Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 687–697

A Necessary and Sufficient Condition for a Local Commutative Algebra to be a Moduli Algebra: Weighted Homogeneous Case

Stephen S.-T. Yau*

Let $\mathcal{O}_{n+1}=\boldsymbol{C}\{z_0, z_1, \cdots, z_n\}$ denote the ring of germs at the origin of holomorphic functions $(\boldsymbol{C}^{n+1}, 0)\to\boldsymbol{C}$. If $(V, 0)$ is a germ at the origin of a hypersurface in $\boldsymbol{C}^{n+1}$, let $I(V)$ be the ideal of functions in $\mathcal{O}_{n+1}$ vanishing on V, and let f be a generator of $I(V)$. It is well known that $V-\{0\}$ is nonsingular if and only if the $\boldsymbol{C}$-vector space

$$A(V)=\mathcal{O}_{n+1}/((f)+\Delta(f))$$

is finite dimensional, where $\Delta(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by the first partial derivatives of f. $A(V)$, provided with the obvious $\boldsymbol{C}$-algebra structure, is called the moduli algebra of V. In [4] the following theorem was proved.

Theorem 1 (Mather-Yau). *Suppose $(V, 0)$ and $(W, 0)$ are germs of hypersurfaces in $\boldsymbol{C}^{n+1}$, and $V-\{0\}$ is nonsingular. Then $(V, 0)$ is biholomorphically equivalent to $(W, 0)$ if and only if $A(V)$ is isomorphic to $A(W)$ as a $\boldsymbol{C}$-algebra.*

It is natural to raise the recognition problem: When a commutative local Artinian algebra is a moduli algebra? How can one construct the singularity $(V, 0)$ explicitly from the moduli algebra $A(V)$. In this short note, we shall answer the above questions in the case $(V, 0)$ is a weighted homogeneous singularity. We thank Herwig Hauser for encouraging us in writing up this note for publication.

Let A be a commutative Noetherian algebra with maximal ideal m. Let $x_1, \cdots, x_k$ be a system of minimal generating set of m such that their images in m/m^2 form a basis. Consider the algebra homomorphism

$$\varphi\colon \boldsymbol{C}\{z_1, \cdots, z_k\}\longrightarrow A$$

Received December 20, 1984.
Revised February 28, 1986.
*) Research partially supported by N.S.F. Grant #DMS-8411477.

where $\varphi(z_i)=x_i$ for all $1\le i\le k$. Let Δ be the kernel of φ. Then $\boldsymbol{C}\{z_1, \cdots, z_k\}/\Delta$ is isomorphic to A. Therefore to determine whether A is a moduli algebra, it suffices to determine when Δ is a moduli ideal, i.e., an ideal of the form $(f(z_1, \cdots, z_k), (\partial f/\partial z_1)(z_1, \cdots, z_k), \cdots, (\partial f/\partial z_k)(z_1, \cdots, z_k))$ in $\mathcal{O}_k$.

Theorem 2. *Let* $\Delta=(g_1(z_1, \cdots, z_k), g_2(z_1, \cdots, z_k), \cdots, g_l(z_1, \cdots, z_k))\mathcal{O}_k$ *be an ideal in* $\mathcal{O}_k$ *with* l *generators where* $1\le l\le k$. *A sufficient condition for* Δ *to be a moduli ideal is the following. There exists a* $k\times l$ *matrix* B *of rank* l *with entries in* $\mathcal{O}_k$ *such that*

$$\frac{\partial F_i}{\partial z_j}=\frac{\partial F_j}{\partial z_i} \qquad \forall 1\le i, j\le k$$

where

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_k \end{bmatrix}=\begin{bmatrix} b_{11} & b_{12}\cdots b_{1l} \\ b_{21} & b_{22}\cdots b_{2l} \\ \cdots & \cdots \\ b_{k1} & b_{k2}\cdots b_{kl} \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{bmatrix}$$

and $(F_1, F_2, \cdots, F_k)\mathcal{O}_k$ *is a weighted homogeneous ideal, i.e.,* $\exists d_1, d_2, \cdots, d_k, l_1, \cdots, l_k \in \boldsymbol{Z}$ *such that for any* $1\le i\le k$

$$F_j(t^{d_1}z_1, \cdots, t^{d_k}z_k)=t^{l_j}F_j(z_1, \cdots, z_k) \quad \forall(z_1, \cdots, z_k)\in \boldsymbol{C}_k \quad t\in \boldsymbol{C}-\{0\}.$$

Proof. $$\frac{\partial F_i}{\partial z_j}=\frac{\partial F_j}{\partial z_i} \qquad \forall 1\le i, j\le k$$

$\Rightarrow \omega=F_1(z_1, \cdots, z_k)dz_1+\cdots+F_k(z_1, \cdots, z_k)dz_k$ is a d-closed holomorphic 1-form

$\Rightarrow \omega=df$ for some $f\in\mathcal{O}_k$ by the Poincaré Lemma

$$\Rightarrow \frac{\partial f}{\partial z_1}=F_1, \ \frac{\partial f}{\partial z_2}=F_2, \cdots, \ \frac{\partial f}{\partial z_k}=F_k$$

$\Rightarrow \Delta(f)\subseteq(g_1, g_2, \cdots, g_l)$.

On the other hand, the fact that the $k\times l$ matrix B is of rank l implies that

$$(g_1, g_2, \cdots, g_l)\mathcal{O}_k\subseteq(F_1, F_2, \cdots, F_k)\mathcal{O}_k=\Delta(f).$$

Hence $(g_1, g_2, \cdots, g_l)\mathcal{O}_k=\Delta(f)$. In order to prove that $(g_1, g_2, \cdots, g_l)\mathcal{O}_k$ is a moduli ideal, it suffices to prove that f is in $\Delta(f)$.

$$f(z_1, z_2, \cdots, z_k)=\int_0^1 \frac{d}{dt} f(t^{d_1}z_1, \cdots, t^{d_k}z_n)dt$$

$$=\int_0^1 \Big[d_1 t^{d_1-1}z_1 \frac{\partial f}{\partial z_1}(t^{d_1}z_1, \cdots, t^{d_k}z_k)+\cdots$$

$$+d_k t^{d_k-1}z_k \frac{\partial f}{\partial z_k}(t^{d_1}z_1, \cdots, t^{d_k}z_k)\Big]dt$$

$$=\int_0^1 \Big[d_1 t^{d_1-1}z_1 F_1(t^{d_1}z_1, \cdots, t^{d_k}z_k)+\cdots+d_k t^{d_k-1}z_k F_k(t^{d_1}z_1, \cdots, t^{d_k}z_k)\Big]dt$$

$$=\int_0^1 [d_1 t^{d_1+l_1-1}z_1 F_1(z_1, \cdots, z_k)+\cdots+d_k t^{d_k+l_k-1}z_k F_k(z_1, \cdots, z_k)]dt$$

$$=\frac{d_1}{d_1+l_1} z_1 F_1(z_1, \cdots, z_k)+\cdots+\frac{d_k}{d_k+l_k} z_k F_k(z_1, \cdots, z_k)$$

$$=\frac{d_1}{d_1+l_1} z_1 \frac{\partial f}{\partial z_1}(z_1, \cdots, z_k)+\cdots+\frac{d_k}{d_k+l_k} z_k \frac{\partial f}{\partial z_k}(z_1, \cdots, z_k). \qquad \text{Q.E.D.}$$

Theorem 3. *Let*

$$\Delta=(g_1(z_1, \cdots, z_k), g_2(z_1, \cdots, z_k), \cdots, g_l(z_1, \cdots, z_k))\mathcal{O}_k$$

be an ideal in $\mathcal{O}_k$ with l generators where $1 \leq l \leq k$. A necessary and sufficient condition for Δ to be a moduli ideal of a weighted homogeneous function is the following. There exists a $k \times l$ matrix B of rank l with entries in $\mathcal{O}_k$ such that

$$\frac{\partial F_i}{\partial z_j}=\frac{\partial F_j}{\partial z_i} \qquad \forall 1 \leq i, j \leq k$$

where

$$\begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_k \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots b_{1l} \\ b_{21} & b_{22} & \cdots b_{2l} \\ \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots b_{kl} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{bmatrix}$$

and there exist $d_1, \cdots, d_k, d \in \boldsymbol{Z}$ such that $\forall 1 \leq i \leq k$

$$F_i(t^{d_1}z_1, \cdots, t^{d_k}z_k)=t^{d-d_i}F_i(z_1, \cdots, z_k) \quad \forall (z_1, \cdots, z_k) \in \boldsymbol{C}^k \quad t \in \boldsymbol{C}-\{0\}.$$

We shall need the following lemma.

Lemma 4. *Let $l \leq k$ be two positive integers. Let A be a $l \times k$ matrix and B be a $k \times l$ matrix with entries in $\boldsymbol{C}$. Then there exists a $k \times l$ matrix C with entries in $\boldsymbol{C}$ such that the matrix*

$$C(I-AB)+B$$

has rank l, *where* I *is the identity matrix of* rank l.

Proof. Let $\alpha\colon \boldsymbol{C}^k \to \boldsymbol{C}^l$ and $\beta\colon \boldsymbol{C}^l \to \boldsymbol{C}^k$ be the linear transformation corresponding to A and B respectively. Choose a basis $e_1, \cdots, e_l$ of $\boldsymbol{C}^l$ such that $\beta e_i = 0$, $i \geq r+1$, where r is the rank of β. Choose $e'_{r+1}, \cdots, e'_k$ in $\boldsymbol{C}^k$ such that $\beta e_1, \cdots, \beta e_r, e'_{r+1}, \cdots, e'_k$ is a basis of $\boldsymbol{C}^k$. Let $\gamma\colon \boldsymbol{C}^l \to \boldsymbol{C}^k$ be the linear transformation defined by $\gamma e_i = 0$, $1 \leq i \leq r$ and $\gamma e_i = e'_i$, $r+1 \leq i \leq l$. Then

$$[\gamma(1-\alpha\beta)+\beta](e_i) = \begin{cases} \beta e_i + \sum_{j=r+1}^{l} d_{ij} e'_j & \text{if } 1 \leq i \leq r \\ e'_i & \text{if } r+1 \leq i \leq l \end{cases}$$

so $\gamma(1-\alpha\beta)+\beta$ has maximal rank l. This proves the lemma, where we take for C the matrix corresponding to γ.

Proof of Theorem 3. Necessary condition: Suppose $\Delta = (g_1, \cdots, g_l)\mathcal{O}_k$ is a moduli ideal of a weighted homogeneous function, i.e., there exist $d_1, \cdots, d_k, d \in \boldsymbol{Z}$ such that

$$f(t^{d_1}z_1, \cdots, t^{d_k}z_k) = t^d f(z_1, \cdots, z_k) \quad \forall (z_1, \cdots, z_k) \in \boldsymbol{C}^k \quad t \in \boldsymbol{C}-\{0\}.$$

Since f is in the Jacobian ideal of f, we have

$$\left(\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_k}\right)\mathcal{O}_k = (g_1, \cdots, g_l)\mathcal{O}_k.$$

There exist $l \times k$ matrix $\tilde{A}$ and $k \times l$ matrix $\tilde{B}$ with entries in $\mathcal{O}_k$ such that

$$\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{bmatrix} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \cdots \tilde{b}_{1l} \\ \tilde{b}_{21} & \tilde{b}_{22} \cdots \tilde{b}_{2l} \\ \cdots & \cdots \\ \tilde{b}_{k1} & \tilde{b}_{k2} \cdots \tilde{b}_{kl} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{bmatrix}$$

and

$$\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \cdots \tilde{a}_{1k} \\ \tilde{a}_{21} & \tilde{a}_{22} \cdots \tilde{a}_{2k} \\ \cdots & \cdots \\ \tilde{a}_{l1} & \tilde{a}_{l2} \cdots \tilde{a}_{lk} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{bmatrix}.$$

Apply Lemma 5 to the matrices $\tilde{A}(0)$ and $\tilde{B}(0)$, we fined a $k \times l$ matrix C such that

$$C(I-\tilde{A}(0)\tilde{B}(0))+\tilde{B}(0)$$

has rank l.

Now we take $F_i=\partial f/\partial z_i$, $1\leq i\leq k$; and $B=C(I-\tilde{A}\tilde{B})+\tilde{B}$. Then clearly $\partial F_i/\partial z_j=\partial F_j/\partial z_i$ $\forall 1\leq i,j\leq k$ and

$$F_i(t^{d_1}z_1,\cdots,t^{d_k}z_k)=t^{d-d_i}F_i(z_1,\cdots,z_k)\quad \forall(z_1,\cdots,z_k)\in \boldsymbol{C}^k\quad t\in \boldsymbol{C}-\{0\}.$$

It remains to check $(F)=B(G)$ where

$$(F)=\begin{bmatrix}F_1\\F_2\\ \vdots\\ F_k\end{bmatrix}\quad\text{and}\quad (G)=\begin{bmatrix}g_1\\g_2\\ \vdots\\ g_l\end{bmatrix}$$

$$\begin{aligned}B(G)&=[C(I-\tilde{A}\tilde{B})+\tilde{B}](G)=C[(G)-\tilde{A}\tilde{B}(G)]+\tilde{B}(G)\\&=C[(G)-\tilde{A}(F)]+(F)\\&=C[(G)-(G)]+(F)\\&=(F).\end{aligned}$$

Sufficient condition: By the proof of Theorem 2, we know that $(g_1,\cdots,g_l)\mathcal{O}_k$ is a moduli ideal of a function f which satisfies the following equation.

$$f(z_1,z_2,\cdots,z_k)=\frac{d_1}{d}z_1F_1(z_1,\cdots,z_k)+\cdots+\frac{d_k}{d}z_kF_k(z_1,\cdots,z_k)$$

$$\begin{aligned}\Rightarrow f(t^{d_1}z_1,\cdots,t^{d_k}z_k)&=\frac{d_1}{d}(t^{d_1}z_1)F_1(t^{d_1}z_1,\cdots,t^{d_k}z_k)+\cdots\\&\quad+\frac{d_k}{d}(t^{d_k}z)F_k(t^{d_1}z_1,\cdots,t^{d_k}z_k)\\&=\frac{d_1}{d}t^dz_1F_1(z_1,\cdots,z_k)+\cdots+\frac{d_k}{d}t^dz_kF_k(z_1,\cdots,z_k)\\&=t^df(z_1,z_2,\cdots,z_k)\quad\forall t\in \boldsymbol{C}-\{0\}\quad(z_1,\cdots,z_k)\in \boldsymbol{C}^k.\end{aligned}$$

Therefore f is a weighted homogeneous function. Q.E.D.

Theorem 5. *Let $\Delta=(g_1(z_1,\cdots,z_k),\ g_2(z_1,\cdots,z_k),\cdots,g_l(z_1,\cdots,z_k))\mathcal{O}_k$ be an ideal in $\mathcal{O}_k$ with l generators where $1\leq l\leq k$. Suppose $g_1(z_1,\cdots,z_k),\cdots,g_l(z_1,\cdots,z_k)$ are homogeneous polynomial of the same degree d. Then a necessary and sufficient condition for Δ to be a moduli ideal is the following. There exists a $k\times l$ matrix B of rank l with entries in $\boldsymbol{C}$ such that*

$$\frac{\partial F_i}{\partial z_j}=\frac{\partial F_j}{\partial z_i} \qquad \forall 1\leq i, j\leq k$$

where

$$\begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_k \end{pmatrix}=\begin{bmatrix} b_{11} & b_{12}\cdots b_{1l} \\ b_{21} & b_{22}\cdots b_{2l} \\ \cdots & \cdots \\ b_{k1} & b_{k2}\cdots b_{kl} \end{bmatrix}\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_l \end{pmatrix}.$$

In fact if Δ is a moduli ideal, it must be a moduli ideal of a homogeneous polynomial of degree $d+1$.

Proof. In view of Theorem 4, it is sufficient to prove the last statement.

Suppose Δ is the moduli ideal of a function f. Write

$$f=\sum_{i=m+1}^{\infty} f_i$$

where f_i is a homogeneous polynomial of degree i and $m+1$ is the multiplicity of f. The fact that $\Delta=$moduli ideal of f implies $d=m$. Since Δ is a homogeneous ideal and

$$\frac{\partial f}{\partial z_j}=\sum_{i=m+1}^{\infty}\frac{\partial f_i}{\partial z_j}\in\Delta.$$

We have $\partial f_{m+1}/\partial z_j\in\Delta\ \forall 1\leq j\leq k$. So $(\partial f_{m+1}/\partial z_1,\cdots,\partial f_{m+1}/\partial z_k)\mathcal{O}_k\subseteq\Delta$. On the other hand, for any $1\leq a\leq l$,

$$g_a=\sum_{j=1}^{k} h_{aj}\frac{\partial f}{\partial z_j} \qquad \text{where} \quad h_{aj}\in\mathcal{O}_k$$

$$=\sum_{j=1}^{k}\sum_{i=d+1}^{\infty} h_{aj}\frac{\partial f_i}{\partial z_j}.$$

Since the degree of g_a is d, by degree consideration, we have

$$g_a=\sum_{j=1}^{k} h_{aj}(0)\frac{\partial f_{d+1}}{\partial z_j}.$$

Therefore

$$\left(\frac{\partial f_{m+1}}{\partial z_1},\frac{\partial f_{m+1}}{\partial z_2},\cdots,\frac{\partial f_{m+1}}{\partial z_k}\right)\mathcal{O}_k=\Delta \qquad \text{Q.E.D.}$$

Remark. To find f explicitly, we simply use the standard method in Advanced Calculus.

Example 1. *Let* $\Delta=(3x_2^2-4x_1x_3,\ x_2x_3-2x_1x_4,\ x_3^2-x_2x_4-2x_1x_5,\ x_3x_4-3x_2x_5,\ x_4^2-2x_3x_5)\mathcal{O}_5$.

Is Δ a moduli ideal? We shall follow the above described procedure and try to find f explicitly.

$$\frac{\partial f}{\partial x_1}=a_{11}(3x_2^2-4x_1x_3)+a_{12}(x_2x_3-2x_1x_4)+a_{13}(x_3^2-x_2x_4-2x_1x_5)$$
$$+a_{14}(x_3x_4-3x_2x_5)+a_{15}(x_4^2-2x_3x_5)$$
$$\frac{\partial f}{\partial x_2}=a_{21}(3x_2^2-4x_1x_3)+a_{22}(x_2x_3-2x_1x_4)+a_{23}(x_3^2-x_2x_4-2x_1x_5)$$
$$+a_{24}(x_3x_4-3x_2x_5)+a_{25}(x_4^2-2x_3x_5)$$
$$\frac{\partial f}{\partial x_3}=a_{31}(3x_2^2-4x_1x_3)+a_{32}(x_2x_3-2x_1x_4)+a_{33}(x_3^2-x_2x_4-2x_1x_5)$$
$$+a_{34}(x_3x_4-3x_2x_5)+a_{35}(x_4^2-2x_3x_5)$$
$$\frac{\partial f}{\partial x_4}=a_{41}(3x_2^2-4x_1x_3)+a_{42}(x_2x_3-2x_1x_4)+a_{43}(x_3^2-x_2x_4-2x_1x_5)$$
$$+a_{44}(x_3x_4-3x_2x_5)+a_{45}(x_4^2-2x_3x_5)$$
$$\frac{\partial f}{\partial x_5}=a_{51}(3x_2^2-4x_1x_3)+a_{52}(x_2x_3-2x_1x_4)+a_{53}(x_3^2-x_2x_4-2x_1x_5)$$
$$+a_{54}(x_3x_4-3x_2x_5)+a_{55}(x_4^2-2x_3x_5)$$

$$\frac{\partial^2 f}{\partial x_2\partial x_1}=a_{11}(6x_2)+a_{12}x_3+a_{13}(-x_4)+a_{14}(-3x_5)$$
$$\frac{\partial^2 f}{\partial x_1\partial x_2}=a_{21}(-4x_3)+a_{22}(-2x_4)+a_{23}(-2x_5)$$

(1) $$\frac{\partial^2 f}{\partial x_2\partial x_1}=\frac{\partial^2 f}{\partial x_1\partial x_2}\Rightarrow a_{11}=0$$
$$a_{12}=-4a_{21}$$
$$a_{13}=2a_{22}$$
$$a_{14}=\tfrac{2}{3}a_{23}$$

$$\frac{\partial^2 f}{\partial x_3\partial x_1}=a_{11}(-4x_1)+a_{12}(x_2)+a_{13}(2x_3)+a_{14}(x_4)+a_{15}(-2x_5)$$

(2) $$\frac{\partial^2 f}{\partial x_1\partial x_3}=a_{31}(-4x_3)+a_{32}(-2x_4)+a_{33}(-2x_5)$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_3} \Rightarrow a_{11} = 0 = a_{12}$$

$$a_{13} = -2a_{31}$$

$$a_{14} = -2a_{32}$$

$$a_{15} = a_{33}$$

$$\frac{\partial^2 f}{\partial x_4 \partial x_1} = a_{12}(-2x_1) + a_{13}(-x_2) + a_{14}(x_3) + a_{15}(2x_4)$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_4} = a_{41}(-4x_3) + a_{42}(-2x_4) + a_{43}(-2x_5)$$

(3)
$$\frac{\partial^2 f}{\partial x_4 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_4} \Rightarrow a_{12} = 0 = a_{13} = a_{43}$$

$$a_{14} = -4a_{41}$$

$$a_{15} = -a_{42}$$

$$\frac{\partial^2 f}{\partial x_5 \partial x_1} = a_{13}(-2x_1) + a_{14}(-3x_2) + a_{15}(-2x_3)$$

(4)
$$\frac{\partial^2 f}{\partial x_1 \partial x_5} = a_{51}(-4x_3) + a_{52}(-2x_4) + a_{53}(-2x_5)$$

$$\frac{\partial^2 f}{\partial x_5 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_5} \Rightarrow a_{13} = a_{14} = 0 = \alpha_{52} = a_{53}$$

$$a_{15} = a_{51}$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = a_{21}(-4x_1) + a_{22}(x_2) + a_{23}(2x_3) + a_{24}(x_4) + a_{25}(-2x_5)$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_3} = a_{31}(6x_2) + a_{32}(x_3) + a_{33}(-x_4) + a_{34}(-3x_5)$$

(5)
$$\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_3} \Rightarrow a_{21} = 0$$

$$a_{22} = 6a_{31}$$

$$a_{23} = \tfrac{1}{2}a_{32}$$

$$a_{24} = -a_{33}$$

$$a_{25} = \tfrac{3}{2}a_{34}$$

$$\frac{\partial^2 f}{\partial x_4 \partial x_2} = a_{22}(-2x_1) + a_{23}(-x_2) + a_{24}(x_3) + a_{25}(2x_4)$$

(6)
$$\frac{\partial^2 f}{\partial x_2 \partial x_4} = a_{41}(6x_2) + a_{42}(x_3) + a_{43}(-x_4) + a_{44}(-3x_5)$$

$$\frac{\partial^2 f}{\partial x_4 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_4} \Rightarrow a_{22} = 0 = a_{44}$$

$$a_{23} = -a_{41}$$

$$a_{24} = a_{42}$$

$$a_{25} = -\tfrac{1}{2} a_{43}$$

$$\frac{\partial^2 f}{\partial x_5 \partial x_2} = a_{23}(-2x_1) + a_{24}(-3x_2) + a_{25}(-2x_3)$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_5} = a_{51}(6x_2) + a_{52}(x_3) + a_{53}(-x_4) + a_{54}(-3x_5)$$

(7) $$\frac{\partial^2 f}{\partial x_5 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_5} \Rightarrow a_{23} = 0 = a_{54} = a_{53}$$

$$a_{24} = -2a_{51}$$

$$a_{25} = -\tfrac{1}{2} a_{52}$$

$$\frac{\partial^2 f}{\partial x_4 \partial x_3} = a_{32}(-2x_1) + a_{33}(-x_2) + a_{34}(x_3) + a_{35}(2x_4)$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_4} = a_{41}(-4x_1) + a_{42}(x_2) + a_{43}(2x_3) + a_{44}(x_4) + a_{45}(-2x_5)$$

(8) $$\frac{\partial^2 f}{\partial x_4 \partial x_3} = \frac{\partial^2 f}{\partial x_3 \partial x_4} \Rightarrow a_{45} = 0$$

$$a_{32} = 2a_{41}$$

$$a_{33} = -a_{42}$$

$$a_{34} = 2a_{43}$$

$$a_{35} = \tfrac{1}{2} a_{44}$$

$$\frac{\partial^2 f}{\partial x_5 \partial x_3} = a_{33}(-2x_1) + a_{34}(-3x_2) + a_{35}(-2x_3)$$

$$\frac{\partial^2 f}{\partial x_3 \partial x_5} = a_{51}(-4x_1) + a_{52}(x_2) + a_{53}(2x_3) + a_{54}(x_4) + a_{55}(-2x_5)$$

(9) $$\frac{\partial^2 f}{\partial x_5 \partial x_3} = \frac{\partial^2 f}{\partial x_3 \partial x_5} \Rightarrow a_{54} = a_{55} = 0$$

$$a_{33} = 2a_{51}$$

$$a_{34} = -\tfrac{1}{3} a_{52}$$

$$a_{35} = -a_{53}$$

$$\frac{\partial^2 f}{\partial x_5 \partial x_4} = a_{43}(-2x_1) + a_{44}(-3x_2) + a_{45}(-2x_3)$$

$$\frac{\partial^2 f}{\partial x_4 \partial x_5} = a_{52}(-2x_1) + a_{53}(-x_2) + a_{54}(x_3) + a_{55}(2x_4)$$

(10) $$\frac{\partial^2 f}{\partial x_5 \partial x_4} = \frac{\partial^2 f}{\partial x_4 \partial x_5} \Rightarrow a_{55} = 0$$

$$a_{43} = a_{52}$$

$$a_{44} = \tfrac{1}{3} a_{53}$$

$$a_{45} = -\tfrac{1}{2} a_{54}$$

$$(1), (2), \cdots, (10) \Rightarrow \begin{cases} a_{15} = a_{33} = -a_{42} = -a_{24} = 2a_{51} = c \\ a_{ij} = 0 \qquad \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial x_1} = c(x_4^2 - 2x_3 x_5)$$

$$\frac{\partial f}{\partial x_2} = -c(x_3 x_4 - 3x_2 x_5)$$

$$\frac{\partial f}{\partial x_3} = c(x_3^2 - x_2 x_4 - 2x_1 x_5)$$

$$\frac{\partial f}{\partial x_4} = c(x_2 x_3 - 2x_1 x_4)$$

$$\frac{\partial f}{\partial x_5} = \frac{c}{2}(3x_2^2 - 4x_1 x_3)$$

$$\Rightarrow f = c(x_1 x_4^2 - 2x_1 x_3 x_5) + h_1(x_2, x_3, x_4, x_5)$$

$$\Rightarrow \frac{\partial f}{\partial x_2} = \frac{\partial h_1}{\partial x_2}$$

$$\Rightarrow h_1(x_2, x_3, x_4, x_5) = -c x_2 x_3 x_4 + \frac{3c}{2} x_2^2 x_5 + h_2(x_3, x_4, x_5)$$

$$\therefore \quad f = c(x_1 x_4^2 - 2x_1 x_3 x_5) - c x_2 x_3 x_4 + \frac{3c}{2} x_2^2 x_5 + h_2(x_3, x_4, x_5)$$

$$\Rightarrow \frac{\partial f}{\partial x_3} = -2c x_1 x_5 - c x_2 x_4 + \frac{\partial h_2}{\partial x_3}(x_3, x_4, x_5)$$

$$\Rightarrow \frac{\partial h_2}{\partial x_3}(x_3, x_4, x_5) = c x_3^2$$

$$\Rightarrow h_2(x_3, x_4, x_5) = \frac{c x_3^3}{3} + h_3(x_4, x_5)$$

$$\therefore \quad f=c(x_1x_4^2-2x_1x_3x_5)-cx_2x_3x_4+\frac{3c}{2}x_2^2x_5+\frac{cx_3^3}{3}+h_3(x_4, x_5)$$

$$\Rightarrow \frac{\partial f}{\partial x_4}=2cx_1x_4-cx_2x_3+\frac{\partial h_3}{\partial x_4}(x_4, x_5)$$

$$\Rightarrow \frac{\partial h_3}{\partial x_4}(x_4, x_5)=0$$

$$\Rightarrow h_3(x_4, x_5)=h_4(x_5)$$

$$\therefore \quad f=c(x_1x_4^2-2x_1x_3x_5)-cx_2x_3x_4+\frac{3c}{2}x_2^2x_5+\frac{c}{3}x_3^3+h_4(x_5)$$

$$\Rightarrow \frac{\partial f}{\partial x_5}=-2cx_1x_3+\frac{3c}{2}x_2^2+\frac{dh_4}{dx_5}(x_5)$$

$$\Rightarrow \frac{dh_4}{dx_5}(x_5)=0$$

$$\Rightarrow h_4(x_5)=0$$

$$\Rightarrow f=c\left(x_1x_4^2-2x_1x_3x_5-x_2x_3x_4+\frac{3}{2}x_2^2x_5+\frac{x_3^3}{3}\right)$$

$\therefore$ Δ is a moduli ideal of the homogeneous polynomial $x_1x_4^2-2x_1x_3x_5$ $-x_2x_3x_4+\frac{3}{2}x_2^2x_5+\frac{x_3^3}{3}$.

Example 2. Let $\Delta=(3x^3+2y^2, yz-3xw, z^2-2yw)\mathcal{O}_4$. It is an easy exercise to prove that Δ is not a moduli ideal.

References

[1] Benson, M., Analytic equivalence of isolated hypersurface singularities defined by homogenous polynomials, Proc. Symp. Pure Math., Vol. 40, Amer. Math. Soc. (1983).

[2] Benson, M., Ph.D. Thesis, Harvard University.

[3] Mather, J., Yau, S.S.-T., Criterion for biholomorphic equivalence of isolated hypersurface singularities, Proc. Nat. Acad. Sci., **78** (1981), 5946–5947.

[4] ——, Classification of isolated hypersurface singularities by their moduli algebra, Invent. Math., **69** (1982), 243–251.

[5] Shoshitaishivili, A. N., Functions with isomorphic Jacobian ideals (translated from Russian), Functional Anal. Appl., **10** (1976), 128–133.

[6] Yau, S. S.-T., Criteria for Right-Left Equivalence and Right Equivalence of Holomorphic Functions with Disolated Critical Points, Proc. Symp. Pure Math., Vol. 41, Amer. Math. Soc. (1984).

Department of Mathematics
University of Illinois at Chicago
Chicago, Illinois 60680
U.S.A.