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On Coxeter Arrangements and the Coxeter Number

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Abstract

Let (G, V) be an irreducible Coxeter group and let $\mathscr{A}$ be the corresponding Coxeter arrangement. Let $H \in \mathscr{A}$ be a hyperplane and let $\mathscr{A}^H$ be the restriction of $\mathscr{A}$ to H. Let h be the Coxeter number. We prove that

$$|\mathscr{A}^H| = |\mathscr{A}| - h + 1$$

and show that $\mathscr{A}^H$ is a free arrangement whose degrees are $m_1, \cdots, m_{l-1}$ the first $l-1$ exponents G.

§ 1. Introduction

Let V be Euclidean space of dimension l with positive definite inner product written $(\ ,\ )$. A hyperplane in V is a vector subspace of codimension 1. An *arrangement* $\mathscr{A} = (\mathscr{A}, V)$ in V is a finite set of hyperplanes.

Let $S = S(V^*)$ be the symmetric algebra of the dual space V^*. We may view S as the graded algebra of polynomial functions on V. Let $\mathrm{Der}(S)$ be the S-module of all derivations of S. A non-zero element $\theta \in \mathrm{Der}(S)$ is said to be *homogeneous of degree* b, written $\deg \theta = b$, if $\theta(S_p) \subset S_{p+b}$. This makes $\mathrm{Der}(S)$ a graded S-module. Choose, for each $H \in \mathscr{A}$, a linear form $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Define $Q \in S$ by

$$Q = \prod_{H \in \mathscr{A}} \alpha_H. \tag{1.1}$$

The polynomial Q is uniquely determined by $\mathscr{A}$ up to a constant multiple. Define

$$D(\mathscr{A}) = \{\theta \in \mathrm{Der}(S) \mid \theta Q \in QS\}. \tag{1.2}$$

Then $D(\mathscr{A})$ is a graded S-submodule of $\mathrm{Der}(S)$.

(1.3) **Definition** [14]. The arrangement $\mathscr{A}$ is *free* if $D(\mathscr{A})$ is a free S-module.

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If $\mathscr{A}$ is free then [11, Theorem 1.11.i] there exists a basis for $D(\mathscr{A})$ as S-module consisting of l homogeneous elements $\theta_1, \cdots, \theta_l$. Let $b_i = \deg \theta_i + 1$. The integers $b_1, \cdots, b_l$ depend only on $\mathscr{A}$. We call them the *degrees* of $\mathscr{A}$ and write $\deg(\mathscr{A}) = \{b_1, \cdots, b_l\}$. The main theorem in [15] asserts that the integers $b_1, \cdots, b_l$ are determined by combinatorial data. To make this precise we need the notion of characteristic polynomial of an arrangement. Let $L = L(\mathscr{A})$ be the set of intersections of elements of $\mathscr{A}$. Partially order L by reverse inclusion so that L has V as its minimal element and $\mathscr{A}$ as its set of atoms. The poset L is a finite geometric lattice with rank function $r(X) = \operatorname{codim}_V X$, $X \in L$. The *Möbius function* μ of L is an integer-valued function defined recursively as follows [9, p. 344].

$$(1.4) \qquad \begin{aligned} &\mu(X, X) = 1 && \text{if} \quad X \in L \\ &\mu(X, Y) = -\sum_{\substack{Z \in L \\ X \leq Z < Y}} \mu(X, Z) && \text{if} \quad X, Y \in L \quad \text{and} \quad X < Y. \end{aligned}$$

It is convenient to define

$$(1.5) \qquad \mu(X) = \mu(V, X) \qquad \text{if} \quad X \in L.$$

The *characteristic polynomial* $\chi(\mathscr{A}, t)$ is defined by

$$(1.6) \qquad \chi(\mathscr{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X}.$$

It is an important invariant to the arrangement. The main theorem in [15] is:

(1.7) **Theorem.** *If $\mathscr{A}$ is a free arrangement with* $\deg \mathscr{A} = \{b_1, \cdots, b_l\}$ *then*

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{l} (t - b_i).$$

Let $O(V)$ be the orthogonal group and let G be a finite subgroup of $O(V)$ which is generated by reflections. Then G is a Coxeter group [1, Ch. 5, § 3.2, Th. 1]. The group G acts naturally as a group of automorphisms of S. Chevalley's theorem [1, Ch. 5, § 5.5, Th. 4] asserts that there are algebraically independent homogeneous elements $f_1, \cdots, f_l$ in the invariant subring S^G such that $S^G = \boldsymbol{R}[f_1, \cdots, f_l]$. The polynomials $f_1, \cdots, f_l$ are called *basic invariants. We agree to number them so that* $\deg f_1 \leq \cdots \leq \deg f_l$. The integers $m_i = \deg f_i - 1$ are the *exponents* of G[1, Ch. 5, § 6.2]. Let $\mathscr{A} = \mathscr{A}(G)$ be the set of reflecting hyperplanes of G. We call $\mathscr{A}$ a *Coxeter arrangement*. K. Saito has shown [10, 4.8] that a Coxeter arrangement $\mathscr{A}$ is free and that $\deg(\mathscr{A}) = \{m_1, \cdots, m_l\}$. It follows from (1.7) and Saito's theorem that

$$\chi(\mathscr{A}, t) = \prod_{i=1}^{l} (t - m_i) \tag{1.8}$$

a formula which was proved in [5, Th. 4.8] in the more general context of unitary reflection groups.

(1.9) **Definition.** Let $\mathscr{A}$ be an arrangement. If $X \in L(\mathscr{A})$ let $\mathscr{A}^X$ be the arrangement in X defined by

$$\mathscr{A}^X = \{X \cap H \mid H \in \mathscr{A} \text{ and } X \cap H \neq X\}.$$

We call $\mathscr{A}^X$ the *restriction* of $\mathscr{A}$ to X.

(1.10) **Conjecture.** If $\mathscr{A}$ is free and $X \in L(\mathscr{A})$ then $\mathscr{A}^X$ is free.

If (1.10) is true and $\deg(\mathscr{A}^X) = \{b_1^X, \cdots, b_k^X\}$ where $k = \dim X$ then (1.7) implies

$$\chi(\mathscr{A}^X, t) = \prod_{i=1}^{k} (t - b_i^X). \tag{1.11}$$

It is shown in [6], using the classification of finite reflection groups, that if $\mathscr{A}$ is a Coxeter arrangement then there is a factorization of type (1.11) for every $X \in L(\mathscr{A})$. By using the Addition Theorem of [14, p. 305] we have checked in all irreducible groups except type E_8 that (1.10) holds for all $X \in L(\mathscr{A})$ and that the degrees b_i^X are the roots of $\chi(\mathscr{A}^X, t)$ computed in [6]. To prove (1.10) it would suffice, arguing by induction on $\operatorname{codim}_V X$ to prove it in case $X = H$ is a hyperplane. The main result of this paper is the following theorem.

(1.12) **Theorem.** *Let $G \subset O(V)$ be a finite irreducible reflection group. Let $\mathscr{A} = \mathscr{A}(G)$ be the Coxeter arrangement and let $H \in \mathscr{A}$. Then the restriction $\mathscr{A}^H$ is a free arrangement and* $\deg(\mathscr{A}^H) = \{m_1, \cdots, m_{l-1}\}$ *the first $l-1$ exponents of G.*

A slightly stronger statement is proved in (3.9). In fact $\mathscr{A}^H$ is free for all Coxeter arrangements $\mathscr{A}(G)$ whether G is irreducible or not. There is an easy reduction to the irreducible case. Unfortunately this does not settle (1.10) for Coxeter arrangement $\mathscr{A}$ because $\mathscr{A}^H$ is not in general a Coxeter arrangement. Theorems (1.7) and (1.12) have the following corollary analogous to (1.8).

(1.13) **Corollary.** *Let $G \subset O(V)$ be a finite irreducible reflection group. Let $\mathscr{A} = \mathscr{A}(G)$ be the Coxeter arrangement and let $H \in \mathscr{A}$. Then*

$$\chi(\mathscr{A}^H, t)=\prod_{i=1}^{l-1}(t-m_i).$$

In particular, $\chi(\mathscr{A}^H, t)$ *is independent of* H.

The numerical results in [6] suggest an assertion which is more general than the one in (1.12):

(1.14) **Conjecture.** *Let* $G\subset O(V)$ *be a finite irreducible reflection group. Let* $\mathscr{A}=\mathscr{A}(G)$ *be the Coxeter arrangement. There exists for each* k *with* $1\leq k\leq l$ *a subspace* $X\in L(\mathscr{A})$ *of codimension* k *such that* $\deg(\mathscr{A}^X)=\{m_1, \cdots, m_k\}$ *the first* k *exponents of* G.

Here is an outline of this paper. In Section 2 we prove some general facts about restricted arrangements $\mathscr{A}^X$. Although our main result is (1.12) this increase in generality costs no additional effort and points in the direction of (1.14). We prove Theorem (1.12) in Section 3. As an application of (1.12) we prove a purely algebraic result concerning the basic invariants:

(1.15) $$f_l \in \alpha_H S+f_1S+\cdots+f_{l-1}S.$$

On the geometric side we show as a consequence of (1.12) that the number of $l-2$ simplexes in the Coxeter complex which lie in a hyperplane $H\in\mathscr{A}$ is $|G|/h$. This is a theorem of Steinberg [13, Cor. 4.2]. In Section 4 we use (1.12) to prove a character formula. Let φ be the character of G afforded by the representation of G on V. If $g\in G$ let $k(g)$ be the dimension of the fixed point set of g. Then

(1.16) $$\sum_{g\in G}\varphi(g)t^{k(g)}=l(t-1)(t+m_1)\cdots(t+m_{l-1}).$$

This may be viewed as an analogue of the Shephard-Todd formula [12, Th. 5.3]

(1.17) $$\sum_{g\in G}t^{k(g)}=(t+m_1)(t+m_2)\cdots(t+m_l).$$

§ 2. Arrangements and their restrictions

Let V be Euclidean space with inner product $(\,,)$ and let $\mathscr{A}=(\mathscr{A}, V)$ be an arrangement. Let $S=S(V^*)$ be the symmetric algebra of V^*. As in the Introduction, choose for each $H\in\mathscr{A}$ a linear form $\alpha_H\in V^*$ with $\ker(\alpha_H)=H$, let $Q=\prod_{H\in\mathscr{A}}\alpha_H$ and let $D(\mathscr{A})$ be the set of all $\theta\in \mathrm{Der}(S)$ such that $\theta Q\in QS$. Let $x_1, \cdots, x_l$ be a basis for V^*. Then $S=\boldsymbol{R}[x_1, \cdots, x_l]$. For $\theta_1, \cdots, \theta_l\in D(\mathscr{A})$ define

$$\theta_1 \wedge \cdots \wedge \theta_l = \det\,[\theta_j(x_i)]_{1 \le i,\, j \le l}.$$

Then $\theta_1 \wedge \cdots \wedge \theta_l$ is independent of the choice of basis, up to multiplication by a constant. The following theorem is due to K. Saito [11].

(2.1) **Theorem.** *Suppose* $\theta_1, \cdots, \theta_l \in D(\mathscr{A})$. *Then*
(i) $\theta_1 \wedge \cdots \wedge \theta_l \in QS$.
(ii) *Suppose* $\theta_1, \cdots, \theta_l$ *are homogeneous. Then* $\{\theta_1, \cdots, \theta_l\}$ *is a basis for* $D(\mathscr{A})$ *if and only if* $\theta_1 \wedge \cdots \wedge \theta_l \neq 0$ *and* $\sum_{i=1}^{l} (\deg \theta_i + 1) = |\mathscr{A}|$.

In Saito's paper the first assertion is proved in (1.5.iii) for the ring of germs of holomorphic functions rather than the polynomial ring S but the arguments in our case are almost the same. The second assertion is proved in (1.11.i) of Saito's paper.

We will use the following theorem in field theory [2, Ch. I, 11.4].

(2.2) **Theorem.** *Let* $x_1, \cdots, x_l$ *be indeterminates. Let* $f_1, \cdots, f_l \in \boldsymbol{R}(x_1, \cdots, x_l)$. *The extension* $\boldsymbol{R}(x_1, \cdots, x_l)/\boldsymbol{R}(f_1, \cdots, f_l)$ *is algebraic if and only if the Jacobian* $\partial(f_1, \cdots, f_l)/\partial(x_1, \cdots, x_l)$ *is not zero.*

The inner product on V induces an inner product on V^* which we also denote by $(\ ,\ )$. Every $\boldsymbol{R}$-linear map $V^* \to S$ can be extended uniquely to a derivation of S. Thus we can make the following two definitions (2.3) and (2.4):

(2.3) **Definition.** Let $x \in V^*$. Define $\theta_x \in \mathrm{Der}(S)$ by $\theta_x(y) = (x, y)$ for any $y \in V^*$.

Note that the correspondence $x \to \theta_x$ gives an $\boldsymbol{R}$-linear map $V^* \to \mathrm{Der}(S)$.

(2.4) **Definition.** Let $f \in S$. Define $\theta_f \in \mathrm{Der}(S)$ by $\theta_f(x) = \theta_x(f)$ for any $x \in V^*$.

Then (2.4) is consistent with (2.3) when $f \in V^*$. The group $O(V)$ acts naturally on V^* and as a group of automorphisms of S.

(2.5) **Lemma.** *Let* $g \in O(V)$. *Then* $g(\theta_f(p)) = \theta_{g(f)}(g(p))$ *for all* $f, p \in S$.

Proof. We begin by proving the special case

$$g(\theta_x(p)) = \theta_{g(x)}(g(p)) \qquad x \in V^*, \quad p \in S. \tag{2.6}$$

Take $x \in V^*$ arbitrarily and fix it. Consider the subset

$$T=\{p \in S \mid g(\theta_x(p))=\theta_{g(x)}(g(p))\}$$

of S. We shall show $T=S$. Assume that $p, q \in T$. Since θ_x is a derivation of S and G acts as a group of automorphisms of S, direct computation shows that

$$g(\theta_x(p+q))=\theta_{g(x)}(g(p+q)),$$

and

$$g(\theta_x(pq))=\theta_{g(x)}(g(pq)),$$

for any $g \in G$. Thus $p+q \in T$ and $pq \in T$. Therefore it suffices to show $V^* \subseteq T$. Let $p \in V^*$. Then

$$g(\theta_x(p))=g(x, p)=(g(x), g(p))=\theta_{g(x)}(g(p)).$$

This shows $p \in T$ and proves (2.6). A similar argument proves that it suffices to show (2.5) when $p \in V^*$. Let $p \in V^*$. Then by (2.4) and (2.6) we have

$$g(\theta_f(p))=g(\theta_p(f))=\theta_{g(p)}(g(f))=\theta_{g(f)}(g(p)). \qquad \square$$

(2.7) **Proposition.** *Suppose there exist homogeneous polynomials $f_1, \cdots, f_l \in S$ such that*

(i) $\theta_{f_i} \in D(\mathscr{A})$ *for* $1 \leq i \leq l$,

(ii) $f_1, \cdots, f_l$ *are algebraically independent*,

(iii) $\sum_{i=1}^{l} (\deg f_i - 1)=|\mathscr{A}|$.

Then $\mathscr{A}$ is free and $\{\theta_{f_1}, \cdots, \theta_{f_l}\}$ is a basis for $D(\mathscr{A})$.

Proof. Choose an orthonormal basis $x_1, \cdots, x_l$ for V^*. Then $\theta_{x_i} = \partial/\partial x_i$ because $\theta_{x_i}(x_j)=(x_i, x_j)=\delta_{ij}=\partial x_j/\partial x_i$. Let $\theta_i=\theta_{f_i}$. Then

$$\begin{aligned}\theta_1 \wedge \cdots \wedge \theta_l &= \det[\theta_i(x_j)]_{1 \leq i, j \leq l} \\ &= \det[\theta_{x_j}(f_i)]_{1 \leq i, j \leq l} \\ &= \partial(f_1, \cdots, f_l)/\partial(x_1, \cdots, x_l).\end{aligned}$$

Since $f_1, \cdots, f_l$ are algebraically independent (2.2) shows that $\partial(f_1, \cdots, f_l)/\partial(x_1, \cdots, x_l) \neq 0$ and thus $\theta_1 \wedge \cdots \wedge \theta_l \neq 0$. Since $\deg \theta_i = \deg f_i - 2$ the assertion follows from (2.1.ii). $\square$

The following proposition gives a useful criterion for an element to lie in $D(\mathscr{A})$.

(2.8) **Proposition.** *Let $\theta \in$ Der(S). For θ to be in $D(\mathscr{A})$ it is necessary* *and sufficient that $\theta(\alpha_H) \in \alpha_H S$ for each $H \in \mathscr{A}$.*

Proof. Let $H \in \mathscr{A}$ and write $Q=\alpha_H Q_H$ where $Q_H \in S$. Since $\theta \in \operatorname{Der}(S)$ we have $\theta(Q)=\theta(\alpha_H)Q_H+\alpha_H\theta(Q_H)$. If $\theta(Q) \in QS$ it follows, because Q_H and α_H are coprime, that $\theta(\alpha_H) \in \alpha_H S$. Conversely suppose that $\theta(\alpha_H) \in \alpha_H S$ for all $H \in \mathscr{A}$. We may assume, arguing by induction on $|\mathscr{A}|$ that $\theta(Q_H) \in Q_H S$. If $\theta(\alpha_H) \in \alpha_H S$ then $\theta(Q)=\theta(\alpha_H)Q_H+\alpha_H\theta(Q_H) \in QS$. □

(2.9) **Corollary.** *If $\mathscr{B} \subseteq \mathscr{A}$, then $D(\mathscr{B}) \supseteq D(\mathscr{A})$.*

Fix $X \in L(\mathscr{A})$ and recall from (1.9) that we have defined a restricted arrangement $\mathscr{A}^X$ in X.

(2.10) **Definition.** For $X \in L(\mathscr{A})$ define an arrangement

$$\mathscr{A}_X=\{H \in \mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}.$$

(2.11) **Definition.** Let

$$I=I(X)=\sum_{H \in \mathscr{A}_X} \alpha_H S$$

be the ideal of S generated by the linear forms α_H with $H \in \mathscr{A}_X$.

If we view S as the set of polynomial functions on V then I is exactly the subset of S whose elements vanish identically on X. Define the factor ring $\bar{S}=S/I$. Let $\pi\colon S \to \bar{S}$ be the canonical projection. Usually we write $\bar{f}$ instead of $\pi(f) \in \bar{S}$ for $f \in S$. The dual space X^* of X can be canonically identified with the quotient space $V^*/I \cap V^*=\pi(V^*)$. Identify $\bar{S}$ with the symmetric algebra of X^*.

(2.12) **Lemma.** i) *For $\theta \in D(\mathscr{A}_x)$, there is a unique element $\bar{\theta} \in \operatorname{Der}(\bar{S})$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\theta} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \bar{S} & \xrightarrow{\bar{\theta}} & \bar{S} \end{array}$$

commutes. In other words, $\bar{\theta}(\bar{f})=\overline{\theta(f)}$ for all $f \in S$. ii) *If $\theta \in D(\mathscr{A})$, then $\bar{\theta} \in D(\mathscr{A}^X)$.*

Proof. i) By (2.8) we have $\theta(\alpha_H) \in \alpha_H S$ for all $H \in \mathscr{A}_X$ and thus $\theta(I) \subseteq \mathrm{I}$. This shows that there exists $\bar{\theta} \in \operatorname{Der}(\bar{S})$ such that the diagram commutes. The uniqueness is obvious.

ii) Note that $D(\mathscr{A})\subseteq D(\mathscr{A}_X)$ by (2.9). Thus $\bar{\theta}\in \mathrm{Der}(\bar{S})$ is defined. If $H\in\mathscr{A}$ and $H\cap X\in\mathscr{A}^X$ then $H\cap X=\ker(\bar{\alpha}_H)$. Note that

$$\bar{\theta}(\overline{\alpha_H})=\overline{\theta(\alpha_H)}\in\pi(\alpha_H S)=\overline{\alpha_H}\bar{S}.$$

This shows that $\bar{\theta}\in D(\mathscr{A}^X)$ by (2.8) applied to $\mathscr{A}^X$. □

The derivation $\bar{\theta}$ is called the restriction of $\theta\in D(\mathscr{A}_X)$ to X. Note that $\bar{\theta}=0$ or $\deg\bar{\theta}=\deg\theta$ if θ is homogeneous. To avoid an interruption in the proof of the next lemma we make an elementary remark about linear algebra. Since X is a subspace of V it inherits a positive definite inner product, which in turn induces a positive definite inner product on X^*. Let $X^\circ\subseteq V^*$ be the annihilator of X and let $(X^\circ)^\perp$ be its orthogonal complement in V^*. Then the restriction to $(X^\circ)^\perp$ of the projection $\pi\colon V^*\to X^*$ is an isometry

$$\sigma\colon (X^\circ)^\perp\simeq X^*. \tag{2.13}$$

(2.14) **Lemma.** *Let $f\in S$ and suppose $\theta_f\in D(\mathscr{A}_X)$. Then $\bar{\theta}_f=\theta_{\bar{f}}$.*

Proof. Note that (2.12.i) shows that $\bar{\theta}_f$ is defined. Since $\bar{\theta}_f$ and $\theta_{\bar{f}}$ are derivations of $\bar{S}$ it suffices, in view of (2.13), to show that

$$\bar{\theta}_f(\bar{x})=\theta_{\bar{f}}(\bar{x}) \qquad \text{for} \quad x\in(X^\circ)^\perp. \tag{2.15}$$

If $x\in(X^\circ)^\perp$ and $H\in\mathscr{A}_X$ then $\alpha_H\in X^\circ$ so $\theta_x(\alpha_H)=(x,\alpha_H)=0$. Thus $\theta_x\in D(\mathscr{A}_X)$ so $\bar{\theta}_x$ is defined. By (2.12.i) and (2.4) we have $\bar{\theta}_f(\bar{x})=\overline{\theta_f(x)}=\overline{\theta_x(f)}=\bar{\theta}_x(\bar{f})$ and by (2.4) we have $\theta_{\bar{f}}(\bar{x})=\theta_{\bar{x}}(\bar{f})$. Thus to prove (2.15) it suffices to show that $\bar{\theta}_x=\theta_{\bar{x}}$ for $x\in(X^\circ)^\perp$. Since $\bar{\theta}_x$ and $\theta_{\bar{x}}$ are derivations of $\bar{S}$ it suffices, in view of (2.13) to show that $\bar{\theta}_x(\bar{y})=\theta_{\bar{x}}(\bar{y})$ for all $x,y\in(X^\circ)^\perp$. Since the map σ in (2.13) is an isometry we have

$$\bar{\theta}_x(\bar{y})=\overline{\theta_x(y)}=\overline{(x,y)}=(x,y)=(\bar{x},\bar{y})=\theta_{\bar{x}}(\bar{y}).$$ □

(2.16) **Proposition.** *Let $\mathscr{A}$ be an arrangement and let $X\in L(\mathscr{A})$ with $\dim X=k$. Suppose there exist homogeneous polynomials $f_1,\cdots,f_k\in S$ such that*

(i) $\theta_{f_i}\in D(\mathscr{A})$ *for* $1\le i\le k$,

(ii) $\bar{f}_1,\cdots,\bar{f}_k$ *are algebraically independent,*

(iii) $\sum_{i=1}^k \deg f_i-1=|\mathscr{A}^X|$.

Then $\mathscr{A}^X$ is free and $\{\bar{\theta}_{f_1},\cdots,\bar{\theta}_{f_k}\}$ is a basis for $D(\mathscr{A}^X)$.

Proof. Since $\theta_{f_i}\in D(\mathscr{A})$ we have $\bar{\theta}_{f_i}\in D(\mathscr{A}^X)$ by (2.12.ii). Now (2.14) shows $\bar{\theta}_{f_i}=\theta_{\bar{f}_i}$ for $1\le i\le k$. The Proposition follows from (2.7) applied with $\bar{S}$ in place of S. □

§ 3. The restriction of a Coxeter arrangement to a hyperplane

Let $G \subset O(V)$ be a finite reflection group and let $\mathscr{A}=\mathscr{A}(G)$ be the corresponding Coxeter arrangement of reflecting hyperplanes. Let S^G denote the ring of G-invariant polynomials.

(3.1) **Proposition.** *Let $f \in S^G$. Then $\theta_f \in D(\mathscr{A})$.*

Proof. Recall that $S^G Q$ is the set of anti-invariants of G:

$$S^G Q=\{f \in S \mid g(f)=(\deg g)^{-1} f \text{ for all } g \in G\}$$

by [1, Ch. 5, §5.4, Prop. 5(i)]. On the other hand,

$$g(\theta_f(Q))=\theta_{g(f)}(g(Q))=\theta_f(g(Q))=(\det g)^{-1}\theta_f(Q)$$

for all $g \in G$ by (2.5). This shows $\theta_f(Q) \in S^G Q$ and thus $\theta_f \in D(\mathscr{A})$. □

Choose homogeneous basic invariants $f_1, \cdots, f_l \in S$ with $\deg f_1 \leq \cdots \leq \deg f_l$. Then $S^G=\boldsymbol{R}[f_1, \cdots, f_l]$ and $m_i=\deg f_i-1$ $(i=1, \cdots, l)$ are the exponents of G. The following theorem is due to K. Saito [10, 4.8].

(3.2) **Theorem.** *Let $\mathscr{A}$ be a Coxeter arrangement. Then $\{\theta_{f_1}, \cdots, \theta_{f_l}\}$ is a basis for $D(\mathscr{A})$. Thus $\mathscr{A}$ is free and* $\deg(\mathscr{A})=\{m_1, \cdots, m_l\}$.

Proof. By [1, Ch. 5, §5.4, Prop. 6.(ii)] we have $\sum_{i=1}^{l} m_i=|\mathscr{A}|$. Apply (2.7). □

(3.3) **Corollary.** *Let $\mathscr{A}$ be a Coxeter arrangement. Then*

$$\chi(\mathscr{A}, t)=\prod_{i=1}^{l}(t-m_i).$$

Proof. Apply (1.7) and (3.2). □

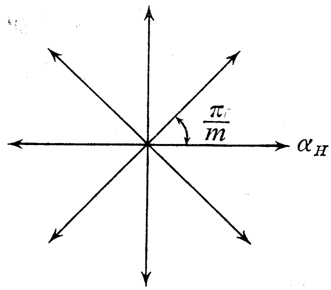
Choose a hyperplane $H \in \mathscr{A}$ and hold it fixed. We want to apply Proposition (2.16) to prove that the arrangement $\mathscr{A}^H$ is free in case (G, V) is an irreducible group. For convenience we agree to choose the linear forms α_K, $K \in \mathscr{A}$ so that $(\alpha_K, \alpha_K)=1$. If $X \in L(\mathscr{A})$ define the fixer

$$G_X=\{g \in G \mid g \text{ fixes all points of } X\}. \tag{3.4}$$

(3.6) **Proposition.** *Let $\mathscr{A}$ be a Coxeter arrangement. Then*

$$|\mathscr{A}|-|\mathscr{A}^H|+1=2\sum_{K \in \mathscr{A}}(\alpha_H, \alpha_K)^2.$$

Proof. If $\dim V=1$ then $\mathscr{A}^H$ is void and both sides of the formula equal 2. Next we prove the Proposition in case dim $V=2$ where $|\mathscr{A}^H|=1$ and the assertion is $|\mathscr{A}|=2\sum_{K\in\mathscr{A}}(\alpha_H,\alpha_K)^2$. In this case the group G is dihedral of order $2m$ where $m=|\mathscr{A}|$. Let $R=\{\pm\alpha_K|K\in\mathscr{A}\}$. We may picture R as



and we may assume that the α_K for $K\neq H$ lie in the upper half plane. Then

$$\begin{aligned}2\sum_{K\in\mathscr{A}}(\alpha_H,\alpha_K)^2&=2\sum_{k=0}^{m-1}\cos^2(k\pi/m)\\&=\sum_{k=0}^{m-1}(1+\cos(2k\pi/m))\\&=m=|\mathscr{A}|.\end{aligned}$$

This proves the assertion in case dim $V=2$. Now suppose that dim $V\geq 2$. Choose $X\in\mathscr{A}^H$. Then $\mathscr{B}=\{K\cap X^\perp|K\in\mathscr{A}_X\}$ is a Coxeter arrangement in the two-dimensional space $X^\perp$. The corresponding group is, by [1, Ch. 5, §3.3, Prop. 2], the restriction of G_X to $X^\perp$. Apply the two-dimensional case to the arrangement $(\mathscr{B},X^\perp)$. If we identify $(X^\perp)^*$ with X° as in (2.13) this gives

$$|\mathscr{A}_X|=2\sum_{K\in\mathscr{A}_X}(\alpha_H,\alpha_K)^2. \tag{3.6}$$

Write $\mathscr{A}^H=\{X_1,\cdots,X_n\}$ where $n=|\mathscr{A}^H|$. Let $\mathscr{A}_i=\mathscr{A}_{X_i}$. Then $\mathscr{A}=\mathscr{A}_1\cup\cdots\cup\mathscr{A}_n$ and $\mathscr{A}_i\cap\mathscr{A}_j=\{H\}$ for $i\neq j$. Thus

$$|\mathscr{A}|=\sum_{i=1}^n|\mathscr{A}_i|-(n-1).$$

It follows from (3.6) that

$$2\sum_{K\in\mathscr{A}}(\alpha_H,\alpha_K)^2=\sum_{i=1}^n\sum_{K\in\mathscr{A}_i}2(\alpha_H,\alpha_K)^2-2(n-1)$$

$$= \sum_{i=1}^{n} |\mathscr{A}_i| - 2(n-1)$$
$$= |\mathscr{A}| - (n-1)$$
$$= |\mathscr{A}| - |\mathscr{A}^H| + 1. \qquad \square$$

(3.7) **Theorem.** *Let $G \subset O(V)$ be a finite irreducible reflection group. Let h be the Coxeter number. Let $\mathscr{A} = \mathscr{A}(G)$ and let $H \in \mathscr{A}$. Then*

$$|\mathscr{A}| - |\mathscr{A}^H| + 1 = h.$$

In particular $|\mathscr{A}^H|$ does not depend on $H \in \mathscr{A}$.

Proof. This follows at once from (3.5) and the known formula [1, Ch. 5, §6.2, Th. 1, Cor.]

$$2 \sum_{K \in \mathscr{A}} (\alpha_H, \alpha_K)^2 = h.$$

Note that the summation in [1] is over a set of cardinality $2|\mathscr{A}|$ and that we have assumed $(\alpha_K, \alpha_K) = 1$ for all $K \in \mathscr{A}$. $\square$

Recall the canonical projection $\pi\colon S \to \bar{S}$ and the notation $\bar{f} = \pi(f) \in \bar{S}$ for $f \in S$.

(3.8) **Proposition.** *Let $G \subset O(V)$ be a finite irreducible reflection group. Let $\mathscr{A} = \mathscr{A}(G)$ and let $H \in \mathscr{A}$. Then $\bar{f}_1, \cdots, \bar{f}_{l-1}$ are algebraically independent.*

Proof. Let $x_1, \cdots, x_l$ be a basis for V^* such that $H = \ker x_l$. Let $E = \boldsymbol{R}(\bar{x}_1, \cdots, \bar{x}_l) = \boldsymbol{R}(\bar{x}_1, \cdots, \bar{x}_{l-1})$ and let $F = \boldsymbol{R}(\bar{f}_1, \cdots, \bar{f}_{l-1})$. K. Saito has shown [10, Lemma 3.1] that

$$Q^2 = cf_l^l + \delta_1(f_1, \cdots, f_{l-1}) f_l^{l-1} + \cdots + \delta_l(f_1, \cdots, f_{l-1})$$

where c is a nonzero constant and $\delta_i \in \boldsymbol{R}[f_1, \cdots, f_{l-1}]$ for $i = 1, \cdots, l$. Since $Q \in x_l S = \ker(\pi)$ we have

$$0 = \pi(Q^2) = c\bar{f}_l^l + \bar{\delta}_1 \bar{f}_l^{l-1} + \cdots + \bar{\delta}_l.$$

This shows that $\bar{f}_l$ is algebraic over F. On the other hand, E is algebraic over $F(\bar{f}_l)$ because $\boldsymbol{R}(x_1, \cdots, x_l)$ is algebraic over $\boldsymbol{R}(f_1, \cdots, f_l)$. Since $E/F(\bar{f}_l)$ and $F(\bar{f}_l)/F$ are both algebraic, so is E/F. Since $\bar{x}_1, \cdots, \bar{x}_{l-1}$ are algebraically independent it follows that $\bar{f}_1, \cdots, \bar{f}_{l-1}$ are algebraically independent. $\square$

(3.9) **Theorem.** *Let $G\subset O(V)$ be a finite irreducible reflection group. Let $\mathscr{A}=\mathscr{A}(G)$ be the Coxeter arrangement and let $H\in\mathscr{A}$. Then the derivations $\bar{\theta}_{f_1},\cdots,\bar{\theta}_{f_{l-1}}$ are a basis for $D(\mathscr{A}^H)$. Thus $\mathscr{A}^H$ is a free arrangement and* $\deg\mathscr{A}^H=\{m_1,\cdots,m_{l-1}\}$ *the first $l-1$ exponents of G.*

Proof. Apply (2.16). It is known [1, Ch. 5, §6.2] that $h=m_l+1$ and that $m_1+\cdots+m_l=|\mathscr{A}|$. It follows from (3.7) that $|\mathscr{A}^H|=m_1+\cdots+m_{l-1}$. This shows that hypothesis (iii) of (2.16) is satisfied. Hypotheses (i) and (ii) of (2.16) are satisfied in view of (3.1) and (3.8). □

(3.10) **Corollary.** $\chi(\mathscr{A}^H,t)=\prod_{i=1}^{l-1}(t-m_i)$.

Proof. Apply (1.7) and (3.9). □

(3.11) **Corollary.** $f_l\in\alpha_H S+f_1S+\cdots+f_{l-1}S$.

Proof. Put $f=f_l$ for simplicity. It suffices to show that

$$\bar{f}\in\bar{f}_1\bar{S}+\cdots+\bar{f}_{l-1}\bar{S}$$

in $\bar{S}=S/\alpha_H S$. Note that, $\bar{\theta}_f\in D(\mathscr{A}^H)$ by (2.12.ii) and (3.1). Since (3.9) asserts that $\bar{\theta}_{f_1},\cdots,\bar{\theta}_{f_{l-1}}$ are a basis for $D(\mathscr{A}^H)$, we have

$$\bar{\theta}_f\in\bar{\theta}_{f_1}\bar{S}+\cdots+\bar{\theta}_{f_{l-1}}\bar{S}.$$

Choose an orthonormal basis $x_1,\cdots,x_l$ for V^*. Recall from the proof of (2.7) that $\theta_{x_i}=\partial/\partial x_i$. Define $q=\sum_{i=1}^{l}x_i^2\in S$. Using (2.4) we get

$$\begin{aligned}\theta_f(q)&=\sum_{i=1}^{l}\theta_f(x_i^2)=2\sum_{i=1}^{l}x_i\theta_f(x_i)\\&=2\sum_{i=1}^{l}x_i\theta_{x_i}(f)=2\sum_{i=1}^{l}x_i(\partial f/\partial x_i)=2(\deg f)f.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}2(\deg f)\bar{f}&=\overline{\theta_f(q)}=\bar{\theta}_f(\bar{q})\in\bar{\theta}_{f_1}(\bar{q})\bar{S}+\cdots+\bar{\theta}_{f_{l-1}}(\bar{q})\bar{S}\\&=\bar{f}_1\bar{S}+\cdots+\bar{f}_{l-1}\bar{S}.\end{aligned}$$ □

The *Coxeter complex* is the simplicial complex Γ cut out on the unit sphere $S^{l-1}\subset V$ by the hyperplanes in the arrangement $\mathscr{A}=\mathscr{A}(G)$. The following corollary of (3.10) is due to Steinberg [13, Cor. 4.2].

(3.12) **Corollary.** *The number of $l-2$ simplexes in the Coxeter complex Γ which lie in a hyperplane $H\in\mathscr{A}$ is $|G|/h$.*

Proof. If $X\in L$ let $\Gamma_X=\{\sigma\in\Gamma\,|\,\sigma\subseteq X\}$. Then Γ_X is a subcomplex of Γ of dimension $k-1$ where $k=\dim X$. Let Θ_X be the set of simplexes of Γ_X of dimension $k-1$. It is shown in [7.19] that

$$(3.13)\qquad (-1)^{k-1}|\Theta_X| = \sum_{\substack{Y\in L\\ Y\geq X}} (-1)^{\dim Y}\mu(X, Y).$$

Since $\{Y \in L \mid Y \geq X\} = L(\mathscr{A}^X)$, the characteristic polynomial of the restricted arrangement $\mathscr{A}^X$ is, from the definition (1.6),

$$(3.14)\qquad \chi(\mathscr{A}^X, t) = \sum_{\substack{Y\in L\\ Y\geq X}} \mu(X, Y)t^{\dim Y}.$$

Thus from (3.13) and (3.14) we have

$$(3.15)\qquad (-1)^{k-1}|\Theta_X| = \chi(\mathscr{A}^X, -1).$$

Now take $X=H$, $k=l-1$ and apply (3.10) with $t=-1$. This gives

$$(3.16)\qquad |\Theta_H| = \prod_{i=1}^{l-1}(m_i+1).$$

The assertion follows since $|G| = \prod_{i=1}^{l}(m_i+1)$ and $h=m_l+1$ by [1, Ch. 5, §6.2]. $\square$

(3.17) **Remark.** If $\mathscr{A}$ is a Coxeter arrangement of type A_l or B_l then the hypotheses of (2.16) are satisfied for all $X \in L$ using basic invariants $f_1, \cdots, f_k$ where $k=\dim X$. In this case (2.16) leads us to analogues of (3.9)–(3.12) for all $X \in L$.

§ 4. A character formula

Let (G, V) be a finite reflection group. Let $\mathscr{A}=\mathscr{A}(G)$ be the Coxeter arrangement and let $L=L(\mathscr{A})$. If $g \in G$ let $\mathrm{Fix}(g)=\{v \in V \mid gv=v\}$ and let $k(g)=\dim \mathrm{Fix}(g)$. If ψ, φ are characters of G let

$$(4.1)\qquad (\psi, \varphi) = \frac{1}{|G|}\sum_{g\in G}\psi(g)\varphi(g^{-1})$$

be the usual inner product of characters. If $X \in L$ let ψ_X denote the restriction of ψ to the subgroup G_X. In particular, let 1_X denote the principal character of G_X. Without risk of confusion we also let $(\ ,\ )$ denote the inner product of characters of G_X.

(4.2) **Definition.** If ψ is a character of G let

$$S_\psi = \sum_{g\in G}\psi(g)t^{k(g)}.$$

The following Proposition allows us to compute the sum S_ψ in terms of the characteristic polynomials $\chi(\mathscr{A}^X, t)$ of restricted arrangements.

(4.3) **Proposition.** *Let ψ be a character of G. Then*

$$S_\psi = \sum_{X \in L} \chi(\mathscr{A}^X, t) |G_X| (\psi_X, 1_X).$$

Proof. If $X \in L$ let $F_X = \{g \in G \mid \mathrm{Fix}(g) = X\}$. Then

$$G_X = \biguplus_{\substack{Y \in L \\ Y \leq X}} F_Y \quad \text{(disjoint)}, \quad X \in L. \tag{4.4}$$

To see this note that if $g \in F_Y$ where $Y \leq X$ then $Y \supseteq X$ so g fixes X and thus $g \in G_X$. On the other hand if $g \in G_X$ then $Y = \mathrm{Fix}(g)$ is a subspace of V which includes X. It is known as [5, Lemma 4.4] that if F_Y is non-empty then $Y \in L(\mathscr{A})$. This proves (4.4). In particular, taking $X = \bigcap_{H \in \mathscr{A}} H$ we have $G = \biguplus_{X \in L} F_Y$. If $g \in F_Y$ then $k(g) = \dim Y$. Thus it follows from the definition (4.2) that

$$S_\psi = \sum_{Y \in L} \Big(\sum_{g \in F_Y} \psi(g)\Big) t^{\dim Y}. \tag{4.5}$$

For $X \in L$ let

$$\alpha(X) = \sum_{g \in F_X} \psi(g) \qquad \beta(X) = \sum_{g \in G_X} \psi(g).$$

It follows from (4.4) that

$$\beta(X) = \sum_{\substack{Y \in L \\ Y \leq X}} \alpha(Y)$$

and hence by the Möbius inversion formula [9, Prop. 3.2] that

$$\alpha(Y) = \sum_{\substack{X \in L \\ X \leq Y}} \mu(X, Y) \beta(X).$$

Thus (4.5) gives

$$\begin{aligned} S_\psi &= \sum_{Y \in L} \alpha(Y) t^{\dim Y} \\ &= \sum_{Y \in L} \Big(\sum_{\substack{X \in L \\ X \leq Y}} \mu(X, Y) \beta(X)\Big) t^{\dim Y} \\ &= \sum_{X \in L} \Big(\sum_{\substack{Y \in L \\ Y \geq X}} \mu(X, Y) t^{\dim Y}\Big) \beta(X). \end{aligned}$$

By (3.14) the inner sum is $\chi(\mathscr{A}^X, t)$ and by (4.1) we have

$$\beta(X)=|G_X|(\psi_X, 1_X).$$ □

To apply (4.3) we must find characters ψ such that $(\psi_X, 1_X)=0$ for most $X \in L$ so that the sum on the right involves few characteristic polynomials $\chi(\mathscr{A}^X, t)$. As a first example we show that (3.3) and (4.3) yield the Shephard-Todd formula [12, Prop. 5.3] for orthogonal reflection groups.

(4.6) **Theorem** (Shephard and Todd). *Let $G \subset O(V)$ be a finite reflection group. Then*

$$\sum_{g \in G} t^{k(g)}=(t+m_1)\cdots(t+m_l).$$

Proof. If $g \in G$ let $\delta(g)$ be the determinant of g. Then δ: $G \to \boldsymbol{R}$ is a character of G. Suppose $X \in L$ and $X \neq V$. Then $X \subseteq H$ for some $H \in \mathscr{A}$ so G_X contains a reflection s. Since $\delta(s)=-1$ it follows that $\delta_X \neq 1_X$ and thus $(\delta_X, 1_X)=0$. Now (3.3) and (4.3) yield

$$\sum_{g \in G} \delta(g) t^{k(g)}=\chi(\mathscr{A}, t)=\prod_{i=1}^{l}(t-m_i). \tag{4.7}$$

Since $g \in O(V)$ we have

$$\delta(g)=(-1)^{l-k(g)}. \tag{4.8}$$

The theorem follows if we replace t by $-t$ in (4.7) and use (4.8). □

(4.9) **Lemma.** *Let φ be the character of the natural representation of G in V and let δ be the determinant character. Let $\psi=\delta\varphi$. Then*

$$(\psi_X, 1_X)=\begin{cases} l & \text{if } X=V \\ 1 & \text{if } X \in \mathscr{A} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $X \in L$. Let

$$Y=\{v \in V \mid gv=\delta(g)v \text{ for all } g \in G_X\}.$$

Since (φ_X, δ_X) is the multiplicity of δ_X as an irreducible constituent of the natural representation of G_X on V we have

$$\dim Y=(\varphi_X, \delta_X)=(\psi_X, 1_X)$$

where the second equality follows from (4.1). Suppose that $X \neq V$ and $(\psi_X, 1_X) \neq 0$. Then $\dim Y \neq 0$. We must prove that $X \in \mathscr{A}$ and $\dim Y=1$. Since $X \neq V$ it follows that $\mathscr{A}_X$ is non-empty. Choose $H \in \mathscr{A}_X$. Let $s \in G$ be the relection fixing H. Then $s \in G_X$ and $sv-v \in H^\perp$ for all $v \in V$. If

$v \in Y$ then $sv=\delta(s)v=-v$ and thus $-2v=sv-v \in H^{\perp}$. This proves that $Y \subseteq H^{\perp}$. But dim $H^{\perp}=1$ so $Y=H^{\perp}$ and dim $Y=1$. Furthermore $H=Y^{\perp}$ is uniquely determined by X. This proves that $|\mathscr{A}_X|=1$. Since X is an intersection of elements of $\mathscr{A}$ we must have $X \in \mathscr{A}$. □

(4.10) **Theorem.** *Let $G \subset O(V)$ be a finite irreducible reflection group. Let φ be the character of the natural representation of G on V. Then*

$$\sum_{g \in G} \varphi(g) t^{k(g)} = l(t-1) \prod_{i=1}^{l-1} (t+m_i).$$

Proof. Let $\psi=\delta\varphi$. By (4.9) we have $(\psi_X, 1_X)=0$ unless $X=V$ or $X \in \mathscr{A}$. Thus (4.3) gives

$$S_{\psi}=\chi(\mathscr{A}, t)|G_V|(\psi_V, 1_X)+\sum_{H \in \mathscr{A}} \chi(\mathscr{A}^H, t)|G_H|(\psi_H, 1_H).$$

We have $|G_V|=1$, $(\psi_V, 1_V)=l$, $|G_H|=2$ and $(\psi_H, 1_H)=1$ where the last equality follows from (4.9). We know from (3.3) that $\chi(\mathscr{A}, t)=\prod_{i=1}^{l}(t-m_i)$ and from (3.10) that $\chi(\mathscr{A}^H, t)=\sum_{i=1}^{l-1}(t-m_i)$. Thus

$$S_{\psi}=(l(t-m_l)+2|\mathscr{A}|) \prod_{i=1}^{l-1} (t-m_i).$$

But $2|\mathscr{A}|=lh=l(m_l+1)$ by [1, Ch. V, §6.2]. Thus

$$\sum_{g \in G} \delta(g)\varphi(g) t^{k(g)} = l(t+1) \prod_{i=1}^{l-1} (t-m_i).$$

The theorem follows if we replace t by $-t$ and use (4.8). □

(4.11) **Remark.** If G is a Coxeter group of type A_l or B_l the polynomials S_{ψ} have integer roots for all irreducible characters ψ. This was proved by D. E. Littlewood and A. R. Richardson [3, p. 56] in the case of A_l and by V. F. Molchanov [4] and W. Ostertag [8] in the case of B_l. In both cases there is a pleasant formula for the roots of S_{ψ} in terms of Young diagrams.

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