Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 123–128

Classifying Space Constructions in the Rational Homotopy Theory

Katsuyuki Shibata

§ 1. Rational homotopy theory

For simplicity's sake, we restrict our descriptions to the 1-connected case. Namely, let TOP_1 be the category of the 1-connected topological spaces, DGL_1 be that of the 1-reduced (*i.e.* $L_i=0$ for $i<1$) differential graded rational Lie algebras, and DGA^1 be that of the 1-connected (*i.e.* $A^i=0$ for $i<0$ and $i=1$, and $A^0\cong \boldsymbol{Q}$) differential graded rational algebras. Throughout these categories the notion of the (rational) weak equivalence is defined as follows.

Definition. (i) A continuous map $f: X\to Y$ is *a* (*rational*) *weak isomorphism* if the induced homomorphism on the rational homotopy groups

$$f_\#: \pi_*(X)\otimes \boldsymbol{Q}\longrightarrow \pi_*(X)\otimes \boldsymbol{Q}$$

is an isomorphism.

(ii) DGL_1-homomorphism $g: (L_*, \partial)\to(L'_*, \partial')$ is *a weak isomorphism* if the induced homomorphism on the homology Lie algebras

$$g_*: H_*(L_*, \partial)\longrightarrow H_*(L'_*, \partial')$$

is an isomorphism.

(iii) DGA^1-homomorphism $h: (A^*, d)\to(B^*, d')$ is *a weak isomorphism* if the induced homomorphism on the cohomology algebras

$$h^*: H^*(A^*, d)\longrightarrow H^*(B^*, d')$$

is an isomorphism.

(iv) In each of the three cases above, the equivalence relation generated by the weak isomorphisms is called *the* (*rational*) *weak equivalence* and the corresponding quotient category of this equivalence relation is denoted by $\mathscr{H}o_{\boldsymbol{Q}}(\_\_)$.

Received January 1, 1985.

The fundamental theorem of Quillen states as follows.

Theorem (Quillen [3]). *There exists a functor* λ: $\mathrm{TOP}_1 \to \mathrm{DGL}_1$ *which induces an equivalence of the quotient categories*:

$$\bar{\lambda}: \mathscr{H}o_Q(\mathrm{TOP}_1) \longrightarrow \mathscr{H}o_Q(\mathrm{DGL}_1).$$

The similar statement in DGA^1 case requires the finite type conditions. Namely, let TOP_1^f be the full subcategory of TOP_1 whose objects consist of the 1-connected spaces having finite dimensional rational cohomology group in each degree, and $\mathrm{DGA}^1_{\pi.f}$ be the full subcategory of DGA^1 whose objects consist of the 1-connected DGA having a finite type minimal model.

Theorem (Sullivan [7], Bousfield-Gugenheim [1]). *The simplicial polynomial form functor* A_{PL}: TOP$\to$DGA *induces an equivalence of the categories*

$$\bar{A}_{PL}: \mathscr{H}o_Q(\mathrm{TOP}_1^f) \longrightarrow \mathscr{H}o_Q(\mathrm{DGA}^1_{\pi.f}).$$

According to the two theorems above, spaces and differential graded (Lie) algebras are in bijective correspondences up to the weak equivalences. The aim of this report is to explain about the algebraic versions of the classifying space constructions for a given fiber.

An algebraic model for a fibration (under the 1-connectedness assumption) can be described in three ways; (a) in DGA^1, (b) in DGL_1, and (c) in the mixed type, i.e. the fiber in DGL_1 and the base in DGA^1 (Shibata [5]). The DGL_1 case was studied by Tanré [9] in details. So we omit this case. Tanré also gave an exposition [8] on the DGA^1 case along the line of Sullivan's original idea [7] about $B_{\mathrm{Aut}(F)}$. We are going to give here an alternative proof of the classification theorem for DGA^1 fibrations, following the classical way of classifying fiber bundles (Steenrod [6], p. 53) with the use of the integration operator (Griffith-Morgan [2]). Finally we will give a brief account on the mixed type case.

I am indebted to H. Shiga for various suggestions.

§ 2. DGA^1_f fibrations

For a fibration

$$F \xrightarrow{\ i\ } E \xrightarrow{\ f\ } B$$

with F, E and B objects of TOP_1^f, there is a commutative diagram

$$\begin{array}{ccccc} A_{PL}(B) & \xrightarrow{f^*} & A_{PL}(E) & \xrightarrow{i^*} & A_{PL}(F) \\ \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \eta} \\ (A^*, d_A) & \xrightarrow{\iota} & (A^* \otimes \mathcal{M}^*, \hat{d}) & \xrightarrow{\pi} & (\mathcal{M}^*, d) \end{array}$$

where ξ is an arbitrarily given finite type DGA[1] model for $A_{PL}(B)$, η a minimal model for $A_{PL}(F)$, and ψ a model for $A_{PL}(E)$ extending $f^* \circ \xi$. If we choose a homogeneous additive base $\{1, x_{(1)}, x_{(2)}, \cdots, x_{(i)}, \cdots\}$ for A^*, the twisted differential $\hat{d}$ on $A^* \otimes \mathcal{M}^*$ can be described as follows.

$$\hat{d}(a \otimes 1) = d_A(a) \otimes 1 \quad (a \in A^*), \quad \text{and}$$
$$\hat{d}(1 \otimes v) = 1 \otimes d(v) + \sum_{i \geqq 1} x_{(i)} \otimes \theta_{(i)}(v) \quad (v \in \mathcal{M}^*).$$

A tensor product of two finite type objects of DGA equipped with a twisted differential (which is always assumed to be a derivation) of the type above is called *a* DGA_f *fibration.* If, furthermore, the base and the fiber are 1-connected, the fibration is called *a* DGA^1_f *fibration.*

Denoting the dual space $\text{Hom}_{\boldsymbol{Q}}(A^+, \boldsymbol{Q})$ by ${}^\sharp A^+$ for brevity, a linear map $\varphi\colon {}^\sharp A^+ \to \text{Der}^-(\mathcal{M}^*)$ is defined by

$$\varphi(f)(v) = \langle f \otimes 1, \hat{d}(1 \otimes v) \rangle = \sum_{i \geqq 1} f(x_{(i)}) \otimes \theta_{(i)}(v) \in \boldsymbol{Q} \otimes \mathcal{M}^* \cong \mathcal{M}^*,$$

where $\text{Der}^-(\mathcal{M}^*)$ is the set of all the negative degree derivations of $\mathcal{M}^*$. Composing the suspension (the shift of degree by $+1$) and taking the dual homomorphism, we obtain a map of graded vector spaces ${}^\sharp(s\varphi)\colon {}^\sharp s\,\text{Der}^-(\mathcal{M}^*) \to {}^\sharp({}^\sharp A^*) \cong A^*$ and its free algebra extension

$$\Lambda({}^\sharp s\varphi)\colon \Lambda({}^\sharp s\,\text{Der}^-(\mathcal{M}^*)) \longrightarrow A^*.$$

Now that $\text{Der}^-(\mathcal{M}^*)$ admits a differential graded Lie algebra structure by $[\theta, \nu] = \theta \circ \nu - (-1)^{|\theta| \cdot |\nu|} \nu \circ \theta$ and $D(\theta) = [d, \theta] = d \circ \theta - (-1)^{|\theta|} \theta \circ d$, the free algebra $\Lambda({}^\sharp s\,\text{Der}^-(\mathcal{M}^*))$ admits a differential which makes it the Koszul complex of the DGL $\text{Der}^-(\mathcal{M}^*)$. By explicit computations, we can verify that the equality $(\hat{d})^2 = 0$ implies the compatibility of the homomorphism $\Lambda({}^\sharp s\varphi)$ with the differentials on both sides.

Conversely if we are given a DGA homomorphism $\psi\colon C^*(\text{Der}^-(\mathcal{M}^*)) \to A^*$, we can define a twisted differential $\hat{d}_\psi$ on $A^* \otimes \mathcal{M}^*$ by

$$\hat{d}_\psi(1 \otimes v) = 1 \otimes d(v) + \sum_j \psi({}^\sharp s\nu_{(j)}) \otimes \nu_{(j)}(v) \quad (v \in \mathcal{M}^*),$$

where $\{\nu_{(j)}\}$ is a basis of the subspace $D_{|v|} \subset \text{Der}^-(\mathcal{M}^*)$ of the derivations annihilating the elements of degree $> |v|$.

It is easily seen that the two constructions above are mutually adjoint, and that the DGA_f^1 fibration structures on $A^*\otimes\mathcal{M}^*$ are in the bijective correspondence with the DGA homomorphisms from $C^*(\mathrm{Der}^-(\mathcal{M}^*))$ to A^*.

We are going to prove the "homotopy invariance" of this correspondence.

An algebraic model for the interval [0, 1] is given by the free DGA $\Lambda(t, dt)$ with $|t|=0$. Two DGA homomorphisms $f_i\colon (A^*, d)\to(B^*, d')$ $(i=0, 1)$ are *homotopic* if there is a DGA homomorphism $F\colon (A^*, d)\to (B^*, d')\otimes\Lambda(t, dt)$ such that $p_i\circ F=f_i$ $(i=0, 1)$, where the $p_i\colon (B^*, d')\otimes\Lambda(t, dt)\to(B^*, d')$ are the homomorphisms defined by $p_i(t)=i$ and $p_i(dt)=0$. Two DGA_f fibration structures $(A^*\otimes F^*, \hat{d}_{(i)})$ $(i=0, 1)$ are *unipotently isomorphic* if there is a DGA isomorphism ψ over A^* which makes the following diagram commutative;

$$\begin{array}{ccccc} & & (A^*\otimes F^*, \hat{d}_{(0)}) & & \\ & \overset{i}{\nearrow} & & \overset{j}{\searrow} & \\ (A^*, d) & & \downarrow{\scriptstyle\psi} & & (F^*, \bar{d}) \\ & \underset{i}{\searrow} & & \underset{j}{\nearrow} & \\ & & (A^*\otimes F^*, \hat{d}_{(1)}) & & \end{array}$$

Proposition. *Let $((A^*\otimes\Lambda(t, dt))\otimes\mathcal{M}^*, \hat{d})$ be a DGA_f fibration with A^* 1-connected and finite type and $\mathcal{M}^*$ 1-connected and minimal. Then it is unipotently isomorphic to the product fibration $((A^*\otimes\Lambda(t, dt))\otimes\mathcal{M}^*, \hat{d}_0)$, where $\hat{d}_0(1\otimes v)=(i_A\otimes 1_{\mathcal{M}})\circ p_0\circ\hat{d}(v)$ for $v\in\mathcal{M}^*$ with $i_A\colon A^*\to A^*\otimes\Lambda(t, dt)$. Consequently, $(A^*\otimes\mathcal{M}^*, p_1\circ\hat{d})$ is unipotently isomorphic to $(A^*\otimes\mathcal{M}^*, p_0\circ\hat{d})$.*

Proof. Since $(\mathcal{M}^*, d)$ is minimal, $\mathcal{M}^*$ is a free algebra $\Lambda(\oplus_k V^k)\cong\otimes\Lambda(V^k)$. We can apply the homotopy extension (lifting) theorem for Hirsch extensions (Griffith-Morgan [2], Proposition 10.4) inductively on k to the diagram

$$\begin{array}{ccc} ((A^*\otimes\Lambda(t, dt))\otimes\Lambda(V^{(k-1)}), \hat{d}|) & \xrightarrow{f_{k-1}} & (A^*\otimes\Lambda(V^*), p_0\circ\hat{d}) \\ \downarrow{\scriptstyle i_{(k)}} & \overset{f_k}{\nearrow} & \downarrow{\scriptstyle \mathrm{id}} \\ ((A^*\otimes\Lambda(t, dt))\otimes\Lambda(V^{k-1}))\otimes_{\hat{d}}\Lambda(V^k) & \xrightarrow{p_0} & (A^*\otimes\Lambda(V^*), \hat{d}_0) \end{array}$$

and a homotopy $H_{k-1}\colon ((A^*\otimes\Lambda(t, dt))\otimes\Lambda(V^{(k-1)}), \hat{d}|)\to(A^*\otimes\Lambda(V^*), p_0\circ\hat{d})\otimes\Lambda(t, dt)\cong((A^*\otimes\Lambda(t, dt))\otimes\Lambda(V^*), \hat{d}_0)$ from $p_0\circ i_{(k)}$ to f_{k-1}, where f_{k-1} is an inductively constructed extension of p_1. So we obtain an extension $f_k\colon (A^*\otimes\Lambda(t, dt))\otimes\Lambda(V^{(k)})\to(A^*\otimes\Lambda(V^*), p_0\circ\hat{d})$ of f_{k-1} and an extension H_k of H_{k-1} to a homotopy from p_0 to f_k. This H_k is explicitly defined by the formula

$$H_k(v)=v+\int_0^t H_{k-1}(\hat{d}(v)) \quad (v \in V^k)$$

where $\int_0^t$ denotes the integration operator with respect to $\Lambda(t, dt)$ (Griffiths-Morgan [2], p. 120). The inductive argument on k shows that $H_k(v)\equiv v$ modulo $A^+\otimes\Lambda(t, dt)\otimes\Lambda(V^*)$. This implies that

$$H=\lim H_k\colon ((A^*\otimes\Lambda\,(t,dt))\otimes\Lambda(V^*), \hat{d})\longrightarrow(A^*\otimes\Lambda(V^*), p_0\circ\hat{d})\otimes\Lambda(t, dt)$$

is a unipotent isomorphism. The ideal $(t-1, dt)$ is evidently preserved by H, and so the quotient homomorphism $\bar{H}$ is a unipotent isomorphism from $(A^*\otimes\Lambda(V^*), p_1\circ\hat{d})$ to $(A^*\otimes\Lambda(V^*), p_0\circ\hat{d})$. Q.E.D.

The proposition above implies that homotopic classifying homomorphisms yield unipotently isomorphic DGA^1_f fibrations. Conversely, if $(A^*\otimes\Lambda(V^*), d_0)$ and $(A^*\otimes\Lambda(V^*), d_1)$ are unipotently isomorphic by θ, then $\psi=\log\theta$ is a nilpotent derivation and θ is expressed as $\exp\psi$. Using the product differential $\hat{d}_0=d_0\otimes 1_t+1\otimes d_t$ on $(A^*\otimes\mathcal{M}^*)\otimes\Lambda(t, dt)$, we define a new differential $\hat{d}$ by $\hat{d}=\exp(t\psi)\circ\hat{d}_0\circ\exp(-t\psi)$. The classifying homomorphism of this DGA_f fibration is a homotopy between those of d_0 and d_1. Henceforth we conclude as follows.

Theorem. *The unipotent isomorphism classes of* DGA^1_f *fibrations* $(A^*\otimes\mathcal{M}^*, \hat{d})$ *are in the bijective correspondence with the homotopy classes of* DGA *homomorphisms from* $C^*(\mathrm{Der}^-(\mathcal{M}^*))$ *to* A^*.

§ 3. Mixed type fibrations

A mixed type fibration over a DGA^1 (A^*, d) is a triple (L_*, χ, D), where L_* is a geraded Lie algebra over (A^*, d) and is as well a finite type free A^*-module, χ an element of L_{-2}, and D is a Lie derivation of degree -1 satisfying $D(\chi)=0$ and $D^2(y)+[\chi, y]=0$ for every $y \in L_*$. Here $A^*=A_{-*}$ is regarded as nonpositively graded and the quotient DGL $\bar{L}_*$ is assumed 1-reduced (Shibata [5]). In such a presentation, χ is classified by $s\bar{L}_*$, the suspension of $\bar{L}_*$ considered as an abelian Lie algebra, and D by by $\mathrm{Der}^+(\bar{L}_*)$, the set of the positive degree derivations of $\bar{L}_*$. Combining these, mixed type fibrations (L_*, χ, D) over (A^*, d) with fiber $\bar{L}_*$ are in a bijective correspondence with the DGA homomorphisms from the Koszul complex $C^*(s\bar{L}_*\oplus\mathrm{Der}^+(\bar{L}_*))$ to (A^*, d). The conditions $D(\chi)=0$ and $D^2(y)+[\chi, y]=0$ for every $y \in L_*$ require that the twisted product structure of DGL on $s\bar{L}_*\oplus\mathrm{Der}^+(\bar{L}_*)$ should be as follows: $[s\bar{x}, \theta]=-(-1)^{|\bar{x}|\cdot|\theta|}s\theta(\bar{x})$, and $\partial(s\bar{x})=-s\bar{D}(\bar{x})\oplus\mathrm{ad}\,(\bar{x})$. This DGL was first discovered by Schlessinger-Stasheff [4].

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Department of Mathematics
Saitama University
Urawa, 338 Japan