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Three-Dimensional Cusp Singularities

Hiroyasu Tsuchihashi

Introduction

2-dimensional cusp singularities are characterized by the exceptional sets of their resolutions. Namely, a 2-dimensional isolated singularity (V, p) is a cusp singularity, if and only if it has a resolution $(U, X) \to (V, p)$ whose exceptional set X is a cycle of rational curves or a rational curve with a node. (See, for instance, Laufer [2].) Thus assigning to each 2-dimensional cusp singularity the weighted dual graph of the exceptional set X of its unique minimal resolution, we have a one-to-one correspondence between the set of all isomorphism classes of 2-dimensional cusp singularities and the set of all weighted triangulations of circles whose weights are not greater than -2 and at least one of whose weights is strictly smaller than -2. Here the weighted dual graph of X is the dual graph of X, in the usual sense, to each vertex of which is attached the self-intersection number of the corresponding rational curve as the weight, when X is a cycle of rational curves. When X is a rational curve with a node, the weighted dual graph of X is a circle with one vertex, to which is attached X^2-2 as the weight.

The purpose of this paper is to find out the 3-dimensional analogues.

First, we define toric divisors as analogues of the exceptional sets of resolutions of 2-dimensional cusp singularities (Definition 1.1). For example, all the isolated cusp singularities which appear in compactifications of quotient spaces of tube domains have resolutions whose exceptional sets are toric divisors (see [1]). Then we define a "cusp" singularity to be a 3-dimensional isolated singularity having a resolution whose exceptional set is a toric divisor. Next, we define the weighted dual graph of a toric divisor, which is a graph on a compact topological surface with two integers attached to each edge as weights (Definition 1.2). Then we have the following result (Theorem 1.6), which we prove in Section 2:

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An arbitrary weighted graph on a compact topological surface is a weighted dual graph of a toric divisor arising as the exceptional set of a resolution of a 3-dimensional isolated singularity, if and only if it satisfies the monodromy condition (Definition 1.3) and the convexity condition (Definition 1.5).

However, we cannot expect to have a natural one-to-one correspondence between the set of all isomorphism classes of 3-dimensional "cusp" singularities and the set of all the weighted graphs on compact topological surfaces satisfying these two conditions. For one thing, a 3-dimensional isolated singularity does not have a unique minimal resolution. For another, there may be many 3-dimensional "cusp" singularities with resolutions, the weighted dual graph of whose exceptional sets coincide with a given weighted graph on a compact topological surface. However, we have the following commutative diagram.

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{3-dimensional ``cusp''}\\ \text{singularities}\end{array}\right\} & \left\{\begin{array}{l}\text{Weighted graphs on compact topological}\\ \text{surfaces satisfying the monodromy condi-}\\ \text{tion and the convexity condition}\end{array}\right\} \\
\left\{\begin{array}{l}\text{Isomorphism classes of pairs of a}\\ \text{3-dimensional complex manifold}\\ \text{and a toric divisor on it contract-}\\ \text{ible to a point}\end{array}\right\} \arrow[u, "h"] \arrow[ur, "i"] \arrow[r, "k"] & \left\{\begin{array}{l}\text{Isomorphism classes of toric}\\ \text{divisors as reduced 2-dimen-}\\ \text{sional analytic spaces}\end{array}\right\} \arrow[u, "j"']
\end{tikzcd}$$

The map h is surjective by definition. The map i is also surjective by the above result, but is not injective.

In Section 3, we examine the inverse image under j of a weighted graph Δ on a compact topological surface $|\Delta|$ satisfying the monodromy and convexity conditions, i.e., the set χ_Δ of isomorphism classes of toric divisors as reduced analytic spaces which have a fixed weighted dual graph Δ. We show that χ_Δ is parametrized by a subset of the first cohomology group $H^1(\Gamma, T)$ of the fundamental group Γ of $|\Delta|$ acting on a 3-dimensional algebraic torus $T=(\boldsymbol{C}^*)^3$ (Theorem 3.6). In particular, if $|\Delta|$ is isomorphic to a 2-dimensional real torus, i.e., the singularity is a 3-dimensional Hilbert modular cusp singularity (see [4, p. 625]), then $\chi_\Delta=H^1(\Gamma, T)$ and is a finite group.

§ 1. The weighted dual graph of a toric divisor

We first recall the definition of torus embeddings. A torus embedding Z is an algebraic variety containing an algebraic torus $T=(\boldsymbol{C}^*)^r$ as a Zariski open set such that T acts on Z extending the natural action on itself defined by multiplication. We call a 2-dimensional torus embedding a toric surface. The union of the closure of the 1-dimensional orbits of a compact toric surface under the action of the algebraic torus is a cycle of rational curves.

Definition 1.1. A (reduced effective) divisor $X=X_1+X_2+\cdots+X_s$ of a 3-dimensional complex manifold U is said to be toric if X has only normal crossings as singularities, each irreducible component X_j of X is isomorphic to a compact toric surface and the union $\bigcup_{k\neq j} X_j\cap X_k$ of the double curves $X_j\cap X_k$ on X_j coincides with the union of the closure of all the 1-dimensional orbits on X_j.

Let X be a 2-dimensional toric divisor and let Δ be its dual graph. Namely, to each irreducible component X_j of X corresponds a vertex v_j of Δ so that X_j and X_k (resp. X_j, X_k and X_l) intersect along a curve (resp. at a point) if and only if v_j and v_k are joined by an edge of Δ (resp. v_j, v_k and v_l form a triangle in Δ).

If X has only simple normal crossings, then by the definition of toric divisors, the dual graph Δ of X is a triangulation of a compact topological surface $S=|\Delta|$. In general, Δ is not a triangulation, because an irreducible component of X may have self-intersection. However, there exists a triangulation $\tilde{\Delta}$ of the universal covering surface $\tilde{S}$ of S invariant under the action of the fundamental group $\pi_1(S)$ such that $\Delta=\tilde{\Delta}/\pi_1(S)$. We call $\tilde{\Delta}$ the triangulation associated to the dual graph Δ.

Definition 1.2. The weighted dual graph WDG(X) of a 2-dimensional toric divisor X is its dual graph Δ as above with a pair of integers attached as weights on both ends of each edge of Δ in the following way: Let X_j and X_k be irreducible components of X intersecting along a double curve E. Let v_j, v_k and e be the vertices and the edge of Δ corresponding to X_j, X_k and E, respectively. Then we attach the self-intersection numbers $(E|_{X_j})^2=(X_j\cdot X_k^2)$ and $(E|_{X_k})^2=(X_j^2\cdot X_k)$ of E on X_j and X_k to the sides of the vertices v_k and v_j of the edge e, respectively (see Figure 1.1).

The above definition is not exact, if $X_j=X_k$. So in this case, we need the following correction: Take a finite unramified covering space S' of $S=|\Delta|$ with a covering transformation group G so that the inverse image Δ' of Δ is a triangulation of S'. Then there exists an unramified covering space X' of X such that the dual graph of X' coincides with Δ' (cf. Section 3 for the construction of X'). Since X' has only simple normal crossings, the weighted dual graph WDG(X') of X' is well-defined and is clearly G-invariant. Then the weighted dual graph WDG(X) of X

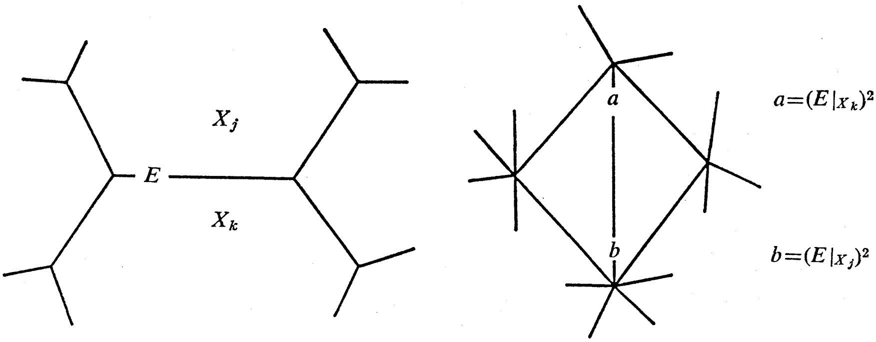


Figure 1.1

is the image WDG$(X')/G$ of WDG(X'). Although we mostly treat X below as if it has only simple normal crossings for simplicity, the same results hold for general X after this modification.

Lifting to each edge of $\tilde{\varDelta}$ two integers on its image, we naturally have a $\pi_1(|\varDelta|)$-invariant weighting on the triangulation $\tilde{\varDelta}$ associated to $\varDelta$. We call $\tilde{\varDelta}$ with this weighting the weighted triangulation associated to WDG(X).

More generally, we consider weighted graphs which need not be the weighted dual graphs of toric divisors. Namely, a weighted graph $\varDelta$ is a graph on a compact topological surface S, on both ends of whose edges are attached integers and which is the image $\tilde{\varDelta}/\pi_1(S)$ of a $\pi_1(S)$-invariant triangulation $\tilde{\varDelta}$ of the universal covering space $\tilde{S}$ of S. Each edge of $\tilde{\varDelta}$ has as its natural $\pi_1(S)$-invariant weights two integers which are inverse images of those on its image. We call $\tilde{\varDelta}$ the weighted triangulation associated to the weighted graph $\varDelta$.

Definition 1.3. A weighted graph $\varDelta$, or the weighted triangulation $\tilde{\varDelta}$ associated to it, is said to satisfy the monodromy condition at a vertex v of $\tilde{\varDelta}$, if the following hold: Let $v_1, v_2, \cdots, v_s$ be the vertices of the link of v going around v in this order. We first attach three elements n, n_1 and n_2 of a basis $\{n, n_1, n_2\}$ of $\boldsymbol{Z}^3$ to the vertices v, v_1 and v_2, respectively. Then we can determine the elements $n_3, \cdots, n_s, n_{s+1}$ and n_{s+2} of $\boldsymbol{Z}^3$ attached to the vertices $v_3, \cdots, v_s, v_1$ and v_2, respectively, by the equalities

$$(*) \quad n_{i-1}+a_i n_i+n_{i+1}+b_i n=0, \quad i=3, \cdots, s+1,$$

where a_i (resp. b_i) is the weight on the side of v_i (resp. v) of the edge joining v and v_i and $a_{s+1}=a_1$ (resp. $b_{s+1}=b_1$). Then we require that $n_{s+1}=n_1$, $n_{s+2}=n_2$ and that $p(n_1), p(n_2), \cdots, p(n_s)$ go around $p(n)$ exactly once in this order, where $p: \boldsymbol{R}^3\backslash\{0\}\to S^2$ is the natural projection onto the sphere $S^2=(\boldsymbol{R}^3\backslash\{0\})/\boldsymbol{R}_{>0}$ (see Figure 1.2).

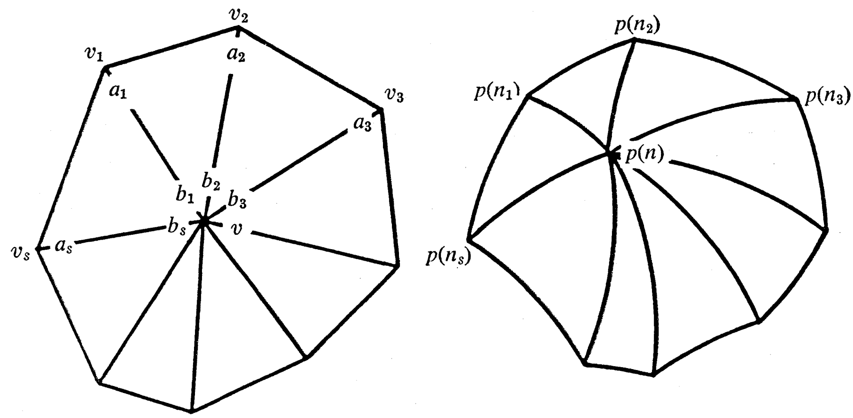


Figure 1.2

Δ or $\tilde{\Delta}$ is said to satisfy the monodromy condition, if it satisfies the monodromy condition at each vertex.

Proposition 1.4. *Let the notations be the same as in Definition* 1.3. *Then $\tilde{\Delta}$ satisfies the monodromy condition at the vertex v, if and only if the following two conditions are satisfied.*

(i) $\sum_{i=1}^{s} a_i = 12-3s$.

(ii) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & -a_s & 0 \\ 0 & -b_s & 1 \end{bmatrix} \cdots \begin{bmatrix} 0 & -1 & 0 \\ 1 & -a_2 & 0 \\ 0 & -b_2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & -a_1 & 0 \\ 0 & -b_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Proof. Let $\{n_1, n_2, n\}$ be a basis of $\boldsymbol{Z}^3$ and let $n_3, \cdots, n_{s+1}, n_{s+2}$ be the elements of $\boldsymbol{Z}^3$ obtained as in Definition 1.3. Then by (*), we have

$$[n_{i+1}, n_i, n] \begin{bmatrix} 0 & -1 & 0 \\ 1 & -a_i & 0 \\ 0 & -b_i & 1 \end{bmatrix} = [n_i, n_{i-1}, n].$$

Hence $n_{s+1} = n_1$ and $n_{s+2} = n_2$, if and only if the condition (ii) is satisfied. From now on, we assume that this condition is satisfied. For an element $l = (l^1, l^2, l^3)$ of $\boldsymbol{Z}^3$, let $l^o = (l^1, l^2) \in \boldsymbol{Z}^2$. We may assume that $n^o = (0, 0)$. Then the relation $n_{i-1}^o + a_i n_i^o + n_{i+1}^o = 0$ holds and $\{n_i^o, n_{i+1}^o\}$ is a basis of $\boldsymbol{Z}^2$, for each i in $\boldsymbol{Z}/s\boldsymbol{Z}$. Moreover, $n_1^o, n_2^o, \cdots$ and n_s^o go around the origin exactly t times, if and only if $p(n_1), p(n_2), \cdots$ and $p(n_s)$ go around $p(n)$ exactly t times. When $t=1$, {faces of $\boldsymbol{R}_{\geq 0} n_i^o + \boldsymbol{R}_{\geq 0} n_{i+1}^o \mid i \in \boldsymbol{Z}/s\boldsymbol{Z}$} is an r.p.p. decomposition associated to a nonsingular compact toric surface. Hence by [3, § 8], we have $\sum_{i=1}^{s} a_i = 12-3s$. On the other hand, when

$t>1$, using [3, Corollary 8.5] and by an easy calculation, we have $\sum_{i=1}^{s} a_i = 12t-3s$. Hence $t=1$, if and only if the condition (i) is satisfied.

q.e.d.

Definition 1.5. A weighted graph Δ or the weighted triangulation $\tilde{\Delta}$ associated to it, is said to satisfy the convexity condition, if we can associate a positive integer $m_{[v]}$ to each vertex $[v]$ of Δ so that the following two conditions are satisfied.

(i) For each edge of $\tilde{\Delta}$, the inequality

$$a\cdot m_{[u]}+b\cdot m_{[v]}+m_{[w]}+m_{[t]}\leqq 0$$

holds, where two triples (u, v, w) and (u, v, t) form two adjacent triangles, a and b are the weights on the u-side and v-side of the edge, respectively, and $[v]$ is the image of v under the projection $\tilde{\Delta}\to\Delta=\tilde{\Delta}/\pi_1(S)$.

(ii) If we delete from $\tilde{\Delta}$ all edges for which the equalities in (i) hold, then we still have a polygonal decomposition $\tilde{\square}$ of $\tilde{S}$.

Clearly, $\tilde{\square}$ is $\pi_1(S)$-invariant. Hence we have a cell division $\square=\tilde{\square}/\pi_1(S)$ of S. The convexity condition is a generalization of the condition (3) of [4, Proposition 4.6].

Theorem 1.6. *Let Δ be a weighted graph. There exist a 3-dimensional isolated singularity (V, p) and its resolution $(U, X)\to(V, p)$ whose exceptional set X is a toric divisor with* $\mathrm{WDG}(X)=\Delta$, *if and only if Δ satisfies the monodromy condition and the convexity condition.*

We devote Section 2 to the proof of this theorem.

§ 2. Proof of Theorem 1.6

First, we show that for a weighted graph Δ satisfying the monodromy condition and the convexity condition, a cusp singularity and its resolution as in the theorem exist. Choose a basis $\{n_1, n_2, n_3\}$ of $\boldsymbol{Z}^3$ and a triangle of $\tilde{\Delta}$. Let $\sigma(v_1)=n_1$, $\sigma(v_2)=n_2$ and $\sigma(v_3)=n_3$, where v_1, v_2 and v_3 are the vertices of the triangle. Then by the equalities (*) of the monodromy condition, we obtain a map σ: {all vertices of $\tilde{\Delta}$}$\to\boldsymbol{Z}^3$. Moreover, we have a unique homomorphism $\rho: \pi_1(S)\to\mathrm{GL}(3, \boldsymbol{Z})$ satisfying $\rho(\gamma)\sigma(v)=\sigma(\gamma v)$ for each element γ of $\pi_1(S)$ and for each vertex v of $\tilde{\Delta}$. Let $\bar{\sigma}(v):=m_{[v]}^{-1}\cdot\sigma(v)\in\boldsymbol{Q}^3$ for each vertex v of $\tilde{\Delta}$. Let u, v, w and t be vertices of $\tilde{\Delta}$ such that two triples (u, v, w) and (u, v, t) form two adjacent triangles. Then we have $a\cdot m_{[u]}\cdot\bar{\sigma}(u)+b\cdot m_{[v]}\cdot\bar{\sigma}(v)+m_{[w]}\cdot\bar{\sigma}(w)+m_{[t]}\cdot\bar{\sigma}(t)=0$, because $a\cdot\sigma(u)+b\cdot\sigma(v)+\sigma(w)+\sigma(t)=0$. Hence the point $\bar{\sigma}(t)$

is either on the plane passing through the points $\bar{\sigma}(u)$, $\bar{\sigma}(v)$ and $\bar{\sigma}(w)$ or in the half space not containing 0 and determined by the plane, according as $a\cdot m_{[u]}+b\cdot m_{[v]}+m_{[w]}+m_{[t]}$ is zero or negative. Thus in the same way as in [4, Proposition 4.3 and Theorem 4.5], we have a cusp singularity $(V, p)=\mathrm{Cusp}(C, \Gamma)$ and its resolution $(U, X)\to(V, p)$ such that the exceptional set X is a toric divisor with $\mathrm{WDG}(X)=\Delta$, where C is the smallest open convex cone containing the image of σ and where $\Gamma=\rho(\pi_1(S))$. (For the definition of $\mathrm{Cusp}(C, \Gamma)$, see [4].)

Next, we show that the weighted dual graph $\Delta=\mathrm{WDG}(X)$ of a toric divisor X of a 3-dimensional complex manifold U satisfies the monodromy condition. Clearly, we have:

Lemma 2.1. *Let X be a toric divisor of a 3-dimensional complex manifold U and let X_j and X_k be two irreducible components of X intersecting along a double curve E. Then*

$$(E|_{X_j})^2=\deg(\mathcal{O}_E(X_k)) \quad \textit{and} \quad (E|_{X_k})^2=\deg(\mathcal{O}_E(X_j)),$$

where $\mathcal{O}_E(X_k)$, for instance, is the invertible sheaf on E, which is the restriction of the invertible sheaf $\mathcal{O}_U(X_k)$ on U.

Lemma 2.2. *Let X be a toric divisor of a 3-dimensional complex manifold. Then the weighted dual graph of X satisfies the monodromy condition.*

Proof. Let Z be an irreducible component of X and let $D=D_1+D_2+\cdots+D_s$ be the double curves on Z, which form a cycle of rational curves. Here, we may assume that D_i and D_{i+1} intersect at a point for each i in $\boldsymbol{Z}/s\boldsymbol{Z}$. Since Z is a compact toric surface and D is the union of its 1-dimensional orbits, we have $\mathcal{O}_Z(Z)=\mathcal{O}_Z(d_1D_1+d_2D_2+\cdots+d_sD_s)$ for some integers $d_1, d_2, \cdots, d_s$ by [3, Proposition 6.1]. Then we have $b_i=\deg(\mathcal{O}_{D_i}(Z))=(\mathcal{O}_Z(Z)\cdot\mathcal{O}_Z(D_i))=d_ia_i+d_{i-1}+d_{i+1}$ by Lemma 2.1, where $b_i=(D_i|_{X_i})^2$ and $a_i=(D_i|_Z)^2$. Here X_i is the irreducible component of X intersecting Z along D_i. On the other hand, associated to the nonsingular compact toric surface Z, we have s elements $n_1, n_2, \cdots$ and n_s of $\boldsymbol{Z}^2$, going around the origin exactly once in this order, each adjacent pair $\{n_i, n_{i+1}\}$ of which form a basis of $\boldsymbol{Z}^2$ with the relations $n_{i-1}+a_in_i+n_{i+1}=0$ (see [3, § 8]). Consider the elements $\bar{n}_i=(n_i, d_i)$ and $n=(0, 0, -1)$ of $\boldsymbol{Z}^3=\boldsymbol{Z}^2\oplus\boldsymbol{Z}$. Then $\bar{n}_{i-1}+\bar{n}_{i+1}+a_i\bar{n}_i+b_i\bar{n}=0$ and $\{\bar{n}_i, \bar{n}_{i+1}, \bar{n}\}$ is a basis of $\boldsymbol{Z}^3$ for each i in $\boldsymbol{Z}/s\boldsymbol{Z}$. This is nothing but the monodromy condition at the vertices of $\tilde{\Delta}$ which are inverse images under the projection $\pi\colon \tilde{\Delta}\to\Delta$ of the vertex corresponding to Z. q.e.d.

Finally, we show that Δ satisfies the convexity condition, if X is the exceptional set of a resolution $\pi: (U, X) \to (V, p)$ of a 3-dimensional isolated singularity. Let $[\pi^* f]$ be the zero divisor of the pulled-back holomorphic function $\pi^* f$ on U for a holomorphic function f on V vanishing at p. Then we can write

$$[\pi^* f] = \sum m_i(f) X_i + Y$$

with an effective divisor Y on U so that the support of each irreducible component of Y is not contained in X. Let m_i be the smallest number among $m_i(f)$'s with f running through all elements of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_V$ at p, and let $X^+ = \sum m_i X_i$. Then $\pi^* f$ is a section in $H^0(U, \mathcal{O}_U(-X^+))$ and we have a canonical linear map

$$h: \mathfrak{m}/\mathfrak{m}^2 \longrightarrow H^0(X, \mathcal{O}_X(-X^+)),$$

sending $f + \mathfrak{m}^2$ to the restriction $(\pi^* f)|_X$ of $\pi^* f$ to X. To show that the above positive integers m_i satisfy the two conditions of Definition 1.5, we need some lemmas.

Lemma 2.3. *Let the notations and the assumptions be as above. After blowing up U along double curves and triple points on X and then replacing X by its total transform, we may assume that no double curve of X is contained in the fixed locus of the image of h: $\mathfrak{m}/\mathfrak{m}^2 \to H^0(X, \mathcal{O}_X(-X^+))$.*

Proof. Let $I: (V, p) \to (B, 0)$ be an embedding of (V, p) into an open set B of $\boldsymbol{C}^N$ with $I(p) = 0$ and let $F = \mathrm{Hom}(\boldsymbol{C}^N, \boldsymbol{C})$ be the vector space consisting of the linear functions on $\boldsymbol{C}^N$. We show that there exist a nonempty open set C of F and a composite $\Pi: W \to U$ of blowing ups along double curves and triple points on (the total transform of) X satisfying the following property: For any function f such that $f + \mathfrak{m}^2$ is in the image $\bar{C}$ of C under the surjective restriction map $F \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$, we have

$$\text{(P)} \quad (\pi \circ \Pi|_{W(p)})^* f = y_1^{a_1} \cdot y_2^{a_2} \cdot y_3^{a_3} \cdot f_p \quad \text{with} \quad f_p(0) \neq 0,$$

$a_1, a_2, a_3 \in \boldsymbol{Z}_{>0}$, on some neighborhood $W(p)$ of each triple point p of $X' = \Pi^{-1}(X)$ with a local coordinate (y_1, y_2, y_3) such that $W(p) \cap X'$ is defined by $y_1 \cdot y_2 \cdot y_3 = 0$. Take a divisor $X^!$ on W and a homomorphism h': $\mathfrak{m}/\mathfrak{m}^2 \to H^0(X', \mathcal{O}_{X'}(-X^!))$ in the same way as we take X^+ on U and h. Then the zero-loci of $h'(f + \mathfrak{m}^2)$ contain no triple point and hence no double curve of X' for all $f + \mathfrak{m}^2$ in $\bar{C}$. Thus we have the assertion of the lemma.

For the germ $f = \sum_n c_n x^n$ of a holomorphic function f on $\boldsymbol{C}^3$ at 0 with $n=(n_1, n_2, n_3)$ running through $\boldsymbol{Z}^3_{\geqq 0}$, where $c_n \in \boldsymbol{C}$ and $x^n=x_1^{n_1}x_2^{n_2}x_3^{n_3}$, we denote by $\Gamma_+(f)$ its Newton polyhedron, i.e., the convex hull of $\bigcup_{n\in K}(n+\boldsymbol{R}^3_{\geqq 0})$ with $K:=\{n \in \boldsymbol{Z}^3 \mid c_n \neq 0\}$.

Lemma (Varchenko [5, Lemma 2.13]). *For any Newton polyhedron Γ, there exists a finite nonsingular r.p.p. decomposition $(\boldsymbol{Z}^3, \Sigma)$ of $\boldsymbol{Z}^3$ with $|\Sigma|:=\bigcup_{\sigma\in\Sigma}\sigma=\boldsymbol{R}^3_{\geqq 0}$ such that we can write $\tau_\sigma^* f=y_1^{a_1}\cdot y_2^{a_2}\cdot y_3^{a_3}\cdot f_\sigma$ with $f_\sigma(0)\neq 0$, for any 3-dimensional cone $\sigma=\boldsymbol{R}_{\geqq 0}\cdot n_1+\boldsymbol{R}_{\geqq 0}\cdot n_2+\boldsymbol{R}_{\geqq 0}\cdot n_3$ of Σ and for any holomorphic function f with $\Gamma_+(f)=\Gamma$. Here τ_σ is the restriction of $\tau\colon T\mathrm{emb}(\Sigma)\to T\mathrm{emb}(\{faces\ of\ \boldsymbol{R}^3_{\geqq 0}\})=\boldsymbol{C}^3$ to the open neighborhood U_σ of the point* orb(σ) *and (y_1, y_2, y_3) is the coordinate of U_σ such that y_i is the holomorphic function obtained as the extension of $m_i\otimes\boldsymbol{C}^*$ for the basis $\{m_1, m_2, m_3\}$ of* Hom$(\boldsymbol{Z}^3, \boldsymbol{Z})$ *dual to $\{n_1, n_2, n_3\}$.*

Here we note that this lemma remains to be true, even if we replace Σ by any nonsingular subdivision of Σ. Let $t_1, t_2, \cdots, t_l$ be the triple points of X and let U_i be an open neighborhood of t_i with local coordinate (z_i^1, z_i^2, z_i^3) such that $X\cap U_i$ is defined by $z_i^1\cdot z_i^2\cdot z_i^3=0$. We choose an open set C of F so that the Newton polyhedron $\Gamma_+(\pi^*f)$ with respect to the coordinate (z_i^1, z_i^2, z_i^3) coincide for all $f+\mathfrak{m}^2$ in $\bar{C}$, on each open set U_i. Assume that we have a composite $\Pi_{j-1}\colon W_{j-1}\to U$ of blowing ups along double curves and triple points on (the total transforms of) X satisfying the property (P) at each triple point of $X^{(j-1)}=\Pi_{j-1}^{-1}(X)$ contained in $\Pi_{j-1}^{-1}(U_k)$ with $k\leqq j-1$. Then we have an r.p.p. decomposition $(\boldsymbol{Z}^3, \Lambda)$ with $|\Lambda|=\boldsymbol{R}^3_{\geqq 0}$ so that the following diagram is commutative:

$$\begin{array}{ccc}
\Pi_{j-1}^{-1}(U_j) & \longrightarrow & U_j \\
\cap\downarrow & & \cap\downarrow \\
T\mathrm{emb}(\Lambda) & \longrightarrow & T\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^3_{\geqq 0}\})=\boldsymbol{C}^3.
\end{array}$$

Let Σ_o be an r.p.p. decomposition of $\boldsymbol{R}^3$ as in the above lemma for $\Gamma_+(\pi|_{U_j}{}^*f)$ with $f+\mathfrak{m}^2$ in $\bar{C}$. We have a subdivision Σ of Σ_o obtained by successive subdivisions of Λ corresponding to blowing ups along double curves and triple points on the total transform of the union of the 2-dimensional orbits of Temb(Λ) (see [3, Corollary 7.6 and § 8]). Then associated to the above subdivision Σ of Λ, we have a composite $\Pi_j'\colon W_j\to W_{j-1}$ of blowing ups of W_{j-1} so that the following diagram is commutative:

$$\begin{array}{ccc}
(\Pi_{j-1}\circ\Pi_j')^{-1}(U_j) & \longrightarrow & \Pi_{j-1}^{-1}(U_j) \\
\cap\downarrow & & \cap\downarrow \\
T\mathrm{emb}(\Sigma) & \longrightarrow & T\mathrm{emb}(\Lambda).
\end{array}$$

Then the composite $\Pi_j = \Pi_{j-1} \circ \Pi'_j$ of Π'_j and Π_{j-1} satisfies (P) at each triple points of $X^{(j)} = \Pi_j^{-1}(X)$ contained in $\Pi_j^{-1}(U_k)$ with $k \leqq j$. Thus we have a desired map $\Pi = \Pi_l : W = W_l \rightarrow U$, because each triple point of $X' = \Pi^{-1}(X)$ is contained in one of $\Pi^{-1}(U_j)$. q.e.d.

Remark. Let $U' \rightarrow U$ be a blowing up along a double curve or a triple point of X and let X' be the total transform of X. Then X' is also a toric divisor and the weighted triangulation $\tilde{\Delta}'$ associated to $\Delta' = \mathrm{WDG}(X')$ is a $\pi_1(|\Delta'|)$-invariant subdivision of the weighted triangulation $\tilde{\Delta}$ associated to $\Delta = \mathrm{WDG}(X)$. We easily see that the weights on $\tilde{\Delta}'$ are obtained from those on $\tilde{\Delta}$, as in Figure 2.1, where (i) (resp. (ii)) is the case of a blowing

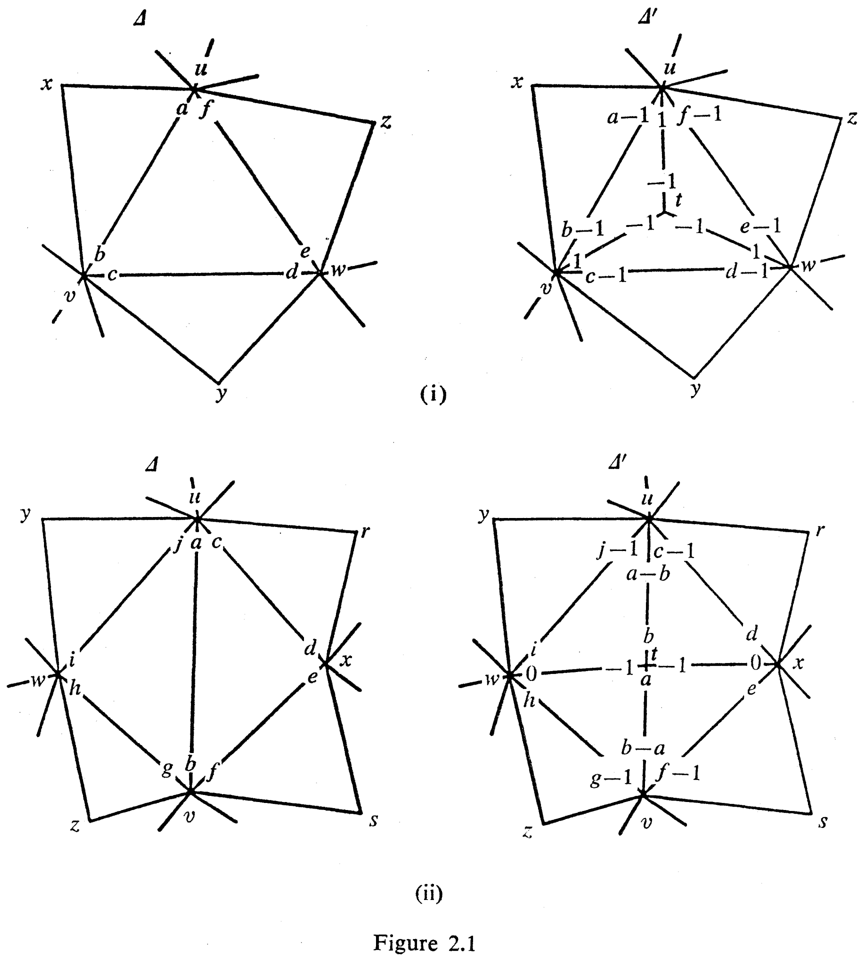


Figure 2.1

up along a triple point (resp. a double curve). The new irreducible component of X' which is the inverse image of the center of the blowing up, corresponds to the vertex t in the figure. The weights of $\tilde{\Delta}'$ which do not appear in the figure, are the same as those of $\tilde{\Delta}$.

Lemma 2.4. *With the above notations, we have*

$$a\cdot m_i+b\cdot m_j+m_k+m_l\leqq 0$$

for each double curve $D=X_i\cap X_j$, where D transversally intersects irreducible components X_k and X_l of X, $a=(D|_{X_j})^2$ and $b=(D|_{X_i})^2$. Moreover, the equality holds if and only if $\deg\mathcal{O}_D(-X^+)=0$. *Here $\mathcal{O}_D(-X^+)$ is the restriction to D of the invertible sheaf $\mathcal{O}_X(-X^+)$ on X (see Figure 2.2).*

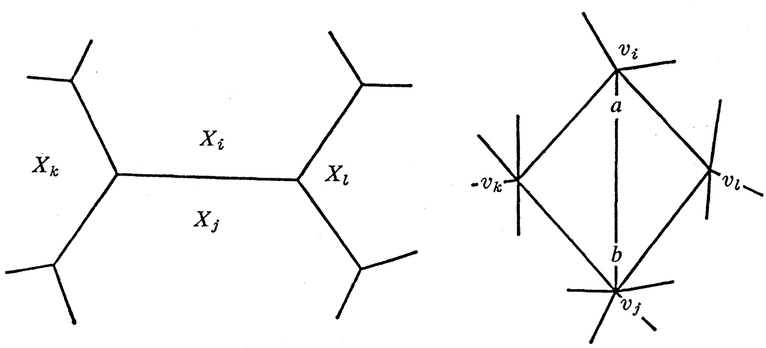


Figure 2.2

Proof. The second assertion follows from the equality

$$\begin{aligned}\deg\mathcal{O}_D(-X^+)&=(-X^+\cdot D)=-(m_iX_i+m_jX_j+m_kX_k+m_lX_l)\cdot D\\&=-(m_ia+m_jb+m_k+m_l),\end{aligned}$$

which we obtain by Lemma 2.1. Moreover, under the assumption of Lemma 2.3, we have $\deg\mathcal{O}_D(-X^+)\geqq 0$ and hence the inequality of the lemma. Also in the general case, we have the inequality by Lemma 2.3 and by the following observation. In the case (i) (resp. (ii)) of the above remark, we easily obtain the inequalities

$$a\cdot m_{[u]}+b\cdot m_{[v]}+m_{[x]}+m_{[w]}\leqq 0 \quad \text{and so on,}$$

$$(\text{resp. } a\cdot m_{[u]}+b\cdot m_{[v]}+m_{[x]}+m_{[w]}\leqq 0,$$

$$c\cdot m_{[u]}+d\cdot m_{[x]}+m_{[v]}+m_{[r]}\leqq 0 \text{ and so on,})$$

from the inequalities

$$m_{[u]}+m_{[v]}+m_{[w]}-m_{[t]}\leqq 0,$$
$$(a-1)\cdot m_{[u]}+(b-1)\cdot m_{[v]}+m_{[x]}+m_{[t]}\leqq 0 \quad \text{and so on}$$

$$(\text{resp. } m_{[u]}+m_{[v]}-m_{[t]}\leqq 0,$$
$$(a-b)\cdot m_{[u]}+b\cdot m_{[t]}+m_{[w]}+m_{[x]}\leqq 0,$$
$$(c-1)\cdot m_{[u]}+d\cdot m_{[x]}+m_{[t]}+m_{[r]}\leqq 0 \text{ and so on}).$$

Therefore, if the inequalities of the lemma hold for all edge of $\tilde{\varDelta}'$, then they hold also for those of $\tilde{\varDelta}$. q.e.d.

We call an edge of $\tilde{\varDelta}$, a *proper edge*, if the strict inequality in the above lemma holds for the edge. Suppose that equalities in the above lemma hold for all edges of $\tilde{\varDelta}$. Then $\deg \mathcal{O}_D(-X^+)=0$, for all double curves D of X by Lemma 2.4. However, the zero-locus of the image $h(f)$ under h of a generic f in $\mathfrak{m}/\mathfrak{m}^2$ must intersect at least one of the double curves, because the complement of the double curves on each irreducible component of X is an algebraic torus and hence contains no compact divisor. Hence by Lemma 2.3, $\tilde{\varDelta}$ has at least one proper edge. By Lemma 2.2, we have a map σ: {all vertices of $\tilde{\varDelta}$}$\rightarrow \boldsymbol{Z}^3$ and a homomorphism $\rho\colon \pi_1(S)\rightarrow \mathrm{GL}(3, \boldsymbol{Z})$ satisfying $\sigma(\gamma v)=\rho(\gamma)\sigma(v)$ for all γ of $\pi_1(S)$ and for all vertices v of $\tilde{\varDelta}$, as in the beginning of this section. Let $\bar{\sigma}(v)=m_{[v]}^{-1}\cdot\sigma(v)$ for each vertex v of $\tilde{\varDelta}$, where $m_{[v]}=m_i$ if its image $[v]$ in $\varDelta$ corresponds to the irreducible component X_i. Take an extension τ of $\bar{\sigma}$ to $\tilde{S}$ so that τ maps each edge (resp. triangle) of $\tilde{\varDelta}$ onto a segment on a line (resp. a triangle on a plane) of $\boldsymbol{R}^3$ and that τ is $\pi_1(S)$-equivariant through ρ. Then τ is locally injective by Lemma 2.2. Let two triples (u, v, w) and (u, v, x) of vertices of $\tilde{\varDelta}$ form two adjacent triangles. Then as we observed before, $\tau(x)$ is in the half space not containing 0 and determined by the plane H passing through $\tau(u)$, $\tau(v)$ and $\tau(w)$, if the edge e joining u and v is a proper edge. On the other hand, by Lemma 2.4, $\tau(x)$ is on H, if e is not a proper edge.

Observations. Let $v_1, v_2, \cdots$ and v_s be the vertices of the link of a vertex v of $\tilde{\varDelta}$ going around v in this order.

1. Suppose that the edge joining v and v_1 is a proper edge and that the other edges meeting at the vertex v are not proper edges. Then $\tau(v_3)$ is on the plane H passing through $\tau(v)$, $\tau(v_1)$ and $\tau(v_2)$. Moreover, we successively see that $\tau(v_4)$, $\tau(v_5)$, $\cdots$ and $\tau(v_s)$ are on H. Hence $\tau(v_s)$, $\tau(v_1)$, $\tau(v_2)$ and $\tau(v)$ are on the same plane H. It contradicts our assumption that the edge joining v_1 and v is a proper edge. Therefore, the number of the proper edges meeting at a vertex cannot be one.

2. Assume that two edges e (joining v and v_1) and e' (joining v

and v_j) are proper edges and the other edges meeting at the vertex v are not proper edges. In a way similar to that above, we see that $\tau(v)$, $\tau(v_1)$, $\tau(v_2)$, $\cdots$ and $\tau(v_j)$ are on a plane H and that $\tau(v)$, $\tau(v_j)$, $\cdots$, $\tau(v_s)$ and $\tau(v_1)$ are on a plane H'. Here we note that H' is different from H, because the edge e is a proper edge. Since $\tau(v_1)$, $\tau(v)$ and $\tau(v_j)$ must be on the intersection of H and H', the images $\tau(e)$ and $\tau(e')$ under τ of the two proper edges are on a line.

3. Let v_{i_1}, v_{i_2}, $\cdots$ and v_{i_t} ($i_1 < i_2 < \cdots < i_t$ with $t > 2$) be the vertices which are the other ends of the proper edges meeting at v. Then $\tau(v_{i_j})$, $\tau(v_{i_j+1})$, $\cdots$, $\tau(v_{i_{j+1}})$ are on a plane H_j and $\tau(v_{i_j-1})$ and $\tau(v_{i_{j+1}+1})$ are in the half-space not containing 0 and determined by the plane H_j. Hence we see that $\tau(v_{i_j+1})$, $\tau(v_{i_j+2})$, $\cdots$ and $\tau(v_{i_{j+1}})$ are on one side of a half-plane of H_j determined by the line passing through $\tau(v_{i_j})$ and $\tau(v)$.

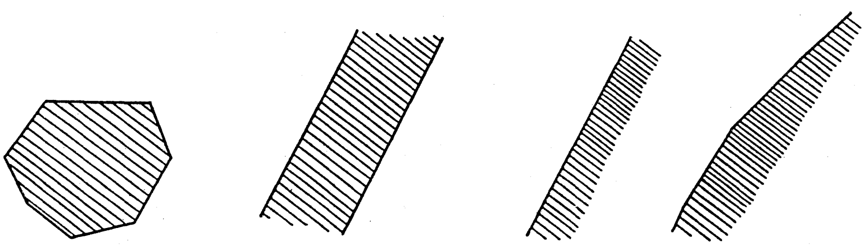
Taking Observation 1 above into account, we classify the vertices of $\tilde{\Delta}$ into three types i), ii) and iii) according as there is no, exactly two or more than two proper edges meeting at a vertex. Now we have a "cell" division $\tilde{\square}$ of $\tilde{S}$, of which $\tilde{\Delta}$ is a subdivision, in the following way: A 0-dimensional "cell" is a vertex of type iii). A 1-dimensional "cell" is a connected union of proper edges such that each vertex of $\tilde{\Delta}$ on it is of type iii) (resp. of type ii)) if and only if the vertex is an end (resp. a relative interior point) of the "cell". Since $\tilde{\Delta}$ has proper edges, $\tilde{\square}$ has at least one 1-dimensional "cell", even though it has no 0-dimensional "cell". A 2-dimensional "cell" is the closure of a connected component of the complement of all the 1-dimensional "cells" on $\tilde{S}$. Note that the image under τ of each 2-dimensional "cell" (resp. 1-dimensional "cell") is on a plane (resp. a line) of $\boldsymbol{R}^3$. In the following, we show that this "cell" division $\tilde{\square}$ is indeed a polygonal decomposition of $\tilde{S}$.

Lemma 2.5. *The restriction $\tau|_\alpha$ of τ to each 1-dimensional "cell" α is injective and the image $\tau(\alpha)$ is a segment or an entire line.*

Proof. Since $\tau(\alpha)$ is on a line L and τ is locally injective, $\tau|_\alpha$ is injective. If α has at least one end, then it has exactly two ends, because the image of α under the projection $\tilde{S} \to S$ is the compact quotient of α by a subgroup of $\pi_1(S)$. In this case, $\tau(\alpha)$ is clearly a segment. When α has no end, $\tau(\alpha)$ agrees with the entire line L, because the images under τ of all the vertices of $\tilde{\Delta}$ on α are contained in $m^{-1} \cdot \boldsymbol{Z}^3 \cap L$, where $m =$ l.c.m. $\{m_{[v]} \mid \text{all vertices } [v] \text{ of } \Delta\}$. q.e.d.

Lemma 2.6. *The restriction $\tau|_\alpha$ of τ to each 2-dimensional "cell" α is injective and the image $\tau(\alpha)$ is a polygon or a domain bounded by two parallel lines of a plane.*

Proof. As we noted above, $\tau(\alpha)$ is on a plane H of $\boldsymbol{R}^3$. Fix a point p in the interior of α and let T be the set of all the points of α joined with p by curves in α which are mapped under τ isomorphically onto segments of lines on H. First, we show that T is an open set of α. Let t be a point of T and let t and p be joined by a curve l in α such that $\tau(l)$ is a segment of a line. Then by Observation 3, we see that $(l\backslash\{t\})\cap\partial\alpha=\phi$, where $\partial\alpha$ is the boundary of α, i.e., the union of 1-dimensional "cells" contained in α. Hence any point of a small neighborhood of t in α is joined by a curve which is mapped isomorphically onto a segment of a line. Therefore, T is open. Moreover $\bar{T}=T$, because $\tau(T)$ is star-shaped with p as the center. Hence $\alpha=T$ and $\tau|_\alpha$ is injective, because it is locally homeomorphic. The image $\tau(\alpha)$ of α under τ is convex, because we can choose an arbitrary point in the interior of α as the center p. Thus we see by Lemma 2.5 that $\tau(\alpha)$ is a polygon, a domain of H bounded by two parallel lines or a domain of H bounded by a straight line or a piecewise linear curve (see Figure 2.3).



(i) (ii) (iii)

Figure 2.3

The image of α under the projection $\tilde{S}\to S$ is its compact quotient by a subgroup G of $\pi_1(S)$. For each element g of G, its image $\rho(g)$ under ρ maps H on itself and the restriction $\rho(g)|_H$ of $\rho(g)$ to H is an affine transformation. Hence the third case (iii) cannot occur, because the quotient of $\tau(\alpha)$ by the group $\{\rho(g)|_H\,|\,g\in G\}$ of affine transformations must be compact. q.e.d.

If one of the 2-dimensional "cells" of $\widetilde{\square}$ is a polygon, then so are all the 2-dimensional "cells" by the above lemma and observation 3. Hence $\widetilde{\square}$ is either a polygonal decomposition or a "cell" division whose 2-dimensional "cells" are homeomorphic to the product of a segment and an entire line. We show that the latter case does not occur. In this case, we see by Lemma 2.6 that τ is injective and that $S\simeq\tau(\tilde{S})/\rho(\pi_1(S))$ is homeomorphic to a 2-dimensional real torus or a Klein bottle. Hence

$\rho(\pi_1(S))$ contains a free abelian subgroup Γ of rank 2 and of index 1 or 2 with $\tau(\tilde{S})/\Gamma \simeq S^1 \times S^1$. Let $\{L_j\}$ be the set of the images under τ of 1-dimensional "cells". We may assume that the lines L_j and L_{j+1} are on a plane H and the other lines L_k $(k \neq j, j+1)$ are on the half-space not containing 0 and determined by H, by Lemma 2.6 and the definition of proper edges. Since Γ has no fixed point on $\tau(\tilde{S})$, all lines L_j are γ-invariant, if at least one of them is γ-invariant for an element γ in Γ. Hence we can take generators γ_1 and γ_2 of Γ so that $\gamma_1(L_j)=L_j$ and that $\gamma_2(L_j)\neq L_j$ for all lines L_j. Since the lines are parallel to each other, $L_j=l_j+\boldsymbol{R}n$ for primitive elements l_j and n in N. Then l_0, l_1 and n are linearly independent and $\gamma(n)=n$ for all elements γ in Γ, because $\tau(\tilde{S})/\Gamma$ is orientable. Hence we can take a basis $\{j, k, n\}$ of N so that j and k are linear combinations of l_0 and l_1, that $\gamma_1(j)=j+an$ and that $\gamma_1(k)=k+bn$ for some integers a and b with $(a, b)\neq(0, 0)$. Let $\gamma_2(j)=cj+dk+en$ and let $\gamma_2(k)=fj+gk+hn$. Since $\gamma_1\circ\gamma_2=\gamma_2\circ\gamma_1$, we have $ac+bd=a$ and $af+bg=b$. Hence $(1-c)(1-g)-df=0$. Let $m'=k$ or $m'=fj+(1-c)k$ according as $(f, (1-c))$ is equal to $(0, 0)$ or not. Then m' is a nonzero element in N and $\gamma_1(m'), \gamma_2(m') \in m'+\boldsymbol{R}n$. Hence we can take another basis $\{l, m, n\}$ of N so that $m'=tm$ for some nonzero integer t. Then $\gamma(l) \in l+\boldsymbol{R}m+\boldsymbol{R}n$ and $\gamma(m) \in m+\boldsymbol{R}n$, for all elements γ in Γ, because Γ is contained in $\mathrm{SL}(N)$. Therefore the images $\gamma(L_0)$ of a line $L_0=l_0+\boldsymbol{R}n$ under Γ must be contained in the plane $l_0+\boldsymbol{R}m+\boldsymbol{R}n$. However, $\{\gamma(L_0)\,|\,\gamma \in \Gamma\}$ is contained in $\{L_i\}$, a contradiction. Thus we complete the proof of Theorem 1.6.

§ 3. A classification of the toric divisors with the same weighted dual graph

Throughout this section, we fix a weighted graph Δ satisfying the monodromy condition and the convexity condition. As we saw at the beginning of Section 2, we have a 3-dimensional cusp singularity $(V_o, p_o)=\mathrm{Cusp}\,(C, \Gamma)$, in the sense of [4] and its resolution $\pi_o\colon (U_o, X_o)\to(V_o, p_o)$ whose exceptional set X_o is a toric divisor with $\mathrm{WDG}\,(X_o)=\Delta$. Here $U_o=\tilde{U}_o/\Gamma$ (resp. $X_o=\tilde{X}_o/\Gamma$) is the quotient space under Γ of an open set $\tilde{U}_o$ (resp. the union $\tilde{X}_o$ of the 2-dimensional orbits) of a torus embedding $T\mathrm{emb}(\Sigma)$ (see [4]).

The subgroup Γ of $\mathrm{GL}(3, \boldsymbol{Z})$ acts naturally on $T=\boldsymbol{C}^*\otimes\boldsymbol{Z}^3=(\boldsymbol{C}^*)^3$. For each γ in Γ, let us denote by γ_* the corresponding automorphism of the algebraic torus T. Since the r.p.p. decomposition Σ of $\boldsymbol{R}^3$ is chosen to be Γ-invariant, Γ acts also on the torus embedding $T\mathrm{emb}(\Sigma)$. For each γ in Γ let us denote again by γ_* the corresponding automorphism of $T\mathrm{emb}(\Sigma)$. Obviously, for any element t in T and for any point x of $T\mathrm{emb}(\Sigma)$, we have $\gamma_*(t\cdot x)=\gamma_*(t)\cdot\gamma_*(x)$. Let us denote by $l(t)$ the action of each element t in T on $T\mathrm{emb}(\Sigma)$. Then we have an injective homomorphism l: $T \to \mathrm{Aut}(T\mathrm{emb}(\Sigma))$ to the group $\mathrm{Aut}(T\mathrm{emb}(\Sigma))$ of automorphisms of $T\mathrm{emb}(\Sigma)$ as an algebraic variety. We also have an injective homomorphism $\Gamma \to \mathrm{Aut}(T\mathrm{emb}(\Sigma))$ sending γ to γ_*. By what we saw above, we have $\gamma_* \circ l(t) = l(\gamma_* t) \circ \gamma_*$ for any t in T and for any γ in Γ. It is easy to see that $A(\Gamma) := \{l(t) \circ \gamma_* \mid \gamma \in \Gamma, t \in T\}$ is a subgroup of $\mathrm{Aut}(T\mathrm{emb}(\Sigma))$, which is the semidirect product of the subgroups $\{l(t) \mid t \in T\}$ and $\{\gamma_* \mid \gamma \in \Gamma\}$. Hence we have a split exact sequence

$$1 \longrightarrow T \longrightarrow A(\Gamma) \longrightarrow \Gamma \longrightarrow 1.$$

Note that $\tilde{X}_o = T\mathrm{emb}(\Sigma) \backslash T$ is invariant under T and Γ, so that $\tilde{X}_o$ is $A(\Gamma)$-invariant.

Consider the compact real torus $CT = U(1) \otimes \boldsymbol{Z}^3 = U(1)^3$ in T, where $U(1) = \{z \in \boldsymbol{C} \mid |z| = 1\}$. Then we have a split exact sequence

$$1 \longrightarrow CT \longrightarrow T \xrightarrow{\mathrm{ord}} \boldsymbol{R}^3 \longrightarrow 0,$$

where ord is induced by the homomorphism $\boldsymbol{C}^* \ni z \mapsto -\log|z| \in \boldsymbol{R}$. Define a subgroup of $A(\Gamma)$ by $CA(\Gamma) := \{l(t) \circ \gamma_* \mid t \in CT, \gamma \in \Gamma\}$. Then we also have a split exact sequence

$$1 \longrightarrow CT \longrightarrow CA(\Gamma) \longrightarrow \Gamma \longrightarrow 1.$$

The open set $\tilde{U}_o := \mathrm{ord}^{-1}(C) \cup \tilde{X}_o$ of $T\mathrm{emb}(\Sigma)$ is invariant under Γ and CT. So $\tilde{U}_o$ is $CA(\Gamma)$-invariant.

Via the action of Γ on T, we can consider the first group cohomology group $H^1(\Gamma, T) = Z^1(\Gamma, T)/B^1(\Gamma, T)$, where $Z^1(\Gamma, T)$ consists of the crossed homomorphisms φ: $\Gamma \to T$ satisfying $\varphi(\gamma\gamma') = \varphi(\gamma) \cdot \gamma_*(\varphi(\gamma'))$, while for an element t in T, we define the element δt of $B^1(\Gamma, T)$ to be the map $\Gamma \to T$ sending γ to $t^{-1} \cdot \gamma_*(t)$. Similarly, we can define $H^1(\Gamma, CT)$ and $H^1(\Gamma, \boldsymbol{R}^3)$. By the split exact sequence $1 \to CT \to T \to \boldsymbol{R}^3 \to 0$, we have a split exact sequence

$$1 \longrightarrow H^1(\Gamma, CT) \longrightarrow H^1(\Gamma, T) \longrightarrow H^1(\Gamma, \boldsymbol{R}^3) \longrightarrow 0.$$

The following is well-known:

Lemma 3.1. $H^1(\Gamma, T)$ (resp. $H^1(\Gamma, CT)$) *is in one-to-one correspondence with the splitting of the exact sequence* $1 \to T \to A(\Gamma) \to \Gamma \to 1$ (*resp.* $1 \to CT \to CA(\Gamma) \to \Gamma \to 1$) *up to conjugacy in* $A(\Gamma)$ (*resp.* $CA(\Gamma)$).

The relevance of the first group cohomology groups $H^1(\Gamma, T)$ and $H^1(\Gamma, CT)$ to our problem in this paper is explained as follows: For φ

in $Z^1(\Gamma, T)$, the group $\Gamma_\varphi := \{l(\varphi(\gamma)) \circ \gamma_* \mid \gamma \in \Gamma\}$ acts on $\tilde{X}_o$. If φ is in $Z^1(\Gamma, CT)$, then Γ_φ acts on $\tilde{U}_o$ properly discontinuously and without fixed points, since the induced action on $(\tilde{U}_o \backslash \tilde{X}_o)/CT = C$ coincides with that of Γ on $C \subset \boldsymbol{R}^3$. If $\varphi' = \varphi \cdot \delta t$ for an element t in T (resp. CT), then we have an isomorphism of quotients

$$\tilde{X}_o/\Gamma_{\varphi'} \simeq \tilde{X}_o/\Gamma_\varphi \quad (\text{resp. } \tilde{U}_o/\Gamma_{\varphi'} \simeq \tilde{U}_o/\Gamma_\varphi).$$

For the cohomology class $[\varphi]$ of $H^1(\Gamma, T)$ (resp. $H^1(\Gamma, CT)$), let us denote the isomorphism class consisting of these analytic spaces (resp. 3-dimensional complex manifolds) by

$$X_{[\varphi]} := \tilde{X}_o/\Gamma_\varphi \quad (\text{resp. } U_{[\varphi]} := \tilde{U}_o/\Gamma_\varphi).$$

If $[\varphi]$ is in $H^1(\Gamma, CT)$, then $X_{[\varphi]}$ is a toric divisor on $U_{[\varphi]}$. Since $l(t)$ maps irreducible components, double curves and triple points of $\tilde{X}_o$ onto themselves, for each t in T, the weighted dual graph of $X_{[\varphi]}$ coincides with that of X_o, i.e., $\mathrm{WDG}(X_{[\varphi]}) = \varDelta$. We can prove the following proposition in the same way as [4, Proposition 1.5].

Proposition 3.2. *If the cohomology class $[\varphi]$ is in $H^1(\Gamma, CT)$, then $X_{[\varphi]}$ is contractible to a point. Hence we obtain a 3-dimensional isolated singularity $V_{[\varphi]}$ and its resolution $(U_{[\varphi]}, X_{[\varphi]}) \to (V_{[\varphi]}, p)$ whose exceptional set $X_{[\varphi]}$ is a toric divisor with* $\mathrm{WDG}(X_{[\varphi]}) = \varDelta$.

In the following, we show conversely that any toric divisor X, with $\mathrm{WDG}(X) = \varDelta$, of a 3-dimensional complex manifold U is isomorphic to $X_{[\varphi]}$ for some cohomology class $[\varphi]$ of $H^1(\Gamma, T)$. Recall that the weighted triangulation $\tilde{\varDelta}$ associated to $\varDelta$ is a $\pi_1(S)$-invariant triangulation of the universal covering space $\tilde{S}$ of $S = |\varDelta|$. We can take an unramified covering space $\tilde{X} \to X$ of X so that the dual graph of $\tilde{X}$ coincides with $\tilde{\varDelta}$ as follows: For each vertex v of $\tilde{\varDelta}$, take a copy X_v of the normalization of the irreducible component $X_{[v]}$ of X corresponding to the image $[v]$ of v under the projection $\tilde{\varDelta} \to \varDelta$, and let $j_v \colon X_v \to X_{[v]}$ be the normalization map. We identify the points x of X_v and y of X_w, if and only if v and w are joined by an edge of $\tilde{\varDelta}$ and $i_{[v]}(j_v(x)) = i_{[w]}(j_w(y))$, where $i_{[v]} \colon X_{[v]} \to X$ is the inclusion map. Then we obtain an analytic space $\tilde{X} = \bigcup_{v \in \{\text{vertices of } \tilde{\varDelta}\}} X_v$ and a map $\tilde{X} \to X$ sending a point x of X_v to $i_{[v]}(j_v(x))$. Here we note that the weighted dual graph of $\tilde{X}_o$ also coincides with $\tilde{\varDelta}$ (see [4, § 4]).

Proposition 3.3. *Under the above notations, we have an isomorphism:* $\tilde{X} \simeq\to \tilde{X}_o$.

Proof. For each vertex v of $\tilde{\varDelta}$, let $(X_o(v), D_o(v))$ (resp. $(X(v), D(v))$)

be the pair of the irreducible component of $\tilde{X}_o$ (resp. $\tilde{X}$) corresponding to v and the union of the double curves on it. Since each irreducible component of a toric divisor X and the double curves on it are uniquely determined up to isomorphism from the weighted dual graph WDG(X) of X (see [3, § 8]), we have an isomorphism $(X(v), D(v))\simeq\rightarrow(X_o(v), D_o(v))$ of pairs. Take a vertex v of $\tilde{\Delta}$ and fix one such isomorphism

$$I(v)\colon\ (X(v), D(v))\simeq\rightarrow(X_o(v), D_o(v)).$$

Let w be a vertex which is connected to v by an edge e of $\tilde{\Delta}$. Let $D_o=X_o(v)\cap X_o(w)$ and $D=X(v)\cap X(w)$ be the double curves of $X_o(v)$ and $X(v)$, respectively, corresponding to e. Then we take an isomorphism

$$I(w)\colon\ (X(w), D(w))\simeq\rightarrow(X_o(w), D_o(w))$$

in such a way that the restriction $I(w)|_D$ of $I(w)$ to D is equal to that of $I(v)$, i.e., $I(w)|_D=I(v)|_D$. Next let u be a vertex of $\tilde{\Delta}$ such that u, v and w are vertices of a triangle. We easily see that for g, h in $\mathrm{Aut}_o(X(u), D(u))\simeq(C^*)^2$, if $g|_E=h|_E$, then $g=h$, where $E=X(u)\cap X(v)+X(u)\cap X(w)$. Thus we have a unique isomorphism

$$I(u)\colon (X(u), D(u))\simeq\rightarrow(X_o(u), D_o(u))$$

with $I(u)|_{X(u)\cap X(v)}=I(v)|_{X(u)\cap X(v)}$ and $I(u)|_{X(u)\cap X(w)}=I(w)|_{X(u)\cap X(w)}$. Hence the following lemma complete the proof, because $\tilde{\Delta}$ is simply connected.

Lemma 3.4. *Let v be a vertex of $\tilde{\Delta}$ and $w_1, w_2, \cdots, w_s$ be the vertices adjacent to v going around v in this order. Fix isomorphisms $I(v)\colon X(v)\simeq\rightarrow X_o(v)$ and $I(w_1)\colon X(w_1)\simeq\rightarrow X_o(w_1)$ with $I(v)|_{D(1)}=I(w_1)|_{D(1)}$, where $D(1)=X(v)\cap X(w_1)$. We have isomorphisms $I(w_2)\colon X(w_2)\simeq\rightarrow X_o(w_2)$, $I(w_3)\colon X(w_3)\simeq\rightarrow X_o(w_3)$, $\cdots$ and $I(w_s)\colon X(w_s)\simeq\rightarrow X_o(w_s)$, successively, with $I(v)|_{D(j)}=I(w_j)|_{D(j)}$ and $I(w_{j-1})|_{E(j)}=I(w_j)|_{E(j)}$ in the above way, where $D(j)=X(v)\cap X(w_j)$ and $E(j)=X(w_{j-1})\cap X(w_j)$. Then $I(w_s)|_{E(1)}=I(w_1)|_{E(1)}$. Namely, we have an isomorphism from $X(v)+X(w_1)+\cdots+X(w_s)$ to $X_o(v)+X_o(w_1)+\cdots+X_o(w_s)$.*

Proof. By the definition of toric divisors, we can take coordinates (x_i, y_i), (u_i, v_i) and (z_i, w_i) for Zariski open sets of toric surfaces $X(v)$, $X(w_{i-1})$ and $X(w_i)$, respectively, for each i of Z/sZ so that $E(i)$ is defined by $v_i=0$ and also by $w_i=0$, that $D(i)$ is defined by $x_i=0$, by $y_{i+1}=0$, by $z_i=0$ and also by $u_{i+1}=0$ and that the relations

$$(\sharp)\quad x_i=y_{i+1}x_{i+1}^{-a_i},\quad y_i=x_{i+1}^{-1},\quad w_i=v_{i+1}^{-1}\quad\text{and}\quad z_i=u_{i+1}v_{i+1}^{-b_i}$$

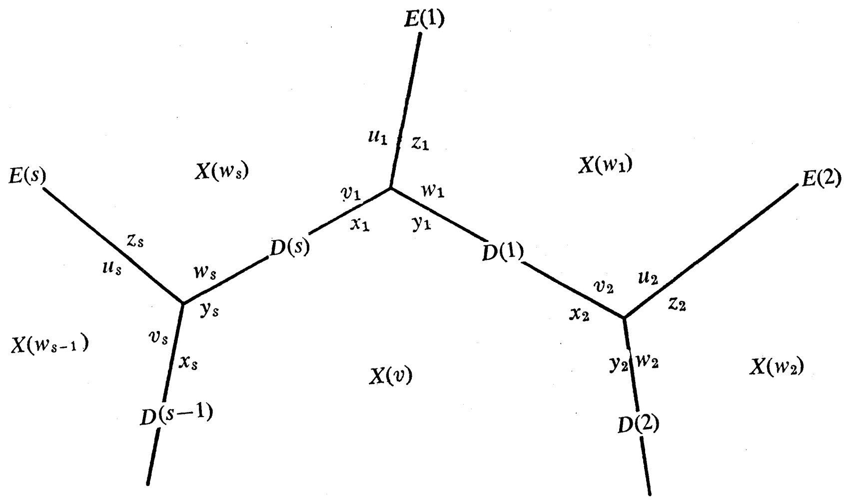


Figure 3.1

hold. (See Figure 3.1.) Here we may assume that $w_i=y_i$ and $v_i=x_i$ on $D(i)$ for each i of $\boldsymbol{Z}/s\boldsymbol{Z}$, multiplying w_i and v_{i+1} by some constants, since $w_i/y_i=x_{i+1}/v_{i+1}$ is a nonzero constant on $D(i)$. Similarly, keeping the relations ($\sharp$), we may assume that $u_i=z_i$ on $E(i)$ for $i=2$ through s. Then $u_1=tz_1$ on $E(1)$ for some nonzero complex number t. It is sufficient to show that $t=1$. Take an open covering $\{U_i\}_{i=1,2,\cdots,s}$ of $X(v)$ so that U_i is a Stein neighborhood in U of the triple point $X(v)\cap X(w_{i-1})\cap X(w_i)$. Let f_i be a defining equation for $X(v)\cap U_i$ on U_i and let $g_{ij}=(f_i/f_j)|_{X(v)}$. Then the collection $\{g_{ij}\}$ of the transition functions g_{ij} defines the normal bundle $\mathcal{O}_{X(v)}(X(v))$ of $X(v)$ in U. On the other hand, the line bundle $\mathcal{O}_{X(v)}(X(v))$ is linearly equivalent to $d_1\cdot D(1)+d_2\cdot D(2)+\cdots+d_s\cdot D(s)$ for some integers d_1, d_2, $\cdots$ and d_s. Hence we have a collection $\{h_i\}$ of nowhere vanishing holomorphic functions h_i on $X(v)\cap U_i$ with

$$\bar{g}_{ij}:=h_i\cdot g_{ij}\cdot h_j^{-1}=(y_i^{d_{i-1}}\cdot x_i^{d_i})\cdot(y_j^{d_{j-1}}\cdot x_j^{d_j})^{-1}.$$

In particular, $\bar{g}_{i,i+1}=x_{i+1}^{-b_i}$, because $b_i=d_{i-1}+a_id_i+d_{i+1}$. Now let $\bar{f}_i=f_i\cdot\bar{h}_i$ for a holomorphic function $\bar{h}_i$ which is an extension of h_i to U_i. Then $(\bar{f}_i/\bar{f}_{i+1})|_{X(v)}=x_{i+1}^{-b_i}$ and hence $\{(\bar{f}_i/z_i)\cdot(\bar{f}_{i+1}/u_{i+1})^{-1}\}|_{D(i)}=1$ for $i=1$ through $s-1$, because $v_{i+1}=x_{i+1}$ on $D(i)$. So $(\bar{f}_i/z_i)$ and $(\bar{f}_{i+1}/u_{i+1})$ define the same nonzero holomorphic function F_i on $D(i)$. Clearly $F_i=F_{i+1}$ on the triple point $D(i)\cap D(i+1)$. Since the double curves $D(i)$ are compact, the collection $\{F_i\}$ defines a nonzero constant function on $D(1)+D(2)+\cdots+D(s)$. Hence $t=u_1/z_1=(\bar{f}_1/z_1)\cdot(\bar{f}_1/u_1)^{-1}=1$. q.e.d.

Let $\mathrm{Aut}_o(\tilde{X}_o)$ be the subgroup of $\mathrm{Aut}(\tilde{X}_o)$ consisting of the automorphisms of $\tilde{X}_o$ which map each irreducible component of $\tilde{X}_o$ to itself, i.e., the induced action of $\mathrm{Aut}_o(\tilde{X}_o)$ on $\tilde{\varDelta}$ is trivial. Clearly T acts on $\tilde{X}_o$ effectively and hence is a subgroup of $\mathrm{Aut}_o(\tilde{X}_o)$.

Proposition 3.5. $\mathrm{Aut}_o(\tilde{X}_o)=T$.

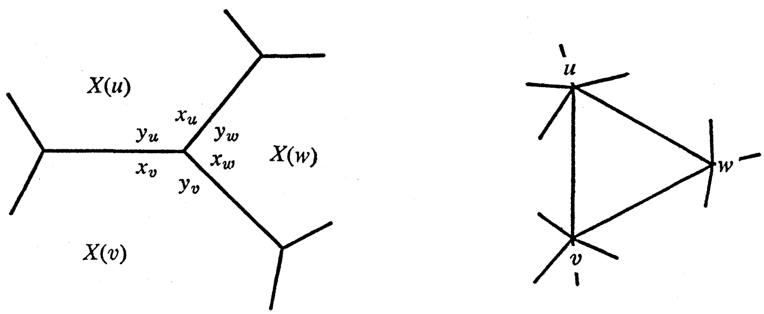


Figure 3.2

Proof. Let u, v and w be the vertices of a triangle of $\tilde{\varDelta}$. We can take a coordinate (z_1, z_2, z_3) on a neighborhood V in $\tilde{U}_o$ of the triple point $X_o(u)\cap X_o(v)\cap X_o(w)$ so that $X_o(u)$, $X_o(v)$ and $X_o(w)$ are defined by $z_1=0$, $z_2=0$ and $z_3=0$, respectively, and that the action of any element c of T can be written as $(z_1, z_2, z_3)\mapsto(c_1z_1, c_2z_2, c_3z_3)$. Then $(x_u, y_u)=(z_2|_{X(u)}, z_3|_{X(u)})$, $(x_v, y_v)=(z_3|_{X(v)}, z_1|_{X(v)})$ and $(x_w, y_w)=(z_1|_{X(w)}, z_2|_{X(w)})$ are coordinates of $V\cap X_o(u)$, $V\cap X_o(v)$ and $V\cap X_o(w)$, respectively (see Figure 3.2). The restriction of any element g of $\mathrm{Aut}_o(\tilde{X}_o)$ to $X_o(t)$ can be written as $(x_t, y_t)\mapsto(a_tx_t, b_ty_t)$ for some nonzero complex numbers a_t and b_t and for $t=u$, v and w. Here clearly, $a_v=b_u$, $a_w=b_v$ and $a_u=b_w$. Hence the restriction of g to $X_o(u)+X_o(v)+X_o(w)$ agrees with that of some element c in T. Then we see that $g=c$ on $\tilde{X}_o$, as in the proof of Proposition 3.3.

q.e.d.

By Proposition 3.3 and 3.5, there exist a subgroup Γ' of $A(\Gamma)\subset \mathrm{Aut}(\tilde{X}_o)$ with $X\simeq\tilde{X}_o/\Gamma'$ and an isomorphism $h\colon \Gamma\simeq\to\Gamma'$ with $h(\gamma)\circ\gamma_*^{-1}\in T$ for any element γ of Γ. Let $\varphi(\gamma)=h(\gamma)\circ\gamma_*^{-1}$. Then by an easy calculation, we see that the map $\varphi\colon \Gamma\to T$ satisfies the cocycle condition $\varphi(\gamma\gamma')=\varphi(\gamma)\cdot\gamma_*(\varphi(\gamma'))$ for any elements γ and γ' of Γ. Hence the map $\varphi\colon \Gamma\to T$ defines an element $[\varphi]$ of $H^1(\Gamma, T)$ and $\Gamma'=\Gamma_\varphi$. Thus we have $X\simeq X_{[\varphi]}$. Suppose that $X_{[\varphi]}\simeq X_{[\psi]}$ for another element $[\psi]$ of $H^1(\Gamma, T)$. Then there exists an element g of $\mathrm{Aut}_o(\tilde{X}_o)$ such that $l(\psi(\gamma))\circ\gamma_*\circ g=g\circ l(\varphi(\gamma))\circ\gamma_*$ for each element γ of Γ. By Proposition 3.5, $g=l(t)$ for some t in T. Hence we have $\varphi(\gamma)\cdot\psi(\gamma)^{-1}=\gamma_*(t)\cdot t^{-1}$. So $[\varphi]=[\psi]$. Thus we obtain:

Theorem 3.6. *There exists an injective map to $H^1(\Gamma, T)$ from the set of isomorphism classes of toric divisors X, with* $\mathrm{WDG}(X)=\Delta$, *of 3-dimensional complex manifolds. The map sends each toric divisor X to the element $[\varphi]$ of $H^1(\Gamma, T)$ with $X \simeq X_{[\varphi]}$ and its image contains $H^1(\Gamma, CT)$.*

When Γ is an abelian group, the calculation of $H^1(\Gamma, T)$ is easy. Γ is isomorphic to the fundamental group $\pi_1(S)$ of a compact topological surface S and S cannot be a Klein bottle [4, Corollary 3.2]. Thus Γ is an abelian group, if and only if S is a 2-dimensional real torus.

Proposition 3.7. *If Γ is an abelian group, then we have $H^1(\Gamma, CT)= H^1(\Gamma, T)$ and it is a finite group.*

Proof. As we saw above, $\Gamma \simeq \boldsymbol{Z}\oplus\boldsymbol{Z}$. Let γ_1 and γ_2 be generators of $\Gamma\subset \mathrm{GL}(3, \boldsymbol{Z})$. Then $\det(\gamma_1-1)\neq 0$, because γ_1 has three eigenvalues none of which are equal to 1 as we saw in the proof of [4, Theorem 3.1]. Hence for any element b of T, there exists an element a of T with $(\gamma_{1*}a)\cdot a^{-1}=b$. Thus for any element $[\varphi]$ of $H^1(\Gamma, T)$, we can take a representative φ with $\varphi(\gamma_1)=1$ in $Z^1(\Gamma, T)$. Since Γ is abelian, we have $(\gamma_{1*}\varphi(\varepsilon))\cdot\varphi(\varepsilon)^{-1}= (\varepsilon_*\varphi(\gamma_1))\cdot\varphi(\gamma_1)^{-1}=1$ by the cocycle condition for any element ε of Γ. Hence $\varphi(\varepsilon)$ is contained in the subgroup K of T consisting of the $(\gamma_1)_*$-fixed elements. Since K is a finite subgroup of CT and since φ is uniquely determined by $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ in K, we are done. q.e.d.

Remark. When Γ is not abelian, $H^1(\Gamma, T)$ and $H^1(\Gamma, CT)$ are infinite. Moreover, $H^1(\Gamma, T)$ (resp. $H^1(\Gamma, CT)$) contains a subgroup isomorphic to an algebraic torus (resp. a real torus), in general. Hence the singularity $\mathrm{Cusp}(C, \Gamma)$ may have a non-trivial deformation. (Freitag and Kiehl showed that Hilbert modular cusp singularities with demensions greater than 2 are rigid.) In a forthcoming paper, we will calculate the cohomology groups $H^1(\Gamma, T)$ and $H^1(\Gamma, CT)$ and study the rigidity of the singularities treated in this paper.

We take this opportunity to correct the following errors in our previous paper [4], which the referee of the present paper kindly pointed out: $\bar{\Theta}$ on p. 609, line 35 from above as well as Θ on p. 610, line 6 from above should both be $\partial\Theta$.

References

[1] A. Ash, D. Mumford, M. Rapoport and Y. Tai, Smooth compactification of locally symmetric varieties, Math. Sci. Press, Brookline, Mass., 1975.

[2] H. Laufer, Normal two-dimensional singularities, Ann. of Math. Stud., No. 71, Princeton U. P., Princeton, N.J., 1971.

[3] T. Oda, Lectures on torus embeddings and applications (based on joint work with K. Miyake), Tata. Inst. of Fund. Res., Bombay, No. 58, Springer-Verlag, Berlin-Heiderberg-New York, 1978.

[4] H. Tsuchihashi, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tohoku Math. J., **35** (1983), 607–639.
[5] A. N. Varchenko, Newton polyhedra and estimation of oscillating integrals, Funct. anal. Appl., **10** (1976), 175–196.
[6] E. Freitag and R. Kiehl, Algebraische Eigenschaften der lokalen Ringe in den spitzen der Hilbertschen Modulgruppen, Invent. math., **24** (1974), 121–148.

Department of General Education
Tohoku Gakuin University
Sendai, 980
Japan