscr{K}^+$ and vary ω_0 together with $\tilde{\omega}$.) Since $R(\omega_0)$ is cohomologous to ω_0, there exists a unique function $f \in C^\infty(X)_{\boldsymbol{R}}$ such that

$$R(\omega_0) = \omega_0 + \sqrt{-1}\,\partial\bar{\partial} f \quad \text{and} \quad \int_X \exp(f)\omega_0{}^n = \int_X \omega_0{}^n.$$

We then consider the following one-parameter families of equations:

$$(0.1) \qquad \log((\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\psi_t)^n/\omega_0{}^n) = -t\psi_t + f, \qquad 0 \leqq t \leqq 1,$$

$$(0.2) \qquad \log((\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t)^n/\omega_0{}^n) = -t\varphi_t - L(0, \varphi_t) + f, \qquad 0 \leqq t \leqq 1,$$

(see Section 1 for the definition of L), where in both cases, the solutions ψ_t and φ_t are required to belong to

$$\mathscr{H} := \{\varphi \in C^\infty(X)_{\boldsymbol{R}} \mid \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi \text{ is positive definite on } X\}.$$

Note that (0.1) above is introduced by Aubin [2] in his study of Einstein Kähler metrics on compact Kähler manifolds with $c_1 > 0$. One can easily pass from the solutions of one of (0.1) and (0.2) to those of the other because for each t, the difference between ψ_t and φ_t is just a constant (which may depend on t) on X. Now a crucial step of the proof of Theorem A is to show the following fact:

(0.3) *Given an orbit* $\boldsymbol{O}$ *in* $\mathscr{E}$, *we can connect* $\boldsymbol{O}$ *with every sufficiently general point* $\tilde{\omega}$ *of* $\mathscr{K}^+$ *by a smooth one-parameter family of solutions* $\{\psi_t \mid 0 \leqq t \leqq 1\}$ (*resp.* $\{\varphi_t \mid 0 \leqq t \leqq 1\}$) *of* (0.1) (*resp.* (0.2)) *such that*

$$\begin{cases} \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\psi_1 = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_1 \in \boldsymbol{O}, \quad \textit{and} \\ \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\psi_0 = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_0 = \tilde{\omega}, \ (\textit{i.e.}, R(\tilde{\omega}) = \omega_0). \end{cases}$$

Once one shows (0.3), the proof of Theorem A proceeds as follows:

(i) Consider the $\mathscr{K}$-energy map $\mu: \mathscr{K} \to \boldsymbol{R}$ of the Kähler manifold (X, ω_0). Recall that μ takes a constant value C on $\boldsymbol{O}$ (cf. [9]). Since $\mu(\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_t)$ is a monotone decreasing function of t (cf. (5.1)), the fact (0.3) above implies $\mu \geqq C$ on a dense subset of $\mathscr{K}^+$. By the continuity of μ, we obtain $\mu \geqq C$ on $\mathscr{K}^+$ (cf. (8.1)).

(ii) Note that (0.2) has a unique solution $\varphi_0 \in \mathscr{H}$ at $t=0$ (cf. (4.3.2)). Hence one can easily show that, over $\{0 \leqq t \leqq 1\}$, only one smooth family of solutions of (0.2) is possible (cf. (5.3), (5.4)). We now fix arbitrary G-orbits $\boldsymbol{O}_1$, $\boldsymbol{O}_2$ in $\mathscr{E}$. In view of (0.3), a sufficiently general $\tilde{\omega} \in \mathscr{K}^+$ can be connected with both $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ by smooth families $\{\varphi_t^{[i]} \mid 0 \leqq t \leqq 1\}$ $(i=1, 2)$ of solutions of (0.2) such that

$$\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi_1^{[i]} \in \boldsymbol{O}_i, \ (i=1, 2), \quad \text{where } R(\tilde{\omega}) = \omega_0.$$

Since these two families must coincide, we have $\varphi_1^{[1]} = \varphi_1^{[2]}$ and therefore $\boldsymbol{O}_1 = \boldsymbol{O}_2$.

We now give an outline of the proof of (0.3). It roughly consists of the following three steps. (For technical reasons, the actual proof is not divided into such steps.)

Step 1: Given a point $\theta \in \boldsymbol{O}$, we can always find a solution ψ_1 of (0.1) at $t=1$ such that $\theta = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\psi_1$. Except the obvious case $H^0(X, \mathcal{O}(T(X))) = \{0\}$, this ψ_1 may fail to extend to a smooth family $\{\psi_t \mid 1-\varepsilon \leqq t \leqq 1\}$ of solutions of (0.1). Because if such a family exists, differentiating (0.1) with respect to t at $t=1$, we have $(\square_\theta + 1)(\dot{\psi}_{t|t=1}) = -\psi_1$. Hence ψ_1 must satisfy

$$(0.4) \qquad \int_X \psi_1 \varphi \theta^n = 0 \qquad \text{for all } \varphi \in H_\theta,$$

where H_θ denotes $\operatorname{Ker}(\square_\theta + 1)$ in $C^\infty(X)_{\boldsymbol{R}}$. We therefore seek $\theta = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\psi_1 \in \boldsymbol{O}$ which satisfies (0.4), and a method to find such a θ will be given in Section 6. However, the condition (0.4) is not enough (cf. (7.2), (7.3)) and a detailed analysis of (0.1) using a bifurcation technique will be effectively employed (cf. Section 7). Finally, since the point $\tilde{\omega} \in \mathscr{K}_+$ is sufficiently general, a suitably chosen ψ_1 (resp. φ_1) extends to a smooth family $\{\psi_t \mid 1-\varepsilon \leqq t \leqq 1\}$ (resp. $\{\varphi_t \mid 1-\varepsilon \leqq t \leqq 1\}$) of solutions of (0.1) (resp. (0.2)).

Step 2: By the monotonicity of $\mu(\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi_t)$ (where we always consider such φ_t's as depending smoothly on t), one has

$$\mu(\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi_t) \geqq \mu(\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi_1) = \mu(\theta) \tag{0.5}$$

along the solutions of (0.2), and the family $\{\varphi_t \mid 1-\varepsilon \leqq t \leqq 1\}$ in Step 1 uniquely extends to a smooth family $\{\varphi_t \mid 0 < t \leqq 1\}$ of solutions of (0.2), because for each $0 < t \leqq 1$, the existence of the lower bound of $\mu(\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi_t)$ gives us a rough a priori estimate for $\|\varphi_t\|_{C^0}$.

Step 3: Now, another difficulty comes up at $t=0$, since the straightforward a priori bound for $\|\varphi_t\|_{C^0}$ obtained from (0.5) tends to infinity as $t \downarrow 0$. In Section 3, we derive a general lower bound of the Green function for the Laplacian from the isoperimetric inequality of Gallot [6]. This bound allows us to overcome the difficulty and thus we complete the whole extension to $\{\varphi_t \mid 0 \leqq t \leqq 1\}$.

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§ 1. Notation, convention and preliminaries

(1.1) Throughout this paper (with the only exception of Section 8) we fix, once for all, an element ω_0 of $\mathscr{K}$. In addition to the notation defined in Introduction, we put

$C^\infty(X)_C$: the space of complex-valued C^∞-functions on X,
$\mathscr{C}$: the space of real d-closed (1, 1)-forms on X in $2\pi c_1(X)_R$,
$\mathscr{V}$: the set of all volume forms on X,

where on X, everywhere positive real $2n$-form is called a volume form. We write an arbitrary element ω of $\mathscr{K}$ as

$$\omega = \sqrt{-1} \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$$

in terms of holomorphic local coordinates $z=(z^1, z^2, \cdots, z^n)$ on X. The corresponding Ricci tensor is denoted by $\sum R(\omega)_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}}$ and we put

$R(\omega):=\sqrt{-1}\sum R(\omega)_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta}$. Then $R(\omega)=\sqrt{-1}\,\bar{\partial}\partial\log\det(g_{\alpha\bar{\beta}})\in\mathscr{C}$. We furthermore denote by $\sigma(\omega)$ (resp. $\square_{\omega}$) the corresponding scalar curvature (resp. Laplacian on functions):

$$\sigma(\omega):=\sum g^{\bar{\beta}\alpha}R(\omega)_{\alpha\bar{\beta}},$$
$$\square_{\omega}:=\sum g^{\bar{\beta}\alpha}\partial^2/\partial z^{\alpha}\partial\bar{z}^{\beta},$$

where $(g^{\bar{\beta}\alpha})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$. For each $\varphi\in C^{\infty}(X)_{\boldsymbol{R}}$, we put

(1.1.1) $$\omega_0(\varphi):=\omega_0+\sqrt{-1}\,\partial\bar{\partial}\varphi,$$

(1.1.2) $$\Omega_0(\varphi):=\exp(-\varphi)\tilde{\omega}^n,$$

where $\tilde{\omega}$ is the unique element of $\mathscr{K}_+$ such that $R(\tilde{\omega})=\omega_0$. Recall that the following is a straightforward consequence of Yau's affirmative answer [14] to Calabi's conjecture:

(1.1.3) *The mapping* $\omega\in\mathscr{K}\mapsto R(\omega)\in\mathscr{C}$ *defines a homeomorphism* $R\colon(\mathscr{K},\|\ \|_{C^{k+2,\alpha}})\cong(\mathscr{C},\|\ \|_{C^{k,\alpha}})$ *for each* $(k,\alpha)\in\boldsymbol{Z}\times\boldsymbol{R}$ *with* $k\geqq 0$ *and* $0<\alpha<1$.

Now, (1.1.1) and (1.1.2) above define the mappings $\omega_0\colon C^{\infty}(X)_{\boldsymbol{R}}\ni\varphi\mapsto\omega_0(\varphi)\in\mathscr{C}$ and $\Omega_0\colon C^{\infty}(X)_{\boldsymbol{R}}\ni\varphi\mapsto\Omega_0(\varphi)\in\mathscr{V}$. Let

$$\mathscr{H}:=\{\varphi\in C^{\infty}(X)_{\boldsymbol{R}}\mid\omega_0(\varphi)\in\mathscr{K}\}$$

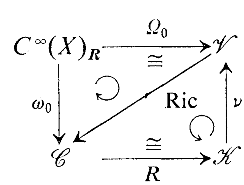
as in Section 0. Then the natural map

$$\mathscr{H}\longrightarrow\mathscr{K},\qquad\varphi\longmapsto\omega_0(\varphi)$$

is surjective. To each $\omega_0(\varphi)\in\mathscr{K}$, the corresponding $\square_{\omega_0(\varphi)}$, $\sigma(\omega_0(\varphi))$, $R(\omega_0(\varphi))$ will be denoted respectively by $\square_{\varphi}$, $\sigma(\varphi)$, $R(\varphi)$ for simplicity. Finally, we define the mappings $\nu\colon\mathscr{K}\to\mathscr{V}$ and $\mathrm{Ric}\colon\mathscr{V}\to\mathscr{C}$ by

$$\nu(\omega):=\omega^n\qquad(\omega\in\mathscr{K}),$$
$$\mathrm{Ric}(\Omega):=\sqrt{-1}\,\bar{\partial}\partial\log v\quad(\Omega\in\mathscr{V}),$$

where we write $\Omega=v(z)\prod_{\alpha=1}^{n}(\sqrt{-1}\,dz^{\alpha}\wedge d\bar{z}^{\alpha})$ in terms of holomorphic local coordinates $z=(z^1,z^2,\cdots,z^n)$ on X. Then the following diagram commutes:



(1.2) Let I be a (not necessarily open or closed) interval in $\boldsymbol{R}$, and S be either I or a product $I\times I\times\cdots\times I$ of I. A family $\{\varphi_s \mid s\in S\}$ of functions in $C^\infty(X)_{\boldsymbol{R}}$ is said to be *smooth* if the map

$$S\times X\longrightarrow \boldsymbol{R},$$
$$(s, x)\longmapsto \varphi_s(x)$$

is a C^∞-mapping. Any one-parameter family $\{\varphi_s \mid s\in I\}$ of functions in $C^\infty(X)_{\boldsymbol{R}}$ is called a *path*, and for every smooth path $\{\varphi_t \mid t\in I\}$, the function $\partial\varphi_t/\partial t\in C^\infty(X)_{\boldsymbol{R}}$ is denoted by $\dot{\varphi}_t$.

(1.3) Let $\mathscr{S}$ be a non-empty set. Then a mapping $H\colon \mathscr{S}\times\mathscr{S}\to\boldsymbol{R}$ is said to satisfy the 1-*cocycle condition* if

(i) $H(\sigma_1, \sigma_2)+H(\sigma_2, \sigma_1)=0$, and

(ii) $H(\sigma_1, \sigma_2)+H(\sigma_2, \sigma_3)+H(\sigma_3, \sigma_1)=0$

for all $\sigma_1, \sigma_2, \sigma_3\in\mathscr{S}$.

(1.4) (cf. [9]). Let V_0 be the volume $\int_X \omega_0{}^n/n!$ of the Kähler manifold (X, ω_0). We put $V:=n!V_0$. To each pair $(\varphi', \varphi'')\in C^\infty(X)_{\boldsymbol{R}}\times C^\infty(X)_{\boldsymbol{R}}$ (resp. $(\varphi', \varphi'')\in\mathscr{H}\times\mathscr{H}$), we associate a real number $L(\varphi', \varphi'')$ (resp. $M(\varphi', \varphi'')$) by

$$\text{(1.4.1)}\qquad L(\varphi', \varphi''):=\int_a^b\left(\int_X \dot{\varphi}_t\omega_0(\varphi_t)^n/V\right)dt,$$

$$\text{(1.4.2)}\qquad \left(\text{resp. } M(\varphi', \varphi''):=-\int_a^b\left\{\int_X \dot{\varphi}_t(\sigma(\varphi_t)-n)\omega_0(\varphi_t)^n/V\right\}dt\right),$$

where $\{\varphi_t \mid a\leqq t\leqq b\}$ is an arbitrary piecewise smooth path in $C^\infty(X)_{\boldsymbol{R}}$ (resp. $\mathscr{H}$) such that $\varphi_a=\varphi'$ and $\varphi_b=\varphi''$. Then $L(\varphi', \varphi'')$ (resp. $M(\varphi', \varphi'')$) is independent of the choice of the path $\{\varphi_t \mid a\leqq t\leqq b\}$ and therefore well-defined. Recall that L (resp. M) satisfies the 1-cocycle condition. Furthermore,

$$\text{(1.4.3)}\qquad L(\varphi_1, \varphi_2+C)=L(\varphi_1-C, \varphi_2)=L(\varphi_1, \varphi_2)+C,$$

$$\text{(1.4.4)}\qquad (\text{resp. } M(\varphi_1+C_1, \varphi_2+C_2)=M(\varphi_1, \varphi_2)),$$

for all $\varphi_1, \varphi_2\in C^\infty(X)_{\boldsymbol{R}}$ (resp. $\varphi_1, \varphi_2\in\mathscr{H}$) and all $C\in\boldsymbol{R}$ (resp. $C_1, C_2\in\boldsymbol{R}$). In view of (1.4.4) above, $M\colon \mathscr{H}\times\mathscr{H}\to\boldsymbol{R}$ factors through $\mathscr{K}\times\mathscr{K}$. Hence we can define the mapping $M\colon \mathscr{K}\times\mathscr{K}\to\boldsymbol{R}$ (denoted by the same M) by

$$M(\omega', \omega''):=M(\varphi', \varphi'')\qquad (\omega', \omega''\in\mathscr{K}),$$

where φ', φ'' are elements of $\mathscr{H}$ such that $\omega_0(\varphi')=\omega'$ and $\omega_0(\varphi'')=\omega''$. Then the mapping

$$\mu\colon \mathscr{K} \longrightarrow \boldsymbol{R}, \qquad \omega \longmapsto \mu(\omega) := M(\omega_0, \omega)$$

is called the *$\mathscr{K}$-energy map* of the Kähler manifold (X, ω_0). We now put $\mathscr{H}_0 := \{\varphi \in \mathscr{H} \mid L(0, \varphi) = 0\}$. The mapping $\mathscr{H}_0 \ni \varphi \mapsto \omega_0(\varphi) \in \mathscr{K}$ enables us to identify $\mathscr{H}_0$ with $\mathscr{K}$, and we have the following commutative diagram:

$$\begin{array}{ccc}
C^\infty(X)_{R} & \xrightarrow[\cong]{\Omega_0} & \mathscr{V} \\
\cup & \overset{\omega_0}{\searrow} \quad \circlearrowright & \big\downarrow \mathrm{Ric} \\
\mathscr{H} & \circlearrowleft & \mathscr{C} \\
\cup & & \cup \\
\mathscr{H}_0 & \xrightarrow{\cong} & \mathscr{K}.
\end{array}$$

(1.5) We regard L as a function on $\mathscr{V} \times \mathscr{V}$ via the identification $\Omega_0\colon C^\infty(X)_{R} \cong \mathscr{V}$. Let $N\colon \mathscr{K} \times \mathscr{K} \to \boldsymbol{R}$ be the pull-back $-\nu^* L$ of $-L$ by $\nu\colon \mathscr{K} \to \mathscr{V}$. Then this N is characterized by the following commutative diagram:

$$\begin{array}{ccc}
\mathscr{K} \times \mathscr{K} & \longrightarrow & \boldsymbol{R} \\
\big\downarrow \nu \times \nu & & \big\uparrow -L \\
\mathscr{V} \times \mathscr{V} & \cong & C^\infty(X)_{\boldsymbol{R}} \times C^\infty(X)_{\boldsymbol{R}}.
\end{array}$$

Since L satisfies the 1-cocycle condition, so does N. A straightforward computation shows that, for each pair $(\omega', \omega'') \in \mathscr{K} \times \mathscr{K}$, the number $N(\omega', \omega'')$ is given by

(1.5.1) $$N(\omega', \omega'') = \int_a^b \left\{ \int_X (\square_{\varphi_t} \dot{\varphi}_t) R(\varphi_t)^n / V \right\} dt,$$

where $\{\varphi_t \mid a \leqq t \leqq b\}$ is an arbitrary piecewise smooth path in $\mathscr{H}$ such that $\omega_0(\varphi_a) = \omega'$ and $\omega_0(\varphi_b) = \omega''$.

Remark (1.5.2). Several generalizations of L, M, N (which were announced in [8], [9] to appear in this paper) will be given separately in [3] as a self-contained article.

(1.6) (cf. Aubin [2]). For each pair $(\varphi', \varphi'') \in \mathscr{H} \times \mathscr{H}$, we put

(1.6.1) $$I(\varphi', \varphi'') := \int_X (\varphi'' - \varphi')(\omega_0(\varphi')^n - \omega_0(\varphi'')^n)/V,$$

(1.6.2) $$J(\varphi', \varphi'') := -L(\varphi', \varphi'') + \int_X (\varphi'' - \varphi')\omega_0(\varphi')^n / V.$$

Then both I: $\mathcal{H}\times\mathcal{H}\to\boldsymbol{R}$ and J: $\mathcal{H}\times\mathcal{H}\to\boldsymbol{R}$ factor through $\mathcal{K}\times\mathcal{K}$, i.e., I and J are regarded as functions on $\mathcal{K}\times\mathcal{K}$ by

$$I(\omega', \omega''):=I(\varphi', \varphi'') \quad \text{and} \quad J(\omega', \omega''):=J(\varphi', \varphi'')$$

for all $\omega', \omega'' \in \mathcal{K}$, where $\varphi', \varphi'' \in \mathcal{H}$ are such that $\omega_0(\varphi')=\omega'$ and $\omega_0(\varphi'')=\omega''$. We later need the following properties of I and J:

(1.6.3) $J(\varphi', \varphi'')\geqq 0$ *and the equality holds if and only if* $\varphi'=\varphi''+$ *constant.*

(1.6.4) $0\leqq I(\varphi', \varphi'')\leqq(n+1)(I(\varphi', \varphi'')-J(\varphi', \varphi''))\leqq nI(\varphi', \varphi'')$ *for all* φ', φ''.

These follow from Aubin's result [2; p. 146] and the identity $J(\varphi', \varphi'')+J(\varphi'', \varphi')=I(\varphi', \varphi'')=I(\varphi'', \varphi')$. We now take an arbitrary smooth path $\{\varphi_t \mid a\leqq t\leqq b\}$ in $\mathcal{H}$. Then a simple calculation shows that

$$\text{(1.6.5)} \qquad \frac{d}{dt}(I(0, \varphi_t)-J(0, \varphi_t))=-\int_X \varphi_t(\Box_{\varphi_t}\dot{\varphi}_t)\omega_0(\varphi_t)^n/V.$$

(1.7) Throughout this paper, we always denote by f the function in $C^\infty(X)_{\boldsymbol{R}}$ defined by

$$\text{(1.7.1)} \qquad R(\omega_0)=\omega_0+\sqrt{-1}\,\partial\bar{\partial}f \quad \text{and} \quad \int_X \exp(f)\omega_0{}^n=V \text{ (cf. Section 0).}$$

To each $\varphi \in \mathcal{H}_0$, we can similarly associate a function $f_\varphi \in C^\infty(X)_{\boldsymbol{R}}$ with the following properties (cf. [9]):

(1.7.2) $R(\varphi)=\omega_0(\varphi)+\sqrt{-1}\,\partial\bar{\partial}f_\varphi$,

(1.7.3) $\mu(\omega_0(\varphi))=-\int_X f_\varphi\omega_0(\varphi)^n/V$,

(1.7.4) $\frac{\partial}{\partial t}f_{\varphi_t}=-(\Box_{\varphi_t}+1)\dot{\varphi}_t$ *for every smooth path* $\{\varphi_t \mid a\leqq t\leqq b\}$ *in* $\mathcal{H}_0$.

(1.8) We shall now show that

(1.8.1) $N(\omega', \omega'')-M(\omega', \omega'') = J(\omega'', R(\omega''))-J(\omega', R(\omega'))$ *for all* $\omega', \omega'' \in \mathcal{K}$.

Proof. Choose $\varphi', \varphi'' \in \mathcal{H}_0$ so that $\omega_0(\varphi')=\omega'$ and $\omega_0(\varphi'')=\omega''$. Let $\varphi_t:=\varphi'+t(\varphi''-\varphi')+C_t \in \mathcal{H}_0$, $0\leqq t\leqq 1$, and we denote each $\omega_0(\varphi_t), f_{\varphi_t}, \Box_{\varphi_t}$ respectively by $\omega^{(t)}, f_t, \Box_t$. Then by (1.7.2)$\sim$(1.7.4) and (1.5.1),

$$\frac{d}{dt}M(\omega_0, \omega^{(t)})=\frac{d}{dt}\mu(\omega^{(t)})=\frac{d}{dt}\left(-\int_X f_t(\omega^{(t)})^n/V\right),$$

$$\frac{d}{dt}N(\omega_0, \omega^{(t)}) = \int_X (\square_t \dot{\varphi}_t) R(\varphi_t)^n / V = -\int_X (\dot{\varphi}_t + \dot{f}_t)\omega_0(\varphi_t + f_t)^n / V$$
$$= -\frac{d}{dt} L(0, \varphi_t + f_t).$$

Since $L(0, \varphi_t) = 0$ for all t, we have

$$\frac{d}{dt}\{N(\omega_0, \omega^{(t)}) - M(\omega_0, \omega^{(t)})\} = \frac{d}{dt}\left\{\int_X f_t(\omega^{(t)})^n / V - L(\varphi_t, \varphi_t + f_t)\right\}$$
$$= \frac{d}{dt} J(\omega^{(t)}, R(\varphi_t)).$$

Integrating this over the interval [0, 1], we obtain (1.8.1).

§ 2. Matsushima's theorem and some identities on Einstein Kähler manifolds

Throughout this section, we assume $\mathscr{E} \neq \phi$, and then fix an arbitrary element $\theta = \sqrt{-1} \sum \theta_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta$ of $\mathscr{E}$. By quoting the well-known theorem of Matsushima [11], we shall introduce several notations on Einstein Kähler manifolds. Some technical identities on such manifolds will also be proven for later purposes.

(2.1) Let $\mathfrak{g}$ be the space $H^0(X, \mathcal{O}(T(X)))$ of all holomorphic vector fields on X. For each $Y \in \mathfrak{g}$, let $Y_{\boldsymbol{R}}$ denote the real vector field $Y + \bar{Y}$ and we set $\mathfrak{g}_{\text{real}} := \{Y_{\boldsymbol{R}} \mid Y \in \mathfrak{g}\}$. Then $Y \mapsto Y_{\boldsymbol{R}}$ defines an isomorphism of the complex Lie algebras $(\mathfrak{g}, \sqrt{-1}) \cong (\mathfrak{g}_{\text{real}}, J)$, where J is the complex structure of X. We now consider the G-orbit $\boldsymbol{O}$ through θ in $\mathscr{E}$. This is written as

$$\boldsymbol{O} \cong G/K_\theta$$

in terms of the isotropy subgroup K_θ of G at θ. Let $\mathfrak{k}_\theta$ be the set of all Killing vector fields on X with respect to the Kähler metric θ, where each Killing vector field is regarded as an element of $\mathfrak{g}$ via the identification $\mathfrak{g} \cong \mathfrak{g}_{\text{real}}$. Then $\mathfrak{k}_\theta$ is the Lie subalgebra of $\mathfrak{g}$ corresponding to K_θ in G. For each $\varphi \in C^\infty(X)_{\boldsymbol{C}}$, we define the vector field Y_θ^φ on X by

(2.1.1)
$$Y_\theta^\varphi := \frac{1}{2} \sum \varphi^\alpha \frac{\partial}{\partial z^\alpha},$$

where $\varphi^\alpha = \sum \theta^{\bar{\beta}\alpha} \partial_{\bar{\beta}} \varphi$, $(\theta^{\bar{\beta}\alpha})$ being the inverse matrix of $(\theta_{\alpha\bar{\beta}})$. Take the one-parameter group $y_{\theta t}^\varphi := \exp(tY_{\theta\boldsymbol{R}}^\varphi)$, $t \in \boldsymbol{R}$, on X generated by $Y_{\theta\boldsymbol{R}}^\varphi$. We now have the following theorem of Matsushima [11]:

Theorem (2.2). *Let* $H_\theta := \{\varphi \in C^\infty(X)_R \mid (\Box_\theta + 1)\varphi = 0\}$ *and we set* $\mathfrak{p}_\theta := \sqrt{-1}\,\mathfrak{k}_\theta$ *and* $H_\theta^C := H_\theta \otimes_R C \subset C^\infty(X)_C$. *Then*

(2.2.1) $\mathfrak{k}_\theta = \{Y_\theta^\varphi \mid \varphi \in \sqrt{-1}\,H_\theta\}$ *and* $\mathfrak{p}_\theta = \{Y_\theta^\varphi \mid \varphi \in H_\theta\}$;

(2.2.2) $\varphi \in H_\theta^C \mapsto Y_\theta^\varphi \in \mathfrak{g}$ *defines an isomorphism* $H_\theta^C \cong \mathfrak{g}$ *and hence* $\mathfrak{g} = \mathfrak{k}_\theta + \mathfrak{p}_\theta$ *and* $\mathfrak{k}_\theta \cap \mathfrak{p}_\theta = \{0\}$.

This theorem in particular implies the following identification:

$$(2.2.3) \qquad \begin{array}{ccccc} T_\theta(O) \cong T_e(G/K_\theta) = & \mathfrak{p}_\theta & \cong & H_\theta \\ \dfrac{\partial}{\partial t}((y_{\theta t}^\varphi)^*\theta)|_{t=0} \longleftrightarrow & Y_\theta^\varphi & \longleftrightarrow & \varphi. \end{array}$$

We put $\tilde{Y} := Y_{\theta R}^\varphi$ for brevity. Then by the next computation, the left-hand side of (2.2.3) is shown to have a very nice description:

$$(2.2.3) \qquad \begin{aligned} \frac{\partial}{\partial t}((y_{\theta t}^\varphi)^*\theta)|_{t=0} &= L_{\tilde{Y}}\theta = (d \circ i_{\tilde{Y}} + i_{\tilde{Y}} \circ d)\theta \\ &= \frac{\sqrt{-1}}{2} d(-\partial\varphi + \bar{\partial}\varphi) = \sqrt{-1}\,\partial\bar{\partial}\varphi. \end{aligned}$$

We shall now prove the following technical Lemma:

Lemma (2.3). *Let* $\langle\ ,\ \rangle_\theta$: $\{p\text{-forms on } X\} \times \{p\text{-forms on } X\} \to C^\infty(X)_C$, $p = 1, 2, \cdots$, *be the natural Hermitian pairings induced from the Kähler metric* θ. *Then for all* $\varphi, \psi \in H_\theta$ *and all* $\zeta \in C^\infty(X)_R$, *we have*

$$(2.3.1) \qquad \Box_\theta \langle \partial\zeta, \partial\varphi \rangle_\theta = \langle \partial\bar{\partial}\zeta, \partial\bar{\partial}\varphi \rangle_\theta + \langle \partial(\Box_\theta \zeta), \partial\varphi \rangle_\theta.$$

In particular $(\Box_\theta + 1)\langle \partial\psi, \partial\varphi \rangle_\theta = \langle \partial\bar{\partial}\psi, \partial\bar{\partial}\varphi \rangle_\theta = (\Box_\theta + 1)\langle \partial\varphi, \partial\psi \rangle_\theta$ *and*

$$(2.3.2) \qquad -\int_X \varphi \langle \partial\bar{\partial}\zeta, \partial\bar{\partial}\psi \rangle_\theta \theta^n = \int_X (\varphi\psi - \langle \partial\varphi, \partial\psi \rangle_\theta)\{(\Box_\theta + 1)\zeta\}\theta^n.$$

Proof. Fix an arbitrary point x of X, and choose holomorphic local coordinates $(z^1, z^2, \cdots, z^n)$ centered at x such that $\theta_{\alpha\bar{\beta}}(x) = \delta_{\alpha\beta}$ and $(d\theta_{\alpha\bar{\beta}})(x) = 0$ for all α and β. Note that φ^α, $\alpha = 1, 2, \cdots, n$, are all holomorphic (cf. (2.1.1), (2.2.2)). Therefore, at the point x,

$$\begin{aligned} \Box_\theta \langle \partial\zeta, \partial\varphi \rangle_\theta = \Box_\theta (\textstyle\sum \zeta_\alpha \varphi^\alpha) &= \sum_{\alpha,\beta} \zeta_{\alpha\beta}\varphi^\alpha{}_\beta + \sum_{\alpha,\beta} \zeta_{\alpha\beta\bar{\beta}}\varphi^\alpha \\ &= \langle \partial\bar{\partial}\zeta, \partial\bar{\partial}\varphi \rangle_\theta + \langle \partial(\Box_\theta\zeta), \partial\varphi \rangle_\theta, \end{aligned}$$

which proves (2.3.1). For (2.3.2), let $\xi := (\Box_\theta + 1)\zeta$. Then

$$\int_X (\varphi\psi - \langle\partial\varphi, \partial\psi\rangle_\theta)\xi\theta^n$$

$$= -\sqrt{-1}\int_X (\varphi\partial\bar{\partial}\psi + \partial\varphi\wedge\bar{\partial}\psi)\xi\wedge n\theta^{n-1} \quad (\text{because } \psi = -\Box_\theta\psi)$$

$$= -\sqrt{-1}\int_X \xi\partial(\varphi\bar{\partial}\psi)\wedge n\theta^{n-1} = \sqrt{-1}\int_X \varphi\partial\xi\wedge\bar{\partial}\psi\wedge n\theta^{n-1}$$

$$= \int_X \varphi\langle\partial\xi, \partial\psi\rangle_\theta\theta^n = \int_X \varphi\{\Box_\theta\langle\partial\zeta, \partial\psi\rangle_\theta - \langle\partial\bar{\partial}\zeta, \partial\bar{\partial}\psi\rangle_\theta\}\theta^n$$

$$+\int_X \varphi\langle\partial\zeta, \partial\psi\rangle_\theta\theta^n \qquad (\text{cf. } (2.3.1))$$

$$= -\int_X \varphi\langle\partial\bar{\partial}\zeta, \partial\bar{\partial}\psi\rangle_\theta\theta^n \qquad (\text{because } \varphi = \Box_\theta\varphi).$$

§ 3. Lower bounds for the Green function of the Laplacian

In this section, using the isoperimetric inequality of Gallot [6], we shall construct some lower bound for the Green function of the Laplacian on a compact Riemannian manifold. This bound applies to our compact Kähler situation and allows us to obtain an inequality which turns out to be crucial in our later investigation.

(3.1) Let (M, g) be an m-dimensional compact Riemannian manifold. The corresponding Ricci tensor, volume, volume form and diameter are denoted respectively by r_g, V_g, dM_g and D_g. We then set

$$a_g := D_g^2 \operatorname{Inf}\{r_g(r, r)/(n-1);\ \|r\|_g = 1\},$$

where the infimum is taken over all unit tangent vectors in $T(M)$. Let Δ_g be the Laplacian of (M, g) (we choose Δ_g so that it always has nonpositive eigenvalues), and $G_g \in C^\infty(M\times M - (\text{diagonal}))_{\boldsymbol{R}}$ be the corresponding Green function (with the well-known prescribed singularity along the diagonal) characterized by the following properties:

(i) $\varphi(x) = V_g^{-1}\int_M \varphi(y)dM_g(y) + \int_M G_g(x, y)(-\Delta_g\varphi)(y)dM_g(y)$,

(ii) $\int_M G_g(x, y)dM_g(y) = 0$,

for all $x \in M$ and $\varphi \in C^\infty(M)_{\boldsymbol{R}}$.

Theorem (3.2). *Let $(m, \alpha) \in \boldsymbol{Z}\times\boldsymbol{R}$ be an arbitrary pair satisfying $m \geqq 2$ and $\alpha \geqq 0$. Then there exists a positive constant $\gamma = \gamma(m, \alpha)$ depending only on m and α such that, for every m-dimensional compact connected Riemannian manifold (M, g) with $a_g \geqq -\alpha^2$,*

$$G_g(x, y) \geqq -\gamma(m, \alpha) D_g^2 / V_g$$

for all $x, y \in M$ *with* $x \neq y$.

Remark (3.3). If $\alpha = 0$, the number $\gamma(m, \alpha)$ is easily computed. For instance, $\gamma(2, 0) = 24$.

(3.4) *Proof of* (3.2). Let W_0 be the space of the functions f in $L_1^2(X)$ which satisfy $\int_M f dM_g = 0$. Then a combination of Theorems of Gallot [6; (1.3), (2.7)] shows that, there exists a positive constant $\kappa(m, \alpha)$ depending only on m and α such that, for every $f \in W_0$, the number $C := \kappa(m, \alpha) V_g^{1/m} D_g^{-1}$ satisfies

(3.4.1) $$\|df\|_{L^2(M,g)} \geqq C \|f\|_{L^{2m/(m-2)}(M,g)} \qquad (\text{if } m \geqq 3);$$

(3.4.2) $$\|df\|_{L^2(M,g)} \geqq C V_g^{-1/4} \|f\|_{L^4(M,g)} \qquad (\text{if } m = 2).$$

Let $H(x, y, t)$ be the heat kernel of (M, g), and we set

$$H_0(x, y, t) := H(x, y, t) - V_g^{-1}.$$

The proof is now divided into two cases:

(*Case* 1) $m \geqq 3$. A result of Cheng and Li [5; (2.9)] says that (3.4.1) implies

$$0 < H_0(x, x, t) \leqq 4(2tC^2/m)^{-m/2},$$

where $(x, t) \in M \times \boldsymbol{R}$ with $t > 0$. Hence for all x and y,

$$|H_0(x, y, t)| \leqq H_0(x, x, t)^{1/2} H_0(y, y, t)^{1/2} \leqq 4(2tC^2/m)^{-m/2}.$$

Together with $H_0(x, y, t) \geqq -V_g^{-1}$, we obtain

$$G_g(x, y) = \int_0^\infty H_0(x, y, t) dt \geqq -\int_0^\tau V_g^{-1} dt - \int_\tau^\infty 4(2tC^2/m)^{-m/2} dt$$

for each $\tau > 0$. If we set $\tau := 2^{4/m} m \kappa(m, \alpha)^{-2} D_g^2 / 2$, the right-hand side of this inequality is written as $-\gamma(m, \alpha) D_g^2 / V_g$ for some constant $\gamma(m, \alpha)$ depending only on m and α, as required.

(*Case* 2)[*] $m = 2$. For each $(x, t) \in M \times \boldsymbol{R}$, we put $M_{x,t} := \{x\} \times M \times \{t\}$, which is a submanifold ($\cong M$) of $M \times M \times \boldsymbol{R}$. For C^∞-functions

[*] G_g ($= G_{g,M}$) is written in terms of the Green function $G_{g \times g, M \times M}$, which provides us with a very simple proof for this case by reduction to $m = 4$. However, the estimate thus obtained is not so sharp (for instance, $\gamma(2, 0)$ would exceed 24).

$\varphi(x, y, t)$ defined on an open subset of $M \times M \times \boldsymbol{R}$, we denote by d_y: $\varphi(x, y, t) \mapsto d_y\varphi(x, y, t)$ the d-operator coming only from the second factor. Then the same argument as in Cheng and Li [5; (2.7)] together with (3.4.2) yields

$$(3.4.3) \qquad \begin{aligned}(\partial H_0/\partial t)(x, x, t) &= -\|d_y H_0(x, y, t/2)\|^2_{L^2(M_x, t/2, g)} \\ &\leqq -C^2 V_g^{-1/2}\|H_0(x, y, t/2)\|^2_{L^4(M_x, t/2, g)}\end{aligned}$$

for each $(x, t) \in M \times \boldsymbol{R}$ with $t > 0$. On the other hand, in view of

$$\int_M |H_0(x, y, t)| dM_g(y) \leqq \int_M (H(x, y, t) + V_g^{-1}) dM_g(y) = 2$$

and the Hölder inequality

$$\begin{aligned}\| |H_0(x, y, t/2)|^{2/3}\|_{L^{3/2}(M_x, t/2, g)} &\| |H_0(x, y, t/2)|^{4/3}\|_{L^3(M_x, t/2, g)} \\ &\geqq \int_M |H_0(x, y, t/2)|^2 dM_g(y) \quad (= H_0(x, x, t)),\end{aligned}$$

we obtain $\|H_0(x, y, t/2)\|^2_{L^4(M_x, t/2, g)} \geqq H_0(x, x, t)^{3/2}/2$. This combined with (3.4.3) shows that

$$(\partial H_0/\partial t)(x, x, t) \leqq -(C^2/2) V_g^{-1/2} H_0(x, x, t)^{3/2}.$$

Then the same argument as in Cheng and Li [5; (2.9)] again applies. Thus,

$$H_0(x, x, t) \leqq (tC^2/4)^{-2} V_g.$$

Finally, similar to Case 1 above, it follows that

$$\begin{aligned}G_g(x, y) = \int_0^\infty H_0(x, y, t) dt &\geqq -\int_0^\tau V_g^{-1} dt - \int_\tau^\infty (tC^2/4)^{-2} V_g dt \\ &\geqq -8/C^2 = -8\kappa(2, \alpha)^{-2} D_g^2 / V_g\end{aligned}$$

by setting $\tau = 4V_g/C^2$.

(3.5) We now return to our original compact Kähler situation. In terms of the notation in (3.2) above, let $\beta(n) := \gamma(2n, 0)$, which is a constant depending on n alone. Furthermore, for each $\varphi \in \mathscr{H}$, let Δ_φ (resp. Δ_0) denote the real Laplacian $2\Box_\varphi$ (resp. $2\Box_{\omega_0}$) of the compact Kähler manifold $(X, \omega_0(\varphi))$ (resp. (X, ω_0)), and G_φ (resp. G_0) be its corresponding Green function as is defined in (3.1). We denote by $-K_\varphi$ (resp. $-K_0$) the infimum of G_φ (resp. G_0) on $X \times X -$(diagonal). Finally, for each $t \in \boldsymbol{R}$, we put

$$\mathscr{H}^{(t)} := \{\varphi \in \mathscr{H} \mid R(\varphi) - t\omega_0(\varphi) \text{ is positive semi-definite}\}.$$

Theorem (3.2) now has the following important implication.

Proposition (3.6). *Let* $t>0$ *be arbitrary. Then for every* $\varphi \in \mathscr{H}^{(t)}$, *its oscillation* $\mathrm{Osc}\,\varphi := \mathrm{Max}_X\,\varphi - \mathrm{Min}_X\,\varphi$ *satisfies*

$$\mathrm{Osc}\,\varphi \leqq I(0, \varphi) + 2n(K_0V_0 + (n-1)\beta(n)\pi^2 t^{-1}).$$

Proof. We observe, by virtue of the identity $\omega_0(\varphi) = \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$, that the following inequalities hold:

$$-\Delta_0\varphi \leqq 2n \quad \text{and} \quad -\Delta_\varphi\varphi \geqq -2n.$$

Hence we have

$$\begin{aligned}\varphi(x) &= V_0^{-1}\int_X \varphi\omega_0{}^n/n! + \int_X (G_0(x, y) + K_0)(-\Delta_0\varphi)(y)\omega_0{}^n(y)/n!\\ &\leqq V_0^{-1}\int_X \varphi\omega_0{}^n/n! + 2nK_0V_0\end{aligned}$$

and

$$\begin{aligned}\varphi(x) &= V_0^{-1}\int_X \varphi\omega_0(\varphi)^n/n! + \int_X (G_\varphi(x, y) + K_\varphi)(-\Delta_\varphi\varphi)(y)\omega_0(\varphi)^n(y)/n!\\ &\geqq V_0^{-1}\int_X \varphi\omega_0(\varphi)^n/n! - 2nK_\varphi V_0.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Osc}\,\varphi &\leqq V_0^{-1}\int_X \varphi(\omega_0{}^n - \omega_0(\varphi)^n)/n! + 2n(K_0V_0 + K_\varphi V_0)\\ &= I(0, \varphi) + 2n(K_0V_0 + K_\varphi V_0) \quad \text{(cf. (1.6.1))}.\end{aligned}$$

Let D_φ be the diameter of $(X, \omega_0(\varphi))$. Since $R(\varphi) \geqq t\omega_0(\varphi)$, Meyer's theorem asserts that $D_\varphi \leqq \pi((n-1)/t)^{1/2}$. We now conclude from Theorem (3.2) that

$$K_\varphi V_0 \leqq \beta(n)D_\varphi{}^2 \leqq (n-1)\beta(n)\pi^2 t^{-1}.$$

§ 4. Generalized Aubin's equations

In a recent paper [2], Aubin introduced a very interesting one-parameter family of non-linear equations in applying the continuity method to the existence of Einstein Kähler metrics on some compact Kähler manifolds with $c_1>0$. In this section, we shall consider a slightly modified family of equations so that it fits our purpose. Elementary properties of such a family will also be given.

(4.1) We define the mapping $A: \mathscr{H} \to C^\infty(X)_R$ by

$$A(\varphi) := \log(\omega_0(\varphi)^n/\omega_0{}^n) \quad (\varphi \in \mathscr{H}),$$

and then consider the following one-parameter families of equations:

$$A(\varphi_t) = -t\varphi_t - L(0, \varphi_t) + f; \quad 0 \leqq t \leqq 1, \tag{4.1.1}$$

$$A(\psi_t) = -t\psi_t + f; \quad 0 \leqq t \leqq 1, \tag{4.1.2}$$

where solutions φ_t and ψ_t are both required to belong to $\mathscr{H}$. We call the former the family of *generalized Aubin's equations* of the Kähler manifold (X, ω_0), while the latter is the original family introduced by Aubin.

Remark (4.1.3). At each point t of $(0, 1]$ (resp. $[0, 1]$), we put

$$j'(\varphi_t) := \varphi_t + t^{-1}L(0, \varphi_t) \quad (\text{resp. } j''(\psi_t) := \psi_t - (t+1)^{-1}L(0, \psi_t))$$

for every φ_t (resp. ψ_t) satisfying (4.1.1) (resp. (4.1.2)). Then by substituting $j'(\varphi_t)$ (resp. $j''(\psi_t)$) for ψ_t (resp. φ_t), one can easily see that $j'(\varphi_t)$ (resp. $j''(\psi_t)$) satisfies (4.1.2) (resp. (4.1.1)). Furthermore, $j' \circ j'' = \mathrm{id}$ and $j'' \circ j' = \mathrm{id}$. Hence in finding solutions for $t \neq 0$, there is no difference between (4.1.1) and (4.1.2).

(4.2) Choose an arbitrary $t \in [0, 1]$. Let φ_t (resp. ψ_t) be a solution of (4.1.1) (resp. (4.1.2)), and we set $\omega^{(t)} := \omega_0(\varphi_t)$ (resp. $\omega^{(t)} := \omega_0(\psi_t)$). Then $\omega^{(t)}$ satisfies

$$R(\omega^{(t)}) = (1-t)\omega_0 + t\omega^{(t)}, \tag{4.2.1}$$

(cf. Aubin [2; p. 149]), and in particular φ_t (resp. ψ_t) belongs to $\mathscr{H}^{(t)}$ (cf. (3.5)). On the other hand, one can easily pass from the solutions of (4.2.1) to those of (4.1.1), though we do not go into details.

(4.3) We shall next study the solutions of (4.1.1) and (4.1.2) at $t=0$. Recall the following affirmative answer to Calabi's conjecture:

Theorem (4.3.1) (Yau [14]). *If* $t=0$, *then* (4.1.2) *has a solution which is unique up to an additive constant.*

This in particular implies:

Corollary (4.3.2). *For* $t=0$, *the equation* (4.1.1) *has a unique solution* φ_0. *Moreover,* $L(0, \varphi_0) = 0$ *and* $R(\omega_0(\varphi_0)) = \omega_0$.

Proof. The existence of a solution is straightforward from (4.1.3) and (4.3.1). For uniqueness, let φ_0 be a solution of (4.1.1) at $t=0$. Then

$$\int_X \omega_0{}^n = \int_X \exp(A(\varphi_0))\omega_0{}^n = \exp(-L(0, \varphi_0)) \int_X \exp(f)\omega_0{}^n.$$

Since $\int_X \exp(f)\omega_0{}^n = \int_X \omega_0{}^n$ (cf. (1.7.1)), we obtain $L(0, \varphi_0)=0$. Therefore, φ_0 is a solution of (4.1.2) at $t=0$. The required uniqueness now follows from a combination of Theorem (4.3.1) and $L(0, \varphi_0)=0$. $R(\omega_0(\varphi_0))=\omega_0$ is an immediate consequence of (1.7.1) and $A(\varphi_0)=f$.

Remark (4.3.3). Suppose that $\{\varphi_t \mid 0 \leqq t \leqq \tau\}$ $(\tau>0)$ is a smooth one-parameter family of solutions of (4.1.1). By (4.3.2) above, $L(0, \varphi_0)=0$, and hence by setting $\psi_t := j'(\varphi_t) = \varphi_t + t^{-1}L(0, \varphi_t)$ (cf. (4.1.3)), we see that $\{\psi_t \mid 0 \leqq t \leqq \tau\}$ is a smooth family of solutions of (4.1.2). (For similar arguments, see Aubin [2; p. 149].)

(4.4) Let $\mathscr{H}^{k,\alpha}$ (where $2 \leqq k \in \boldsymbol{Z}$ and $0<\alpha<1$) be the set of all $\varphi \in C^{k,\alpha}(X)_{\boldsymbol{R}}$ with positive definite $\omega_0(\varphi)$. Note that $\mathscr{H}^{k,\alpha}$ is an open subset of $C^{k,\alpha}(X)_{\boldsymbol{R}}$. We now conclude this section by showing the following local extension property of solutions of (4.1.1) for $0 \leqq t < 1$.

Proposition (4.4.1) (cf. Aubin [2]). *Let $2 \leqq k \in \boldsymbol{Z}$ and fix $\alpha \in \boldsymbol{R}$ with $0<\alpha<1$. Let $0 \leqq \tau < 1$. Suppose, moreover, that* (4.1.1) *has a solution φ_τ at $t=\tau$. Then for some $\varepsilon>0$, φ_τ uniquely extends to a smooth one-parameter family*

$$\{\varphi_t \mid t \in [0, 1) \cap [\tau-\varepsilon, \tau+\varepsilon]\}$$

of solutions of (4.1.1) *in $\mathscr{H}$, and furthermore, if $(\varphi, t) \in \mathscr{H}^{k,\alpha} \times [0, 1)$ satisfies the conditions $\|\varphi-\varphi_\tau\|_{C^{k,\alpha}} \leqq \varepsilon$, $|t-\tau| \leqq \varepsilon$ and $A(\varphi) = -t\varphi - L(0, \varphi) + f$, then φ coincides with φ_t.*

Proof. Consider the mapping $\Gamma: \mathscr{H}^{k,\alpha} \times \boldsymbol{R} \to C^{k-2,\alpha}(X)_{\boldsymbol{R}}$ defined by

$$\Gamma(\varphi, t) := A(\varphi) + t\varphi + L(0, \varphi) - f, \qquad (\varphi, t) \in \mathscr{H}^{k,\alpha} \times \boldsymbol{R}.$$

Then its Fréchet derivative $D_\varphi \Gamma: C^{k,\alpha}(X)_{\boldsymbol{R}} \to C^{k-2,\alpha}(X)_{\boldsymbol{R}}$ (at (φ, t)) with respect to the first factor is given by

$$D_\varphi \Gamma(\psi) = (\square_\varphi + t)\psi + \int_X \psi \omega_0(\varphi)^n / V, \qquad \psi \in C^{k,\alpha}(X)_{\boldsymbol{R}}.$$

Note that, by the well-known regularity theorem, we have $\varphi \in \mathscr{H}$ for every $(\varphi, t) \in \mathscr{H}^{k,\alpha} \times \boldsymbol{R}$, whenever $\Gamma(\varphi, t)=0$. Since $\Gamma(\varphi_\tau, \tau)=0$, an application of the implicit function theorem now reduces the proof to showing that $D_\varphi \Gamma$ is invertible at (φ_τ, τ). The following cases are possible.

(*Case* 1) $\tau=0$. Then $D_\varphi\Gamma\,|_{(\varphi_0,0)}$ is the mapping

$$C^{k,\alpha}(X)_{\mathbf{R}} \ni \psi \longmapsto \square_{\varphi_0}\psi + \int_X \psi\omega_0(\varphi_0)^n/V \in C^{k-2,\alpha}(X)_{\mathbf{R}},$$

which is invertible.

(*Case* 2) $\tau\neq 0$. Since $R(\varphi_\tau)-\tau\omega_0(\varphi_\tau)$ is positive definite (cf. (4.2)), a theorem of Lichnerowicz [7] asserts that τ is less than the first (positive) eigenvalue of $-\square_{\varphi_\tau}$. Hence $D_\varphi\Gamma|_{(\varphi_\tau,\tau)}$ is invertible.

Remark (4.4.2). Proposition (4.4.1) is valid even if φ_t and (4.1.1) are replaced respectively by ψ_t and (4.1.2). This is the original local extension theorem proved by Aubin [2].

§ 5. The $\mathscr{K}$-energy map along the solutions of generalized Aubin's equations

Recall that (4.1.1) has a unique solution φ_0 at $t=0$ (see (4.3.2)). By using an explicit description (cf. (5.1)) of the $\mathscr{K}$-energy map μ along the solutions of (4.1.1), we shall show that any φ_τ satisfying (4.1.1) at $t=\tau$ ($\tau\neq 1$) uniquely extends to a smooth one-parameter family $\{\varphi_t\,|\,0\leqq t\leqq\tau\}$ of solutions of (4.1.1). Note that this fact in particular shows that (4.1.1) admits at most one solution at $t=\tau$ for $0\leqq\tau<1$ (cf. (5.3)). The same technique enables us to show that if μ is bounded from below, then φ_0 uniquely extends to a smooth one-parameter family $\{\varphi_t\,|\,0\leqq t<1\}$ of solutions of (4.1.1) (cf. (5.7)).

Theorem (5.1). *Let* $\{\varphi_t\,|\,a\leqq t\leqq b\}$ *be an arbitrary smooth one-parameter family of solutions of* (4.1.1) *in* $\mathscr{H}$. *For brevity, we put*

$$\omega^{(t)}:=\omega_0(\varphi_t),\quad I_t:=I(\omega_0,\omega^{(t)})\ (=I(0,\varphi_t)),$$
$$J_t:=J(\omega_0,\omega^{(t)})\ (=J(0,\varphi_t)).$$

Then on $[a, b]$,

$$\frac{d\mu(\omega^{(t)})}{dt}=-(1-t)\frac{d}{dt}(I_t-J_t)\leqq 0.$$

Proof. By $R(\varphi_t)=(1-t)\omega_0+t\omega^{(t)}=\omega_0(\varphi_t)-\sqrt{-1}(1-t)\partial\bar\partial\varphi_t$ (cf. (4.2.1)), we have $\sigma(\varphi_t)=n-(1-t)\square_{\varphi_t}\varphi_t$. Hence,

$$\frac{d}{dt}\mu(\omega^{(t)})=\frac{d}{dt}M(0,\varphi_t)=\int_X(1-t)\dot\varphi_t(\square_{\varphi_t}\varphi_t)\omega_0(\varphi_t)^n/V$$
$$=-(1-t)\frac{d}{dt}(I_t-J_t)\quad \text{(cf. (1.6.5))}.$$

On the other hand, differentiating (4.1.1) with respect to t, we obtain

$$\Box_{\varphi_t}\dot{\varphi}_t + t\dot{\varphi}_t + \varphi_t + C_t = 0$$

for some constant $C_t \in \boldsymbol{R}$ on X. Combining this with (1.6.5), we now see that

$$\frac{d}{dt}(I_t - J_t) = \int_X (\Box_{\varphi_t}\dot{\varphi}_t + t\dot{\varphi}_t)(\Box_{\varphi_t}\dot{\varphi}_t)\omega_0(\varphi_t)^n/V \geqq 0,$$

where the last inequality is a straightforward consequence of (the first eigenvalue of $-\Box_{\varphi_t}$)$>t$.

(5.2) (i) Fix $\alpha \in \boldsymbol{R}$ with $0<\alpha<1$. Let φ_τ be a solution of (4.1.1) at $t=\tau$ (where $\tau \neq 0, 1$). A smooth family $\{\varphi_t \mid \sigma < t \leqq \tau\}$ (resp. $\{\varphi_t \mid \tau \leqq t < \sigma\}$) of solutions of (4.1.1) is said to be *maximal* if for any sequence $t_j \in (\sigma, \tau]$ (resp. $[\tau, \sigma)$) $(j=1, 2, \cdots)$ with $\lim t_j = \sigma$, the corresponding sequence $\{\varphi_{t_j}\}$ in $\mathscr{H}$ does not converge to any point of $\mathscr{H}^{2,\alpha}$ in the $C^{2,\alpha}$-norm.

(ii) Suppose $\mathscr{E} \neq \phi$. Then to each $\theta \in \mathscr{E}$, we can uniquely associate a function $\lambda_\theta \in \mathscr{H}$ such that $\theta = \omega_0(\lambda_\theta)$ and that λ_θ satisfies (4.1.2) at $t=1$, i.e., $A(\lambda_\theta) = -\lambda_\theta + f$. An element θ of $\mathscr{E}$ is said to be *excellent* on the Kähler manifold (X, ω_0) if for some $\varepsilon>0$, there exists a smooth family $\{\psi_t \mid 1-\varepsilon \leqq t \leqq 1\}$ of solutions of (4.1.2) such that $\psi_1 = \lambda_\theta$.

Theorem (5.3). *Let* $0<\tau<1$. *Then any solution* φ_τ *of* (4.1.1) *at* $t=\tau$ *uniquely extends to a smooth family* $\{\varphi_t \mid 0 \leqq t \leqq \tau\}$ *of solutions of* (4.1.1). *In particular* (4.1.1) *admits at most one solution in* $\mathscr{H}$ *at* $t=\tau$.

Corollary (5.4). (i) *There exists at most one* $\theta \in \mathscr{E}$ *which is excellent on the Kähler manifold* (X, ω_0).

(ii) *Suppose that* $\theta \in \mathscr{E}$ *is excellent on the Kähler manifold* (X, ω_0). *Then* $M(\theta, \omega_0(\varphi_0)) \geqq 0$.

(5.5) *Proof of* (5.3). The required uniqueness is immediate from (4.4.1), once the existence of an extension is proven. We therefore assume, for contradiction, that any such extension is impossible. Then by (4.4.1), we have a maximal smooth family $\{\varphi_t \mid \sigma < t \leqq \tau\}$ of solutions of (4.1.1) for some $0 \leqq \sigma \in \boldsymbol{R}$. In this proof, we always denote by $t \in \boldsymbol{R}$ an arbitrary number satisfying $\sigma < t \leqq \tau$, and where a constant occurs, it denotes a positive real number which depends neither on t nor $x \in X$. The proof is now divided into two steps.

Step 1: By (1.6.4) and Theorem (5.1),

$$0 \leqq I_t \leqq (n+1)(I_t - J_t) \leqq (n+1)(I_\tau - J_\tau). \tag{5.5.1}$$

We put $F_t := -t\varphi_t - L(0, \varphi_t) + f \in C^\infty(X)_{\boldsymbol{R}}$. Then by (4.1.1),

$$\int_X \omega_0{}^n = \int_X \exp(A(\varphi_t))\omega_0{}^n = \int_X \exp(F_t)\omega_0{}^n.$$

Applying the mean value theorem, we have $F_t(x_t)=0$ for some $x_t \in X$. Therefore for every $x \in X$,

$$\begin{aligned}|F_t(x)| &= |F_t(x) - F_t(x_t)| = |-t(\varphi_t(x) - \varphi_t(x_t)) + f(x) - f(x_t)| \\ &\leqq t(\operatorname{Osc}\varphi_t) + 2\|f\|_{C^0} \\ &\leqq tI_t + 2n(tK_0V_0 + (n-1)\beta(n)\pi^2) + 2\|f\|_{C^0} \quad \text{(cf. (3.6), (4.2)).}\end{aligned}$$

Hence by (5.5.1), there exists a constant K_1 such that

$$\|F_t\|_{C^0} \leqq K_1. \tag{5.5.2}$$

Since $A(\varphi_t) = F_t$, a result of Yau [14] (see also Bourguignon et al. [4; VII]) now asserts that

$$\operatorname{Osc}\varphi_t \leqq K_2 \tag{5.5.3}$$

for some constant K_2. Put $\tilde{\varphi}_t := \varphi_t - \varphi_t(x_t) \in C^\infty(X)_{\boldsymbol{R}}$. In view of

$$0 = F_t(x_t) = -(1+t)\varphi_t(x_t) - L(0, \tilde{\varphi}_t) + f(x_t) \quad \text{(cf. (1.4.3)),}$$

we obtain

$$|\varphi_t(x_t)| \leqq |L(0, \tilde{\varphi}_t)| + \|f\|_{C^0}. \tag{5.5.4}$$

Since $\|\tilde{\varphi}_t\|_{C^0} \leqq K_2$, it follows that

$$\begin{aligned}|L(0, \tilde{\varphi}_t)| &= \left|\int_0^1 \left(\int_X \tilde{\varphi}_t \omega_0(s\tilde{\varphi}_t)^n / V\right) ds\right| \\ &\leqq \int_0^1 \left(\int_X K_2 \omega_0(s\tilde{\varphi}_t)^n / V\right) ds = K_2.\end{aligned} \tag{5.5.5}$$

Hence by (5.5.3) and (5.5.4),

$$\|\varphi_t\|_{C^0} \leqq K_3 \tag{5.5.6}$$

for some constant K_3. Since $L(0, \varphi_t) = \varphi_t(x_t) + L(0, \tilde{\varphi}_t)$, a combination of (5.5.5) and (5.5.6) now provides us with a constant K_4 such that

$$\|-L(0, \varphi_t) + f\|_{C^3} \leqq K_4. \tag{5.5.7}$$

Step 2: Recall that $A(\varphi_t) + t\varphi_t = -L(0, \varphi_t) + f$. By (5.5.6) and (5.5.7), we have constants K_5, K_6, K_7 such that

$$\begin{cases} \|\varphi_t\|_{C^{2,\alpha'}} \leqq K_5 & \text{for all } \alpha' \text{ with } \alpha < \alpha' < 1, \\ K_6\omega_0 \leqq \omega_0(\varphi_t) \leqq K_7\omega_0 & \text{(cf. Aubin [1; pp. 151–154]).} \end{cases}$$

We now choose an arbitrary decreasing sequence $t_j \in (\sigma, \tau]$, $j=1, 2, \cdots$ such that $\lim t_j = \sigma$. Then by Ascoli's theorem, there exists a convergent subsequence of $\{\varphi_{t_j}\}$ in $C^{2,\alpha}$, which leads to a contradiction to the maximality of $\{\varphi_t \mid \sigma < t \leqq \tau\}$.

(5.6) *Proof of* (5.4). Let $\theta \in \mathscr{E}$ be excellent on the Kähler manifold (X, ω_0). Then in view of Remark (4.1.3), there exists a smooth family $\{\varphi_t \mid 1-\varepsilon \leqq t \leqq 1\}$ of solutions of (4.1.1) such that $\theta = \omega_0(\varphi_1)$. By (5.3) above, $\{\varphi_t \mid 1-\varepsilon \leqq t < 1\}$ uniquely extends to *the* smooth one-parameter family $\{\varphi_t \mid 0 \leqq t < 1\}$ of solutions of (4.1.1). Then (i) immediately follows from

$$\varphi_1 = \lim_{t\to 1} \varphi_t \text{ in } C^k(X)_{\boldsymbol{R}} \quad (k \geqq 0),$$

and (ii) from $M(\theta, \omega_0(\varphi_0)) = \mu(\omega_0(\varphi_0)) - \mu(\omega_0(\varphi_1)) \geqq 0$ (cf. (5.1)).

The following theorem, which we do not need later, is of some interest in understanding the $\mathscr{K}$-energy map μ. We therefore give it together with a proof.

Theorem (5.7). *Let* $\mathscr{S} := \{\omega_0(\varphi) \mid \varphi \in \mathscr{H}$ *satisfies* $A(\varphi) = -t\varphi - L(0, \varphi) + f$ *for some* $t \in [0, 1)\}$. *Suppose that* μ *is bounded from below on* $\mathscr{S}$. *Then* φ_0 *uniquely extends to a smooth one-parameter family* $\{\varphi_t \mid 0 \leqq t < 1\}$ *of solutions of* (4.1.1).

Proof. We assume, for contradiction, that there exists a maximal smooth family $\{\varphi_t \mid 0 \leqq t < \sigma\}$ of solutions of (4.1.1) for some $\sigma < 1$. Let $K \in \boldsymbol{R}$ be the infimum of μ on $\mathscr{S}$. Then whenever $0 \leqq t < \sigma$, $\omega^{(t)} := \omega_0(\varphi_t)$ belongs to $\mathscr{S}$ and in particular $\mu(\omega^{(t)}) \geqq K$. For each such t, we infer from Theorem (5.1) that

$$\begin{aligned} I_t - J_t &= \int_0^t \frac{-1}{1-s} \frac{d\mu(\omega^{(s)})}{ds} ds + (I_0 - J_0) \\ &\leqq \int_0^t \frac{-1}{1-\sigma} \frac{d\mu(\omega^{(s)})}{ds} ds + (I_0 - J_0) \\ &\leqq \frac{1}{1-\sigma} (\mu(\omega_0(\varphi_0)) - K) + (I_0 - J_0), \end{aligned}$$

where we used the notation in (5.1). Thus $I_t - J_t$ $(0 \leqq t < \sigma)$ is bounded from above. The rest of the proof is quite similar to (5.5).

§ 6. Lemmas for choosing a good gauge

(6.1) Throughout this section, we use the same notation as in Section 2, and fix an arbitrary G-orbit $\boldsymbol{O}$ in $\mathscr{E}$, assuming $\mathscr{E}\neq\phi$. To each $\theta\in\boldsymbol{O}$, we can uniquely associate a function $\lambda_\theta\in\mathscr{H}$ such that $\theta=\omega_0(\lambda_\theta)$ and that $A(\lambda_\theta)=-\lambda_\theta+f$ (cf. (ii) of (5.2)). Then $\theta\mapsto\lambda_\theta$ defines a bijection between $\boldsymbol{O}$ and $\bar{\boldsymbol{O}}:=\{\lambda_\theta\,|\,\theta\in\boldsymbol{O}\}$. We endow $\bar{\boldsymbol{O}}$ with the topology naturally induced from the $C^{2,\alpha}$-norm of $\mathscr{H}$. This defines a topology on $\boldsymbol{O}$, in terms of which the G-action on $\boldsymbol{O}$ is clearly continuous. Hence our topology on $\boldsymbol{O}$ coincides with the natural topology of the homogeneous space G/K_θ via the identification $\boldsymbol{O}\cong G/K_\theta$ (cf. (2.1)). Recall that for each $\varphi\in H_\theta$, we have the corresponding one-parameter group $y^\varphi_{\theta t}=\exp(tY^\varphi_{\theta\boldsymbol{R}})$, $t\in\boldsymbol{R}$. For simplicity, we put $\theta(t):=(y^\varphi_{\theta t})^*\theta$ and $\lambda(t):=\lambda_{\theta(t)}$. Then by (2.2.4) and $\omega_0(\lambda(t))=\theta(t)$, we have $\dot\lambda(0)=\varphi+C$ for some $C\in\boldsymbol{R}$. On the other hand, differentiating the identity $A(\lambda(t))=-\lambda(t)+f$ with respect to t at $t=0$, we obtain $\Box_\theta(\dot\lambda(0))=-\dot\lambda(0)$. Hence $\int_X\dot\lambda(0)\theta^n=0=\int_X\varphi\theta^n$ and this implies $\dot\lambda(0)=\varphi$. Thus we established the following identification (cf. (2.2.3)):

$$(6.1.1)\qquad \begin{array}{ccccc} T_{\lambda_\theta}(\bar{\boldsymbol{O}}) & \cong & T_\theta(\boldsymbol{O}) & \cong & H_\theta \\ \dot\lambda(0)=\varphi & \longleftrightarrow & \dot\theta(0)=\sqrt{-1}\,\partial\bar\partial\varphi & \longleftrightarrow & \varphi. \end{array}$$

The purpose of this section is to prove the following lemmas:

Lemma (6.2). *The C^∞-function ι defined by*

$$\iota\colon \boldsymbol{O}\longrightarrow\boldsymbol{R},\quad \theta\longmapsto\iota(\theta):=I(\omega_0,\theta)-J(\omega_0,\theta)\quad(\geqq 0)$$

is a proper map. In particular, its minimum is always attained at some point of the orbit $\boldsymbol{O}$.

Lemma (6.3). *Let $\theta\in\boldsymbol{O}$. Then the following are equivalent.*

(i) *θ is a critical point for ι;*

(ii) *$\int_X\lambda_\theta\varphi\theta^n=0$ for all $\varphi\in H_\theta$;*

(iii) *θ is expressible as $\omega_0(\psi)$ for some function $\psi\in\mathscr{H}$ such that $\int_X\varphi\psi\theta^n=0$ for all $\varphi\in H_\theta$.*

Lemma (6.4). *Let $\theta\in\boldsymbol{O}$ be a critical point for ι. Then the Hessian* $(\mathrm{Hess}\,\iota)_\theta$ *of ι at the point θ is given by*

$$(\mathrm{Hess}\,\iota)_\theta(\varphi',\varphi'')=\int_X\left(1+\frac{1}{2}\Box_\theta\lambda_\theta\right)\varphi'\varphi''\theta^n/V$$

for all $\varphi',\varphi''\in H_\theta$ $(\cong T_\theta(\boldsymbol{O}))$.

(6.5) *Proof of* (6.2). By the well-known regularity theorem applied to the equation $A(\psi)=-\psi+f$, the proof is reduced*) to showing

(6.5.1) *given a real number* $r\geqq 0$, *one can always find positive numbers* $K_r, K_r', K_r''\in \boldsymbol{R}$ *such that*

$$\|\psi\|_{C^{2,\alpha}}\leqq K_r \quad \textit{and} \quad K_r'\omega_0\leqq\omega_0(\psi)\leqq K_r''\omega_0$$

hold simultaneously for all $\psi\in\bar{\boldsymbol{O}}$ *satisfying* $|\iota_\theta(\omega_0(\psi))|\leqq r$.

Fix an arbitrary element ψ of $\bar{\boldsymbol{O}}$ as in (6.5.1). Then by (1.6.4), $I(0,\psi)\leqq (n+1)r$. In view of (4.2), we have $\psi\in\mathscr{H}^{(1)}$ and hence by (3.6),

(6.5.2) $\operatorname{Osc}\psi\leqq K$, where $K:=(n+1)r+2n(K_0V_0+(n-1)\beta(n)\pi^2)$.

On the other hand, from $A(\psi)=-\psi+f$, we obtain

$$\int_X\omega_0{}^n=\int_X\exp(A(\psi))\omega_0{}^n=\int_X\exp(-\psi+f)\omega_0{}^n.$$

Therefore by the mean value theorem, there exists a point $x\in X$ such that $\psi(x)=f(x)$. Together with (6.5.2), we have

$$\|\psi\|_{C^0}\leqq K+\|f\|_{C^0}.$$

Then applying standard arguments (cf. Aubin [1; pp. 151–154]), we obtain K_r, K_r', K_r'' as required in (6.5.1).

(6.6) *Proof of* (6.3). (ii) and (iii) are clearly equivalent. To see the equivalence of (i) and (ii), we fix an arbitrary $\varphi\in H_\theta(\cong T_\theta(\boldsymbol{O}))$ with its corresponding one-parameter families $\{\theta(t)\in\boldsymbol{O}\,|\,t\in\boldsymbol{R}\}$ and $\{\lambda(t)\in\bar{\boldsymbol{O}}\,|\,t\in\boldsymbol{R}\}$ as in (6.1). Then in view of (1.6.5),

$$\begin{aligned}\frac{d}{dt}\iota(\theta(t))\Big|_{t=0}&=\frac{d}{dt}(I(0,\lambda(t))-J(0,\lambda(t)))\Big|_{t=0}\\&=-\int_X\lambda_\theta\square_\theta(\dot{\lambda}(0))\theta^n/V=-\int_X\lambda_\theta(\square_\theta\varphi)\theta^n/V \quad (\text{cf. } (6.1.1))\\&=\int_X\lambda_\theta\varphi\theta^n/V \quad (\text{because } \varphi\in H_\theta).\end{aligned}$$

The required equivalence is now straightforward.

(6.7) *Proof of* (6.4). Let $\{\lambda_{s,t}\,|\,(s,t)\in[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]\}$ be a smooth

*) This reduction is easily obtained from the following standard fact: $\boldsymbol{O}$ is a connected component of $\mathscr{E}$ (see Calabi's article "Extremal Kähler metrics II" in "Differential Geometry and Complex Analysis" dedicated to H. E. Rauch, Springer-Verlag, 1985).

family of functions in $\bar{O}$ satisfying the following conditions:

(i) $\lambda_{0,0}=\lambda_\theta$,

(ii) $\left(\frac{\partial}{\partial s}\lambda_{s,t}\right)\Big|_{(s,t)=(0,0)}=\varphi'$,

(iii) $\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)\Big|_{(s,t)=(0,0)}=\varphi''$.

We shall denote $\square_{\lambda_{s,t}}$ (resp. $\omega_0(\lambda_{s,t})$) by $\square_{s,t}$ (resp. $\theta_{s,t}$) for brevity. Differentiating the identity $A(\lambda_{s,t})+\lambda_{s,t}-f=0$ with respect to t, we obtain

$$\square_{s,t}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)+\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)=0. \tag{6.7.1}$$

Further differentiation with respect to s yields

$$-\left\langle\partial\bar{\partial}\left(\frac{\partial}{\partial s}\lambda_{s,t}\right), \partial\bar{\partial}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)\right\rangle_{s,t}+(\square_{s,t}+1)\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right)=0,$$

where we denote $\langle\ ,\ \rangle_{\theta_{s,t}}$ (cf. (2.3)) simply by $\langle\ ,\ \rangle_{s,t}$. Evaluating this at $(s,t)=(0,0)$, we obtain

$$(\square_\theta+1)\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right)\Big|_{(0,0)}=\langle\partial\bar{\partial}\varphi', \partial\bar{\partial}\varphi''\rangle_\theta.$$

Together with (2.3.1), it then follows that

$$\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right)\Big|_{(0,0)}\equiv\langle\partial\varphi', \partial\varphi''\rangle_\theta\equiv\langle\partial\varphi'', \partial\varphi'\rangle_\theta \quad (\text{modulo } H_\theta^{\mathbf{C}}). \tag{6.7.2}$$

We can now finish the proof by the following computation:

$$\begin{aligned}
(\mathrm{Hess}\ \iota)_\theta(\varphi', \varphi'')&=\frac{\partial^2}{\partial s\partial t}(I(0, \lambda_{s,t})-J(0, \lambda_{s,t}))\Big|_{(0,0)}\\
&=-\frac{\partial}{\partial s}\left\{\int_X \lambda_{s,t}\square_{s,t}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)(\theta_{s,t})^n/V\right\}\Big|_{(0,0)}\\
&=\frac{\partial}{\partial s}\left\{\int_X \lambda_{s,t}\left(\frac{\partial}{\partial t}\lambda_{s,t}\right)(\theta_{s,t})^n/V\right\}\Big|_{(0,0)} \quad (\text{cf. } (6.7.1))\\
&=\int_X\left\{\varphi'\varphi''+\lambda_\theta\left(\frac{\partial^2}{\partial s\partial t}\lambda_{s,t}\right)\Big|_{(0,0)}+\lambda_\theta\varphi''(\square_\theta\varphi')\right\}\theta^n/V\\
&=\int_X\{\varphi'\varphi''+2^{-1}(\langle\partial\varphi', \partial\varphi''\rangle_\theta+\langle\partial\varphi'', \partial\varphi'\rangle_\theta)\lambda_\theta-\lambda_\theta\varphi'\varphi''\}\theta^n/V\\
&\qquad (\text{cf. } (6.7.2))
\end{aligned}$$

$$=\int_X \{\varphi'\varphi''+2^{-1}((\square_\theta\varphi')\varphi''+(\square_\theta\varphi'')\varphi'$$
$$+\langle\partial\varphi',\partial\varphi''\rangle_\theta+\langle\partial\varphi'',\partial\varphi'\rangle_\theta)\lambda_\theta\}\theta^n/V$$
$$=\int_X\left\{\varphi'\varphi''+\frac{1}{2}\lambda_\theta\square_\theta(\varphi'\varphi'')\right\}\theta^n/V=\int_X\left(1+\frac{1}{2}\square_\theta\lambda_\theta\right)\varphi'\varphi''\theta^n/V.$$

§ 7. Unfolding the singularity at $t=1$ by a bifurcation technique

We again assume that $\mathscr{E}\neq\phi$ and fix an arbitrary G-orbit $\boldsymbol{O}$ in $\mathscr{E}$. Using the same notation as in Section 6, we fix a critical point θ for the mapping ι: $\boldsymbol{O}\to\boldsymbol{R}$ (cf. (6.2)). The purpose of this section is to find out a good sufficient condition for θ to be excellent (cf. (ii) of (5.2)). Fixing $\alpha\in\boldsymbol{R}$ with $0<\alpha<1$, we set

$$H^{\perp}_{\theta,k}:=\left\{\psi\in C^{k,\alpha}(X)_{\boldsymbol{R}}\,\middle|\,\int_X\varphi\psi\theta^n=0 \text{ for all } \varphi\in H_\theta\right\},\quad k=0,1,2,\cdots.$$

Recall that, corresponding to θ, we have the function $\lambda_\theta\in\mathscr{H}$ with the following properties (cf. (6.1), (6.3)):

(i) $\theta=\omega_0(\lambda_\theta)$,
(ii) $A(\lambda_\theta)=-\lambda_\theta+f$,
(iii) $\lambda_\theta\in H^{\perp}_{\theta,k}$.

Let $k\geqq 2$, and consider the mapping

$$\Phi:\boldsymbol{R}\times C^{k,\alpha}(X)_{\boldsymbol{R}}\longrightarrow C^{k-2,\alpha}(X)_{\boldsymbol{R}},\quad (t,u)\longmapsto\Phi(t,u):=A(u)+tu-f.$$

Note that, by the well-known regularity theorem, any $v\in\mathscr{H}^{k,\alpha}$ (cf. (4.4)) satisfying $\Phi(t,v)=0$ for some t is automatically in $\mathscr{H}$. Let

$$P: C^{0,\alpha}(X)_{\boldsymbol{R}}(\cong H_\theta\oplus H^{\perp}_{\theta,0})\longrightarrow H_\theta$$

be the natural projection to the first factor. For each $u\in C^{k,\alpha}(X)_{\boldsymbol{R}}$, we write

$$u=\lambda_\theta+\varphi+\psi, \tag{7.1.1}$$

with $\varphi:=P(u-\lambda_\theta)\in H_\theta$ and $\psi:=(1-P)(u-\lambda_\theta)\in H^{\perp}_{\theta,k}$. Now the equation

$$\Phi(t,u)=0 \tag{7.1.2}$$

is written in the form

$$P\Phi(t,\lambda_\theta+\varphi+\psi)=0\quad\text{and}\quad\Psi(t,\varphi,\psi)=0,$$

where Ψ: $\boldsymbol{R}\times H_\theta\times H^{\perp}_{\theta,k}\to H^{\perp}_{\theta,k-2}$ is the mapping defined by

$$\Psi(t, \varphi, \psi) := (1-P)\Phi(t, \lambda_\theta+\varphi+\psi) \quad ((t, \varphi, \psi) \in \boldsymbol{R}\times H_\theta \times H_{\theta,k}^\perp).$$

Then $\Psi(1, 0, 0)=0$ and the Fréchet derivative $D_\psi\Psi|_{(1,0,0)}$ of Ψ with respect to ψ at $(t, \varphi, \psi)=(1, 0, 0)$ is

$$H_{\theta,k}^\perp \ni \psi' \longmapsto (D_\psi\Psi)|_{(1,0,0)}(\psi') = (\square_\theta+1)\psi' \in H_{\theta,k-2}^\perp,$$

which is invertible. Therefore the implicit function theorem enables us to obtain a smooth mapping $U \ni (t, \varphi) \mapsto \psi_{t,\varphi} \in H_{\theta,k}^\perp$ of a small neighbourhood U of $(1, 0)$ in $\boldsymbol{R}\times H_\theta$ to the Banach space $H_{\theta,k}^\perp$ such that

(i) $\psi_{1,0}=0$,

(ii) $\|\psi_{t,\varphi}\|_{C^{k,\alpha}} \leqq \delta$ on U for some $\delta>0$, and

(iii) $\Psi(t, \varphi, \psi)=0$ (where $\|\psi\|_{C^{k,\alpha}} \leqq \delta$) is, as an equation in $\psi \in C^{k,\alpha}(X)_{\boldsymbol{R}}$, uniquely solvable in the form $\psi=\psi_{t,\varphi}$ on U.

Differentiating the identity $\Psi(t, \varphi, \psi_{t,\varphi})=0$ at $(t, \varphi)=(1, 0)$, we obtain

$$\text{(7.1.3)} \qquad (\square_\theta+1)\left(\frac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\right) = -\lambda_\theta,$$

$$\text{(7.1.4)} \qquad (D_\varphi\psi_{t,\varphi})|_{(1,0)}(\varphi')=0 \qquad \text{for all } \varphi' \in H_\theta,$$

where $(D_\varphi\psi_{t,\varphi})|_{(1,0)}: H_\theta \to H_{\theta,k}^\perp$ denotes the Fréchet derivative of $\psi_{t,\varphi}$ with respect to φ at the point $(t, \varphi)=(1, 0)$. Then the equation (7.1.2), on a small neighbourhood of λ_θ, reduces to

$$\text{(7.1.5)} \qquad \Phi_0(t, \varphi)=0 \quad (\text{with } u=\lambda_\theta+\varphi+\psi_{t,\varphi}),$$

where we put $\Phi_0(t, \varphi) := P\Phi(t, \lambda_\theta+\varphi+\psi_{t,\varphi})$ for $(t, \varphi) \in U$. Recall that $\Phi(1, u)=0$ for all $u \in \bar{\boldsymbol{O}}$. Hence $\Phi_0=0$ on $\{t=1\}$ and therefore the mapping

$$U|_{\{t\neq 1\}} \ni (t, \varphi) \longmapsto \Phi_1(t, \varphi) := \Phi_0(t, \varphi)/(t-1) \in H_\theta$$

naturally extends to a smooth map: $U \to H_\theta$ (denoted by the same Φ_1) of finite dimensional sets. In view of (7.1.3), we obtain

$$\Phi_1(1, 0) = (\partial\Phi_0/\partial t)(1, 0) = 0.$$

Furthermore, we shall later show that the Fréchet derivative $D_\varphi\Phi_1|_{(1,0)}: H_\theta \to H_\theta$ of Φ_1 with respect to φ at $(t, \varphi)=(1, 0)$ is written in the following form:

Lemma (7.2). *For all $\varphi', \varphi'' \in H_\theta$ (where θ is a critical point of ι),*

$$(D_\varphi\Phi_1|_{(1,0)}(\varphi'), \varphi'')_{L^2(X,\theta)} = \int_X \left(1+\frac{1}{2}\square_\theta\lambda_\theta\right)\varphi'\varphi''\theta^n/n!$$

$$= V_0(\text{Hess } \iota)_\theta(\varphi', \varphi'').$$

Suppose now that $(\text{Hess }\iota)_\theta$: $H_\theta \times H_\theta \to \boldsymbol{R}$ is a nondegenerate bilinear form. Then by this lemma, $D_\varphi \Phi_1|_{(1,0)}$ is invertible, and the implicit function theorem shows that the equation $\Phi_1(t, \varphi)=0$ in φ is uniquely solvable in a neighbourhood of (1, 0) to produce a smooth curve $\{\varphi(t)|1-\varepsilon \leqq t \leqq 1\}$ $(\varepsilon>0)$ in H_θ such that (i) $\varphi(1)=0$ and (ii) $\Phi_1(t, \varphi(t))=0$ $(1-\varepsilon \leqq t \leqq 1)$. Therefore, in view of (7.1.5), we have $\Phi(t, \lambda_\theta+\varphi(t)+\psi_{t,\varphi(t)})=0$ $(1-\varepsilon \leqq t \leqq 1)$, and hence $\{\psi_t := \lambda_\theta+\varphi(t)+\psi_{t,\varphi(t)} \mid 1-\varepsilon \leqq t \leqq 1\}$ is a smooth one-parameter family of solutions of (4.1.2) in $\mathscr{H}$ with $\psi_1=\lambda_\theta$, i.e., θ is excellent on the Kähler manifold (X, ω_0) (cf. (ii) of (5.2)). Thus we obtain:

Theorem (7.3). *Every critical point θ of ι with non-degenerate Hessian is excellent on the Kähler manifold (X, ω_0).*

We shall finally show Lemma (7.2).

(7.4) *Proof of* (7.2). By (7.1.4), using the notation in (2.3), we have

$$D_\varphi \Phi_1|_{(1,0)}(\varphi') = \left(D_\varphi \frac{\partial}{\partial t}\Phi_0\right)\Big|_{(1,0)}(\varphi')$$
$$= \varphi' - P\left\langle \partial\bar\partial\left(\frac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\right), \partial\bar\partial\varphi'\right\rangle_\theta .$$

Hence, it follows that

$$(D_\varphi \Phi_1|_{(1,0)}(\varphi'), \varphi'')_{L^2(X,\theta)}$$
$$= \int_X \left\{\varphi'\varphi'' - \varphi''\left\langle \partial\bar\partial\left(\frac{\partial}{\partial t}\psi_{t,\varphi}|_{(1,0)}\right), \partial\bar\partial\varphi'\right\rangle_\theta\right\}\theta^n/n!$$
$$= \int_X \{\varphi'\varphi'' - (\varphi'\varphi'' - \langle\partial\varphi'', \partial\varphi'\rangle_\theta)\lambda_\theta\}\theta^n/n! \quad \text{(cf. (2.3.2), (7.1.3))}$$
$$= \int_X \{\varphi'\varphi'' + 2^{-1}(\langle\partial\varphi', \partial\varphi''\rangle_\theta + \langle\partial\varphi'', \partial\varphi'\rangle_\theta)\lambda_\theta - \lambda_\theta\varphi'\varphi''\}\theta^n/n! \quad \text{(cf. (6.7.2))}$$
$$= \int_X \left(1+\frac{1}{2}\square_\theta\lambda_\theta\right)\varphi'\varphi''\theta^n/n! \quad \text{(see the end of (6.7))}.$$

Since $V=n!V_0$, this completes the proof (cf. (1.4), (6.4)).

Remark (7.4.1). If $\theta \in \boldsymbol{O}$ is a point where ι attains its minimum (cf. (6.2)), then $(\text{Hess }\iota)_\theta$ is positive semidefinite. In the next section, we shall realize a critical point for ι with positive definite Hessian via a small change of our presently fixed ω_0.

§ 8. Proof of Main Theorem

(8.1) *Proof of* (i) *of Theorem* A. Fix an element $\tilde{\omega}$ of $\mathscr{K}^+$ and a G-orbit $\boldsymbol{O}$ in $\mathscr{E}$ arbitrarily. In this section, we write

$$\omega_0 = \omega_0^{\varepsilon}$$

regarding ω_0 as a function of $\varepsilon \in [0, 1]$. Hence the corresponding f, φ_0, ι, $\mathscr{H}$, $\omega_0(\varphi)$ and $A(\varphi)$ (where $\varphi \in C^{\infty}(X)_{\boldsymbol{R}}$) will be written respectively as f_ε, $\varphi_{0;\varepsilon}$, ι_ε, $\mathscr{H}^{\varepsilon}$, $\omega_0^{\varepsilon}(\varphi)$ and $A^{\varepsilon}(\varphi)$ (see (1.7.1), (4.3.2), (6.2), (1.1), (1.1.1) and (4.1)). We first consider the special case $\varepsilon=0$ and then go to the general situation $\varepsilon>0$.

Case 1: $\varepsilon=0$. Put $\omega_0^0 := R(\tilde{\omega})$. Then $\iota_0\colon \boldsymbol{O}\to\boldsymbol{R}$ takes its minimum at some point θ of $\boldsymbol{O}$ (cf. (6.2)). Corresponding to this θ, there uniquely exists a function $\lambda_{\theta;0} \in \mathscr{H}^0$ such that $\theta=\omega_0^0(\lambda_{\theta;0})$ and that $A^0(\lambda_{\theta;0}) = -\lambda_{\theta;0} + f_0$ (cf. (ii) of (5.2)). Recall that H_θ is Ker $(\Box_\theta + 1)$ in $C^{\infty}(X)_{\boldsymbol{R}}$. Then by (6.3),

(8.1.1) $$\int_X \lambda_{\theta;0}\varphi\theta^n = 0 \qquad \text{for all } \varphi \in H_\theta,$$

and the bilinear form $(\text{Hess }\iota_0)_\theta\colon H_\theta \times H_\theta \to \boldsymbol{R}$ is positive semidefinite.

Case 2: $\varepsilon>0$. In this case, we set $\omega_0^{\varepsilon} := (1-\varepsilon)\omega_0^0 + \varepsilon\theta = \omega_0^0(\varepsilon\lambda_{\theta;0})$. Again by (ii) of (5.2), one obtains a function $\lambda_{\theta;\varepsilon} \in \mathscr{H}^{\varepsilon}$ uniquely determined by the identities $\theta=\omega_0^{\varepsilon}(\lambda_{\theta;\varepsilon})$ and $A^{\varepsilon}(\lambda_{\theta;\varepsilon}) = -\lambda_{\theta;\varepsilon} + f_\varepsilon$. Then in view of $\omega_0^0(\lambda_{\theta;0}) = \theta = \omega_0^0(\lambda_{\theta;\varepsilon} + \varepsilon\lambda_{\theta;0})$, we have

(8.1.2) $$\lambda_{\theta;\varepsilon} = (1-\varepsilon)\lambda_{\theta;0} + C_\varepsilon \qquad \text{for some } C_\varepsilon \in \boldsymbol{R}.$$

Hence $\int_X \lambda_{\theta;\varepsilon}\varphi\theta^n = \int_X (1-\varepsilon)\lambda_{\theta;0}\varphi\theta^n = 0$ if $\varphi \in H_\theta$ (see (8.1.1)). Therefore by (6.3), θ is a critical point for $\iota_\varepsilon\colon \boldsymbol{O}\to\boldsymbol{R}$. Moreover for all $0\neq\varphi \in H_\theta$,

$$\begin{aligned}(\text{Hess }\iota_\varepsilon)_\theta(\varphi,\varphi) &= \int_X \left(1+\frac{1}{2}\Box_\theta\lambda_{\theta;\varepsilon}\right)\varphi^2\theta^n/V \quad \text{(cf. (6.4))}\\ &= (1-\varepsilon)\int_X \left(1+\frac{1}{2}\Box_\theta\lambda_{\theta;0}\right)\varphi^2\theta^n/V + \varepsilon\int_X \varphi^2\theta^n/V \quad \text{(cf. (8.1.2))}\\ &= (1-\varepsilon)(\text{Hess }\iota_0)_\theta(\varphi,\varphi) + \varepsilon\int_X \varphi^2\theta^n/V > 0.\end{aligned}$$

Theorem (7.3) now shows that θ is excellent on the Kähler manifold $(X, \omega_0^{\varepsilon})$. In particular, by (ii) of (5.4),

(8.1.3) $$M(\theta, \omega_0^{\varepsilon}(\varphi_{0;\varepsilon})) \geqq 0.$$

By (4.3.2), $R(\omega_0^\varepsilon(\varphi_{0;\varepsilon}))=\omega_0^\varepsilon$. We also have $R(\tilde{\omega})=\omega_0^0$. Note that $\omega_0^\varepsilon \to \omega_0^0$ in $C^{0,\alpha}$ (as $\varepsilon\downarrow 0$). Then by (1.1.3), $\omega_0^\varepsilon(\varphi_{0;\varepsilon})\to\tilde{\omega}$ in $C^{2,\alpha}$. Let $\varepsilon\downarrow 0$ in (8.1.3). By the continuity of M, we have

$$M(\theta, \tilde{\omega})\geqq 0, \quad \text{i.e.,} \quad \mu^+(\theta)\leqq\mu^+(\tilde{\omega}).$$

Recall that μ^+ is a constant function on $\boldsymbol{O}$ (cf. [9]). Since both $\tilde{\omega}\in\mathscr{K}^+$ and the G-orbit $\boldsymbol{O}$ in $\mathscr{E}$ are arbitrary, it now follows that $\mu^+\colon \mathscr{K}^+\to \boldsymbol{R}$ takes its absolute minimum C on $\mathscr{E}$. Note that $\mathscr{E}$ is the set of all critical points of μ^+ (cf. [9]). Hence

$$\{\omega\in\mathscr{K}^+ \mid \mu^+(\omega)=C\}=\mathscr{E},$$

because, otherwise, at some point $\tilde{\omega}$ of $\mathscr{K}^+$ with $\tilde{\omega}\notin\mathscr{E}$, the function μ^+ would take its critical value C in contradiction to $\tilde{\omega}\notin\mathscr{E}$.

(8.2) *Proof of* (ii) *of Theorem* A. Let $\boldsymbol{O}'$ and $\boldsymbol{O}''$ be arbitrary G-orbits in $\mathscr{E}$. Then from the argument of (8.1) applied to the orbit $\boldsymbol{O}'$, we see the following:

(8.2.1) For a suitable choice of $\omega_0'\in\mathscr{K}$, the function $\iota'\colon \boldsymbol{O}'\ni\omega\mapsto \iota'(\omega):=I(\omega_0', \omega)-J(\omega_0', \omega)\in\boldsymbol{R}$ has a critical point $\theta'\in\boldsymbol{O}'$ with positive definite Hessian.

Recall that the function $\iota''\colon \boldsymbol{O}''\ni\omega\mapsto\iota''(\omega):=I(\omega_0', \omega)-J(\omega_0', \omega)\in\boldsymbol{R}$ takes its minimum at some point $\theta''\in\boldsymbol{O}''$ (cf. (6.2)). We now put $\omega_0^\varepsilon:=(1-\varepsilon)\omega_0'+\varepsilon\theta''$, $(0\leqq\varepsilon\leqq 1)$. Again by the argument of (8.1) applied to $\boldsymbol{O}''$, we have:

(8.2.2) θ'' is excellent on the Kähler manifold $(X, \omega_0^\varepsilon)$ whenever $0<\varepsilon\leqq 1$.

We finally define $\iota_\varepsilon'\colon \boldsymbol{O}'\to\boldsymbol{R}$ by

$$\iota_\varepsilon'(\omega):=I(\omega_0^\varepsilon, \omega)-J(\omega_0^\varepsilon, \omega), \qquad (\omega\in\boldsymbol{O}').$$

Note that ι_ε' converges to ι', say in $C^{2,\alpha}$, as ε tends to zero. Fix a sufficiently small $\varepsilon>0$. Then by (8.2.1), the function ι_ε' takes its local minimum with positive definite Hessian at some point θ_ε' of $\boldsymbol{O}'$ near θ'. In view of Theorem (7.3), one finds that θ_ε' is also excellent on the Kähler manifold $(X, \omega_0^\varepsilon)$. Combining this with (8.2.2), we conclude from (i) of (5.4) that $\theta_\varepsilon'=\theta''$. Thus, $\boldsymbol{O}'=\boldsymbol{O}''$ and the proof is now complete.

Theorem A is valid even when $M\colon \mathscr{K}\times\mathscr{K}\to\boldsymbol{R}$ is replaced by $N\colon \mathscr{K}\times\mathscr{K}\to\boldsymbol{R}$ (cf. (1.5)). We conclude this section by showing:

Corollary (8.3). *Under the same assumption as in Theorem* A, *the mapping*

$$\nu^+ : \mathscr{K}^+ \longrightarrow \boldsymbol{R}, \qquad \omega \longmapsto \nu^+(\omega) := N(\omega_1, \omega)$$

is bounded from below and takes its absolute minimum exactly on $\mathscr{E}$.

Proof. By (1.8.1), $\nu^+(\omega) = \mu^+(\omega) + J(\omega, R(\omega)) - J(\omega_1, R(\omega_1))$ for every $\omega \in \mathscr{K}^+$. Since both μ^+ and $J(\omega, R(\omega))$ $(\omega \in \mathscr{K}^+)$ take their minima exactly on $\mathscr{E}$ (see (1.6.3) and Theorem A), so does ν^+.

§ 9. Proof of Theorems B and C

(9.1) *Proof of Theorem* B. Recall that the natural Riemannian metric on $\mathscr{E}$ is characterized in terms of lengths of smooth paths in $\mathscr{E}$ as follows (cf. [10]):

For every smooth path $\Gamma = \{\tilde{\gamma}_t \mid a \leqq t \leqq b\}$ *in* $\mathscr{E}$, *let* $\{\gamma_t \mid a \leqq t \leqq b\}$ *be the corresponding smooth path in* $\mathscr{H}_0$ (cf. (1.4)) *uniquely determined by* $\omega_0(\gamma_t) = \tilde{\gamma}_t$ $(t \in [a, b])$. *Then the length* $\mathscr{L}(\Gamma)$ *of the path* Γ *in terms of the metric of* $\mathscr{E}$ *is defined by*

$$\mathscr{L}(\Gamma) := \int_a^b \left(\int_X (\dot{\gamma}_t)^2 \tilde{\gamma}_t{}^n / V \right)^{1/2} dt.$$

In view of Theorem A, the proof is reduced to showing that $\mathrm{Aut}(X)$ acts isometrically on $\mathscr{E}$. Hence it suffices to show

$$\mathscr{L}(g^*\Gamma) = \mathscr{L}(\Gamma) \qquad (g \in \mathrm{Aut}(X))$$

for every smooth path $\Gamma = \{\tilde{\gamma}_t \mid a \leqq t \leqq b\}$ in $\mathscr{E}$. Then even if $g \notin \mathrm{Aut}^0(X)$, the same proof as in [10] goes through as follows:

Let φ_g be the function in $\mathscr{H}$ uniquely determined by the properties $g^*\omega_0 = \omega_0(\varphi_g)$ and $\varphi_g + g^*\gamma_a \in \mathscr{H}_0$. We put $\eta_t := \varphi_g + g^*\gamma_t$ $(a \leqq t \leqq b)$. Then $g^*\tilde{\gamma}_t = \omega_0(\eta_t)$ and $\eta_t \in \mathscr{H}_0$ for all t. Hence

$$\begin{aligned} \mathscr{L}(g^*\Gamma) &= \int_a^b \left(\int_X (\dot{\eta}_t)^2 \omega_0(\eta_t)^n / V \right)^{1/2} dt \\ &= \int_a^b \left(\int_X (g^*\dot{\gamma}_t)^2 (g^*\tilde{\gamma}_t)^n / V \right)^{1/2} dt = \mathscr{L}(\Gamma). \end{aligned}$$

(9.2) *Proof of Theorem* C. By Theorem B, $\mathscr{E}$ is isometric to the Riemannian symmetric space G/K without compact factors. In particular, $\mathscr{E}$ is a simply connected Riemannian manifold with nonpositive sectional curvature. Since the compact group H acts isometrically on $\mathscr{E}$, it always has a fixed point in $\mathscr{E}$.

Remark (9.3). If X admits no nonzero holomorphic vector fields, then Theorems A and C assert the following:

Einstein Kähler metrics on X are, if any, unique up to constant multiple. Moreover, they are invariant under the action of Aut (X).

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