On Uniformizations of Orbifolds

Mitsuyoshi Kato*

§ 1. Introduction

By a *transformation group*, we shall mean a pair (G, M) of a connected paracompact n-manifold M and a group G of topological transformations of M. For a second transformation group (H, N), by a *covering map* $(\phi, f): (G, M) \to (H, N)$, we shall mean a pair of an epimorphism $\phi: G \to H$ and a regular covering map $f: M \to N$ which is ϕ-equivariant, i.e., $f(g \cdot z) = \phi(g) \cdot f(z)$ for $(g, z) \in G \times M$. In particular, if f is a homeomorphism, then (ϕ, f) is called an *isomorphism* and (G, M) and (H, N) are said to be *isomorphic*, written $(G, M) \cong (H, N)$. We shall say that (G, M) is a *proper transformation group*, if a track of the action mapping $G \times M \to M \times M; (g, z) \mapsto (z, g \cdot z)$ is proper, where G has the discrete topology. In other words, the orbit space $X = \{G \cdot z \mid z \in M\}$ is Hausdorff and (G, M) is discontinuous, i.e., for each point z of M the isotropy subgroup $G_z = \{g \in G \mid g(z) = z\}$ of G at z is finite and there is a neighborhood U_z, referred to as a *G-equivariant neighborhood*, of z in M such that $g \cdot U_z = U_z$ for $g \in G_z$ and $g \cdot U_z \cap U_z = \phi$ for $g \in G - G_z$. A proper transformation group (G, M) is *locally smooth*, if for each point z of M, there is a G-equivariant neighborhood M_z of z in M such that (G_z, M_z) is isomorphic with a finite orthogonal transformation group (G'_z, E_z), i.e., E_z is euclidean n-space R^n or closed half n-space H_n and G'_z is a finite subgroup of $O(n)$.

In this paper, we concern ourselves with *the classification of locally smooth proper transformation groups*. Thus by a transformation group (G, M) we shall mean a locally smooth proper one, unless otherwise mentioned.

Now for a transformation group (G, M), we have the orbit space X, which is a connected separable Hausdorff space, and a function $b: X \to N$

defined by $b(x) = \sharp G_z$ for $x \in X$ and for $z \in M$ such that $G \cdot z = x$.

Throughout this paper, we denote $G \backslash M = (X, b)$ and $|G \backslash M| = X$.

Conversely, suppose that we are given a pair (X, b) referred to as a *b-space*, of a connected separable Hausdorff space X and a function b: $X \to N$. A homeomorphism $h: X \to X'$ is an *isomorphism* $h: (X, b) \to (X', b')$, if $b = b' \circ h$. In order to classify our transformation groups, a basic problem is

Uniformization Problem. *Under what condition on a b-space* (X, b), *does there exist a transformation group* (G, M) *such that* $G \backslash M = (X, b)$?

If this holds, then (G, M) is called a *uniformization* of (X, b) and (X, b) is said to be *uniformizable* or *good* (á là Thurston). Otherwise, (X, b) is said to be *bad*. An obviously necessary condition for (X, b) to be good is

Local Uniformizability Condition. *For each point* x *of* X, *there is an open neighborhood* X_x *of* x *in* X *such that* $(X_x, b \,|\, X_x)$ *can be uniformized by a finite orthogonal transformation group* (G_x, E_x); $G_x \backslash E_x = (X_x, b \,|\, X_x)$, *where* $E_x = \mathbf{R}^n$ *or* H_n *and* G_x *is a finite subgroup of* $O(n)$.

Following Thurston [14], we shall refer to a locally uniformizable *b*-space as an *n-orbifold* or more precisely as a *locally smooth n-orbifold*.

Our Theorems 1 and 2 answer the uniformization problem for orbifolds. In fact, we give a necessary and sufficient condition for an orbifold to be uniformizable. To do this, we define the *branch set* of (X, b) as $\Sigma X (= \Sigma(X, b)) = \{x \in X \,|\, b(x) \geq 2\}$ and the *stratification* $\mathscr{S}$ of (X, b) in an obvious manner (see § 2). An *n-orbifold* (X, b) is said to be an *n-branchfold*, if $\dim \Sigma X \leq n - 2$. Then the necessary and sufficient condition, called *completeness condition*, for an *n*-branchfold (X, b) to be uniformizable will be described by a property of homomorphisms from local fundamental groups $\pi_1(X_x - \Sigma X_x)$ $(x \in \Sigma X)$ into the fundamental group $\pi_1(X - \Sigma X)$ concerning the normal closures of some loops surrounding $(n-2)$-strata of the stratification $\mathscr{S}$ of $(X_x, b \,|\, X_x)$ and $(X, b \,|\, X)$, respectively.

In case $\dim \Sigma X = n - 1$, a double trick (Theorem 2) reduces the problem to Theorem 1. It is known that there is a one to one correspondence of isomorphism classes of good *n*-orbifolds and isomorphism classes of transformation groups on simply connected *n*-manifolds, called *the universal transformation groups*. Actually, the universal transformation group $(\hat{G}, \hat{M})$ uniformizing (X, b) is the universal uniformization of (X, b) in the sense that $(\hat{G}, \hat{M})$ covers any uniformization of (X, b). (Refer to [14], § 13). It is a formal consequence of Theorem 1 that *there is a one to one correspondence of isomorphism classes of uniformizations of a branchfold*

(X, b) *and some normal subgroups, called b-complete normal subgroups, of* $\pi_1(X - \Sigma X)$.

On one hand, Theorem 1 answers Fox's question ([5], p. 252, line 3rd from the bottom) in the case of locally smooth regular branched coverings.

On the other hand, Theorems 1 and 2 are regarded as a topological version of the generalized Poincaré's Theorem for fundamental polygons formulated and proved in our previous paper [6]. (Refer to also [14], § 13, Proposition 13.3.2).

From our topological view points, it is also possible to study two basic transformation groups; (topological) reflection groups and (topological) rotation groups. We introduce the notions of reflection orbifolds and rotation orbifolds to show that the orbifolds of reflection groups and rotation groups are characterized as regular reflection orbifolds (Theorem 3) and as good rotation orbifolds (Theorem 4), respectively.

Meanwhile, Thurston conjectured that a bad 3-orbifold should contain some bad 2-suborbifold. This conjecture is generalized in dimension $n \geqq 3$ as the bad orbifold conjecture.

As an implication of Theorem 3, we prove the bad orbifold conjecture for reflection n-orbifolds and their doubles providing that $n \neq 4, 5$. On the contrary to this, we construct some rotation n-orbifolds for $n \geqq 4$ as counterexamples to the bad orbifold conjecture, (Theorem 6). The last examples also show that the completeness condition can not be weakened by a simpler condition, called niceness condition, in the higher dimension $n \geqq 4$, though it can be done in dimension $n = 3$ (Theorem 5).

It would be a deep problem in the combinatorial group theory to detect, in general, a given 3-branchfold being complete. Actually the affirmative answer to the bad orbifold conjecture for the doubles of reflection orbifolds is based on some combinatorial results on Coxeter systems furnished in Bourbaki ([2], Chap. IV, V)

In Section 5, we add some remarks which follow from the consideration of orbifolds and our theorems.

At this point, we would like to remark that our completeness condition can be formulated for locally triangulable (not necessary locally smooth) orbifolds so that Theorems 1 and 2 hold for their locally triangulable uniformizations, and that the uniformization problem of orbifolds with various structures over locally triangulable orbifolds is essentially reduced to the topological one presented here.

§ 2. Statements of Theorems

Let E denote $\mathbf{R}^n$ or $H_n = \mathbf{R}^{n-1} \times [0, \infty[$, S^{n-1} the unit $(n-1)$-sphere in $\mathbf{R}^n$ and $\dot{E} = E \cap S^{n-1}$. If (G, E) is an orthogonal transformation group,

i.e. $G \subset O(n)$, then $\dot{E}$ is obviously invariant under G and (G, E) is an open cone of $(G, \dot{E})$ in the sense that $g \cdot (t\ u) = t(g \cdot u)$ for all $(g, u) \in G \times E$ and $t \geq 0$. Recall that a b-space (X, b) is an n-orbifold, if for each point x of X, there is an open neighborhood X_x of x in X such that $(X_x, b | X_x) = G_x \backslash E_x$ for some finite orthogonal transformation group (G_x, E_x). Thus X_x is an open cone of $\dot{X}_x = |G_x \backslash \dot{E}_x|$ from the center x. The quotient map $q_x : E_x \to X_x$ or $(G_x, E_x) \to (X_x, q | X_x)$ is called an *orthogonal local chart* of (X, b) at x. In case $E_x = H_n$ and $q_x^{-1}(x) \subset \partial H_n = R^{n-1} \times \{0\}$, then x is called a *boundary point* of (X, b) and the set of boundary points of (X, b) is called the boundary of (X, b), written $\partial(X, b)$ or $\partial^* X$.

For a b-space (X, b), a stratification $\mathscr{S}$, which consists of connected manifolds (possibly with boundary), of X is a *stratification of* (X, b), if the following conditions are satisfied:

(1) b is constant on each stratum of $\mathscr{S}$ and

(2)   if C and D are distinct strata of $\mathscr{S}$ such that $C \subset \bar{D}$, then $b(C) > b(D)$.

If (X, b) is an orbifold, then since for each point x of X, there is a unique stratification of $(X_x, b | X_x) = G_x \backslash E_x$ which is stratified, we have a unique stratification of (X, b).

Thus we may speak of the stratification of an orbifold (X, b). For the stratification $\mathscr{S}$ of (X, b), let $\{A_i | i \in I\}$ and $\{B_j | j \in J\}$ be the sets of all $(n-1)$-strata and all $(n-2)$-strata of $\mathscr{S}$, respectively. Then the closure of $\bigcup_{i \in I} A_i$ in X is called the mirror boundary of (X, b) and denoted by $\partial_* X$. We define $\partial X = \partial^* X \cup \partial_* X$. For the branch set $\Sigma X = \{x \in X | b(x) \geq 2\}$ of (X, b), we put $X_0 = X - \Sigma X$. Then X_0 is clearly the unique top dimensional stratum of $\mathscr{S}$. Let $\Sigma' X = \Sigma X - \bigcup_{i \in I} A_i - \bigcup_{j \in J} B_j$ and $X' = X - \Sigma' X$. It is also clear that X' is a connected n-manifold and each B_j $(j \in J)$ is a locally flat $(n-2)$-submanifold of either X' or $\partial_* X' = \partial_* X \cap X'$. An n-orbifold (X, b) is said to be an *n-branchfold*, if $\dim \Sigma X \leq n-2$, i.e., $\partial_* X = \phi$. The reason why we call so is that if a uniformization (G, M) of (X, b) exists, then the quotient map $q: M \to X$ is nothing but a regular branched covering with deck transformation group G in the sense of Fox [5], i.e., $q_0 = q | M_0 : M_0 \to X_0$ is a regular covering which is associated with a normal subgroup K of $\pi_1(X_0)$ such that $\pi_1(X_0)/K = G$, where $M_0 = M - q^{-1}(\Sigma X)$. In this sense, we shall say that (G, M) is associated with $K \subset \pi_1(X_0)$.

Conversely, according to Fox, for a normal subgroup K of $\pi_1(X_0)$, there is a unique regular branched covering $q: M \to X$ with deck transformation group $G = \pi_1(X_0)/K$ as a completion of a covering spread $q_0: M_0 \to X_0 \subset X$ which is associated with K, (refer to [ibid], Existence, Extension and Uniqueness Theorems). Note that $\pi_1(X_0)$ is countable, because X_0 is a connected n-manifold which has always the homotopy type of a locally

finite simplicial complex, refer to [7]. Thus M is a separable Hausdorff space. However, it should be noted that, in general, M often fails to be a manifold. We shall refer to such a pair (G, M) as the *uniformization of* $(X, \Sigma X)$ *associated with* K. In order to state the completeness condition for (X, b), we consider of the fundamental group $H = \pi_1(X_0)$ of $X_0 = X - \Sigma X$ up to inner automorphism so that we are free from the choice of the base points and we specify a normal loop $\mu_j \in H$ to each $(n-2)$-stratum B_j $(j \in J)$ in X_0; μ_j is a boundary loop of a disk in X which meets ΣX transversally at exactly one point of B_j. Putting $b_j = b(B_j)$ $(j \in J)$ and $\mu^b = \{\mu_j^{b_j} \mid j \in J\}$, let $H[\mu^b]$ be the normal closure of μ^b in H. In the same manner, for each $x \in \Sigma X$, we have $X_{0,x} = X_x \cap X_0$, $\dot{X}_{0,x} = \dot{X}_x \cap X_0$, and $H_x = \pi_1(X_{0,x}) \cong \pi_1(\dot{X}_{0,x})$, and define μ_x^b for $(X_x, b \mid X_x)$ and $H_x[\mu_x^b]$. Let $i_x : H_x \to H$ be a homomorphism induced by an inclusion map $X_{0,x} \hookrightarrow X_0$.

Definition. We shall say that a subgroup K of H is *b-complete*, if for each $x \in \Sigma X$, $i_x^{-1}(K) = H_x[\mu_x^b]$. A branchfold (X, b) is said to be *complete*, if $H[\mu^b]$ is b-complete in H.

Remark. If a normal subgroup K of H is b-complete, then $K \supset H[\mu^b]$. Let $\eta_x = \eta_x(K) : H_x \to H \to H/K = G$ be a composition of the natural homomorphisms. Then K is b-complete if and only if $\mathrm{Ker}\ \eta_x = H_x[\mu_x^b]$. If $\dim \Sigma X \leq n - 3$, then (X, b) is complete if and only if $i_x : H_x \to H$ is injective for $x \in \Sigma X$.

Theorem 1 (Uniformization Theorem). *Let (X, b) be an n-branchfold. Then (X, b) is uniformizable if and only if (X, b) is complete. To be explicit, if (X, b) is complete, then the uniformization $(\hat{G}, \hat{M})$ of $(X, \Sigma X)$ associated with $H[\mu^b]$ is actually the universal uniformization of (X, b). Moreover, for a normal subgroup K of H, there is the uniformization (G, N) of (X, b) associated with K if any only if K is b-complete. In this case, we have that $\pi_1(N) = K/H[\mu^b]$ and that a covering $f_0 : \hat{M}_0 \to N_0$ associated with $H[\mu^b] \subset K$ gives rise to a covering $(\phi, f) : (\hat{G}, \hat{M}) \to (G, N)$, where $\phi : \hat{G} = H/H[\mu^b] \to G = H/K$ is a natural epimorphism.*

Remark. Let $h : (X, b) \to (X, b)$ be an automorphism of a good branchfold (X, b). Since h induces an automorphism of H making $H[\mu^b]$ invariant, it follows that h can be lifted to an automorphism $\hat{h}$ of the universal uniformization $(\hat{G}, \hat{M})$ of (X, b).

Next we consider the uniformization problem of an n-orbifold (X, b) with mirror boundary $\partial_* X \neq \phi$, i.e., $\dim \Sigma X = n - 1$. Let $D_*(X)$ be a double of X along the mirror $\partial_* X$, i.e.

$$D_*(X) = X \times \{0\} \cup X \times \{1\}/(x, 0) \equiv (x, 1) \text{ for } x \in \partial_* X.$$

Let $t \colon D_*(X) \to D_*(X)$ be an involution of $D_*(X)$, called the *reflection of $D_*(X)$*, which is defined by $t(x, 0) = (x, 1)$ and $t(x, 1) = (x, 0)$ for all $x \in X$. Thus we have that $D_*(X) = (X) \cup_{\partial_* X} t(X)$. We define a function $D_*(b) \colon D_*(X) \to N$ by setting

$$D_*(b)(x, \varepsilon) = \begin{cases} \dfrac{b(x)}{2}, & \text{if } x \in \partial_* X \text{ and} \\ b(x), & \text{otherwise,} \end{cases}$$

where $\varepsilon = 0$ or 1.

Since $b(A_i) = 2$ for $i \in I$, if $x \in \bar{A}_i$, then $b(A_i)$ divides $b(x)$ so that $D_*(b)(x) \in N$.

A *b-space* $D_*(X, b) = (D_*(X), D_*(b))$ is referred to as the *mirror double* of (X, b).

We can reduce the uniformization problem in this case to Theorem 1 as follows:

Theorem 2 (Double trick). *Let (X, b) be an n-orbifold with $\partial_* X \neq \phi$. Then (X, b) is uniformizable if and only if the mirror double $D_*(X, b)$ of (X, b) is uniformizable. In fact, $D_*(X, b)$ is always an n-branchfold. Moreover, if $(\hat{G}^+, \hat{M})$ and $(\hat{G}, \tilde{M})$ are the universal transformation groups with $\hat{G}^+ \backslash \hat{M} = D_*(X, b)$ and $\hat{G} \backslash \tilde{M} = (X, b)$, then we may identify $\tilde{M}$ with $\hat{M}$ and $\hat{G}$ with the semi-direct product of $\hat{G}^+$ by Z_2 generated by the reflection t of $D_*(X)$.*

For an orbifold (X, b), $(\{A_i \mid i \in I\}, \{(B_j, b_j) \mid j \in J\})$ is referred to as *signature* of (X, b), where $b_j = b(B_j)$ $(j \in J)$. Compare with signature of a Fuchsian group (for example, see Maskit [9]). In view of Theorems 1 and 2, if $(\bigcup_i A_i) \cup (\bigcup_j B_j)$ is dense in ΣX, then the universal uniformization $(\hat{G}, \hat{M})$ of (X, b) is completely determined by the signature of (X, b).

By Theorem 2, we may think of an orbifold (X, b) with $\partial_* X \neq \phi$ as a branchfold $D_*(X, b)$ together with the reflection t.

On the other hand, let (X, b) be an orbifold with non-empty boundary $\partial^* X$. Then we have a double $D^*(X, b) = (D^*(X), D^*(b))$ of (X, b) along the boundary $\partial^* X$ of (X, b) by setting $D^*(X) = X \cup_{\partial^* X} X$ and $D^*(b)(x) = b(x)$ for all $x \in D^*(X)$. In the same manner as Theorem 2, we may think of an orbifold (X, b) with boundary $\partial^* X \neq \phi$ as an orbifold $D^*(X, b)$ without boundary together with the reflection t^* of $D^*(X)$ along the boundary $\partial^* X$.

In view of this in the following we are concerned with orbifolds without boundary.

Let (X, b) be an n-orbifold without boundary; $\partial^* X = \phi$. We shall say that a subspace Y of X gives rise to an m-*suborbifold* (Y, c) of (X, b), if for each point x of Y, there is an orthogonal local chart $q_x : (G_x, \mathbf{R}^n) \to (X_x, b \mid X_x)$ and a linear m-subspace F_x of $\mathbf{R}^n$ so that for the stabilizer $H_x \subset G_x$ of F_x, we have that $H'_x \backslash F_x = (Y_x, c \mid Y_x)$, where $Y_x = X_x \cap Y$ and H'_x is the effective quotient of H_x for F_x.

In case $m = 2$, then (Y, c) or Y itself is called a *surface* of (X, b). The following conjecture is a generalization of Thurston's conjecture.
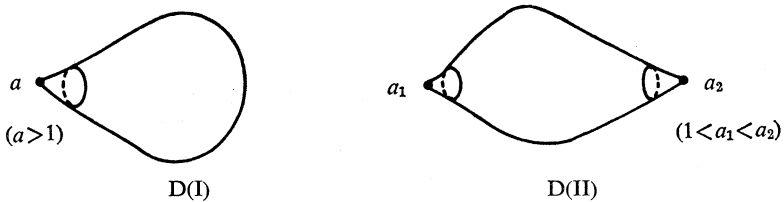
Bad orbifold conjecture (Thurston [14], § 13, p. 13.35). *An n-orbifold* $(n \geq 3)$ *is bad if and only if it contains a bad surface.* (*The proof of "if part" is easy.*)

All bad 2-orbifolds are illustrated as follows:

Figure 1

(I)   (II)
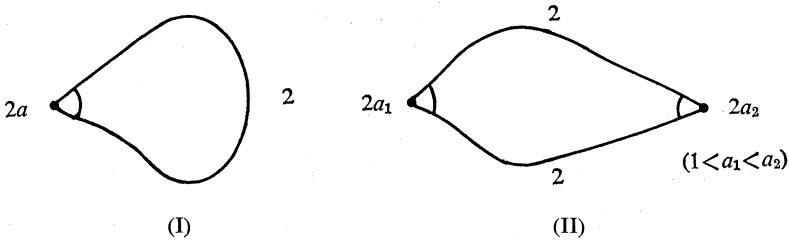
and doubles of these by the mirror;

D(I)   D(II)

where the indicated numbers such as 2, $2a$ and a stand for the values of b at the vertices or the edges.

We examine the bad orbifold conjecture for two basic transformation groups; reflection groups and rotation groups.

Let (G, M) be an n-transformation group. A homeomorphism $r : M \to M$ is a *reflection*, if the fixed point set $F(r)$ of r disconnects M, i.e., $M - F(r)$ is not connected. (Compare with [13]).

Remark. It is not hard to see that if $r \in G$ is a reflection, then

(i) r has order 2,

(ii) $M - F(r)$ has exactly two connected components which are permuted by r, and

(iii)   each connected component of $F(r)$ is an $(n-1)$-manifold.

Definition. We shall say that (G, M) is a *reflection group* or a *reflection uniformization* of an orbifold $(X, b) = G \backslash M$, if G is generated by reflections.

For a reflection group (G, M), let $\mathscr{R}$ be the set of all reflections of G, X the closure of a connected component of $M - \bigcup_{r \in \mathscr{R}} F(r)$,

$$S = \{s \in \mathscr{R} \mid \dim F(s) \cap X = n - 1\} \quad \text{and} \quad S_x = \{s \in S \mid x \in F(s)\}$$

for $x \in X$.   It is known that

(i) $|G \backslash M| = X$, and

(ii) (G, S) and $(G_x, S_x)(x \in X)$ are Coxeter systems.

See Straume ([13], Theorem 1), Davis [15].   Thus (G, M) is completely determined by a system (S, X), called *fundamental system* of (G, M).

Definition. We shall say that an n-orbifold (X, b) is a *reflection orbifold*, if X is an n-manifold and $\partial_* X = \Sigma X$.

A reflection n-orbifold (X, b) is said to be *regular*, if for each $(n-2)$-stratum B of the stratification $\mathscr{S}$ of (X, b) there are distinct $(n-1)$-strata A_k and A_l of $\mathscr{S}$ such that $B \subset \bar{A}_k \cap \bar{A}_l$ and the value $b(B)$, written b_{kl}, depends only on A_k and A_l.

Let (X, b) be a regular reflection orbifold.   Putting

$$K = \{(i, j) \in I \times I \mid \dim \bar{A}_i \cap \bar{A}_j = n - 2 \text{ and } i < j\},$$

we have that each number $b_{ij}((i, j) \in K)$ is an even number.   Thus we have an $I \times I$ Coxeter matrix $(a_{ij})_K$ which is supported by $\{b_{ij}/2 \mid (i, j) \in K\}$ in the following sense: $a_{ii} = 1$, if $i \in I$, $2 \leq a_{ij} = a_{ji} = b_{ij}/2 < \infty$, if $(i, j) \in K$ and $a_{ij} = \infty$, otherwise.

Let $\tilde{G}$ be the Coxeter group determined by $(a_{ij})_K$, i.e.,

$$\tilde{G} = \langle S = \{s_i \mid i \in I\} : \{(s_i s_j)^{a_{ij}} \mid (i. j) \in K\} \cup \{s_i^2 \mid i \in I\} \rangle.$$

Then $(\tilde{G}, S)$ is a Coxeter system.   Refer to Bourbaki ([2], Chap. IV and Chap. V § 4, p. 92, Corollaire).*

---

* In [2], only a Coxeter system with $\sharp S < \infty$ is discussed. However, as pointed out by Iwahori to the author, the arguments there equally work for a Coxeter system with $\sharp S = \infty$

For $x \in \Sigma X$, let $I_x = \{i \in I \mid x \in \bar{A}_i\}$, $S_x = \{s_i \in S \mid i \in I_x\}$ and let $\tilde{G}_x$ be the subgroup of $\tilde{G}$ generated by S_x.   Following Koszul ([8], Chap III, § 3), we construct the formal uniformization $(\tilde{G}, \tilde{M})$ of a regular reflection orbifold (X, b) as follows: Let $\tilde{M}$ be an identification space obtained from $\tilde{G} \times X$ by an equivalence relation; $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1} \circ h \in \tilde{G}_x$, where $\tilde{G}$ has the discrete topology. Then $\tilde{G}$ acts on $\tilde{M}$ by left translation; $h \cdot [g, x] = [h \cdot g, x]$ for $h \in \tilde{G}$ and $[g, x] \in \tilde{M}$, where $[g, x]$ stands for the equivalence class of $(g, x) \in \tilde{G} \times X$ in $\tilde{M}$. We shall refer to $(\tilde{G}, \tilde{M})$ as the formal uniformization of a regular reflection orbifold (X, b).

Theorem 3.   *A reflection orbifold is uniformized by a reflection group if and only if it is regular. To be explicit, the formal uniformization $(\tilde{G}, \tilde{M})$ of a regular reflection orbifold (X, b) is the reflection uniformization of (X, b) with fundamental system (S, X) that is universal among reflection uniformization (G, M) of (X, b) with fundamental system (S, X) in the following sense;   There is a covering $(\tilde{G}, \tilde{M}) \to (G, M)$, where $\tilde{G} \to G$ is an epimorphism between Coxeter systems $(\tilde{G}, S)$ and (G, S).   Moreover, if X is 1-connected, then $\tilde{M}$ is 1-connected so that $(\tilde{G}, \tilde{M})$ is universal and hence any uniformization of (X, b) is a reflection uniformization.*

As an implication of Theorem 3, we have

Corollary to Theorem 3.   *The bad orbifold conjecture holds for reflection n-orbifolds and their doubles, provided that $n \neq 4, 5$.*

Remark.   (i)   The lacks of the proof in the cases $n = 4, 5$ are due to the lacks of the locally smooth Dehn's lemma for a topological manifold of dimension $n = 4, 5$.

(ii)   Note that there is a good reflection orbifold which is not regular. To get such an example, puncture the disk of the bad 2-orbifold with mirror boundary ((I) or (II) in Figure 1).

Definition.   A transformation group (G, M) is *orientable*, if M is orientable and G is orientation preserving.   In this case, an element ρ of G is a *rotation*, if the fixed point set $F(\rho)$ of ρ has codimension 2.

Remark.   A rotation $\rho \in G$ is cyclic transformation of M.   But it should be noted that $F(\rho)$ has possibly a component of dimension less than $n - 2$.   For example, a fixed point free cyclic transformation on the Hopf-fibration of S^3 is naturally extended to a rotation of CP^2 whose fixed point set consists of CP^1 and a point.

Definition.   We shall say that an orientable transformation group (G, M) is a *rotation group* or a *rotation uniformization* of $(X, b) = G \backslash M$,

if G is generated by rotations.   We shall say that a branchfold (X, b) of signature $\{(B_j, b_j) \mid j \in J\}$ is a *rotation orbifold*, if X is orientable, i.e., $X_0 = X - \Sigma X$ is orientable, and an inclusion map $X_0 \hookrightarrow X$ induces an isomorphism $H/H[\mu] \cong \pi_1(X)$, where $H[\mu]$ is the normal closure of $\mu = \{\mu_j \mid j \in J\}$ in H.

Remark.   The condition $\pi_1(X) \cong H/H[\mu]$ is equivalent to saying that an inclusion map $X' = X - \Sigma' X \hookrightarrow X$ induces an isomorphism $\pi_1(X') \cong \pi_1(X)$.   Because we have that $\pi_1(X') \cong H/H[\mu]$, since X' is a manifold and each B_j ($j \in J$) is a locally smooth $(n-2)$ submanifold of X.   Thus if X is an orientable manifold, then a branchfold (X, b) is always a rotation orbifold.   But by the example in the remark above, the converse is not true.   In particular, the double $D_*(X, b)$ of a reflection orbifold (X, b) is a rotation orbifold, provided that X is orientable.

Theorem 4.   *Let (X, b) be an n-orbifold.   If (X, b) is uniformized by a rotation group, then (X, b) is a good rotation orbifold.   Conversely if (X, b) is a good rotation orbifold with $\pi_1(X) = \{1\}$, then the universal uniformization of (X, b) is a rotation uniformization and hence any uniformization of it is a rotation uniformization.*

Definition.   A branchfold (X, b) with signature $\{(B_j, b_j) \mid j \in J\}$ is said to be *nice*, if the image ρ_j of each $\mu_j \in H = (X - \Sigma X)$ ($j \in J$) in $G = H/H[\mu^b]$ has order exactly b_j.

Of course, good branchfolds are nice and the converse is obviously true for 2-branchfolds.   Moreover, for 3-branchfolds we have the following.

Theorem 5.   *Let (X, b) be a 3-branchfold.*
(1)   Then (X, b) is good if and only if it is nice.
(2)   A 3-branchfold (X, b) is a rotation orbifold if and only if X is an orientable 3-manifold.
(3)   If (X, b) is good, then for each point x of ΣX, $i_x: H_x = \pi_1(X_{0,x}) \to H$ is injective so that, when H_x is regarded as a subgroup of H, $H[\mu^b] \cap H_x = H_x[\mu_x^b]$.

Finally we give counterexamples to the higher dimensional bad orbifold conjecture.

Theorem 6.   *There are bad rotation 4-orbifolds $S^4(b_1, b_2, b_3)$ on a 4-sphere S^4 for integers $b_1, b_2, b_3 \geq 2$ such that $1/b_1 + 1/b_2 + 1/b_3 > 1$ that are nice and do not contain any bad surface.*

§ 3. The proof of Theorems 1 and 2

Proof of Theorem 1. The proof is induction on the dimension n. In case $n=1$, we have nothing to prove. Let (X, b) be an n-branchfold. We will show that the uniformization (G, M) of $(X, \Sigma X)$ associated with a b-complete normal subgroup $K \subset H$ is a uniformization of (X, b). Let $q: M \to X$ be the quotient map, $M_0 = M - q^{-1}(\Sigma X)$, $x \in \Sigma X$, $y \in q^{-1}(x)$, M_y the connected component of $q^{-1}(X_x)$ containing y and $M_{0,y} = M_y \cap M_0$. By ([5], Lemma, p. 247), $M_{0,y}$ is a connected component of $q^{-1}(X_{0,x})$. Let $(G)_y$ be the subgroup of G stabilizing $M_{0,y}$ and hence M_y. Then $(G)_y$ is a conjugate of $\eta_x(H_x)$ in $G = H/K$, where $\eta_x = \eta_x(K): H_x \to H \to H/K$ is a natural homomorphism. Thus $((G)_y, M_y)$ is the uniformization of $(X_x, \Sigma X_x)$ associated with $\operatorname{Ker} \eta_x = H_x[\mu_x^b]$.

Let $X_x^0 = X_x - \{x\}$, $M_y^0 = M_y \cap q^{-1}(X_x^0)$ and $\dot{M}_y = M_y \cap q^{-1}(\dot{X}_x)$. Since $(X_x^0, X_{0,x})$ is homeomorphic with $(\dot{X}_x, \dot{X}_{0,x}) \times \mathbf{R}$, it follows that $((G_y), \dot{M}_y)$ is also the uniformization of $(\dot{X}_x, \Sigma \dot{X}_x)$, associated with $\operatorname{Ker} \eta_x$. On the other hand, by the local uniformizability, we have that $(X_x, b \,|\, X_x) = G_x \backslash E_x$ for some finite orthogonal transformation group (G_x, E_x). Putting $E_x^0 = E_x - \{0\}$, we have that $G_x \backslash E_x^0 = (X_x^0, b \,|\, X_x^0) = (\dot{X}_x, b \,|\, \dot{X}_x) \times \mathbf{R}$. In case $n=2$, we have that $x = B_j$ for some $j \in J$ and that $H_x = Z(\mu_j)$ and $H_x[\mu_j^{b_j}] = Z(b_j \cdot \mu_j)$. Since $\dot{X}_x$ is circle and G_x is cyclic of order $b_j = b(x)$, it follows that $((G)_y, M_y^0) \cong (G_x, E_x^0)$.

In case $n \geq 3$, by the induction hypothesis, the uniformization $((G_y), \dot{M}_y)$ of $(\dot{X}_x, \Sigma \dot{X}_x)$ associated with $H_x[\mu_x^b]$ is the universal uniformization of $(\dot{X}_x, b \,|\, \dot{X}_x)$. Hence we have a covering $((G)_y, \dot{M}_y) \to (G_x, \dot{E}_x)$. Since $\dot{E}_x$ is 1-connected in this case $n \geq 3$, it follows that $((G)_y, \dot{M}_y) \cong (G_x, \dot{E}_x)$ and hence $((G)_y, M_y) \cong (G_x, E_x)$. In particular, $(G)_y$ coincides with the isotropy group G_y of G at y so that $\sharp G_y = \sharp G_x = b(x)$. Therefore, we have at once shown that (G, M) is a locally smooth proper transformation group with $G \backslash M = (X, b)$, i.e., a uniformization of (X, b).

Conversely, suppose that there is a uniformization (G, M) of (X, b) associated with $K \subset H$.

Then for each $y \in M$, (G_y, M_y) is regarded as a finite orthogonal transformation group which is an open cone of $(G_y, \dot{M}_y)$. Note that (G_y, M_y) is a uniformization of $(X_x, b \,|\, X_x)$ associated with $\operatorname{Ker} \eta_x$, where $x = G \cdot y$ and $\eta_x = \eta_x(K)$. In case $n=2$, it can be seen directly that $\operatorname{Ker} \eta_x = H_x[\mu_x^{b(x)}]$. In case $n \geq 3$, $\dot{M}_y$ is simply connected. By the induction hypothesis, $(G_y, \dot{M}_y)$ is associated with $H_x[\mu_x^b]$. Therefore, we have that $\operatorname{Ker} \eta_x = H_x[\mu_x^b]$, namely, K is b-complete.

Thus if (X, b) is complete, then the uniformization $(\hat{G}, \hat{M})$ of $(X, \Sigma X)$ associated with $H[\mu^b]$ is a uniformization of (X, b). We will show that $(\hat{G}, \hat{M})$ is the universal one. Let (G, M) be a uniformization of (X, b)

associated with $K \subset H$. Since K contains $H[\mu^b]$, we have a natural homomorphism $\phi: \hat{G} = H/H[\mu^b] \to G = H/K$ and a ϕ-equivariant regular covering map $f_0: \hat{M}_0 \to M_0$ with deck transformation group Ker ϕ. By ([ibid], Extension Theorem), this covering $(\phi, f_0): (\hat{G}, \hat{M}_0) \to (G, M_0)$ can be extended to a ϕ-equivariant map $f: \hat{M} \to M$, which is a regular branched covering associated with $H[\mu^b] \subset K = \pi_1(M_0)$ and whose deck transformation group is Ker $\phi = K/H[\mu^b]$. We will show that Ker ϕ acts on $\hat{M}$ fixed point freely so that $(\phi, f): (\hat{G}, \hat{M}) \to (G, M)$ is a covering. For the quotient maps $\hat{q}: \hat{M} \to X$ and $q: M \to X$ of those uniformizations, we have that $\hat{q} = f \circ q$. For $z \in \hat{M}$, let $y = f(z)$ and $x = q(y)$. Then $\hat{q}$ and q restricted to connected components $\hat{M}_z^{\cdot}$ and $\dot{M}_y$ over $\dot{X}_x$ are the quotient maps of the universal uniformizations $(\hat{G}_z, \hat{M}_z^{\cdot})$ and $(G_y, \dot{M}_y)$ of $(\dot{X}_x, b \mid X_x)$. Hence (ϕ, f) maps $(\hat{G}_z, \hat{M}_z^{\cdot})$ isomorphically onto $(G_y, \dot{M}_y)$. This implies that Ker $\phi \cap \hat{G}_z = \{1\}$ for all $z \in \hat{M}$; Ker ϕ acts on $\hat{M}$ fixed point freely.

Therefore, $(\phi, f): (\hat{G}, \hat{M}) \to (G, M)$ is a covering. Finally, we will show that $\pi_1(M) = K/H[\mu^b]$. Let $\Sigma'X = \Sigma X - \bigcup_{j \in J} B_j$, $\Sigma'M = q^{-1}(\Sigma'X)$ and $M' = M - \Sigma'M$. Since dim $\Sigma'M \leq n-3$ and $(M, \Sigma'M)$ is locally triangulable as a pair, it follows from the general position that $\pi_1(M') \cong \pi_1(M)$. Note that a normal loop to a component of $q^{-1}(B_j)$ is a lift of $\mu_j^{b_j}$ to M_0 up to conjugate. Since $\pi_1(M')$ is obtained from $K = \pi_1(M_0)$ by killing those lifts, we have that $\pi_1(M') = K/K[\mu^b]$. However, K is a normal subgroup of H containing $H[\mu^b]$. By the minimality of the normal closure, we have that $K[\mu^b] = H[\mu^b]$ and hence $\pi_1(M) \cong \pi_1(M') \cong K/H[\mu^b]$.

Proof of Theorem 2. The proof is induction on the dimension n. In case $n = 1$, the proof is trivial. Let (X, b) be an n-orbifold with $\partial_* X \neq \phi$ $(n \geq 2)$. Since $b(A_i) = 2$, we have that dim $\Sigma D_*(X) \leq n-2$. We will show that $D_*(X, b)$ is locally unifomizable, i.e., $D_*(X, b)$ is an n-branchfold. For $x \in X - \partial_* X$, X_x or $t \cdot X_x$ is a neighborhood of x or $t \cdot x$ in $D_*(X)$, respectively. Thus $D_*(X, b)$ is locally uniformizable at x. For $x \in \partial_* X$, note that $D_*(X_x, b \mid X_x)$ is a neighborhood restriction of $D_*(X, b)$ at x. Let (G_x, E_x) be an orthogonal uniformization of $(X_x, b \mid X_x)$. In case $n = 2$, $D_*(X, b)$ is clearly a branchfold. Suppose that $n \geq 3$. Then $\dot{E}_x$ is simply connected. By the induction hypothesis and Theorem 1, G_x is a semi-direct product of G_x^+ by $Z_2(t_x)$ so that $(G_x^+, \dot{E}_x)$ is the universal uniformization of $D_*(X_x, b \mid X_x)$ and t_x is the reflection of $D_*(\dot{X}_x)$. Thus (G_x^+, E_x) is an orthogonal uniformization of $D_*(X_x, b \mid X_x)$ and hence $D_*(X, b)$ is an n-branchfold.

Now suppose that $D_*(X, b)$ is uniformizable. Let $(\hat{G}^+, \hat{M})$ be the universal uniformization of $D_*(X, b)$. Since the reflection t is an automorphism of $D_*(X, b)$, t can be lifted to an automorphism $\hat{t}$ of $(\hat{G}^+, \hat{M})$. Since dim $F(t) \cap (D_*(X) - \Sigma D_*(X)) = n-1$, we can choose $\hat{t}$ so that

dim $F(\hat{t})=n-1$. Then $\hat{t}$ is a reflection of $\hat{M}$, because $\hat{M}$ is 1-connected. Let $\hat{G}$ be the group of transformations of $\hat{M}$ generated by $\hat{t}$ and $\hat{G}^+$. Since t also makes $H_x[\mu_x^{D*(b)}]$ invariant for $x \in \Sigma D_*(X)$, it follows from the induction hypothesis together with the local consideration above that $(\hat{G}, \hat{M})$ is a (locally smooth proper) transformation group. Moreover, we have clearly that $\hat{G}\backslash\hat{M}=(X, b)$ and, $\hat{G}/\hat{G}^+=Z_2(t)$ so that $t \mapsto \hat{t}$ gives a splitting $Z_2(t) \rightarrow \hat{G}$. Thus (X, b) is uniformizable and the universal uniformization $(\hat{G}, \hat{M})$ of it is obtained from the universal uniformization $(\hat{G}^+, \hat{M})$ of $D_*(X, b)$ replacing $\hat{G}^+$ by the semi-direct product of $\hat{G}^+$ by $Z_2(t)$. Conversely, suppose that there is a uniformization (G, M) of (X, b). Let $\tilde{M}$ be the universal covering of M, $\tilde{G}$ the group of transformations of $\tilde{M}$ that consists of lifts of elements of G. Then the deck transformation group $K=\pi_1(M)$ of $\tilde{M}$ is a normal subgroup of $\tilde{G}$ and we have that $\tilde{G}/K =G$ and $|K\backslash\tilde{M}|=M$. Thus $(\tilde{G}, \tilde{M})$ is a uniformization of (X, b) which covers (G, M). We will show that there is an index 2 subgroup $\tilde{G}^+$ of $\tilde{G}$ such that $(\tilde{G}^+, \tilde{M})$ is the universal uniformization of $D_*(X, b)$ and the generator of $Z_2=\tilde{G}/\tilde{G}^+$ is the reflection t of $D_*(X)=|\tilde{G}^+\backslash\tilde{M}|$. Let $\mathscr{R}$ be the set of all reflections of $\tilde{G}$. Since $\tilde{M}$ is 1-connected and dim $\Sigma X=n-1$, we have that $\mathscr{R}=\{r \in G \,|\, \dim F(r)=n-1\} \neq \phi$. Let R be the subgroup of $\tilde{G}$ generated by $\mathscr{R}$, (S, X_R) the fundamental system of a reflection group $(R, \tilde{M})$ and $R\backslash\tilde{M}=(X_R, b_R)$. Then we have the normal subgroup R^+ of R which consists of elements of even length with respect to S. In other words, R^+ is the kernel of a well-defined homomorphism from R to $Z_2= \langle t: t^2=1 \rangle$ which sends each generator $s \in S$ to t. Let $R^+\backslash\tilde{M}=(Z, d)$ and let $q: \tilde{M} \rightarrow Z$ be the quotient map. We have clearly that $q(X_R \cup sX_R)=Z$ for any $s \in S$. Then the generator t acts on Z with the fixed point set $F(t)=q(\bigcup_{r \in \mathscr{R}} F(r))=q(\partial_* X_R)$. Hence $Z-F(t)$ is homeomorphic to a disjoint union of $X_R-\partial_* X_R$ and $s\cdot(X_R-\partial_* X_R)$. Therefore, we have that $R^+\backslash M=D_*(X_R, b_R)$ and the generator t acts on $D_*(X_R, b_R)$ as the reflection. Thus we obtain a splitting $D_*: \tilde{G}/R \rightarrow \tilde{G}/R^+$ of an obvious central extension $\{1\} \rightarrow Z_2(=R/R^+) \rightarrow \tilde{G}/R^+ \rightarrow \tilde{G}/R \rightarrow \{1\}$ by setting

$$D_*(h)(z)=\begin{cases} h(z), & \text{if } z \in X_R \\ t\cdot h\cdot t^{-1}(z), & \text{if } z \in t\cdot X_R \end{cases}$$

for each transformation $h \in \tilde{G}/R$ of X_R. Therefore, we have that $\tilde{G}/R^+ \cong Z_2 \times \tilde{G}/R$. Now $\tilde{G}/R$ is identified with $\{\mathrm{id}\} \times \tilde{G}/R \subset \tilde{G}/R^+$ via D_* and acts on $D_*(X_R)$ making each of X_R and $t\cdot X_R$ invariant. This means that $|\tilde{G}/R\backslash D_*(X_R)|$ is the mirror double $D_*(X)$ of $X=|\tilde{G}/R^+\backslash D_*(X_R)|= |\tilde{G}/R\backslash X_R|$ and the generator t of $Z_2=R/R^+$ is regarded as the reflection of $D_*(X)$. Let G^+ be the kernel of a natural epimorphism $\hat{G} \rightarrow \hat{G}/R^+=Z_2 \times \tilde{G}/R \mapsto Z_2$. Then we have that $\tilde{G}^+\backslash\tilde{M}=D_*(X, b)$ and $\tilde{G}$ is a semi-direct

product of $\widetilde{G}^+$ by Z_2. Since $\widetilde{M}$ is 1-connected, it follows from Theorem 1 that $(\widetilde{G}^+, \widetilde{M})$ is the universal uniformization of $D_*(X, b)$. In particular, $(\widetilde{G}, \widetilde{M})$ is isomorphic with $(\widehat{G}, \widehat{M})$ which has been constructed in the first half of this proof. This completes the proof of Theorem 2.

§ 4. Uniformizing reflection orbifolds and rotation orbifolds

Proof of Theorem 3. First, we will show that if (G, M) is a reflection group, then $G \backslash M = (X, b)$ is a regular reflection orbifold. Let (S, X) be the fundamental system of (G, M). It is clear that $X - \varSigma X$ is an n-manifold. By the property (ii), for each $x \in \varSigma X$, the isotropy transformation group (G_x, M_x) is regarded as a finite orthogonal reflection group with fundamental system (S_x, X_x). Hence it is not hard to see that if x belongs to an $(n-k)$-stratum of the stratification $\mathscr{S}$ of (X, b), then $\sharp S_x = k$ and $G_x \backslash M_x = (X_x, b \,|\, X_x)$ is an open cone of an $(n-k-1)$-suspension of a spherical simplex $(k-1)$-orbifold. (Refer to Coxeter [4] and Thurston [14], § 13). This implies that X is an n-manifold with boundary $\partial_* X = \varSigma X$. To prove the regularity of (X, b), let $x \in B_j$ $(j \in J)$. Then $\dot{X}_x$ is an $(n-3)$-suspension of a 1-simplex. Since $B_j \subset \partial_* X$, there are at most two $(n-1)$-strata A_k and A_l intersecting X_x. Then $S_x = \{s, s' \,|\, F(s) \supset \bar{A}_k$ and $F(s') \supset \bar{A}_l\}$ generates a dihedral group G_x of order $b(x) > b(A_k) = b(A_l) = 2$. This implies that $A_k \neq A_l$, $b(x) = b(B_j)$ is even and depends only on A_k and A_l. Thus (X, b) is a regular reflection orbifold.

Conversely, suppose that (X, b) is a regular reflection n-orbifold. We will show that by induction on the dimension n the formal uniformization $(\widetilde{G}, \widetilde{M})$ is a reflection uniformization of (X, b). In case $n = 1$, the conclusion holds obviously. Suppose that $n \geq 2$. Note that a map $x \mapsto [1, x]$ embeds X into $[1, X] \subset \widetilde{M}$. Thus we identify $x \in X$ with $[1, x] \in [1, X]$. For each point $x \in X$, we have that $g \cdot x \in X$ for some $g \in \widetilde{G}$ if and only if $g \in \widetilde{G}_x$. Hence $\widetilde{G}_x$ is the isotropy subgroup of $\widetilde{G}$ at x and $\widetilde{G}_x \cdot X_x$, denoted by $\widetilde{M}_x$, is a $\widetilde{G}_x$-invariant neighborhood of x in $\widetilde{M}$. Since $(X_x, b \,|\, X_x)$ is an open cone of $(\dot{X}_x, b \,|\, \dot{X}_x)$, we have that for $y \in \dot{X}_x$, $S_y \subset S_x$ and hence $\widetilde{G}_y \subset \widetilde{G}_x$. Hence $\widetilde{M}_x$ is actually a $\widetilde{G}$-equivariant neighborhood of x in $\widetilde{M}$, namely, $g \cdot \widetilde{M}_x = \widetilde{M}_x$ for $g \in \widetilde{G}_x$ and $g \cdot \widetilde{M}_x \cap \widetilde{M}_x = \phi$ for $g \in \widetilde{G} - \widetilde{G}_x$. It is an important theorem for a Coxeter system $(\widetilde{G}, S)$ that for any subset S' of S and the subgroup G' of G generated by S', (G', S') is again a Coxeter system. Refer to Bourbaki ([2], Ch. IV, § 1, Theorem 2). Thus for $x \in X$, $(\widetilde{G}_x, S_x)$ is a Coxeter system. It follows from the construction of $(\widetilde{G}, \widetilde{M})$ that $(\widetilde{G}_x, \widetilde{M}_x)$ is again the formal uniformization of $(X_x, b \,|\, X_x)$ which is an open cone of the formal uniformization $(\widetilde{G}_x, \widetilde{M}_x = \widetilde{G}_x \cdot \dot{X}_x)$ of $(\dot{X}_x, b \,|\, \dot{X}_x)$. On the other hand, by the local uniformizability we have that $(X_x, b \,|\, X_x) = G_x \backslash E_x$ for some finite orthogonal transformation group (G_x, E_x). In

case $n=2$ notice that if $x \in A_i$ for some $i \in I$, then $\sharp G_x = \sharp \widetilde{G}_x = 2$ and hence $(G_x, E_x) \cong (\widetilde{G}_x, \widetilde{M}_x)$, and if $x \in \bar{A}_i \cap \bar{A}_j$ for some $i \neq j$, then G_x and $\widetilde{G}_x$ are both a dihedral group of order $2a_{ij} = b(x)$, so that $(G_x, E_x) \cong (\widetilde{G}_x, \widetilde{M}_x)$. In case $n \geq 3$, since $\dot{X}_x$ and $\dot{E}_x$ are 1-connected, it follows that by the induction hypothesis, $\widetilde{M}_x^{\cdot}$ is 1-connected and $(G_x, \dot{E}_x) \cong (\widetilde{G}_x, \widetilde{M}_x^{\cdot})$. At any rate, we have that $(\widetilde{G}_x, \widetilde{M}_x) \cong (G_x, E_x)$. Thus we have proved that for $x \in X \subset \widetilde{M}$, $(\widetilde{G}, \widetilde{M})$ is locally smooth and proper at x. For any point $y \in \widetilde{M}$, we have that $y = g \cdot x$ for some $(g, x) \in \widetilde{G} \times X$. This implies that the isotropy group $\widetilde{G}_y$ of $\widetilde{G}$ at y is $g \cdot \widetilde{G}_x \cdot g^{-1}$ and $\widetilde{M}_y = g \cdot \widetilde{M}_x = \widetilde{G}_y[g, X_x]$ is a $\widetilde{G}$-equivariant neighborhood of y so that $(\widetilde{G}_x, \widetilde{M}_x) \cong (\widetilde{G}_x, \widetilde{M}_y)$. Since $\sharp \widetilde{G}_y = \sharp \widetilde{G}_x = \sharp G_x = b(x)$, it follows that $(\widetilde{G}, \widetilde{M})$ is a uniformization of (X, b). We have to show that $(\widetilde{G}, \widetilde{M})$ is a reflection group with fundamental system (S, X). For this, it suffices to show that each element s of S is a reflection of $\widetilde{M}$. Suppose that there is an element $s \in S$ which is not a reflection, i.e., the fixed point set $F(s)$ of s does not disconnect $\widetilde{M}$. Let $x \in X - \Sigma X$ and $s \cdot x = y$. Then we can find a path w from x to y in $\widetilde{M} - F(s)$ which is generic in the sense that w does not meet $\widetilde{G} \cdot (\Sigma X - \bigcup_{i \in I} A_i)$ and meets $\widetilde{G} \cdot (\bigcup_{i \in I} A_i)$ transversally. Then w meets connected components $\tilde{A}_1, \cdots, \tilde{A}_k$ of $\widetilde{G} \cdot (\bigcup_{i \in I} A_i)$ in such a way that $\tilde{A}_1 \subset X \cap g_1 \cdot X$, $\tilde{A}_2 \subset g_1 \cdot X \cap g_2 \cdot X, \cdots, \tilde{A}_k \subset g_{k-1} \cdot X \cap g_k \cdot X$, where $g_i = s_1 \cdots s_i$ for some $s_j \in S$ $(1 \leq j \leq i)$ and $g_k \cdot X = s \cdot X$, i.e., $s = g_k$. In particular, we have that $s \cdot g_k^{-1} = s_1 \cdots s_k = r_k \cdots r_1$, where $r_i = g_{i-1} \cdot s_i \cdot g_{i-1}^{-1} (= s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1)$. Notice that $F(r_i) \supset \tilde{A}_i$. Since w does not meet $F(s)$, we have that $s \neq r_i$ for all $i = 1, \cdots, k$. It is known that in a Coxeter system $(\widetilde{G}, S)$, the mod 2 reduction of a number $n(g_k; s) = \sharp\{r_i \mid r_i = s\}$ does not depend on the choice of an expression of g_k by S-symbols; $g_k = s_1 \cdots s_k$, where $r_i = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$, $i = 1, \cdots, k$. ([2], Ch. IV, § 1, p. 13, Lemma 1). Since $s \neq r_i$, $i = 1, \cdots, k$, we have that $n(g_k; s) = 0$. But $g_k = s$ gives us $n(g_k; s) = 1$. This is a contradiction. Therefore, each element $s \in S$ is a reflection. Now we have proved that $(\widetilde{G}, \widetilde{M})$ is a reflection uniformization of (X, b) with fundamental system (S, X). We will show that any reflection uniformization (G, M) of (X, b) with fundamental system (S', X) is covered by $(\widetilde{G}, \widetilde{M})$ so that the epimorphism $\widetilde{G} \to G$ is given by a homomorphism between the Coxeter systems $(\widetilde{G}, S) \to (G, S')$. Notice that by the definition of the fundamental system, A_i is contained in $F(s')$ for a unique element s' of S' for each $i \in I$. Conversely, for any element $s' \in S'$, there is A_i such that $A_i \subset F(s')$, and B_j is contained in $F(s_1') \cap F(s_2')$ for some distinct elements s_1' and s_2' of S' for each $j \in J$. It follows that by sending each $s_i \in S$ determined by A_i $(i \in I)$ to the element $s' \in S'$ such that $A_i \subset F(s')$, we have an epimorphism $\phi: (\widetilde{G}, S) \to (G, S')$ between Coxeter systems. Then we have a well-defined map $f: \widetilde{M} \to M$ by setting $f(y) = \phi(g) \cdot x$ for $y = g \cdot x \in \widetilde{M}$, $(g, x) \in \widetilde{G} \times X$. Since $\widetilde{G} \backslash \widetilde{M} = G \backslash M = (X, b)$ and

(ϕ, f) sends $(\widetilde{G}_y, \widetilde{M}_y)$ isomorphically onto $(G_{f(y)}, M_{f(y)})$ it follows that $f: \widetilde{M} \rightarrow M$ is a covering map with the deck transformation group Ker ϕ. The details will be left to the reader. (Compare with the arguments of Koszul [9].) Thus the formal uniformization $(\widetilde{G}, \widetilde{M})$ of a regular reflection orbifold (X, b) is universal among reflection uniformizations of (X, b). Finally, suppose that X is 1-connected. Let (G, M) be a uniformization of a reflection orbifold (X, b). Let R be the subgroup of G generated by all reflections, and $(X_R, b_R) = R \backslash M$. Then G/R acts on $X_R - \partial(X_R)$ $(\subset X_R$ $-\partial_* X_R)$ fixed point freely so that the quotient map $X_R - \partial(X_R) \rightarrow X - \partial X$ is a covering. Since X is a 1-connected manifold, it follows that this covering is trivial, i.e., $G = R$. Thus (G, M) is a reflection uniformization of (X, b) and $(\widetilde{G}, \widetilde{M})$ is the universal uniformization of (X, b). Therefore, $\widetilde{M}$ should be clearly 1-connected. This completes the proof of Theorem 3.

Proof of Corollary to Theorem 3. Let (X, b) be a bad reflection n-orbifold. We will show that (X, b) contains a bad surface for $n \neq 4, 5$. Let $\pi: \hat{X} \rightarrow X$ be the universal covering of X and $\hat{b} = b \circ \pi$. Then $(\hat{X}, \hat{b})$ is also a bad reflection orbifold. Since $\hat{X}$ is 1-connected, it follows from Theorem 3 that $(\hat{X}, \hat{b})$ is not regular. Thus at least one of the following two cases occurs:

(I) There is a couple $(A; B)$ of an $(n-1)$-stratum A and an $(n-2)$-stratum B of the stratification $\mathcal{S}$ of $(\hat{X}, \hat{b})$ such that B is contained in $\overline{A}$, but not contained in the closures of other $(n-1)$-strata of $\mathcal{S}$. This couple $(A; B)$ is referred to as a *singularity* of type I.

(II) There are two $(n-1)$-strata A, A' and two $(n-2)$-strata B, B' of $\mathcal{S}$ such that $B \cup B' \subset \overline{A} \cap \overline{A}'$ and $b(B) \neq b(B')$. This quadraple $(A, A'; B, B')$ is referred to as a singularity of type II.

We will show that the existence of singularity of type I or II implies the existence of a bad surface of (X, b) of type I or II in Figure 1, respectively. For this, notice that

$$\pi(A; B) = (\pi(A); \pi(B)) \quad \text{and} \quad \pi(A, A'; B, B') = (\pi(A), \pi(A'); \pi(B), \pi(B'))$$

are also singularities of types I and II, respectively, since $\hat{b} = b \circ \pi$. Now first suppose that $(A; B)$ exists on $(\hat{X}, \hat{b})$. Then we have a simple closed curve L, called a *characteristic loop* of $\pi(A; B)$, on $\pi(A \cup B)$ which meets $\pi(B)$ transversally at one point x. Note that L gives rise to a covering translation making $\hat{b}$ invariant and hence sending $(A; B)$ onto itself. If L is essential in X, then a lift $\hat{L}$ of L starting from point z of B is not a loop and has an end point z' on B. Since B is connected, there is a path w on B from z to z'. Note that $\pi \circ (w * \hat{L})$ is inessential in X. Thus we can change L in a neighborhood of $\pi(B)$ in $\pi(A \cup B)$ to a characteristic loop of

$(A; B)$ which is inessential in X. In case $n=3$, by the Dehn's Lemma [12] and the smoothing theory, we may choose L so that it bounds a locally smooth 2-disk D in X. In case $n \geq 6$, by the handle straightening method [7] we can choose L so that it has a normal $(n-2)$-disk bundle in $\pi(A \cup B)$ which intersects $\pi(B)$ fiberwise. Then it is not hard to see that by the handle straightening method together with general position argument L bounds a locally smooth 2-disk D in X. Clearly D is a bad surface of (X, b) of type I in Figure 1. If a singularity of type II exists in (X, b), then by the same procedure as above we can construct a bad surface of (X, b) of type II in Figure 1, completing the proof.

Proof of Theorem 4. For an orientable uniformization (G, M) of (X, b), let G_0 be the normal subgroup of G generated by those elements which have fixed points and G_ρ the normal subgroup of G generated by rotations. Notice that the universal uniformization $(\hat{G}, \hat{M})$ of an orientable branchfold (X, b) is orientable and there is a one to one correspondence of rotations of $\hat{G}$ and conjugates of the image of μ_j $(j \in J)$ in $\hat{G}$. Thus we have that $\hat{G} = H/H[\mu^b] \supset \hat{G}_0 \supset \hat{G}_\rho = H[\mu]/H[\mu^b]$. Now suppose that there is a rotation uniformization (G, M) of an n-orbifold (X, b). Since (G, M) is orientable, it is immediately seen that (X, b) is an orientable branchfold. Let K be the deck transformation group of the universal covering $(\hat{G}, \hat{M}) \rightarrow (G, M)$ and $(\hat{X}, \hat{b}) = \hat{G}_\rho \backslash \hat{M}$. Since $G_\rho = G$, we have that $\hat{G} = \hat{G}_\rho \cdot K$, namely, $\hat{G}$ is generated by its normal subgroups $\hat{G}_\rho$ and K. By Armstrong [1], $\hat{X}$ is 1-connected. Moreover, $\hat{G}/\hat{G}_\rho = K/(K \cap \hat{G}_\rho)$ acts on $\hat{X}$ fixed point freely so that $|\hat{G}/\hat{G}_\rho \backslash \hat{X}| = |\hat{G} \backslash \hat{M}| = X$. Hence $\hat{X}$ is the universal covering space of X and $\pi_1(X) = \hat{G}/\hat{G}_\rho = H/H[\mu]$. Thus (X, b) is a good rotation orbifold. Conversely, suppose that (X, b) is a good rotation orbifold. Since X is orientable, the universal uniformization $(\hat{G}, \hat{M})$ of (X, b) is orientable. Let $(\hat{X}, \hat{b}) = \hat{G}_0 \backslash \hat{M}$. Again by Armstrong, $(\hat{X}, \hat{b})$ is the universal covering of (X, b) and $\pi_1(X) = \hat{G}/\hat{G}_0$. Then the condition $\pi_1(X) = H/H[\mu]$ implies that $\hat{G}_0 = H[\mu]/H[\mu^b] = \hat{G}_\rho$. In particular, if $\pi_1(X) = \{1\}$, then the universal uniformization of (X, b) is a rotation uniformization, completing the proof of Theorem 4.

Proof of Theorem 5. To prove (1), it suffices to show that if (X, b) is a nice 3-branchfold, then (X, b) is good. Let $p_0 : \hat{M}_0 \rightarrow X_0 = X - \Sigma X$ be a covering of X_0 associated with a subgroup $H[\mu^b] \subset H = \pi_1(X)$. Then by the niceness we can complete this covering to a branched covering $p_1 : \hat{M}_1 \rightarrow X' = X - \Sigma' X$ so that $(\hat{G} = H/H[\mu^b], \hat{M}_1)$ is a uniformization of $(X', b \mid X')$. For each point x of $\Sigma' X$, let $\hat{M}_{1,x}$ be a connected component of $p_1^{-1}(\dot{X}_x)$ and let $\hat{G}_x \subset \hat{G}$ be the stabilizer of $\hat{M}_{1,x}$. Then $(\hat{G}_x, \hat{M}_{1,x})$ is a uniformization of $(\dot{X}_x, b \mid \dot{X}_x)$. On the other hand, we have an orthogonal uni-

formization $(G'_x, \dot{E}_x)$ of $(\dot{X}_x, b \,|\, \dot{X}_x)$ which is universal, since $\dot{E}_x$ is 1-connected. Thus we have a covering $(\phi, f): (G'_x, \dot{E}_x) \to (\hat{G}_x, \hat{M}'_{1,x})$. Suppose that Ker $\phi \neq \{1\}$, then the only possible case is the case where $\dot{E}_x = S^2$ and Ker $\phi = Z_2$ so that $\hat{M}'_{1,x} = RP^2$. But since $\pi_1(\hat{M}_0) = H[\mu^b]$ and $\pi_1(\hat{M}_1) = H[\mu^b]/(H[\mu^b])[\mu^b] = \{1\}$, $\hat{M}_1$ is orientable and M_1 can not contain a two sided $\hat{M}_{1,x} \cong RP^2$. Therefore $\hat{M}'_{1,x}$ should be 1-connected so that $(G'_x, \dot{E}_x) \cong (\hat{G}_x, \hat{M}_{1,x})$ for all $x \in \Sigma X$. Thus $p_1 : \hat{M}_1 \to X'$ can be completed to a branched covering $p : \hat{M} \to X$ so that $(\hat{G}, \hat{M})$ is the universal uniformization of (X, b). In order to prove (2), we can assume without loss of generality that X is 1-connected. Suppose that (X, b) is a rotation 3-orbifold. Then $X' = X - \Sigma' X$ is an orientable 3-manifold. For each point $x \in \Sigma' X$, we have that by Gauss-Bonnet formula, $\dot{X}_x = S^2$ or RP^2. But the case $\dot{X}_x = RP^2$ cannot occur, since X is orientable. Thus X is an orientable 3-manifold. Conversely, if X is an orientable 3-manifold, then we have, that $\pi_1(X') \cong \pi_1(X)$. Thus (X, b) is a rotation orbifold, completing the proof of (2). To prove (3), suppose that there is a point x of ΣX such that $i_x : H_x \to H$ is not injective. Then by the Dehn's lemma [12], we have a properly embedded disk $(D, \partial D)$ in $(X'_0, \dot{X}_{0,x})$ such that ∂D is essential in $\dot{X}_{0,x}$, where $X'_0 = X_0 - (x * \dot{X}_{0,x} - \dot{X}_{0,x})$. In case $x \in B_j$, then $\dot{X}_{0,x} \cong S^1 \times R$ so that ∂D can be identified with the normal loop up to isotopy so that ∂D bounds a normal disk D' to B_j in $x * \dot{X}_{0,x}$. Then $D \cup D'$ gives rise to a bad surface. In case $x \in \Sigma' X = \Sigma X - \bigcup_{j \in J} B_j$, there are two cases:

  (i) $\dot{X}_x = S^2$ and $\Sigma \dot{X}_x =$ three points, and
  (ii) $\dot{X}_x = RP^2$ and either $\Sigma \dot{X}_x =$ one point or $\Sigma \dot{X}_x = \phi$.

In the case (i), ∂D bounds a disk on $\dot{X}_x$ that contains exactly one point of $\dot{X}_x$. By the same construction as above, we have a bad surface of (X, b). In the case (ii), we can assume that ∂D is $RP^1 \subset RP^2$, because ∂D is essential in RP^2-the point or RP^2. Putting, $D' = x * RP^1$, we have also a bad surface $D \cup D'$ of (X, b). We have shown that if $i_x : H_x \to H$ is not injective for some $x \in \Sigma X$, then (X, b) is bad. Thus if (X, b) is good, then $i_x : H_x \to H$ is injective and $H_x \cap H[\mu^b] = H_x[\mu^b_x]$, completing the proof of Theorem 5.

Proof of Thoerem 6.

Construction. Let L_1, L_2 and L_3 be three complex lines in complex plane $C^2(z_1, z_2)$ which are in general position.

For example, let L_1 and L_2 be the z_1- and z_2-axes, respectively and L_3 the complex line through two points $(1, 0)$ and $(0, 1)$. Let $p_0 = (0, 0)$, $p_1 = (1, 0)$, $p_2 = (0, 1)$, $B_1 = L_1 - \{p_0, p_1\}$, $B_2 = L_2 - \{p_0, p_2\}$ and $B_3 = L_3 - \{p_1, p_2\}$. We have $(S^4; \bar{L}_1, \bar{L}_2, \bar{L}_3)$ as the one point compactification $(C^2 \cup \{\infty\}; L_1 \cup \{\infty\}, L_2 \cup \{\infty\}, L_3 \cup \{\infty\})$ of $(C^2; L_1, L_2, L_3)$. Let b_1, b_2 and b_3 be integers

$\geqq 2$. Then we will show that there is an orbifold (X, b), referred to as $S^4(b_1, b_2, b_3)$, on $X = S^4$ with signature $\{(B_j, b_j) \mid j = 1, 2, 3\}$ such that $\Sigma X = \bigcup_{j=1}^3 \bar{B}_j = \bigcup_{j=1}^3 \bar{L}_j$ if and only if $1/b_1 + 1/b_2 + 1/b_3 > 1$. Moreover, indeed the case, we will show that (X, b) is bad, but it is nice and does not contain any bad surface. Let $X = S^4$, $\Sigma X = \bigcup_{j=1}^3 \bar{L}_j$, $X_0 = X - \Sigma X = C^2 - \bigcup_{j=1}^3 L_j$ and $H = \pi_1(X_0)$. Then we have that

$$H = \pi_1(X_0) = Z(\mu_1) + Z(\mu_2) + Z(\mu_3) \quad \text{and}$$
$$H[\mu^b] = Z(b_1\mu_1) + Z(b_2\mu_2) + Z(b_3\mu_3),$$

where μ_j, $j = 1, 2, 3$, is a normal loop to B_j. It is an exercise to see that an orbifold on C^2 determined by a signature $\{(B_j, b_j) \mid j = 1, 2, 3\}$ is universally uniformized by $(Z_{b_1} + Z_{b_2} + Z_{b_3}, F(b_1, b_2, b_3))$, where $F(b_1, b_2, b_3) = \{(z_1, z_2, z_3) \in C^3 \mid z_1^{b_1} + z_2^{b_2} + z_3^{b_3} = 1\}$ and if we regard $\omega_i = \exp(2\pi\sqrt{-1}/b_i)$ as a generator of Z_{b_i}, $i = 1, 2, 3$, then the action of $(\omega_1^{a_1}, \omega_2^{a_2}, \omega_3^{a_3})$ is given by $(z_1, z_2, z_3) \mapsto (\omega_1^{a_1} z_1, \omega_2^{a_2} z_2, \omega_3^{a_3} z_3)$. Let $(\dot{X}_\infty, \Sigma \dot{X}_\infty)$ be a link of ∞ in $(X, \Sigma X)$. Then $\Sigma \dot{X}_\infty$ is a link in $\dot{X}_\infty = S^3$ which consists of three fibers of the Hopf-fibration of S^3. Thus we have that an inclusion map induces an isomorphism $H_1(\dot{X}_\infty - \Sigma \dot{X}_\infty) \cong H_1(X_0) \cong \pi_1(X_0)$ and that the induced $Z_{b_1} + Z_{b_2} + Z_{b_3}$-branched covering space of $\dot{X}_\infty$ is the Brieskorn 3-manifold $K(b_1, b_2, b_3) = F(b_1, b_2, b_3) \cap S^5$. It is known that the universal convering space of $K(b_1, b_2, b_3)$ is S^3 if and only if $1/b_1 + 1/b_2 + 1/b_3 > 1$. Refer to Milnor [11]. This implies that $\{(B_j, b_j) \mid j = 1, 2, 3\}$ determines an orbifold $S^4(b_1, b_2, b_3) = (X, b)$ such that $\Sigma X = \bigcup_{j=1}^3 \bar{L}_j$ if and only if $1/b_1 + 1/b_2 + 1/b_3 > 1$. Indeed in this case, we have already seen that the universal uniformization of $(C^2, b \mid C^2)$ has a non-simply connected end with the fundamental group

$$\pi_1(K(b_1, b_2, b_3)) = \langle \alpha, \beta : \alpha^{b_1} = \beta^{b_2} = (\alpha\beta)^{b_3} \rangle,$$

because $b_1, b_2, b_3 \geqq 2$. This implies that (X, b) is bad. It is clear from the arguments above that (X, b) is nice. Suppose that (X, b) has a bad surface (Y, c). Since $(C^2, b \mid C^2)$ is good, Y must contain ∞. Since each sphere $\bar{L}_i = L_i \cup \{\infty\}$ clearly gives rise to a good surface of (X, b), Y is not contained in ΣX. Hence Y contains ∞ and intersects each $\bar{L}_i$ transversally at more than two points. It is not hard to make sure that for an intersection point $x \in Y \cap \bar{L}_i$, $c(x)$ is divisible by $b_i \geqq 2$ so that $x \in \Sigma Y$. This implies that $\sharp(\Sigma Y) \geqq 3$ and hence (Y, c) is good, contradicting to the hypothesis. Therefore, (X, b) does not contain any bad surface, completing the proof of Theorem 6.

§ 5. Some remarks

(5.1) *Various structures on transformation groups and orbifolds.*

By a $\mathscr{G}$ *n-manifold* M_σ we mean an n-manifold M with a specified $\mathscr{G}$ structure σ. For example,

$\mathscr{G}=$"triangulated"; σ is a triangulation of M,

$\mathscr{G}=$"PL"; σ is a *PL* manifold triangulation of M.

$\mathscr{G}=$"smooth"; σ is a smooth ($=C^\infty$) structure of M,

$\mathscr{G}=$"Riemannian"; σ is a smooth structure of M with a specified Riemannian metric.

A proper, but not necessary locally smooth, transformation group (G, M) is said to be a $\mathscr{G}$ *transformation group*, written (G, M_σ) if each element of G is an $\mathscr{G}$ homeomorphism, i.e., preserves σ. In this case, $(X, b)=G\backslash M$ is said to be $\mathscr{G}$ *uniformizable* and (G, M_σ) is called a $\mathscr{G}$ *uniformization* of a b-space (X, b). A b-space (X, b) is said to be a *locally* $\mathscr{G}$ *orbifold*, if it is locally $\mathscr{G}$ uniformizable in the obvious sense. This condition restricts locally, only the action of G, but not M. Thus we may speak of topological, locally triangulable, locally *PL* and locally smooth orbifolds. Note that the following equivalences and implications hold: *Locally smooth* $\Leftrightarrow$ *Locally Riemannian* $\Leftrightarrow$ *locally orthogonal* $\Rightarrow$ *Locally PL* $\Rightarrow$ *Locally triangulable*. Recall that our orbifolds in the preceding sections have been defined as a locally orthogonal or smooth ones in the definition above.

For an open subset V of X and a $\mathscr{G}$ uniformization (G, U_ρ) of $(V, b\,|\,V)$, the quotient map $q\colon (G, U_\sigma)\to(V, b\,|\,V)$ is called a *local* $\mathscr{G}$ *chart of* (X, b). A second local $\mathscr{G}$ chart $q'\colon (G', U'_\sigma)\to(V', b\,|\,V')$ of (X, b) is $\mathscr{G}$ *related to* q, if for each point $x \in V \cap V'$ and for $u \in q^{-1}(x)$, $v \in q'^{-1}(x)$, there are G-and G'-equivariant neighborhoods W and W', such that $q(W)=q'(W')$ and $q\,|\,(G_u, W)$ and $q'\,|\,(G'_v, W')$ are $\mathscr{G}$ isomorphic. A b-space (X, b) is a $\mathscr{G}$ *orbifold*, if a collection

$$\tilde{\sigma}=\{q_\alpha\colon (G_\alpha, U_{\sigma,\alpha})\to(V_\alpha, b\,|\,V_\alpha)\,|\,\alpha \in I\},$$

called a $\mathscr{G}$ *chart of* (X, b), of mutually $\mathscr{G}$ related local $\mathscr{G}$ charts of (X, b) with $\bigcup_{\alpha \in I} V_\alpha=X$ is specified. Then the maximal $\mathscr{G}$ chart σ containing $\tilde{\sigma}$ is called a $\mathscr{G}$ *structure* of (X, b) and a $\mathscr{G}$ orbifold (X, b) with $\mathscr{G}$ structure σ is denoted by $(X, b)_\sigma$. If (X', b') has a $\mathscr{G}$ structure σ', then an isomorphism $h\colon (X, b)\to(X', b')$ pulls back σ' to a $\mathscr{G}$ structure $h^*\sigma'$ of (X, b). We shall say that an isomorphism $h\colon (X, b)\to(X', b')$ is a $\mathscr{G}$ *isomorphism*, written $h\colon (X, b)_\sigma\to(X', b')_\sigma$, if $h^*\sigma'=\sigma$. A $\mathscr{G}$ *uniformization* (G, M_τ) of a $\mathscr{G}$ orbifold $(X, b)_\sigma$, is a uniformization of (X, b) such that the quotient map $q\colon (G, M_\tau)\to(X, b)$ is contained in the $\mathscr{G}$ structure σ of (X, b). It is not hard to see that

(5.1.1) *a topological orbifold (X, b) is locally triangulable if and only if $(X, \Sigma X)$ is locally triangulable as a pair of spaces.* Refer to [5].

By making use of this fact, for locally triangulable branchfolds we can formulate the local completeness condition by means of local triangulations of $(X, \Sigma X)$ in stead of the stratifications of locally smooth branchfolds, and prove the generalizations of Theorems 1 and 2. Note that if we take $b_1 = 2$, $b_2 = 3$ and $b_3 = 5$ in the proof of Theorem 6, then $S^4(2, 3, 5) \times \boldsymbol{R}$ is a triangulable 5-orbifold but not a PL 5-orbifold. Refer to Edwards and Cannon [3].

(5.1.2) *Let $(X, b)_\sigma$ be a $\mathscr{G}$ orbifold. Suppose that $(X, \Sigma X)$ is locally triangulable. If (X, b) is topologically uniformizable, then $(X, b)_\sigma$ is $\mathscr{G}$ uniformizable and topologically isomorphic $\mathscr{G}$ uniformizations of a $\mathscr{G}$ orbifold (X, b) are $\mathscr{G}$ isomorphic.*

In the proof of this, we make use of the local triangulability of $(X, \Sigma X)$ to prove that for a topological uniformization (G, M) of (X, b) and for each point z of M, there is a small G-invariant neighborhood M_z which is 1-connected. We do not know if this folds for any "topological" transformation group (G, M).

(5.2) *Uniformizing finite punctured branchsurfaces.*

By a *r-punctured* $(0 \leq r < \infty)$ *branchsurface* (X, b) we mean a b-space such that X is a r-punctured closed surface and ΣX is a discrete subset of X. Note that a r-punctured branchsurface is a branchfold. A o-punctured branchsurface (X, b) is called a *closed branchsurface*. In this case, X is compact and hence ΣX is finite. We have immediately the following:

(5.2.1) *A closed branchsurface (X, b) is a double of a regular reflection orbifold if and only if*
(i)   rank $H_1(X; \boldsymbol{Z}_2) = even$ *and*
(ii)   *either* $\sharp \Sigma X \geq 3$ *or* $\Sigma X = \{x_1, x_2\}$ *and* $b(x_1) = b(x_2)$. *If rank $H_1(X; \boldsymbol{Z}_2)$ is odd ≥ 1, then an orientable double cover $(\tilde{X}, \tilde{b})$ of (X, b) satisfies* (i) *and* (ii).

Thus the universal uniformization of a good closed branchsurface (X, b) can be constructed by Theorem 3 via Theorem 2.

In case $r \geq 1$, the set $\mu = \{\mu_j \mid j \in J\}$ of normal loops to $\Sigma X = \{x_j \mid j \in J\}$ forms a part of a free basis of a free group $H = \pi_1(X - \Sigma X)$. It follows that by the homological arguments,

(5.2.2.) *a r-punctured branchsurface (X, b) is always nice and hence good, provided that $r \geq 1$.*

Thus in this case the universal uniformization of (X, b) is given by Theorem 1.

(5.3) *Uniformizations of 3-orbifolds.*
Fitst of all, we have that

(5.3.1) *every 3-orbifold is uniquely smoothable up to isotopy. Namely, a 3-orbifold (X, b) has a smooth structure. If σ and σ' are two smooth structures on (X, b), then the identity map is isotopic to a smooth isomorphism $(X, b)_\sigma \cong (X, b)_{\sigma'}$. In particular, every (proper locally smooth) transformation group (G, M) is uniquely smoothable in the following sense: There is a smooth structure σ of M and a homeomorphism $h: M \to M$ isotopic to the identity such that $h^{-1} \circ G \circ h$ is a diffeomorphism group of M_σ. For two smoothings (G_1, M_{σ_1}) and (G_2, M_{σ_2}) of (G, M), there is a smooth isomorphism $(G_1, M_{\sigma_1}) \cong (G_2, M_{\sigma_2})$ isotopic to the identity.*

We will only outline how to put a Riemannian structure on (X, b). It is not hard to see that $(X, \Sigma X)$ can be triangulated, since $\dot{X}_x$ $(x \in X)$ is D^2, S^2 or RP^2. Then, roughly speaking, a regular neighborhood U of the 1-skeleton $\Gamma^{(1)}$ of a graph $\Gamma = \Sigma X - \bigcup_{i \in I} A_i$ in X can be considered as a 1-handlebody with 0-handles $v * \dot{X}_v$ $(v \in \Gamma^{(0)})$ and 1-handles $x_j * \dot{X}_{x_j}$ $(x_j \in B_j$ (edge of Γ)). Since $(X_x, b \mid X_x)$ is orthogonal for $x \in \Sigma X$, we may think of (X, b) restricted to $v * \dot{X}_v - \dot{X}_v$ and $x_j * \dot{X}_{x_j} - \dot{X}_{x_j}$ as complete hyperbolic 3-orbifolds. Then the Klein-Maskit combination [10] fits together these hyperbolic orbifolds making $(\mathring{U}, b \mid \mathring{U})$ a complete hyperbolic orbifold $(\mathring{U}, b \mid \mathring{U})_\sigma$. Now it is standard to extend this special Riemannian structure σ restricted to an open neighborhood of Γ in U to a Riemannian structure of X so that $\partial_* X - \Gamma$ is its normal boundary. It follows that (X, b) admits a Riemannian structure. This skeletonwise smoothing method can be done up to isotopy and applied also for smoothing a topological isomorphism between smooth 3-orbifolds up to isotopy.

By a *graph* Γ we mean a space homeomorphic to a polyhedron of dimension 1. Let γ be the stratification by connected manifolds without boundary of Γ that is maximal with respect to subdivisoin. Then a 0-stratum and a 1-stratum of γ is called a *vertex* and an *edge* of Γ, respectively. A graph is a *Y-graph*, if each vertex of Γ has valency 3.

(5.3.2) *Let X be a connected 3-manifold without boundary. Then a b-space (X, b) is a branchfold if and only if*
 (i) *ΣX is a Y-graph*
 (ii) *$(X, \Sigma X)$ is triangulable and*
 (iii) *for each vertex v of ΣX and for three branches e_1, e_2, e_3 at v, the values $b_i = b(e_i)$, $i = 1, 2, 3$, satisfy $1/b_1 + 1/b_2 + 1/b_3 > 1$.*

A graph is said to be *doubly-connected*, if the deletion of one edge of a connected component Γ' of it does not disconnect Γ'.

Since a 3-sphere could be homeomorphic to the double of only a 3-ball, it follows from Theorem 3 that

(5.3.3) *a 3-branchfold (X, b) on a 3-sphere $X(=S^3)$ is isomorphic with the double of a regular reflection orbifold if and only if*
(1) ΣX *lies on a locally flat 2-sphere in X,*
(2) ΣX *is doubly-connected and*
(3) $b(e)=b(e')$, *whenever the deletion of two edges e and e' of a connected component Γ of ΣX disconnects Γ.*

Generality of the branch set of a good branchfold is illustrated by the following:

(5.4.3) *Let Γ be a Y-graph PL embedded in a Z_2-homology sphere X which is a closed 3-manifold. Suppose that each connected component of Γ is doubly connected and has no loop. Then there is a good branchfold (X, b) with $\Sigma X = \Gamma$.*

In fact, let $b: X \to N$ be a function defined by

$$b(x) = \begin{cases} 1, & \text{if } x \in X - \Gamma, \\ 2, & \text{if } x \in \text{an edge of } \Gamma, \text{ and} \\ 4, & \text{if } x = \text{vertex of } \Gamma. \end{cases}$$

By (5.3.2), (X, b) is clearly a branchfold with $\Sigma X = \Gamma$. By the hypothesis, each normal loop μ_j does not vanish in $H_1(X - \Gamma; Z_2)$. Since b takes the same value 2 on each edge of Γ, it follows from the mod 2 homology arguments that (X, b) is nice and hence good.

References

[1] M. A. Armstrong, The fundamental group of the orbit space of a discontinuous group, Proc. Camb. Phil. Soc., **64** (1968), 299–301.
[2] N. Bourbaki, Groups et algebres de Lie, Elements de Mathematique, Chaps. IV, V, VI, Herman, Paris (1968).
[3] J. W. Cannon, Shrinking cell-like decompositions of manifolds, Codimension three, Ann. of Math., **110** (1979), 83–112.
[4] H. S. M. Coxeter, Discrete groups generated by reflections, Ann. of Math., **35** (1934), 588–621.
[5] R. H. Fox, Covering spaces with singularities, Algebraic Geometry and Topology, Princeton Univ. Press (1957), 243–257.
[6] M. Kato, On combinatorial space forms, Scient. Papers of College Gen. Ed. Univ. of Tokyo, **30** (1980), 107–146.
[7] R. C. Kirby and L. C. Siebenmann, On the triangulation of manifolds and the Hauptvermutung, Bull. Amer. Math. Soc., **75** (1969), 742–749.

[8] J. L. Koszul, Lectures on groups of transformation. Tata Inst. Fund. Res., (1965).

[9] B. Maskit, Uniformizations of Riemannian surfaces, Contributions to Analysis, Academic Press, (1974), 293–312.

[10] ——, On Klein's combination theorem III, Advances in the Theory of Riemannien surfaces, Princeton Univ. Press, (1971), 297–315.

[11] J. W. Milnor, On the 3-dimensional Brieskorn manifold $M(p, q, r)$; Knots, Groups and 3-Manifolds, Papers dedicated to the Memory of R.H. Fox, Edited by L.P. Newvirth, Ann. Math. Study, **84** (1975), 175–225.

[12] C. Papakyriakopoulos, On Dehn's lemma and the asphericity of knots, Ann. of Math., **66** (1957), 1–26.

[13] E. Straume, The topological version of groups generated by reflections, Math. Z., **176** (1981), 429–446.

[14] W. Thurston, The geometry and topology of three-manifolds, Princeton Univ. (mimeographed notes 1978–1979).

[15] M. W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. of Math., **117** (1983), 293–324.

Department of Mathematics
Kyushu University **33**
Fukuoka, Japan