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Maximal-Ideal-Adic Filtration on $R^1\psi_*O_{\tilde{V}}$ for Normal Two-Dimensional Singularities

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§ 0. Introduction

(0.1) Let (V, p) be a germ of normal two-dimensional algebraic variety over $\boldsymbol{C}$ at a reference point p. We simply call it a normal two-dimensional singularity. Let $\psi: (\tilde{V}, A)\to(V, p)$ be a resolution of the singularity (V, p) with exceptional set A. It is well-known that the coherent O_V-module $R^1\psi_*O_{\tilde{V}}$ is independent of the choice of resolution. The *geometric genus* of the singularity (V, p) is the integer $p_g(V, p)$ defined by: $p_g(V, p)=\dim R^1\psi_*O_{\tilde{V}}$. This number has been studied by many authors from many viewpoints (cf. [9, 10, 11, 12, 15, 17, 19, 20, 21, 22, 23, 24] and the references there).

In this paper, we shall study the O_V-module $R^1\psi_*O_{\tilde{V}}$ itself. More precisely, we shall study the numerical invariants which are related to the following maximal-ideal-adic filtration on $R^1\psi_*O_{\tilde{V}}$:

$$(*)\qquad R^1\psi_*O_{\tilde{V}}\supseteqq m\cdot R^1\psi_*O_{\tilde{V}}\supseteqq\cdots\supseteqq m^L\cdot R^1\psi_*O_{\tilde{V}}=0,$$

where m denotes the maximal ideal of $O_{V,p}$. We define the invariant $L(V, p)$ as the length of the filtration above. Since $m^i\cdot R^1\psi_*O_{\tilde{V}}\neq m^{i+1}\cdot R^1\psi_*O_{\tilde{V}}$ for non-zero $m^i\cdot R^1\psi_*O_{\tilde{V}}$, this integer can be written as follows (see also (0.2) and (2.9)): $L(V, p)=\min\{r\in\boldsymbol{Z}\,|\,r\geqq 0,\ m^r\cdot R^1\psi_*O_{\tilde{V}}=0\}$.

First we shall show the existence of an element f of m such that the equalities $m^r\cdot R^1\psi_*O_{\tilde{V}}=f^r\cdot R^1\psi_*O_{\tilde{V}}$ for $r\geqq 0$ hold. Hence the filtration (*) is determined by the nilpotent endomorphism

$$F: R^1\psi_*O_{\tilde{V}}\longrightarrow R^1\psi_*O_{\tilde{V}};\ \alpha\longmapsto f\cdot\alpha \text{ (Section 1 and (2.3)).}$$

At the same time, by using the divisor $D(m, \psi)$ which is called *maximal ideal cycle* in [24, 17, 19] we can show the equality $(R^1\psi_*O_{\tilde{V}}/m\cdot R^1\psi_*O_{\tilde{V}})\cong H^1(O_{D(m,\psi)})$. In particular, we obtain the equalities $\dim(R^1\psi_*O_{\tilde{V}}/$

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$m \cdot R^1\psi_* O_{\tilde V}) = \dim H^1(O_{Z_0}) = p_a(Z_0)$ when *Artin's fundamental cycle* Z_0 equals the divisor $D(m, \psi)$ for the resolution ψ (cf. (2.2.2)). The virtual arithmetic genus $p_a(Z_0)$ of Z_0 is defined as the integer $1-\chi(O_{Z_0})$ and can be computed arithmetically from the intersection dual graph of the exceptional set A by Laufer's computation sequence method (Section 2 of [11]). Some of the examples of singularities and resolutions ψ with the condition $Z_0 = D(m, \psi)$ are as follows:

(0.1.1) (Artin [2]). *A rational singularity and any resolution.*

(0.1.2) (Theorem 3.12 of Yau [24], Corollary (7.9) of Tomari [19]). *A maximally elliptic singularity and the minimal resolution. In this case, the equality* $L = p_g$ *holds.*

(0.1.3) (Dixon [5]). *Let* (V, p) *be an isolated singularity written as follows*: $(V, p) = (\{(x, y, z) \in C^3 | z^2 - g(x, y) = 0\}, o)$. *If the order of g at o is even or g is irreducible at o, then the condition* $D(m, \psi) = Z_0$ *holds in the minimal resolution* ψ.

Generally two integers $\dim H^1(O_{D(m,\psi)})$ and $p_a(Z_0)$ are independent of the choice of the resolution, and not necessarily the same (cf. Example (2.13)). The study, as in (2.2), of the relations between them still remains open.

Next we shall study another invariant. The *arithmetic genus* $p_a(V, p)$ of the singularity (V, p) is the invariant defined via a resolution $\psi : (\tilde V, A) \to (V, p)$ as follows: $p_a(V, p) = \max\{p_a(D) | D$ is non-zero effective divisor on $\tilde V$ whose support is contained in $A\}$ (Wagreich [20]). Here $p_a(D)$ is the virtual arithmetic genus of D. The invariant $p_a(V, p)$ is also related to the filtration (*) by the following inequality: $p_a(V, p) + L(V, p) - 1 \leq p_g(V, p)$ (cf. Theorem (2.6)).

By using this, we shall show the relation $p_g(V, p) \leq$ *the Cohen-Macaulay type of the local ring* $O_{V,p}$ *under the assumption* $p_a(V, p) = p_g(V, p)$ (Corollary (2.11)). Furthermore, we will rediscover *Yau's criterion for singularity to be elliptic* (Corollary (2.10)).

From the study on $p_a(Z_0)$ above, we obtain the inequality

$$\dim(R^1\psi_* O_{\tilde V} / m \cdot R^1\psi_* O_{\tilde V}) \leq p_a(V, p)$$

when the equality $Z_0 = D(m, \psi)$ holds. Moreover we can prove the same inequality for a singularity with good C^*-action (Theorem (3.4)) by using the general theory of canonical partial resolution with good properties (cf. (3.2)).

In particular, we obtain the following characterizations in the case the singularity has a good C^*-action (Corollary (3.6)):

(0.1.4) $L\leqq 1$ *if and only if* $p_g=p_a$.
(0.1.5) $L=p_g$ *if and only if* $p_a\leqq 1$.

For more studies on the highest direct image sheaf $R^{n-1}\psi_* O_{\tilde V}$ (in the case of $n=\dim O_{V,p}\geqq 2$) by purely ring-theoretic data of the singularity (V,p), we refer to [21, 22].

(0.2) The invariant $L(V,p)$ is originally introduced in [17] for the Gorenstein singularity (V,p) in the following form:

$$L(V,p)=\min\{r\in Z\,|\,r\geqq 0,\ -K_{\tilde V}\leqq r\cdot D(m,\psi)\}$$

in t he terminology of (2.9). In the situation of [17] the geometric genus $p_g(V,p)$, the maximal ideal cycle $D(m,\psi)$, and the canonical divisor $K_{\tilde V}$ are computed explicitly in terms of the numerical data appearing in Zariski's canonical resolution which is decomposed into the composition of blowing-ups with smooth centers. Based on those computations, the information concerning $L(V,p)$ and $p_g(V,p)$ is closely related to precise numerical data in the resolution. Actually the resolution process of the elliptic singularity of multiplicity two is studied by proving the equivalence of the conditions $L=p_g$ and $p_a\leqq 1$ in [17].

Hence our studies of the present paper might be expected to help us to study the resolution process of more general singularities (cf. [18]).

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§ 1. A lemma about $R^{n-1}\psi_* N$

(1.1) Let (V,p) be a reduced n-dimensional singularity and ψ : $(\tilde V, A)\to(V,p)$ a proper modification with relation $A=|\psi^{-1}(p)|$[1] such that $\tilde V$ is normal. After M. Reid, we call ψ a partial resolution. Let I be an O_V-ideal sheaf and N a locally free $O_{\tilde V}$-module sheaf. The purpose of this section is to show the existence of an element f of the stalk I_p at p such that the equality $I\cdot R^{n-1}\psi_* N=f\cdot R^{n-1}\psi_* N$ holds at p (Lemma (1.3)).

(1.2) Before we proceed to discuss the problem above, we prepare some notations and remarks for precise statements.

We denote the critical locus of ψ in $\tilde V$ by E. Since $\dim E\leqq n-1$, the set $\{q\in V\,|\,\dim|\psi^{-1}(q)|\geqq n-1\}$ is a discrete subset of V. Hence we may assume that the support of the sheaf $R^{n-1}\psi_* N$ is contained in $\{p\}$.

[1] For an analytic space D, we denote the support of D by $|D|$.

We decompose the analytic set A as follows: $A=(\bigcup_{j=1}^{m} A_j)\cup A'$, where A_j is a prime divisor of $\tilde{V}$, $j=1,\cdots,m$, and A' the part of codimension greater than or equal 2.

Let the function $v_{A_j}: O_{V,p}\to \boldsymbol{Z}\cup\{\infty\}$ be as follows: $v_{A_j}(\psi^*f)=\infty$ if ψ^*f vanishes identically in the neighborhood of A_j in $\tilde{V}$. In the other case, $v_{A_j}(\psi^*f)=$ the vanishing order of ψ^*f *at the generic point of* A_j.

For I and ψ, we introduce a symbol $D(I,\psi)$ as follows:

$$D(I,\psi)=\sum_{j=1}^{m}\{\inf_{f\in I_p} v_{A_j}(\psi^*f)\}A_j.$$

By definition, there is an element f_j of I_p such that $v_{A_j}(\psi^*f_j)=\inf_{f\in I_p} v_{A_j}(\psi^*f)$ holds, $j=1,\cdots,m$. If we define an element f_α of I_p by $f_\alpha=\sum_{j=1}^{m} a_j f_j$, for $\alpha=(a_j)\in \boldsymbol{C}^m$, the condition $D((f_\alpha),\psi)=D(I,\psi)$ holds for generic $\alpha\in\boldsymbol{C}^m$.

We define the $O_{\tilde{V}}$-ideal sheaf $I_{D(I,\psi)}$ associated to the symbol $D(I,\psi)$ as follows: $I_{D(I,\psi)}=0$ in the neighborhood of A_j such that $v_{A_j}(\psi^*f)=\infty$ for any f in I_p. Otherwise, $I_{D(I,\psi)}=$ the divisorial ideal sheaf $O_{\tilde{V}}(-D(I,\psi))$ in the neighborhood of A_j.

We denote $N/I_{D(I,\psi)}N$ by $N_{D(I,\psi)}$. We shall simply denote $(O_{\tilde{V}})_{D(I,\psi)}$ by $O_{D(I,\psi)}$.

Lemma (1.3). *Let* $\psi:(\tilde{V},A)\to(V,p)$, *$N$ and I be as in* (1.1). *If an element f of I_p satisfies the condition* $D(I,\psi)=D((f),\psi)$, *then*:

(1) $I\cdot R^{n-1}\psi_*N=f\cdot R^{n-1}\psi_*N$.

(2) $R^{n-1}\psi_*N/I\cdot R^{n-1}\psi_*N=R^{n-1}\psi_*(N_{D(I,\psi)})=R^{n-1}\psi_*(N/\psi^{-1}I\cdot N)$ *where $\psi^{-1}I\cdot N$ is a sheaf on $\tilde{V}$ defined as the image of* $\psi^*I\otimes_{O_V}N\to N$.

Proof. (1) We note the following exact sequence.

$$\begin{tikzcd}
R^{n-1}\psi_*((\psi^*f)\cdot N) \arrow[r] \arrow[dr, "a"'] & R^{n-1}\psi_*(I_{D(I,\psi)}\cdot N) \arrow[r] \arrow[d, "b"] & R^{n-1}\psi_*(I_{D(I,\psi)}N/(\psi^*f)\cdot N) \arrow[d, equal, "c"] \\
& R^{n-1}\psi_*N & 0
\end{tikzcd}$$

The vanishing c comes from the fact that $I_{D(I,\psi)}=(\psi^*f)$ if $n=1$ and that the relative dimension of the support of $I_{D(I,\psi)}/(\psi^*f)\leqq n-2$ if $n\geqq 2$. Hence we obtain the equality *Image a=Image b*. Noting the isomorphisms $N\xrightarrow{\cong}(\psi^*f)\otimes_{O_V}N\xrightarrow{\cong}(\psi^*f)N$ by the locally $O_{\tilde{V}}$-freeness of N, we have the following commutative diagram:

$$\begin{tikzcd}
R^{n-1}\psi_*((\psi^*f)N) \arrow[r, "a"] \arrow[d, "\wr"] & R^{n-1}\psi_*N \\
R^{n-1}\psi_*N. \arrow[ur, "\text{multiplication by } f"'] &
\end{tikzcd}$$

Hence we obtain the equality *Image* $a=f\cdot R^{n-1}\psi_*N$. From the commutative diagram

$$\begin{tikzcd} R^{n-1}\psi_*(\psi^{-1}I\cdot N) \arrow[r] \arrow[dr, "d"] & R^{n-1}\psi_*(I_{D(I,\psi)}\cdot N) \arrow[d, "b"] \\ I\otimes R^{n-1}\psi_*N \arrow[u] \arrow[r, "e"', "\text{multiplication}"'] & R^{n-1}\psi_*N, \end{tikzcd}$$

we obtain the relations $f\cdot R^{n-1}\psi_*N\subseteqq I\cdot R^{n-1}\psi_*N=$*Image* $e\subseteqq$*Image* $d\subseteqq$ *Image* b. Furthermore we have already shown the equalities *Image* b $=$*Image* $a=f\cdot R^{n-1}\psi_*N$ above. Therefore the equalities $f\cdot R^{n-1}\psi_*N=$ $I\cdot R^{n-1}\psi_*N=$*Image* $d=$*Image* $b=$*Image* a hold.

(2) From the above, *Coker* $e=$*Coker* $b=$*Coker* d. Q.E.D.

§ 2. The inequality $p_a+L-1\leqq p_g$

(2.1) In the rest of this paper, we assume that (V, p) is a normal two-dimensional singularity over $\boldsymbol{C}$. Let $\psi:(\tilde{V}, A)\to(V, p)$ be a resolution of (V, p) with exceptional set $A=|\psi^{-1}(p)|$. The purpose of this section is to prove the inequality $p_a(V, p)+L(V, p)-1\leqq p_g(V, p)$ and to discuss some corollaries of this.

(2.2) First we shall review some results concerning the invariants p_g and p_a. Let the situation be as in (2.1). Let Z_0 be Artin's fundamental cycle on $(\tilde{V}, A)$. The inequalities

(2.2.1) $0\leqq p_a(Z_0)\leqq p_a\leqq p_g$

were proved in [2,20]. Here the non-negativity of $p_a(Z_0)$ was improved to the following form in Section 2 of [11]:

(2.2.2) $p_a(Z_0)=\dim H^1(O_{Z_0})$ (*that is*, $\dim H^0(O_{Z_0})=1$).

M. Artin [1, 2] characterized the rational singularity as follows:

(2.2.3) *The three conditions* $p_a(Z_0)=0$, $p_a=0$, and $p_g=0$ *are equivalent each other.*

Moreover P. Wagreich [20] pointed out the following (see also Corollary 4.2 [11], Remark (2.2) [17], and Remark (6.5) [19]):

(2.2.4) $p_a(Z_0)=1$ *if and only if* $p_a=1$.

Then he noted

(2.2.5) *There is a singularity with* $p_a=1$ *and with arbitrarily large* p_g.

Furthermore in Remark (4.3) [17], the following fact was proved:

(2.2.6) *For any couple of integers* (β, γ) *such that* $\beta\geqq\gamma\geqq 2$, *there is a singularity with* $p_a\geqq\beta\geqq\gamma=p_a(Z_0)$.

It seems that $p_a(Z_0)$, p_a, and p_g are very different in general. In fact, under the Gorenstein condition, the following fact was proved by several authors independently ([12], Theorem B [24], Proposition 1.3 [9], and Theorem 2.16 [23]):

(2.2.7) *If* (V, p) *is Gorenstein and* $p_a(Z_0)=p_a=p_g$ *hold, then the inequality* $p_g \leqq 1$ *holds.*

In this section, we generalize (2.2.7) in Corollary (2.11).

(2.3) We consider the m-adic filtration of $R^1\psi_*O_{\tilde{V}}$ as in Introduction:

(2.3.1) $R^1\psi_*O_{\tilde{V}} \supseteqq m \cdot R^1\psi_*O_{\tilde{V}} \supseteqq \cdots \supseteqq m^{L(V,p)} \cdot R^1\psi_*O_{\tilde{V}}=0.$

We take an element f of m such that the condition $D(m, \psi)= D((f), \psi)$ holds. Then the equality $D(m^r, \psi)=D((f^r), \psi)$ automatically holds by definition of the symbol $D(\ , \psi)$, for $r \geqq 0$. By Lemma (1.3), we obtain the relation

$$m^r \cdot R^1\psi_*O_{\tilde{V}}=f^r \cdot R^1\psi_*O_{\tilde{V}} \qquad \text{for } r \geqq 0.$$

This means that the m-adic filtration (2.3.1) is determined by the nilpotent endomorphism $F: R^1\psi_*O_{\tilde{V}} \to R^1\psi_*O_{\tilde{V}}$; $\alpha \mapsto f \cdot \alpha$. Since the order of nilpotency of F equals $L(V, p)$ and the dimension of the eigen vector space of F equals the dimension of *Coker F*, Jordan's theorem and Lemma (1.3) implies the following theorem:

Theorem (2.4). *Let* (V, p) *be a normal two-dimensional singularity. Then the following inequalities hold*:

(1) $\dim(R^1\psi_*O_{\tilde{V}}/m \cdot R^1\psi_*O_{\tilde{V}})+L(V, p)-1 \leqq p_g(V, p)$.

(2) $p_g(V, p) \leqq L(V, p) \cdot \dim(R^1\psi_*O_{\tilde{V}}/m \cdot R^1\psi_*O_{\tilde{V}})$.

We need the following lemma for the discussion on p_a:

Lemma (2.5). *Let the situation be as in* (2.1). *Let* D *be a non-zero effective divisor on* $\tilde{V}$ *such that* $|D| \subseteqq A$. *Then the following equality holds*:

$$p_a(D)=\dim R^1\psi_*O_{\tilde{V}}-\dim R^1\psi_*I_D-\dim(m/\psi_*I_D).$$

Proof. We have the following exact sequence on V: $0 \to O_V/\psi_*I_D \to \psi_*O_D \to R^1\psi_*I_D \to R^1\psi_*O_{\tilde{V}} \to R^1\psi_*O_D \to 0$. Hence we obtain: $p_a(D)= 1-\chi(O_D)=1-\dim \psi_*O_D+\dim R^1\psi_*O_D=1+\dim R^1\psi_*O_{\tilde{V}}-\dim R^1\psi_*I_D -\dim(O_V/\psi_*I_D)$. Since $\dim(O_V/m)=1$, the assertion follows. Q.E.D.

Theorem (2.6). *Let* (V, p) *be a normal two-dimensional singularity. Then the following inequality holds*:

$$p_a(V, p)+L(V, p)-1 \leqq p_g(V, p).$$

Proof. We take a resolution $\psi: (\tilde{V}, A) \to (V, p)$ and choose a non-zero effective divisor D on $\tilde{V}$ such that $|D|\subseteqq A$ and that $p_a(D)=p_a(V,p)$. By Lemma (2.5), we obtain the equality $p_g(V,p)-p_a(V,p)=\dim R^1\psi_*I_D+\dim(m/\psi_*I_D)$. Now we set the integers s and t by $s=\dim R^1\psi_*I_D$ and $t=\dim(m/\psi_*I_D)$. By Nakayama's lemma, we can easily show $m^s\cdot R^1\psi_*I_D=0$ and $m^t\cdot(m/\psi_*I_D)=0$ (that is $m^{t+1}\subseteqq\psi_*I_D$). From the commutative diagram

$$\begin{tikzcd} R^1\psi_*I_D \arrow[r, "g"] & R^1\psi_*O_{\tilde{V}} \\ \psi_*I_D\otimes R^1\psi_*O_{\tilde{V}} \arrow[u] \arrow[ur, "\text{multiplication}"'] & \end{tikzcd}$$

we obtain the following relations: $m^{s+t+1}\cdot R^1\psi_*O_{\tilde{V}}\subseteqq m^s\cdot\psi_*I_D\cdot R^1\psi_*O_{\tilde{V}}\subseteqq g(m^s\cdot R^1\psi_*I_D)=0$. Therefore the inequality $1+s+t\geqq L(V,p)$ follows. Q.E.D.

(2.7) We recall some basic facts related to the Serre duality (cf. [10, 11]). We choose a representative V of (V,p) such that $V-\{p\}$ is non-singular and $\tilde{V}$ is strongly pseudoconvex. Then $H^1(\tilde{V},O_{\tilde{V}})$ is a finite dimensional vector space and isomorphic to $R^1\psi_*O_{\tilde{V}}$. By Serre [16], there is a non-degenerate $\boldsymbol{C}$-bilinear pairing $\langle\ ,\ \rangle: H^1(\tilde{V},O_{\tilde{V}})\times H^1_*(\tilde{V},\Omega^2_{\tilde{V}})\to\boldsymbol{C}$ such that the equality $\langle\gamma\alpha,\beta\rangle=\langle\alpha,\gamma\beta\rangle$ holds for any triple (α,β,γ) of $H^1(\tilde{V},O_{\tilde{V}})\times H^1_*(\tilde{V},\Omega^2_{\tilde{V}})\times H^0(\tilde{V},O_{\tilde{V}})$. Here $\Omega^2_{\tilde{V}}$ is the sheaf of holomorphic two-forms on $\tilde{V}$ and $H^1_*(\tilde{V},\ )$ denotes the cohomology with compact supports.

Furthermore Laufer [10] represented $H^1_*(\tilde{V},\Omega^2_{\tilde{V}})$ in the following form: $H^1_*(\tilde{V},\Omega^2_{\tilde{V}})=\omega_V/\psi_*(\Omega^2_{\tilde{V}})$. Here ω_V is the dualizing sheaf of O_V defined by $\omega_V=i_*(\psi_*(\Omega^2_{\tilde{V}-A}))$, where $i: V-\{p\}\hookrightarrow V$ is the inclusion map.

From the characteristic of the pairing $\langle\ ,\ \rangle$, we can easily prove the following lemma:

Lemma (2.8). *Let the situation be as above. Let I be an ideal sheaf of O_V. Then the pairing $\langle\ ,\ \rangle$ above induces the following duality over $\boldsymbol{C}$:*

$$(R^1\psi_*O_{\tilde{V}}/I\cdot R^1\psi_*O_{\tilde{V}})\xleftrightarrow[\text{dual over } \boldsymbol{C}]{}\{\beta\in\omega_V/\psi_*(\Omega^2_{\tilde{V}})\,|\,I\cdot\beta=0\}.$$

(2.9) Therefore the integer $L(V,p)$ is written as follows: Let an element f of m satisfy the condition $D(m,\psi)=D((f),\psi)$. Then:

$$\begin{aligned} L(V,p)&=\min\{r\in\boldsymbol{Z}\,|\,r\geqq 0,\ m^r\cdot(\omega_V/\psi_*(\Omega^2_{\tilde{V}}))=0\}\\ &=\min\{r\in\boldsymbol{Z}\,|\,r\geqq 0,\ f^r\cdot(\omega_V/\psi_*(\Omega^2_{\tilde{V}}))=0\}\\ &=\max\{\dim\boldsymbol{C}[\psi^*f]\cdot\beta\,|\,\beta\in\omega_V/\psi_*(\Omega^2_{\tilde{V}})\}\\ &=\max\{\dim\boldsymbol{C}[\psi^*h]\cdot\beta\,|\,\beta\in\omega_V/\psi_*(\Omega^2_{\tilde{V}}),\ h\in m\}.\end{aligned}$$

Hence if (V, p) is Gorenstein, the integer $L(V, p)$ coincides with the integer $\min\{r \in \boldsymbol{Z} \mid -K_{\tilde{V}} \leqq r \cdot D(m, \psi)\}$ (Proposition 4 [17]). Here $K_{\tilde{V}}$ is the divisor on $\tilde{V}$ such that $|K_{\tilde{V}}| \subseteqq A$ and $\Omega^2_{\tilde{V}} \cdot A_j = K_{\tilde{V}} \cdot A_j$ for any irreducible component A_j of A.

Corollary (2.10) (*Theorem* 3.2 *of Yau* [24]). *Let (V, p) be a normal two-dimensional singularity. If there is a couple of elements (β, f) of $(\omega_V/\psi_*(\Omega^2_{\tilde{V}})) \times m$ such that the set $\{\beta, f \cdot \beta, \cdots, f^{(p_g(V,p)-1)} \cdot \beta\}$ forms a $\boldsymbol{C}$-basis of $\omega_V/\psi_*(\Omega^2_{\tilde{V}})$, then the inequality $p_a(V, p) \leqq 1$ holds.*

Proof. By (2.9), the assumption is equivalent to the condition $L(V, p) = p_g(V, p)$. Therefore the assertion is a corollary of Theorem (2.6). Q.E.D.

The following statement generalizes (2.2.7):

Corollary (2.11). *Let (V, p) be a normal two-dimensional singularity. If the condition $p_g(V, p) = p_a(V, p)$ holds, then $p_g(V, p)$ is not greater than the Cohen-Macaulay type of $O_{V,p}$.*

Here the Cohen-Macaulay type of $O_{V,p}$ is the number of the minimal generators of the dualizing sheaf at p.

Proof. By Theorem (2.6), we obtain the relation $m \cdot R^1\psi_* O_{\tilde{V}} = 0$ for any resolution $\psi : (\tilde{V}, A) \to (V, p)$ of (V, p). Hence the relation $m \cdot (\omega_V/\psi_*(\Omega^2_{\tilde{V}})) = 0$ holds by Lemma (2.8). Therefore

$$p_g(V, p) = \dim \omega_V/\psi_*(\Omega^2_{\tilde{V}}) \leqq \dim \omega_V/m \cdot \omega_V.$$

Q.E.D.

Example (2.12) (K.-i. Watanabe). *A singularity with $p_g = p_a = p_a(Z_0)$ = Cohen-Macaulay type.* Let F be a negative line bundle over a non-singular curve X of genus g and (V, p) the singularity obtained from the construction of zero-section of F.

Then we can represent (V, p) as follows:

$$V = \operatorname{Spec} R,\ R = \bigoplus_{k \geqq 0} H^0(X, F^{-k}) \cdot T^k \subseteqq k(X)[T],$$

where $k(X)$ is the field of rational functions of X and T is an indeterminate (cf. Pinkham [15]). By Goto-Watanabe [8] and Watanabe [21], the canonical module of R is the graded module K_R written by

$$K_R = \bigoplus_{k \in \boldsymbol{Z}} H^0(X, K_X \otimes (F^{-k})) \cdot T^k.$$

If $-\deg(F) \geqq 2 \cdot g + 1$, then $H^1(X, F^{-n})$ and $H^0(X, K_X \otimes (F^n))$ vanish for $n \geqq 1$, and $H^0(X, K_X) \otimes H^0(X, F^{-n}) \to H^0(X, K_X \otimes (F^{-n}))$ is surjective for $n \geqq 0$ by Proposition 1.10 of Fujita [6].

By Pinkham [15], $p_g(V, p) = \sum_{k \geqq 0} \dim H^1(X, F^{-k})$.

Therefore $p_g(V, p) = p_a(V, p) = p_a(Z_0) =$ the Cohen-Macaulay type $= g$, if $-\deg(F) \geqq 2 \cdot g + 1$.

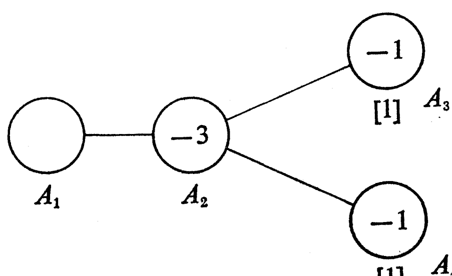
Example (2.13). *A singularity with* $\dim H^1(O_{D(m,\psi)}) \neq p_a(Z_0)$.

We compute $\dim H^1(O_{D(m,\psi)})$ for the minimal resolution $\psi : (\tilde{V}, A) \to (V, p)$ of the singularity $(V, p) = (\{(x, y, z) \in C^3 \mid z^2 = (x+y) \cdot (x^4 + y^6) \cdot (x^6 + y^4)\}, o)$ by our method.

First one can compute the geometric genus of (V, p) by Lemma 2 of [17]: $p_g(V, p) = 8$.

Second we choose a C-basis of $\omega_V / \psi_*(\Omega^2_{\tilde{V}})$ in the following way: We choose a meromorphic two-form ω_0 on $\tilde{V}$ which is holomorphic and nowhere vanishing on $\tilde{V} - A$. Then the relation $\omega_V = O_V \cdot \omega_0$ at p follows by definition of ω_V (2.7).

We write the dual graph of the exceptional set A as follows:



Then the divisors defined by the form and functions ω_0, $\psi^*(x)$, $\psi^*(y)$, $\psi^*(z)$, and $\psi^*(x-y)$ on $\tilde{V}$ are as follows:

(2.13.1) $\quad \operatorname{div}(\omega_0) = -3 \cdot D(m, \psi) - A_3 - A_4,$

(2.13.2) $\quad \operatorname{div}(\psi^*(x)) = D(m, \psi) + 2 \cdot A_3 + D_x,$

(2.13.3) $\quad \operatorname{div}(\psi^*(y)) = D(m, \psi) + 2 \cdot A_4 + D_y,$

(2.13.4) $\quad \operatorname{div}(\psi^*(z)) = 5 \cdot D(m, \psi) + A_3 + A_4 + D_z,$

(2.13.5) $\quad \operatorname{div}(\psi^*(x-y)) = D(m, \psi) + D_{x-y}$, and

(2.13.6) $\quad D(m, \psi) = A_1 + 2 \cdot A_2 + 2 \cdot A_3 + 2 \cdot A_4.$

Here the divisors D_x, D_y, D_z, and D_{x-y} on $\tilde{V}$ are the strict transforms of the divisors $\{x=0\}$, $\{y=0\}$, $\{z=0\}$, and $\{x-y=0\}$ on V by ψ respectively.

These relations can be easily proved by using *Zariski's canonical resolution* in Section 1 of [17].

Hence the meromorphic form $x^i \cdot y^j \cdot z^k \cdot \omega_0$, where i, j and k are non-negative integers, is not holomorphic on $\tilde{V}$ if and only if this is one of the following 8 forms: ω_0, $x_0 \cdot \omega_0$, $y \cdot \omega_0$, $x^2 \cdot \omega_0$, $xy \cdot \omega_0$, $y^2 \cdot \omega_0$, $x^3 \cdot \omega_0$ and $y^3 \cdot \omega_0$. They give a C-basis of $\omega_V / \psi_*(\Omega^2_{\tilde{V}})$. Furthermore (2.13.5) means the condition $D(m, \psi) = D((x-y), \psi)$.

Finally we can easily see the following relations:

$$\begin{aligned}\{\beta \in \omega_V/\psi_*(\Omega^2_{\tilde{V}}) \mid m \cdot \beta = 0\} &= \{\beta \in \omega_V/\psi_*(\Omega^2_{\tilde{V}}) \mid (x-y) \cdot \beta = 0\} \\ &= C \cdot \mathrm{cls}[xy \cdot \omega_0] \oplus C \cdot \mathrm{cls}[x^3 \cdot \omega_0] \oplus C \cdot \mathrm{cls}[y^3 \cdot \omega_0].\end{aligned}$$

Therefore $\dim(R^1\psi_* O_{\tilde{V}} / m \cdot R^1\psi_* O_{\tilde{V}}) = \dim H^1(O_{D(m,\psi)}) = 3$.

On the other hand, the fundamental cycle Z_0 is reduced. Then one can easily see the following equalities (2.2.2): $\dim H^1(O_{Z_0}) = p_a(Z_0) = 2$.

§ 3. Arithmetic genus of normal two-dimensional singularity with good C^*-action

(3.1) The purpose of this section is to prove the inequality

$$\dim(R^1\psi_* O_{\tilde{V}} / m \cdot R^1\psi_* O_{\tilde{V}}) \leqq p_a(V, p)$$

for the normal two-dimensional singularity with good C^*-action. At the same time, we obtain a formula for computation of the arithmetic genus of the singularity with exceptional set of star-shaped dual graph. Then we shall give some corollaries of these.

(3.2) Let (V, p) be a normal two-dimensional singularity with good C^*-action and $\psi : (\tilde{V}, A) \to (V, p)$ the minimal good resolution. Then the dual graph of A is *star-shaped* (Orlik-Wagreich [13]). If the dual graph of A has no center, (V, p) is a cyclic quotient singularity (Brieskorn [3]). In particular it is rational. If the dual graph of A has the central curve E, we can represent (V, p) by using certain Q-divisor D on E as follows (Pinkham [15]): $V = \mathrm{Spec}\, R$, $R = \oplus_{k \geqq 0} H^0(E, O_E(D^{(k)})) \cdot T^k \subseteqq k(E)[T]$, where $k(E)$ is the field of rational function on E, the symbol T is an indeterminate, and the integral divisor $D^{(k)}$ on E is defined by the maximum among the set {integral divisor G on $E \mid G \leqq k \cdot D$} for $k = 0, 1, 2, \cdots$.

By using the representation above, we shall construct the following *canonical partial resolution* ([4, 14, 21, 22]): We denote the scheme $\mathrm{Spec}(\oplus_{k \geqq 0} O_E(D^{(k)}) T^k)$ over E by $C(E, D)$ and the canonical morphism $C(E, D) \to \mathrm{Spec}(\oplus_{k \geqq 0} H^0(E, O_E(D^{(k)})) \cdot T^k) = V$ by ϕ. Then $C(E, D)$ is normal. The exceptional locus of ϕ is isomorphic to E. In fact, as analytic space, $C(E, D)$ is obtained by contracting rational contractible curves from the minimal good resolution.

Lemma (3.3). *Let (V, p) be a normal two-dimensional singularity and $\psi : (\tilde{V}, A) \to (V, p)$ a non-trivial partial resolution such that the singularity of $\tilde{V}$ is rational.*

Then the arithmetic genus $p_a(V, p)$ equals the maximum among the set $\{1-\chi(O_{\tilde{V}}/I) \mid I$ is coherent ideal sheaf of $O_{\tilde{V}}$ such that $I \neq O_{\tilde{V}}$ and the support of $O_{\tilde{V}}/I$ is contained in $A\}$. Moreover this maximum is attained by a divisorial ideal sheaf I.

Proof. Let I be an ideal sheaf of $O_{\tilde{V}}$ such that $I \neq O_{\tilde{V}}$ and the support of $O_{\tilde{V}}/I$ is contained in A. First we shall prove the inequality $p_a(V, p) \geqq 1-\chi(O_{\tilde{V}}/I)$. If the support of $O_{\tilde{V}}/I$ is zero-dimensional, $1-\chi(O_{\tilde{V}}/I) \leqq 0$. Hence we assume the support of $O_{\tilde{V}}/I$ is one-dimensional. Let I^{**} be the reflexive hull of the sheaf I. Then I^{**} is a divisorial proper ideal sheaf of $O_{\tilde{V}}$. From the exact sequence

$$0 \longrightarrow I^{**}/I \longrightarrow O_{\tilde{V}}/I \longrightarrow O_{\tilde{V}}/I^{**} \longrightarrow 0,$$

we obtain: $1-\chi(O_{\tilde{V}}/I) = 1-\chi(O_{\tilde{V}}/I^{**}) - \dim(I^{**}/I)$. Therefore it suffices to show the inequality above in the case divisorial ideal sheaf I. We assume $I = I^{**}$ and take a proper modification $\tau : V' \to \tilde{V}$ such that $\tau^{-1}I$ is $O_{V'}$-invertible and V' is non-singular.

Then the composition $\psi \circ \tau : V' \to V$ is a resolution of (V, p). There is a non-zero effective divisor B on V' such that $\tau^{-1}I = O_{V'}(-B)$. Then $|B| \subseteqq |(\psi \circ \tau)^{-1}(p)|$ by assumption of I. By Proposition 1.8 and Proposition 1.9 of Giraud [7], we obtain the relations $\tau_*(\tau^{-1}I) = I$ and $R^1\tau_*(\tau^{-1}) = 0$. Hence $O_{\tilde{V}}/I = \tau_*(O_B)$. Since $R^1\tau_* O_{\tilde{V}} = 0$, we have $R^1\tau_* O_B = 0$. Hence we obtain the equality $\chi(O_{\tilde{V}}/I) = \chi(O_B)$ by Leray's spectral sequence. Now the relation $1-\chi(O_{\tilde{V}}/I) = p_a(B) \leqq p_a(V, p)$ follows.

We shall complete the proof. Let $\sigma : V'' \to \tilde{V}$ be a proper modification of $\tilde{V}$ induced from a resolution of singularities of $\tilde{V}$. Let D be a non-zero effective divisor on V'' such that $|D| \subseteqq |\sigma^{-1}(A)|$ and $p_a(D) = p_a(V, p)$. By the same argument as that of the proof of Lemma (3.8) of [19], we can prove the equality $p_a(D) = 1-\chi(O_{\tilde{V}}/\sigma_* I_D) - \dim R^1\sigma_* I_D$. Hence we obtain the inequality $p_a(V, p) \leqq 1-\chi(O_{\tilde{V}}/\sigma_* I_D)$.

The remaining assertion is clear from the arguments above.

Theorem (3.4). *Let (V, p) be a normal two-dimensional singularity with good C^*-action and $\psi : (\tilde{V}, A) \to (V, p)$ a resolution of singularity (V, p). Then the following inequality holds:*

$$\dim(R^1\psi_* O_{\tilde{V}} / m \cdot R^1\psi_* O_{\tilde{V}}) \leqq p_a(V, p).$$

Proof. Our assertion is obvious if (V, p) is rational. Hence we assume (V, p) is not rational. We can consider the canonical partial resolution $\phi : (C(E, D), E) \to (V, p)$ as in (3.2). We write the Weil divisor $D(m, \phi)$ on $C(E, D)$ in the form $r_m \cdot E$ by the integer r_m. Then the divisorial ideal sheaf $I_{D(m,\phi)} = O_{C(E,D)}(-r_m \cdot E)$ is isomorphic to the graded O_E-module sheaf $\oplus_{k \geqq r_m} O_E(D^{(k)}) \cdot T^k$ (Remark (1.5) of Watanabe [22]). Hence we obtain the following diagram:

$$\begin{array}{ccccc}
0 \longrightarrow \phi_*(O_{C(E,D)}(-r_m \cdot E)) & \longrightarrow & \phi_* O_{C(E,D)} & \longrightarrow & \phi_* O_{D(m,\phi)} \\
\wr\| & & \wr\| & & \\
m & & O_V & &
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow R^1\phi_*(O_{C(E,D)}(-r_m \cdot E)) & \xrightarrow{\quad h \quad} & R^1\phi_* O_{C(E,D)} \\
\wr\| & & \wr\| \\
\bigoplus\limits_{k \geqq r_m} H^1(E, O_E(D^{(k)})) \cdot T^k & \longrightarrow & \bigoplus\limits_{k \geqq 0} H^1(E, O_E(D^{(k)})) \cdot T^k.
\end{array}$$

Therefore h is injective. Then $\phi_* O_{D(m,\phi)}$ is isomorphic to $\boldsymbol{C}$. By Lemma (1.3) and Lemma (3.3), we obtain the relations

$$\begin{aligned}
\dim(R^1\phi_* O_{C(E,D)}/m \cdot R^1\phi_* O_{D(E,D)}) &= \dim H^1(O_{D(m,\phi)}) \\
&= 1 - \chi(O_{D(m,\phi)}) \leqq p_a(V, p).
\end{aligned}$$

By the Leray spectral sequence, $R^1\phi_* O_{C(E,D)} = R^1\psi_* O_{\tilde{V}}$, since $C(D, E)$ has only rational singularities. Q.E.D.

Combining with Theorem (2.4), we obtain the following:

Corollary (3.5). *Let (V, p) be a normal two-dimensional singularity with good $\boldsymbol{C}^*$-action. Then the following inequality holds:*

$$p_g(V, p) \leqq L(V, p) \cdot p_a(V, p).$$

Furthermore this and Theorem (2.6) imply the following:

Corollary (3.6). *Let (V, p) be a normal two-dimensional singularity with good $\boldsymbol{C}^*$-action. Then:*

(1) $L(V, p) \leqq 1$ *if and only if* $p_a(V, p) = p_g(V, p)$.

(2) $L(V, p) = p_g(V, p)$ *if and only if* $p_a(V, p) \leqq 1$.

(3.7) Next we consider the star-shaped negative definite weighted dual graph with central curve E. By Pinkham [15], we can find a $\boldsymbol{Q}$-divisor D on E such that the dual graph of exceptional set in the minimal good resolution of the singularity $\operatorname{Spec}(\oplus_{k \geqq 0} H^0(E, O_E(D^{(k)})) \cdot T^k)$ at the vertex coincides with the given weighted dual graph. Then such a divisor D is not unique, but the integers $g = genus(E)$ and $\deg(D^{(k)})$, $k \in \boldsymbol{Z}$, are independent of the choice of D. Hence we shall treat these integers as the numerical invariants of weighted dual graph in the following:

Theorem (3.8). *Let (V,p) be a normal two-dimensional singularity and $\psi:(\tilde{V}, A)\to(V,p)$ the minimal good resolution. Assume that the dual graph of A is star-shaped with central curve E. Then the arithmetic genus $p_a(V,p)$ of (V,p) is written as follows:*

$$p_a(V,p)=\max_{1\leqq r}\left\{r(g-1)-\left(\sum_{k=0}^{r-1}\deg(D^{(k)})\right)+1\right\},$$

where the integers g and $\deg(D^{(k)})$, $k\geqq 0$, are defined as in (3.7) *by Pinkham's rule* [15].

Proof. Let (V',p') be a singularity defined by

$$V'=\operatorname{Spec}(\bigoplus_{k\geqq 0} H^0(E, O_E(D^{(k)}))T^k)$$

as in (3.7). Then $p_a(V,p)=p_a(V',p')$. We take the canonical partial resolution $\phi:(C(E,D),E)\to(V',p')$ as in (3.2). Let I be a divisorial ideal sheaf on $C(E,D)$ such that $I\neq O_{C(E,D)}$ and that the support of $O_{C(E,D)}/I$ is contained in E. Then I is written as $I=O_{C(E,D)}(-r\cdot E)$ by the positive integer r and is isomorphic to the graded O_E-module sheaf $\oplus_{k\geqq r}O_E(D^{(k)})\cdot T^k$ (Remark (1.5) of Watanabe [22]).

By Lemma (3.3), we obtain the equality

$$p_a(V',p')=\max_{1\leqq r}\{1-\chi(O_{C(E,D)}/O_{C(E,D)}(-r\cdot E))\}.$$

Here we have

$$\chi(O_{C(E,D)}/O_{C(E,D)}(-r\cdot E))=\sum_{k=0}^{r-1}\chi(O_E(D^{(k)}))=\sum_{k=0}^{r-1}\{\deg(D^{(k)})+1-g\}$$

by the Riemann-Roch formula. Q.E.D.

Corollary (3.9). *Let (V,p) be a normal two-dimensional singularity and $\psi:(\tilde{V}, A)\to(V,p)$ the minimal good resolution. Assume that the dual graph of A is star-shaped. Then:*

(1) $p_a(V,p)=0$ *if and only if one of the conditions* (a) *and* (b) *holds:* (a) *The dual graph of A has no center.* (b) *The dual graph of A has the center $E\cong \boldsymbol{P}^1$ and* $\deg(D^{(k)})\geqq -1$ *for $k\geqq 1$ (cf. Corollary* 5.8 *of Pinkham* [15]).

(2) $p_a(V,p)=1$ *if and only if one of the conditions* (c) *and* (d) *holds:* (c) *The dual graph of A has the center E. E is an elliptic curve and*

$\deg(D^{(1)})\geqq 0$. (d) *The dual graph of A has the center E. $E\cong \boldsymbol{P}^1$ and there are integers γ and δ such that the conditions $1\leqq \gamma<\delta$, $\deg(D^{(\gamma)})=-2$, $\deg(D^{(\delta)})\geqq 0$, and $\deg(D^{(i)})=-1$ for $i\in\{1,\cdots,\delta-1\}\cap\{j|j\neq\gamma\}$ hold.*

Proof. (1) is obvious from Theorem (3.8). (2) Sufficiency of (c) and (d). First we note the inequality $D^{(k+k')}\geqq D^{(k)}+D^{(k')}$ for $k, k'\geqq 0$. In the case (c), $\deg(D^{(k)})\geqq 0$ follows for $k\geqq 0$. Hence the condition $p_a(V,p)=1$ can be easily computed by Theorem (3.8). In the case (d), we can easily show the following inequalities: $\deg(D^{(k)})\geqq 0$ if $k\equiv 0 \mod \delta$, $\deg(D^{(k)})\geqq -2$ if $k\equiv\gamma \mod\delta$, and $\deg(D^{(k)})\geqq -1$ if $k\not\equiv 0, \gamma, \mod\delta$. Therefore the condition $p_a(V,p)=1$ can be computed by Theorem (3.8).

The proof of the necessity is not difficult.

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