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An Upper Semicontinuity Theorem for some Leading Poles of $|f|^{2s}$

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Introduction

In this paper an application is made of certain numerical invariants introduced by Libgober [9], called "quasi-adjoint characters". To each germ of an analytic function f at a singular point p and to any other germ of an analytic function ϕ at p one may define the quasi-adjoint character $\kappa_\phi(p)$ by studying the family of cyclic covers over f and the adjointness properties to these cyclic covers of canonical differentials with ϕ as a coefficient (for precise definitions see (2. 4)). Each $\kappa_\phi(p)$ value is in $[0, 1)$.

The main result of this paper is the

Theorem (3.1). *Let $\{f_t\}$ be any 1-parameter family of germs of analytic functions at the common singular point $\bar{0} \in C^n$. Let ϕ be a germ of an analytic function at $\bar{0}$. Let $\kappa_\phi(t)$ be the quasi-adjoint character associated to f_t and ϕ at $\bar{0}$. Then, if $\kappa_\phi(0) \in (0, 1)$, one has*

$$\kappa_\phi(t) \leq \kappa_\phi(0)$$

for all t sufficiently close to 0.

This is of particular interest because of the following. For each t, let $U_t \Subset U'_t$ be two Milnor balls for a representative of f_t (denoted by f_t). Let ρ be a C^∞ function which is 1 on U_t and 0 off U'_t. Define the generalized functions on $C^\infty(U'_t, C)$ $I_t(s, \psi) = \int_{U'_t} |f_t|^{2s} |\psi|^2 \rho dx d\bar{x}$. This is often denoted by $|f_t|^{2s}$ for short. Let $\beta_\phi(t)$ be the largest pole of $I_t(s, \phi)$. Then there is a simple relation between $\kappa_\phi(t)$ and $\beta_\phi(t)$ given by $\kappa_\phi(t) + 1 = \beta_\phi(t)$ if $\kappa_\phi(t) \in (0, 1)$. Thus, (3.1) implies as a corollary

Corollary (3.8). *If $\kappa_\phi(0) \in (0, 1)$ then $\beta_\phi(t) \leq \beta_\phi(0)$ for t near* 0.

To understand this condition it is helpful to remark that if ϕ is a local unit at $\bar{0}$, then $\kappa_\phi = 0$ iff $\bar{0}$ is a rational singular point of f. More generally, $\kappa_\phi = 0$ iff ϕ is adjoint to f at $\bar{0}$.

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(3.1) and (3.8) when combined with results of Loeser and Varcenko provide an extension of Steenbrink's result on the lower semi-continuity of the spectrum (and of the Arnol'd Index) of a hypersurface singularity which is not a rational singularity. If $\omega=\phi dx$ is a holomorphic n-differential in an open neighborhood U, $U\supset\bigcup_t U'_t$, there is associated to ω an initial exponent $\alpha_t(\omega)$ of significance in the mixed Hodge structure on the vanishing cohomology of the hypersurface f_t. Theorem (4.6) shows that for $\{f_t\}$ as above, if $\alpha_0(\omega)\in(-1,0)$ then $\alpha_t(\omega)\geq\alpha_0(\omega)$ for t near 0. In particular, when the Arnol'd index $\sigma(f_0)$ of f_0 is in (0, 1), this gives the lower semicontinuity of the $\sigma(f_t)$ at 0.

This in turn leads to an alternative basis for Loeser's proof of a conjecture of Teissier. This conjecture states a general property for the largest pole of the generalized function $|f|^{2s}$ by connecting it to the polar invariants of f and its restrictions to generic linear planes of codimension $i=1, 2, \cdots, n-1$. (3.8) can be used to extend this conjecture to obtain estimates for other $\alpha_t(\omega)$ with values in $(-1, 0)$ (cf. (4.16)). Nonetheless, the conjecture is only an approximation, formulated by a pair of inequalities.

On the other hand, recent efforts have provided precise results on the poles of the distribution $|f|_K^s$ where $|\ |_K$ is the norm in any local field K of characteristic 0 and f is a function of two variables defined over K. These results are summarized in Section (1) and serve as a standard to which the results in Sections (4), (5) should be compared. Improvements in the conclusions from Section (4) can hopefully lead to results of analogous precision.

The last section extends Igusa's theory somewhat. It uses the results in Sections (3), (4) to obtain upper bound estimates (lower bounds can not be shown in general, as yet) for the largest poles of the extended zeta function over a local field K, when these poles lie in $(-1, 0)$. The estimates have the same form as those given in Section (4) and are shown by using any complex embedding of K.

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Section 1.

Let K be either (A) a finite extension of Q_p for some prime p, (B) $\boldsymbol{R}$ or (C) $\boldsymbol{C}$. These different possibilities are denoted by cases (A)–(C) in the following.

Let f: $(K^2, \bar{0})\to(K, 0)$ be a germ of a K-analytic function which is irreducible in the ring $\bar{K}\{x_1, x_2\}$. To f is associated a finite sequence of positive integers $(n, \beta_1, \cdots, \beta_g)$ (the characteristic sequence) with $n=$multiplicity of f at $\bar{0}$. The ratios β_i/n are the Puiseux ratios of the branch determined by f in $\bar{K}^2$. We assume $n\geq 2$.

In all three cases, a distribution $|f|_K^s$ determined by f can be defined on an appropriate space of "test functions" and has been studied in [7]. [10, 11], [12, 17]. We summarize here the conclusions of these investigations all of which are based on the explicit canonical resolution of the singularity $\bar{0}$ [10].

The space of test functions. Let $U\subset K^2$ be an open neighborhood of $\bar{0}$ containing no other singularity of $\{f=0\}$ in $\bar{U}$ and on which f is K-analytic.

Case (A). $\mathscr{S}_U(K^2)$ is the Schwartz-Bruhat space of complex valued functions which are locally constant.

Cases (B, C). Define $\mathscr{S}_U(K^2)=\{\phi\colon U\to K\colon \text{supp}(\phi)$ is compact in U and ϕ is a C^∞ function$\}$.

Definition (1.1). (A) For $x\in K$, define $|x|_K=q^{-\text{ord}_K(x)}$ where $q=p^c$, $c=[K\colon Q_p]$ and $\text{ord}(x)=\min\{\ell\colon x\in\mathscr{P}^\ell\}$. Here, $\mathscr{P}$ is the unique maximal ideal of the valuation ring $\mathscr{R}\subset K$ consisting of the elements with K-norm at most 1. If $x\neq 0$, $x\notin\mathscr{R}$, $\text{ord}_K(x)=-\text{ord}_K(1/x)$. $\text{ord}_k(0)=+\infty$.

(B) $|x|_{\boldsymbol{R}}$ is the standard absolute value in $\boldsymbol{R}$.

(C) $|x|_C=x\cdot\bar{x}=\text{mod}(x)^2$.

Remark (1.2). K^2 is a locally compact additive group with norm the supremum norm $|(x,y)|_K=\sup\{|x|_K, |y|_K\}$ and a unique Haar measure $d\mu$ for which

$$\int_{\{(x,y)\colon|(x,y)|_K\leq 1\}} d\mu=1.$$

Remark (1.3). For $\text{Re}(s)>0$ and $\phi\in\mathscr{S}_U(K^2)$ $I_\phi(s)=\int_{K^2}|f|_K^s\phi\, d\mu$ is analytic.

The results of the papers referred to above have investigated the analytic continuation to $\boldsymbol{C}$ of $I_\phi(s)$, subject to the property that $\phi(\bar{0})\neq 0$ (cf. Remark (1.7) below however).

The poles and residues of $I_\phi(s)$, $\phi(\bar{0})\neq 0$, have been determined as follows.

Let $e^{(0)}=n$, $e^{(i)}=\text{g.c.d.}(e^{(i-1)},\beta_i)$.

Set

$$r_i=\frac{\beta_i+n}{e^{(i)}}$$

$$R_i=\frac{\beta_i e^{(i-1)}+\beta_{i-1}(e^{(i-2)}-e^{(i-1)})+\cdots+\beta_1(e^{(0)}-e^{(1)})}{e^{(i)}}.$$

Set $\beta_1(i)=\max\{\beta_{i+1}-\beta_i, e^{(i)}\}$ for $i=1, 2, \cdots, g-1$. Define $\varepsilon_1, \varepsilon_2, \varepsilon_3$, so that

$$\varepsilon_1+1=\frac{r_{i-1}e^{(i-1)}-R_{i-1}}{R_i},$$

$$\varepsilon_2+1=\frac{R_{i-1}+\beta_1^{(i-1)}}{R_i},$$

and

$$\varepsilon_3+1=\frac{R_i-r_ie^{(i)}}{R_i} \qquad \text{for } i=2, 3, \cdots, g.$$

Theorem (1.4). *The numbers* $-r_1/R_1, -r_2/R_2, \cdots, -r_g/R_g$ *are non-integral poles of order* 1 *of* $I_\phi(s)$ *in cases* A, C *and, if* $i\geq 2$ *and* g.c.d. $(r_i, R_i)=1$, $-r_i/R_i$ *is a pole in case* B.

Theorem (1.5). *Up to a non-zero positive constant factor which depends only on* f,

$$\operatorname*{Res}_{s=\frac{-r_i}{R_i}} I_\phi(s)=\phi(\bar{0})\int_{D_i}|x|_K^{\varepsilon_1}|1-x|_K^{\varepsilon_2}\,dx\neq 0$$

if g.c.d. $(r_i, R_i)=1$ *and* $\phi\in S_U(K^2)$. *Here* D_i *is the divisor in the canonical resolution* $\pi\colon X_{\mathrm{res}}\to U$ *such that* $R_i=\operatorname{mult}_{D_i}(f\cdot\pi)$, $r_i-1=\operatorname{mult}_{D_i}(\det d\pi)$. D_i *is a* $\boldsymbol{P}^1(K)$ *in all cases.*

Remark (1.6). As in [10, 11], [17], one can either explicitly evaluate the integral in all three cases to show it is non-zero or, as shown in [7], one can derive that such an integral (when correctly interpreted) is a product of three non-zero gamma functional values (the gamma function in case (A) is that defined by Sally and Taibleson [21]). As such, in all cases, the value of the integral can even be shown to be positive if $i=1$ and negative if $i>1$. Note that this is a conclusion that cannot be obtained from [10] in case (B) but can be from the representation of the value for the residue in [11] in case (C).

Remark (1.7). For an extension of these results to those involving $I_\phi(s)$ when $\phi(\bar{0})=0$ is allowed, see [11].

Open Question. It would be interesting to extend these results to a convenient class of analytically reducible functions. Are there classical analysis type identities which can be used to show that certain ratios arising via resolution data (that is, ratios of form $(\lambda)_D=-(1+\operatorname{mult}_D\det d\pi))/\operatorname{mult}_D(f\circ\pi)$, D an irreducible reduced component of the exceptional locus) are genuine poles of the $I_\phi(s)$? In the irreducible case, and over $\boldsymbol{R}$, the residue at $s=-r_i/R_i$ is a sum of three beta functional values associated to the three intersections of the divisors D_i with other divisors. That the residue is always non-zero implies that these points of intersection do not affect the property that $-r_i/R_i$ is a pole of $I_\phi(s)$. In what way can one detect a non-trivial modification of this in the reducible case. That is, when *would* the positions of intersections of a divisor D with other divisors determine whether the ratio λ_D is or is not a pole of the $I_\phi(s)$ when $\phi(\bar{0})\neq 0$ or $\phi(\bar{0})=0$. The phenomenon of jumping of roots of the local b-function, discovered by Yano [32] is presumably related to this.

Section 2.

Quasi-adjoint characters were introduced by Libgober [9] in his study of the Alexander module for plane curves. Nonetheless, they can be used in other settings. To state their definition we recall for the reader's convenience the definition of the adjoint ideal as given in [19].

Definition (2.1). Let X be a reduced complex analytic hypersurface of dimension n. Let $p\in X$ be an isolated singular point of X (isolatedness is not essential to the definition but in the following it is the only type of singularity that will be considered). Let f be a local defining function of X at p. This defines a local embedding of X into $\boldsymbol{C}^{n+1}$. In complex coordinates $(x_1,\cdots,x_{n+1})$ centered at p, the n-differentials

$$\frac{dx_2\cdots dx_{n+1}}{f_{x_1}},\quad \frac{(-1)dx_1\widehat{dx_2}dx_3\cdots dx_{n+1}}{f_{x_2}},\ \cdots,\ \frac{(-1)^{n-1}dx_1\cdots dx_n}{f_{x_{n+1}}}$$

patch on the nonsingular part X_{sp} of X to give a meromorphic differential on X, denoted by σ_X. As discussed in Section 3, σ_X generates an $\mathcal{O}_X$ module near p, called the dualizing sheaf and denoted ω_X. Let g be a local section of $\mathcal{O}_X$ defined in an open neighborhood W of p in X. The germ of g at p is adjoint to X at p if the n-differential $\omega_1=(g dx_2\cdots dx_{n+1})/f_{x_1}|_W$ satisfies the L^2 condition

$$\int_{U-U\cap\{f_{x_1}=0\}}\omega_1\wedge\bar{\omega}_1<\infty \tag{2.2}$$

for any relatively compact open subset U of W with $p\in U$.

Remark. It follows from Lemma (1.3) of [19] that g is adjoint to X at p if one replaces ω_1 by any of the $\omega_i=(-1)^{i-1}((g dx_1\cdots\widehat{dx_i}\cdots dx_n)/f_{x_i})$. If (2.2) holds for one ω_{i_0} it will hold for all the ω_i.

Now, let $f: (C^n, \bar{0}) \to (C, 0)$ be the germ of a complex analytic function with isolated singular point at $\bar{0}$. Let $\bar{0} \in U \subset C^n$ be an open neighborhood of $\bar{0}$ over which $f: U - \{f=0\} \to C - \{0\}$ is a Milnor fibration. Let $m \in N$. We define the cyclic cover Σ_m of $\{f=0\}$ of degree m by the local equation

$$F_m(x_0, x_1, \cdots, x_n) = x_0^m - f(x_1, \cdots, x_n)$$

in an open neighborhood $X_0 \times U \subset C^{n+1}$ of $(0, \bar{0})$. Here, X_0 is an open neighborhood of 0 in C and is independent of m. Thus, $\Sigma_m \subset X_0 \times U$ for all m.

Let $\phi \in \Gamma(U, \mathcal{O}_U)$. The local section ϕ determines a local section, also denoted by ϕ, of $\mathcal{O}_{\Sigma_m}$ near $(0, \bar{0})$.

Following Libgober we define the function $\psi_\phi(m) = \min\{k: x_0^k \cdot \phi$ is adjoint to Σ_m at $(0, \bar{0})\}$. The basic theorem about $\psi_\phi(m)$ is

Theorem (2.3). *Either $\psi_\phi(m) \equiv 0$ (i.e. ϕ itself is adjoint to Σ_m at $(0, \bar{0})$) or there is a unique rational number κ_ϕ in $(0, 1)$ such that $\psi_\phi(m) = [\![m \cdot \kappa_\phi]\!]$.*

Definition (2.4). κ_ϕ is the quasi-adjoint character associated to ϕ. If the singular point of Σ_m also needs to be specified, we do so by writing $\kappa_\phi(p)$ (here $\kappa_\phi(0, \bar{0})$). For different singular points p the values of $\kappa_\phi(p)$ will be different in general. Similarly, when p is specified, we will use the notation $\psi_\phi(p, m)$.

The interest in the κ_ϕ comes from the

Theorem (2.5). *Let $U' \Subset U$ be two Milnor balls for a representative of f. Let ρ be a C^∞ function which is 1 on U' and 0 off U. Let β_ϕ be the largest pole of the analytic continuation of the generalized function $|f|^{2s}$ evaluated at ϕ, that is,*

$$I(s, \phi) = \int_U |f|^{2s} |\phi|^2 \rho dx_1 d\bar{x}_1 \cdots dx_n d\bar{x}_n.$$

Then, $\kappa_\phi - 1 = \beta_\phi$ if $\kappa_\phi \in (0, 1)$ or, equivalently, if $\beta_\phi > -1$.

Proof. For f, as above, introduce the set of triples $\mathcal{A}_\phi = \{(\mathrm{ord}_D(f \circ \pi), \mathrm{ord}_D(\det d\pi), \mathrm{ord}_D(\phi \circ \pi)): D$ is a component of the exceptional locus $\pi^{-1}(\bar{0})$ where $\pi: X_{\mathrm{res}} \to U$ is a local resolution of $f \cdot \phi$, defined in a neighborhood $U_{\bar{0}}$ of $\bar{0}\}$.

This is the evident generalization of the array of multiplicities [10] whose study is sufficient to detect the largest root of $b_f(s)$, corresponding to ϕ with the property $\phi(\bar{0}) \neq 0$. However, when ϕ also vanishes at $\bar{0}$, it is necessary to include the data $\mathrm{ord}_D(\phi \circ \pi)$ in $\mathcal{A}_\phi$ as well.

Given $\pi: X_{\text{res}} \to U$ a local resolution of $f\cdot\phi$ in U, set

(2.6) $$\beta_\phi = \max_D \left\{ \frac{-(1+\operatorname{ord}_D(\det d\pi)+\operatorname{ord}_D(\phi\circ\pi))}{\operatorname{ord}_D(f\circ\pi)} : D \text{ is a component of } \pi^{-1}(\bar{0}) \right\}.$$

From standard arguments in [11] one can immediately see that β_ϕ is a root of $b_f(s)$, as well as a pole of $|f|^{2s}$, if $\beta_\phi \in (-1, 0)$.

To prove (2.5), at least for all $m \gg 0$ (which evidently suffices), one can use any of three methods. The argument based on the L^2 criterion of [19] can be found in a more general form in (2.13) below. Another short argument can be based on Hodge Theory (applied to the Thom-Sebastiani function defining the cyclic covers Σ_m) and the characterization of adjointness in terms of the values of the exponents in the spectrum of the singularity [14]. However, here we want to give a "classical" type argument, inspired from Zariski's original argument calculating, in effect, the quasi-adjoint character for the cusp [34].

Let σ_{Σ_m} be the generator for the dualizing sheaf of Σ_m (in modern parlance). The criterion for adjointness is [19] that for any compact analytic n-chain γ on Σ_m

(2.7) $$\int_\gamma x_0^k \phi \sigma_{\Sigma_m} < \infty.$$

Let $\bar{0}$ be the fixed singular point of f. Let $U = U_{\bar{0}}$ be a neighborhood of $\bar{0}$ containing no other singular points of f in its closure. One then may think of Σ_m as branched over the hypersurface $\mathscr{X}_0$ defined by f in U.

$$\begin{array}{c} \Sigma_m \\ \eta \big\downarrow \\ \mathscr{X}_0 \subset U \subset C^n. \end{array}$$

Let $\pi: X_{\text{res}} \to U$ be an embedded resolution of $\{f\phi = 0\}$ in U. Let $\Sigma'_m = X_{\text{res}} \times_U \Sigma_m$ and $\tilde{\Sigma}_m \to \Sigma'_m$ the normalization of the analytic space Σ'_m. One has the commutative diagram

$$\begin{array}{ccc} \tilde{\Sigma}_m & \xrightarrow{p_2} & \Sigma_m \\ p_1 \big\downarrow & & \big\downarrow \eta \\ X_{\text{res}} & \xrightarrow{\pi} & U \end{array} .$$

Observe first that if $\operatorname{supp}(\gamma)$ does not contain $(0, \bar{0})$, then (2.7) is clearly satisfied. Next, because $\tilde{\Sigma}_m$ is normal and p_2 is proper and onto, it suffices to check (2.7) in the codimension≤ 1 part of $\tilde{\Sigma}_m$. That is, if $\mathscr{X}=p_1^{-1}(\{f\circ\pi=0\})$, a subvariety of $\tilde{\Sigma}_m$ (non-reduced) it suffices to check (2.7) on $\tilde{\Sigma}_m$ and for compact analytic n-chains $\tilde{\gamma}$ intersecting $\mathscr{X}'=\mathscr{X}_{\text{red}}$ in at most a codimension 1 subvariety of $\mathscr{X}'$. For if $\tilde{\gamma}$ lies in $\mathscr{X}'$ then p_2 blows $\tilde{\gamma}$ down to the point $(0, \bar{0})$. Since $\tilde{\Sigma}_m$ is normal, if $\tilde{\gamma}$ only intersects X' in a codimension ≥ 2 subvariety, then the pullback $p_2^*(x_0^k\phi\sigma_{\Sigma_m})$ can be extended holomorphically across $\tilde{\gamma}$ so that (2.7) is satisfied.

Thus, it suffices to consider the following situation. Let $\mathscr{U}$ be a chart in $\mathscr{X}_{\text{res}}$ with coordinates $(u_1, \cdots, u_n)$ such that

$$\begin{aligned} f\circ\pi(u_1, \cdots, u_n)&=u_1^{N_1}\cdots u_n^{N_n}\\ \phi\circ\pi(u_1, \cdots, u_n)&=u_1^{m_1}\cdots u_n^{m_n}(\text{unit})\\ \det d\pi(u_1, \cdots, u_n)&=u_1^{b_1}\cdots u_n^{b_n}(\text{unit}). \end{aligned}$$

Thus,

$$p_2^*(x_0^{k-m+1}\phi\sigma_{\Sigma_m})|_{p_1^{-1}(\mathscr{U})\cap(\tilde{\Sigma}_m)_{\text{sp}}}=x_0^{k-m+1}\cdot u_1^{m_1+b_1}\cdots u_n^{m_n+b_n}(\text{unit})du_1\cdots du_n.$$

Set $\tilde{\mathscr{U}}=p_1^{-1}(\mathscr{U})\cap(\tilde{\Sigma}_m)_{\text{sp}}$. From the above remarks, it suffices to assume that $\text{supp}(\tilde{\gamma})$ lies in $\bar{\tilde{\mathscr{U}}}$. In $\tilde{\mathscr{U}}$, $x_0^m=f\circ\pi(u)=u_1^{N_1}\cdots u_n^{N_n}$. Let θ: $\tilde{\mathscr{U}}(m)\to p_1^{-1}(\mathscr{U})$ be an m^n-fold cover of $p_1^{-1}(\mathscr{U})$. Let $(v_1, \cdots, v_n)$ be coordinates on $\tilde{\mathscr{U}}(m)$ so that $u_i=v_i^m$. Lifting $p_2^*(x_0^{k-m+1}\phi\sigma_{\Sigma_m})$ to $\tilde{\mathscr{U}}(m)$ one obtains

$$p_2^*(x_0^{k-m+1}\phi\sigma_{\Sigma_m})|_{\tilde{\mathscr{U}}(m)_{\text{sp}}}=m^n\left[\prod_{i=1}^n (v_i)^{kN_i+m(m_i+b_i)+(m-1)(1-N_i)}(\text{unit})\right]dv_1\cdots dv_n.$$

Since θ is proper and onto $p_1^{-1}(\mathscr{U})$, (2.7) can be verified for any chain $\tilde{\gamma}$ with support in $p_1^{-1}(\mathscr{U})$ by verifying it in $\tilde{\mathscr{U}}(m)$. Using the argument in the proof of Lemma 1.3 (i)$\Rightarrow$(iii) [19], it follows that (2.7) will be satisfied if and only if

$$kN_i+m(m_i+b_i)+(m-1)(1-N_i)>-1 \qquad \text{for } i=1, \cdots, n.$$

This holds if and only if

$$k\geq\max_i\left\{m\left(\frac{N_i-(1+m_i+b_i)}{N_i}-\frac{1}{m}\right)\right\}.$$

Thus, in $p_1^{-1}(\mathscr{U})$, the smallest possible k for which

$$k\geq\max_i\left\{m\left(\frac{N_i-(1+m_i+b_i)}{N_i}-\frac{1}{m}\right)\right\}>k-1$$

for all $m\gg 0$ is given by the expression

$$\left[\!\left[m\left(\frac{N_a-(1+m_a+b_a)}{N}\right)\right]\!\right]$$

where $i=a$ is that value at which $-(1+m_i+b_i)/N_i$ is largest. Since $\tilde{\Sigma}_m$ is covered by open sets of the form $p_1^{-1}(\mathscr{U})$, $\mathscr{U}$ a coordinate chart of X_{res} as above, we see that the smallest possible k for which $x_0^k\phi$ is adjoint to Σ_m at $(0, \bar{0})$ is given by

$$[\![m\cdot\kappa_\phi]\!] \qquad \text{for } m\gg 0$$

where $\kappa_\phi=\beta_\phi+1$, β_ϕ defined in (2.6), if $\beta_\phi>-1$ (or equivalently if $\kappa_\phi \in (0, 1)$).

Remark (2.8). It would be very interesting to know how to extend (2.5) to poles β_ϕ less than -1.

Remark (2.9). Let f define a germ of a real analytic function at $\bar{0}$ in $\boldsymbol{R}^n$. Assume $\bar{0}$ is a singular point for f. Let ϕ be a real analytic function defined in a real neighborhood U of $\bar{0}$. Let ρ be a C^∞ function which is identically 1 in a smaller open neighborhood of $\bar{0}$ and is 0 off U. Define

$$I_\phi(s)=\int_U |f|^s\phi\rho dx_1\cdots dx_n.$$

One can define β_ϕ and κ_ϕ as in (2.4), (2.6). However now, one has the relation $\beta_\phi+1\leq\kappa_\phi$ if $\beta_\phi>-1$.

Let A_{Σ_m} be the adjoint ideal sheaf of Σ_m (cf. Section 3 for definition).

Remark (2.10). *Let* $C_{(0,\bar{0})}(m)=(\mathscr{O}_{\Sigma_m}/A_{\Sigma_m})_{(0,\bar{0})}$ be the conductor of Σ_m at $(0, \bar{0})$. $C_{(0,\bar{0})}(m)$ admits a $\boldsymbol{C}\{x_0\}$ module structure by x_0 multiplication.

Let $\bar{\phi}$ denote the class of ϕ in $C_{(0,\bar{0})}(m)$ and N_ϕ the $\boldsymbol{C}\{x_0\}$ submodule $N_\phi=\boldsymbol{C}\{x_0\}\cdot\bar{\phi}$ generated by $\bar{\phi}$. Then it is clear that $\psi_\phi(m)=\text{length}_{\boldsymbol{C}\{x_0\}}N_\phi$.

In this way, the information about the leading pole of $I_\phi(s)$ has been coded into the length of a certain $\boldsymbol{C}\{x_0\}$ module.

If $p_g(m)=\dim_{\boldsymbol{C}} C_{(0,\bar{0})}(m)$ then we see that $p_g(m)=\sum_j \psi_{\phi_j}(m)$ for some finite subset of local sections ϕ_j of $\mathscr{O}_U$.

K. Watanabe has used a notion of "$L^{2/d}$ integrability" (introduced by Sakai [20]) for isolated normal singularities (X, x). When (X, x) is the germ of a cyclic covering of a hypersurface, $(\Sigma_m, (0, \bar{0}))$ in the prior notation) we can use the above considerations to help analyze this notion in more detail.

Following the notation as above, let $\theta\in\Gamma(\Sigma_m-(0, \bar{0}), \mathscr{O}(d\omega_{\Sigma_m}))$ be a section of the sheaf of d-fold tensor products of the dualizing sheaf of Σ_m.

In local coordinates $(x_0, \cdots, x_n)$ defined in a $\boldsymbol{C}^{n+1}$ neighborhood Y

of $(0, \bar{0})$ (using the local embedding of Σ_m into C^{n+1} determined by the choice of defining function F_m), $\theta(x)$ can be written in the open subset $\{\partial F_m/\partial x_0 \neq 0\} \cap \Sigma_m \cap Y$ (denoted from now on as $\{x_0 \neq 0\} \cap \Sigma_m$!) as

$$\theta(x) = \frac{\tilde{\theta}(x)}{(\partial F_m/\partial x_0)^d}(dx_1 \wedge \cdots \wedge dx_n)^{\otimes d} = \tilde{\theta}(x)\sigma_{\Sigma m}^{\otimes d}$$

where $\tilde{\theta}$ is a section of $\mathcal{O}_{\Sigma m}$ defined (at least) in $Y \cap \Sigma_m - \{(0, \bar{0})\}$. θ determines a section $\theta \wedge \bar{\theta}$ of $(\omega_{\Sigma m} \wedge \bar{\omega}_{\Sigma m})^{\otimes d}$, defined in $Y \cap \Sigma_m$.

Observe that

$$\begin{aligned}(2.11) \qquad \theta \wedge \bar{\theta}|_{\{x_0 \neq 0\} \cap \Sigma m} &= |\tilde{\theta}|^2[(\sigma_{\Sigma m} \wedge \bar{\sigma}_{\Sigma m}|_{\{x_0 \neq 0\} \cap \Sigma m}]^{\otimes d} \\ &= \left|\frac{\tilde{\theta}}{(\partial F_m/\partial x_0)^d}\right|^2 \cdot (dx_1\, d\bar{x}_1 \cdots dx_n\, d\bar{x}_n)^{\otimes d}\end{aligned}$$

One interprets this last equality as follows. The smooth manifold $(\Sigma_m)_{\rm sp}$ is a real manifold of dimension $2n$. On it, there is the sheaf of germs of complex valued C^∞ functions, denoted $\mathcal{O}_{\Sigma m}^\infty$. The sheaf $\Omega_{(\Sigma m)\rm sp}^{(n,n)}$ is an invertible $\mathcal{O}_{\Sigma m}^\infty$ module. Similarly, $\omega_{\Sigma m} \wedge \bar{\omega}_{\Sigma m}$ is an invertible $\mathcal{O}_{\Sigma m}^\infty$ module. $\theta \wedge \bar{\theta}$, in the open set $\{x_0 \neq 0\} \cap \Sigma_m$, determines a section of $[\Omega_{(\Sigma m)\rm sp}^{(n,n)}]^{\otimes d}$ on this open set. This is written in (2.11).

It is, of course, independent of which partial derivative of F_m is used to represent $\sigma_{\Sigma m}$ in an open subset of Σ_m.

To $\theta \wedge \bar{\theta}$ is associated a section of the sheaf of continuous (n, n) differentials on $(\Sigma_m)_{\rm sp}$. This is written in $\{x_0 \neq 0\} \cap \Sigma_m$ as

$$(\theta \wedge \bar{\theta})^{1/d} = \left|\frac{\tilde{\theta}(x)}{(\partial F_m/\partial x_0)^d}\right|^{2/d} dx_1\, d\bar{x}_1 \cdots dx_n\, d\bar{x}_n.$$

That it is a section of this sheaf is left to the reader to check. To say θ is $L^{2/d}$ integrable at $(0, \bar{0})$ (i.e. $(\theta \wedge \bar{\theta})^{1/d}$ is a section of the sheaf of $L^{2/d}$ integrable (n, n) differentials on a punctured neighborhood of $(0, \bar{0})$ in $(\Sigma_m)_{\rm sp}$) is to say that $\int_{U - V \cap \{x_0 = 0\}} (\theta \wedge \bar{\theta})^{1/d} < \infty$ for each relatively compact open set $V \subset \Sigma_m$ containing $(0, \bar{0})$.

In analogy to the functions $\psi_\phi(m)$, assume now $\tilde{\theta}$ is a section of $\mathcal{O}_U$. Define the function

$$\zeta_{\tilde{\theta}}(d, m) = \min\left\{k : x_0^k \frac{\tilde{\theta}(x)}{(\partial F_m/\partial x_0)^d}(dx_1 \cdots dx_n)^{\otimes d} = \tilde{\theta}_k \text{ is } L^{2/d} \text{ integrable at } (0, \bar{0})\right\}.$$

$\zeta_{\tilde{\theta}}(d, m)$ can be computed in the same way as is $\psi_\phi(m)$.

Consider the diagram

$$\begin{array}{ccc} \Sigma_m \times_U X_{\text{res}} & \xrightarrow{p_1} & \Sigma_m \\ \Big\downarrow p_2 & & \Big\downarrow p_3 \\ X_{\text{res}} & \xrightarrow[\pi]{} & U \end{array}$$

where π is an embedded local resolution of $\{f\cdot\tilde{\theta}=0\}$ in U and p_i are the natural projections.

Let $\{V_\alpha\}$ be a finite open cover of X_{res} for which there are coordinates $(x_1^{(\alpha)}, \cdots, x_n^{(\alpha)})$ centered at a point $p^{(\alpha)}$ in $\pi^{-1}(f^{-1}(0))$ such that

$$f\circ\pi=(x_1^{(\alpha)})^{M_1(\alpha)}\cdots(x_n^{(\alpha)})^{M_n(\alpha)}\cdot(\text{local unit})$$
$$\tilde{\theta}\circ\pi=(x_1^{(\alpha)})^{b_1(\alpha)}\cdots(x_n^{(\alpha)})^{b_n(\alpha)}\cdot(\text{local unit})$$
$$\det d\pi=(x_1^{(\alpha)})^{m_1(\alpha)}\cdots(x_n^{(\alpha)})^{m_n(\alpha)}\cdot(\text{local unit}).$$

Since $x_0^m=f$ in Σ_m, $|x_0|^k=|f|^{k/m}$ in Σ_m and $|x_0|^k=|f\circ\pi|^{k/m}$ in $\Sigma_m\times_U X_{\text{res}}$.

Consider the pullback of $(\theta_k\wedge\bar{\theta}_k)^{1/d}$ on $p_2^{-1}(V_\alpha)$. One checks that it is given by

$$|x_1^{(\alpha)}|^{c_1(\alpha)}\cdots|x_n^{(\alpha)}|^{c_n(\alpha)}\cdot|\text{local unit in } V_\alpha|\,dx_1d\bar{x}_1\cdots dx_nd\bar{x}_n$$

where

$$C_j(\alpha)=\frac{2}{d}\left[\frac{k}{m}M_j(\alpha)+b_j(\alpha)+m_j(\alpha)-\frac{d(m-1)}{m}M_j(\alpha)\right].$$

$\theta_k\wedge\bar{\theta}_k$ is $L^{2/d}$ integrable iff its pullback to each $p_2^{-1}(V_\alpha)$ is $L^{2/d}$ integrable. This will be so iff k satisfies the inequality

$$kM_j(\alpha)\gneqq d(m-1)M_j(\alpha)-m(b_j(\alpha)+m_j(\alpha))-dm \tag{2.12}$$
for $j=1, \cdots, n$. (That is, $C_j(\alpha)\gneqq -2$ for all j.)

Rearranging, one has that k must satisfy

$$k\gneqq m\left\{\frac{dM_j(\alpha)-(d+b_j(\alpha)+m_j(\alpha))}{M_j(\alpha)}-\frac{d}{mM_j(\alpha)}\right\}\quad\text{for all } j=1, \cdots, n.$$

The least k must therefore be

$$k(\alpha)=\max_j\left[\!\!\left[\left(d-\frac{(d+b_j(\alpha)+m_j(\alpha))}{M_j(\alpha)}\right)\cdot m\right]\!\!\right].$$

Hence,

$$\zeta_{\tilde{\theta}}(d)=\max_{\alpha} k(\alpha)=[\![m\cdot\mu_{\tilde{\theta}}(d)]\!],$$

where

$$\mu_{\tilde{\theta}}(d)=d+\max_{\alpha,j}\left[-\frac{(d+b_j(\alpha)+m_j(\alpha))}{M_j(\alpha)}\right].$$

Define $\beta_{\tilde{\theta}}(d)=\mu_{\tilde{\theta}}(d)-d$.

When $d=1$, we evidently recover the formula of Theorem (2.5) for $\phi=\tilde{\theta}$. For $d\gneq 1$, $\beta_{\tilde{\theta}}(d)$ is not so easily related to a pole of the analytic continuation of

$$I_{\tilde{\theta}}(s)=\int_U |f|^{2s}|\tilde{\theta}|^2\rho dx_1 d\bar{x}_1\cdots dx_n d\bar{x}_n.$$

For this depends on derivatives of $\tilde{\theta}\circ\pi$ of order $d-1$ (as well as other factors which need not be detailed here) along the divisors in X_{res}.

Nonetheless we have shown by the above discussion the

Theorem (2.13). *For* $(\Sigma_m, (0, \bar{0}))$ *the germ of a cyclic cover of an isolated hypersurface singularity the function, introduced by K. Watanabe,* [31]

$$\delta_d(\Sigma_m, (0, \bar{0}))=\dim_C\frac{\Gamma(\Sigma_m-(0, \bar{0}), \mathcal{O}(d\omega_{\Sigma_m}))}{L^{2/d}(\Sigma_m)}$$

(*defined for a neighborhood of* $(0, \bar{0})$ *in* Σ_m) *admits an expression*

$$\delta_d=\sum_{\theta}\zeta_{\tilde{\theta}}(d)$$

where the summation is over some finite subset of sections $\tilde{\theta}$ *of* $\mathcal{O}_{\Sigma_m}$ *whose classes modulo* $L^{2/d}(\Sigma_m)$ *are distinct and non-zero.*

Remark. It would be interesting to know if for each d one had an upper semi-continuity property for the δ_d, in the sense that if f_t is a one-parameter family of germs of hypersurfaces at a common singular point $\bar{0}$, then the function $\delta_d(t)=\delta_d(\Sigma_m(t), (0, \bar{0}))$ is upper semi-continuous at $t=0$ for each d and m.

Section 3.

This section will state and prove the upper semi-continuity theorem described in the Introduction.

Let $U\subset C^n$ be an open neighborhood of $\bar{0}$ as in Section 2. Let $T\subset C$ be an open neighborhood of 0 in C. We are given a 1-parameter family of representatives of germs f_t: $(C^n, \bar{0})\to(C, 0)$, defined in U for each $t\in T$.

We interpret this to mean that there is a complex analytic function F: $U\times T\to \boldsymbol{C}$ so that if π: $\{F=0\}\to T$ is the projection to t, the fiber $\pi^{-1}(t)=f_t^{-1}(0)$ lies in U. We assume f has an isolated singularity at $\bar{0}$ and then by shrinking U, if necessary, we may assume, by the Preparation theorem, that when f_t also has an isolated singularity at $\bar{0}$ for each $t\neq 0$ the fiber $\pi^{-1}(t)$ has only finitely many critical points of $\{f_t=0\}$ in U. Let $x_1(t), \cdots, x_R(t)$ be these critical points. Let $\mu_{x_j(t)}(f_t)$ be the Milnor number of the germ at $x_j(t)$. It follows that $\mu_{\bar{0}}(f)\geq\sum_{j=1}^{R}\mu_{x_j(t)}(f_t)$. [24].

To each t, $x_j(t)$, and each $\phi\in\Gamma(U, \mathcal{O}_U)$, there are quasi-adjoint characters, denoted by $\kappa_\phi(x_j(t))$, as defined in Section 2. For the critical point $\bar{0}$, we use the notation $\bar{0}_t$ resp. $\bar{0}$ to denote $\bar{0}$ as a critical point of f_t resp. $f_0=f$.

Upper semi-continuity will take the following form

Theorem (3.1). *For f_t: $(\boldsymbol{C}^n, \bar{0})\to(\boldsymbol{C}, 0)$ a 1 parameter family of germs of complex analytic functions with isolated singular point at $\bar{0}$ for each t, then for $t\neq 0$ and t sufficiently close to 0, one has for all $\phi\in\Gamma(U, \mathcal{O}_U)$*

$$\kappa_\phi(\bar{0}_t)\leq\kappa_\phi(\bar{0}),$$

where U is an open neighborhood of $\bar{0}$ satisfying the properties described in the second paragraph of the section.

Proof. Following notation of (2.3), recall that $\psi_\phi(x_j(t), m)=[\![m\cdot\kappa_\phi(x_j(t))]\!]=\inf\{k$: $x_0^k\phi$ *is* adjoint to the germ of $\{x_0^m-f_t(x_1, \cdots, x_n)=0\}$ at $(0, x_j(t))\}$. Set $\Sigma_m(t)=\{x_0^m-f_t(x_1, \cdots, x_n)=0\}$. To show the theorem, it suffices to show $\psi_\phi(\bar{0}_t, m)\leq\psi_\phi(\bar{0}, m)$ for t near to 0. This follows from the fact that if $\kappa_\phi(\bar{0}_t)=\alpha(t)/\beta(t)$ and $\kappa_\phi(\bar{0})=a/b$, then for fixed t not zero, set $m=\beta(t)\cdot b$. If $\psi_\phi(\bar{0}_t, m)=b\alpha(t)\leq\psi_\phi(\bar{0}, m)=a\beta(t)$ then $\alpha(t)/\beta(t)\leq a/b$ as wanted.

Remark (2.10) says that the inequality involving values of ψ_ϕ would follow if there was a theorem which showed that $\text{length}_{\boldsymbol{C}\{x_0\}} N_\phi(t)\leq \text{length}_{\boldsymbol{C}\{x_0\}} N_\phi(0)$, where for any t, $N_\phi(t)$ denotes the $\boldsymbol{C}\{x_0\}$ submodule of $(\mathcal{O}_{\Sigma_m}(t)/A_{\Sigma_m}(t))_{(0,\bar{0}_t)}$ generated by ϕ.

Such a theorem was essentially proved by Elkik [3]. Here we summarize those parts of her results pertaining to Theorem (3.1). Necessarily, we use the analytic (derived) category.

Given the open neighborhoods T of $t=0$ in $\boldsymbol{C}$, X_0 of $x_0=0$ in $\boldsymbol{C}$ and U of $\bar{0}$ in $\boldsymbol{C}^n$, set $Z=T\times X_0\times U$. In Z, there is a closed analytic subvariety X defined by the equation $x_0^m-F(t, x_1, \cdots, x_n)=0$. Let π: $X\to T$ be the projection to t. Having chosen U as above, π has "finite relative singular locus" ("lieu singulier relatif fini"). That is, $\pi|_{X_{\text{sing}}}$ is a finite map by the Preparation Theorem. Also, π is flat with fiber dimension n.

To discuss properties of the relative dualizing complex for an analytic morphism $h\colon X\to S$ is helpful at this point. To any such h one can construct in a functorial way a complex $R^{\cdot}_{X/S}$ (or $h^{!}(\mathcal{O}_S)$ in Verdier's notation [30]) satisfying these properties. (All of which will hold in the bounded below derived category $D^{+}_{\mathrm{coh}}(\mathcal{O}_X)$ of complexes with coherent cohomology sheaves).

(3.2) (a) (Base change). Let $\gamma\colon W\to S$ be any analytic morphism. In the diagram

$$\begin{array}{ccc} Y=X\underset{S}{\times}W & \xrightarrow{\tilde{\gamma}} & X \\ \downarrow & & \downarrow \\ W & \xrightarrow[\gamma]{} & S \end{array}$$

$$R^{\cdot}_{Y/W}=\boldsymbol{L}\tilde{\gamma}^{*}R^{\cdot}_{X/S}$$

(b) If $h\colon X\to S$ is a flat morphism with fibers of dimension d then $R^{\cdot}_{X/S}$ is a bounded complex of S flat $\mathcal{O}_X$ coherent sheaves with (Tor) amplitude $[-d, 0]$.

(c) If each fiber of h in (b) is a Cohen-Macaulay variety (i.e. local rings are Cohen-Macaulay rings) then

$$H^{-j}(R^{\cdot}_{X/S})=0 \text{ if } j\neq d.$$

(d) For $\phi\colon X'\to X$ a projective morphism $\boldsymbol{R}\phi_{*}\,\boldsymbol{R}\mathcal{H}om_{\mathcal{O}_{X'}}(F, R^{\cdot}_{X'/S})=\boldsymbol{R}\mathcal{H}om_{\mathcal{O}_X}(\boldsymbol{R}\phi_{*}F, R^{\cdot}_{X/S})$, for any coherent $\mathcal{O}_{X'}$module F.

When S is a point, denote $R^{\cdot}_{X/S}$ by $R^{\cdot}_X$.

When X is a hypersurface defined by an equation G in $\boldsymbol{C}^{m+1}$, the sheaf $H^{-m}(R^{\cdot}_X)=\omega_X$ is an $\mathcal{O}_X$ module generated on X_{sp} by the differential σ_X which, in a $\boldsymbol{C}^{m+1}$ neighborhood of a point p at which $(\partial G/\partial z_j)(p)\neq 0$, is written in the local coordinates $(z_0, \cdots, z_m)$ as

$$\sigma_X=(-1)^{j-1}\left.\frac{dz_0\cdots\widehat{dz_j}\cdots dz_{m+1}}{\partial G/\partial z_j}\right|_X \quad [1, 2, 8].$$

Let $\rho\colon X'\to X$ be a resolution of singularities of the analytic variety X equidimensional of dimension d (equidimensional for simplicity). X' is smooth so $\mathcal{O}_{X',x'}$ is Cohen-Macaulay at x'. The Grauert-Riemenschneider Theorem asserts that $R^{i}\rho_{*}(R^{\cdot}_{X'})=0$ for $i>0$ [4].

Construct the distinguished triangle in $D^{+}_{\mathrm{coh}}(\mathcal{O}_X)$

$$\begin{array}{ccc} & N^{\cdot}_X & \\ {\scriptstyle +1}\swarrow & & \nwarrow \\ \rho_{*}R^{\cdot}_{X'} & \longrightarrow & R^{\cdot}_X \end{array}$$

where the horizontal arrow is that dual to the natural morphism in $D^+_{\mathrm{coh}}(\mathcal{O}_X)$

$$\mathcal{O}_X \longrightarrow R\rho_* \mathcal{O}_{X'}$$

by application of duality ((d) above).

Consider the homology of $N^\bullet_X$. By the long exact sequence of cohomology applied to the triangle,

i) $H^{-i}(N^\bullet_X) = H^{-i}(R^\bullet_X)$ if $i \neq d$

ii) $H^{-d}(N^\bullet_X) = H^{-d}(R^\bullet_X)/\rho_* H^{-d}(R^\bullet_{X'}) = \mathrm{cok}\,(\rho_*\omega_{X'} \to \omega_X)$ where $\omega_X = H^{-d}(R^\bullet_X)$, $\omega_{X'} = H^{-d}(R^\bullet_{X'})$.

iii) $H^i(N^\bullet_X)$ is a coherent $\mathcal{O}_X$ module with support in X_{sing} for all i.

From [2], the adjoint ideal sheaf of X is defined as $A_X = \mathrm{ann}_{\mathcal{O}_X}[\mathrm{cok}\,(\rho_*\omega_{X'} \to \omega_X)]$. Thus, $\mathcal{O}_X/A_X \simeq H^{-d}(N^\bullet_X)$ as $\mathcal{O}_X$ modules supported on X_{sing} with stalks which are finite dimensional $\boldsymbol{C}$ vector spaces. (It is also well known that if x is an isolated singular point of X for which $\mathcal{O}_{X,x}$ is normal and Cohen-Macaulay of dimension d then $\dim_{\boldsymbol{C}}(\mathcal{O}_X/A_X)_x = \dim_{\boldsymbol{C}} R^{d-1}\rho_*(\mathcal{O}_{X'})_x$ [33].)

The discussion needs to be relativized now by consideration of the analogous objects in the fibers of a flat morphism $h: X \to S$ of fiber dimension d (and equidimensional for simplicity) as above. Assume first that $h: X \to S$ admits a simultaneous resolution. Thus, there is a projective morphism $\phi: X' \to X$ such that i) X' is smooth; ii) $(h \circ \phi)^{-1}(t)$ is smooth for each $t \in S$, and iii) $h \circ \phi$ is flat.

We can then construct two distinguished triangles by considering this diagram

(3.3)
$$\begin{array}{ccc} X' \times_S \{t\} = X'_t & \xrightarrow{j'_t} & X' \\ \downarrow{\scriptstyle \phi_t} & & \downarrow{\scriptstyle \phi} \\ X \times \{t\} = X_t & \xrightarrow{j_t} & X \\ \downarrow{\scriptstyle h_t} & & \downarrow{\scriptstyle h} \\ \{t\} & \xrightarrow{i_t} & S \end{array}$$

At first we have the triangle

$$\begin{array}{ccc} & N^\bullet_{X_t} & \\ {\scriptstyle +1}\swarrow & & \nwarrow \\ (\phi_t)_* R^\bullet_{X'_t} & \longrightarrow & R^\bullet_{X_t} \,. \end{array}$$

in $D^+_{\mathrm{coh}}(X_t)$

By property (3.2) (a) we have that $R^{\bullet}_{X_t}=Lj_t^*R^{\bullet}_{X/S}$ and $R^{\bullet}_{X'_t}=L(j'_t)^*R^{\bullet}_{X'/S}$. So, $(\phi_t)_*R^{\bullet}_{X'_t}=L(\phi_t)_*(j'_t)^*R^{\bullet}_{X'/S}=Lj_t^*\phi_*R^{\bullet}_{X'/S}$.

The triangle in $D^+_{\text{coh}}(X)$

$$\begin{array}{ccc} & N^{\bullet}_{X/S} & \\ {}^{+1}\swarrow & & \nwarrow \\ \phi_*R^{\bullet}_{X'/S} & \longrightarrow & R^{\bullet}_{X/S} \end{array}$$

(where $N^{\bullet}_{X/S}$ is the mapping cone for $\phi_*R^{\bullet}_{X'/S}\rightarrow R^{\bullet}_{X/S}$) induces by application of j_t^*, the triangle

$$\begin{array}{ccc} & Lj_t^*N^{\bullet}_{X/S} & \\ {}^{+1}\swarrow & & \nwarrow \\ Lj_t^*\phi_*R^{\bullet}_{X'/S} & \longrightarrow & Lj_t^*R^{\bullet}_{X/S} \end{array} \tag{3.4}$$

in $D^+_{\text{coh}}(X_t)$.

From the above identifications, however, and by the isomorphism of mapping cones in a derived category [5], we therefore have that

$$N^{\bullet}_{X_t}=Lj_t^*N^{\bullet}_{X/S}. \tag{3.5}$$

In general, a simultaneous resolution of h only will exist off a proper analytic subvariety of S. By (3.2) (a), we may then reduce the analysis of the comparison of the complexes $N^{\bullet}_{X_t}$ (and in particular their homology sheaves $H^{-d}(N^{\bullet}_{X_t})$) to the situation where S is an analytic curve in a neighborhood of a distinguished point s such that for $S-\{s\}$, $h\colon X-h^{-1}(s)\rightarrow S-\{s\}$ admits a simultaneous resolution but the fiber $h^{-1}(s)$ possesses an obstruction to the simultaneity.

Let $\phi\colon X'\rightarrow X$ be a resolution of singularities of X. For $t\in S-\{s\}$, we may assume $\phi^{-1}(X_t=h^{-1}(t))=X'_t$ is a desingularization of X_t. To (3.3) we again refer.

We now consider only the fibers of h for $t\neq s$. Again, we obtain that $N^{\bullet}_{X_t}=Lj_t^*N^{\bullet}_{X/S}$ for $t\neq s$. However, we cannot yet extend this to $N^{\bullet}_{X_s}$.

If $\psi\colon Z\rightarrow X_s$ is a desingularization of X_s we of course have the triangle

$$\begin{array}{ccc} & N^{\bullet}_{X_s} & \\ {}^{+1}\swarrow & & \nwarrow \\ \psi_*R^{\bullet}_{Z} & \longrightarrow & R^{\bullet}_{X_s} \end{array} \tag{3.6}$$

(using the absolute dualizing complexes for X_s, Z).

We observe that the fiber X'_s of $(h\circ\phi)$ can contain a desingularization

Z of X_s as follows. If the fiber $(h\circ\phi)^{-1}(s)$ in the smooth variety X' is not already smooth, one can construct an embedded desingularization of $X'_s \hookrightarrow X'$ that is, a smooth variety $\hat{X}'$ and a proper bimeromorphic map $\theta\colon \hat{X}' \to X'$ so that θ is an isomorphism off X'_s and so that $\theta^{-1}(X'_s)$ is locally in normal crossing form. The *strict transform* of X'_s in $\hat{X}'$ is a desingularization of X'_s. Let Z be this strict transform. Then $\psi=\phi\circ\theta\colon Z\to X_s$ can be used in (3.6). So, this says that one can always find a desingularization $\phi\colon X'\to X$ of X for which the strict transform Z of X_s is a desingularization of X_s and such that $\mu_Z\colon Z\hookrightarrow X'_s=\phi^{-1}(X_s)$. Moreover, $\psi\colon Z\to X_s$ equals $\phi_s\circ\mu_Z$ where $\phi_s=\phi|_{X'_s}\colon X'_s\to X_s$.

There is a natural morphism $R^{\cdot}_Z\to R^{\cdot}_{X'_s}$ [5] which induces a morphism $\alpha\colon \psi_* R^{\cdot}_Z\to(\phi_s)_* R^{\cdot}_{X'_s}$ in $D^+_{\mathrm{coh}}(X_s)$ so that one has a commutative diagram of complexes

$$\begin{array}{ccc} \psi_* R^{\cdot}_Z & \longrightarrow & (\phi_s)_* R^{\cdot}_{X'_s} \\ & \searrow \quad \swarrow & \\ & R^{\cdot}_{X_s} & \end{array} .$$

Now consider the analogue of (3.3)

$$\begin{array}{rcl} X'\underset{S}{\times}\{s\}=X'_s & \xrightarrow{j'_s} & X' \\ \downarrow{\scriptstyle \phi_s} & & \downarrow{\scriptstyle \phi} \\ X\underset{S}{\times}\{s\}=X_s & \xrightarrow{j_s} & X \\ \downarrow & & \downarrow{\scriptstyle h} \\ \{s\} & \xrightarrow{i_s} & S \end{array} \qquad (3.7)$$

As before, we have by (3.2) (a) that $R^{\cdot}_{X'_s}=L(j'_s)^* R^{\cdot}_{X'/S}$, so that $(\phi_s)_* R^{\cdot}_{X'_s} = Lj^*_s\phi_* R^{\cdot}_{X'_s}$. Also, $R^{\cdot}_{X_s}=Lj^*_s R^{\cdot}_{X/S}$.

Thus, we have two distinguished triangles in $D^+_{\mathrm{coh}}(X_s)$.

$$\begin{array}{ccc} & N^{\cdot}_{X_s} & \\ {\scriptstyle +1}\swarrow & & \nwarrow \\ \psi_* R^{\cdot}_Z & \xrightarrow[u]{} & R^{\cdot}_{X_s} \end{array} \quad \text{and} \quad \begin{array}{ccc} & Lj^*_s N^{\cdot}_{X/S} & \\ {\scriptstyle +1}\swarrow & & \nwarrow \\ Lj^*_s\phi_* R^{\cdot}_{X'/S} & \xrightarrow[\tilde{u}]{} & Lj^*_s R^{\cdot}_{X/S} \end{array}$$

(Δ_1) (Δ_2)

with morphisms

$$\alpha\colon \psi_* R^{\cdot}_Z \longrightarrow Lj^*_s\phi_* R^{\cdot}_{X'/S}$$

and

$$\beta = \mathrm{id}\colon R^{\bullet}_{X_s} \longrightarrow \boldsymbol{L} j_s^* R^{\bullet}_{X/S}$$

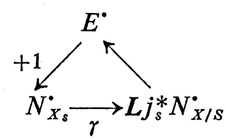
commuting with u and $\tilde{u}$.

Since $D^{+}_{\mathrm{coh}}(X)$ is a triangulated category, there is a morphism

$$\gamma\colon N^{\bullet}_{X_s} \longrightarrow \boldsymbol{L} j_s^* N^{\bullet}_{X/S}$$

so that (α, β, γ) is a morphism of (Δ_1) to (Δ_2).

If one now constructs the mapping cone over γ



one finds that only $H^{-d-1}(E^{\bullet})$ can be non-zero ($d = \dim X_s$).

Hence, $H^{-d}(N^{\bullet}_{X_s}) \to H^{-d}(\boldsymbol{L} j_s^* N^{\bullet}_{X/S})$ is a surjection.

Applied to the situation of interest here this says that for $\pi\colon X \to T$, the family of hypersurfaces $\{x_0^m - F(t, x_1, \cdots, x_n) = 0\} \subset T \times X_0 \times U$ one has

i) For $t \neq 0$, $\oplus_{j=1}^{R} (\mathcal{O}_{\Sigma_m}(t)/\mathrm{A}_{\Sigma_m}(t))_{x_j(t)} \simeq i_t^* R^d \pi_*(N^{\bullet}_{X/T})$ with $x_1(t), \cdots, x_R(t)$ the singular points of $\Sigma_m(t)$, and $i_t\colon \{t\} \to T$ the inclusion as a morphism in the category of analytic spaces.

ii) For $t = 0$, $(\mathcal{O}_{\Sigma_m(0)}/A_{\Sigma_m(0)})_{(0,\bar{0})} \to i_0^* R^d \pi_*(N^{\bullet}_{X/T})$ is a surjection.

The observation that is necessary to make now is that the sheaves $H^i(N^{\bullet}_{X/T})$ are coherent and supported on X_{sing}. Thus, since $\pi\colon X_{\mathrm{sing}} \to T$ is finite, π_* is an exact functor on the category of $\mathcal{O}_X$ modules with support contained in X_{sing}. Moreover, π_* (a coherent $\mathcal{O}_X$ module with support in X_{sing}) is a coherent $\mathcal{O}_T$ module. Thus, the sheaves $R^j \pi_*(N^{\bullet}_{X/T}) = H^j({}^{\bullet}\pi_*(N\boldsymbol{R}^{\bullet}_{X/T}))$ are actually of the form $\pi_* H^j(N^{\bullet}_{X/T})$ and are therefore $\mathcal{O}_T$ coherent modules.

This discussion has sketched Elkik's argument that proved the upper semi-continuity of the dimensions of the conductor $C_{x(t)}$ at isolated singular points $x(t)$ in fibers X_t of a flat equidimensional morphism $X \to S$.

It is also the discussion that allows one to extend her argument to give a proof of Theorem (3.1) as follows.

This is based on the main conclusion from her argument. That is, for each $t \in T$, if $x_1(t), \cdots, x_R(t)$ are the singular points of $\pi^{-1}(t)$ there is a $\boldsymbol{C}$-vector space surjection

$$H_t\colon \bigoplus_j (\mathcal{O}_{\Sigma_m(t)}/A_{\Sigma_m(t)})_{x_j(t)} \longrightarrow [\pi_* H^{-d}(N^{\bullet}_{X/T})]_t.$$

Let $C = \coprod_t (\oplus_j [\mathcal{O}_{\Sigma_m(t)}/A_{\Sigma_m(t)}]_{x_j(t)})$ be a bundle over T. C can be made into a sheaf on T as follows. Think of $\pi_* H^{-d}(N^{\bullet}_{X/T})$ as the total space of the $\mathcal{O}_T$ sheaf $\pi_*(H^{-d}(N^{\bullet}_{X/T}))$ constructed in the above discussion. We then have $\pi_*(H^{-d}(N^{\bullet}_{X/T}))=\coprod_t \pi_*(H^{-d}(N^{\bullet}_{X/T}))_t$. The C-linear surjections H_t: $C_t \to \pi_*(H^{-d}(N^{\bullet}_{X/T}))_t$ induce a topology on C via the topology on the total space $\pi_* H^{-d}(N^{\bullet}_{X/T})$. It is the weakest topology which makes $H=\coprod H_t$ C linear and continuous. In this way, C is provided with a topology which also makes it into a sheaf of $\boldsymbol{C}$ vector spaces over T. (That is, if π_1: $C\to T$ and π_2: $\pi_* H^{-d}(N^{\bullet}_{X/T})\to T$ are the projections then $\pi_1=\pi_2\circ H$ is continuous.)

Given a fixed section $\phi\in\Gamma(U, \mathcal{O}_U)$, there is an evident section of C corresponding to ϕ. That is, σ_ϕ: $t\to\oplus_j \bar{\phi}_{x_j(t)}$, where $\bar{\phi}_{x_j(t)}$ is the class of ϕ in $(\mathcal{O}_{\Sigma_m(t)}/A_{\Sigma_m(t)})_{x_j(t)}$. Then $H\circ\sigma_\phi$ gives a section of $\pi_*(H^{-d}(N^{\bullet}_{X/T}))$. We denote this section as $s(\phi)$.

To obtain an $\mathcal{O}_{X_0}$ module structure on the stalks of $\pi_*(H^{-d}(N^{\bullet}_{X/T}))$ consider the diagram

$$\begin{array}{ccccc}
Y\underset{T\times X_0}{\times}\{t\}\times X_0 & \xrightarrow{\tilde{v}_t} & Y=X\underset{T}{\times}(T\times X_0) & \xrightarrow{\hat{u}} & X \\
\downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\hat{\pi}} & & \downarrow{\scriptstyle\pi} \\
\{t\}\times X_0 & \xhookrightarrow{v_t} & T\times X_0 & \xrightarrow{u} & T
\end{array}$$

$u(t, x_0)=t$.

The morphisms u and $\hat{u}$ are flat. Hence, $\hat{u}^* H^{-d}(N^{\bullet}_{X/F})$ is a coherent $\mathcal{O}_Y$ module and $\hat{\pi}_*\hat{u}^*(H^{-d}(N^{\bullet}_{X/T}))=u^*\pi_*(H^{-d}(N^{\bullet}_{X/T}))$ is a coherent $\mathcal{O}_{T\times X_0}$ module. Its support is contained in $T\times\{0\}$ since for $x_0\neq 0$ and t_0 in T, let W be an open neighborhood of (t_0, x_0) lying outside $T\times\{0\}$. Now,

$$(\hat{\pi}_*\hat{u}^*)(H^{-d}(N^{\bullet}_{X/T}))(W)=\Gamma(\hat{\pi}^{-1}(W), \hat{u}^* H^{-d}(N^{\bullet}_{X/T})|_{\hat{\pi}^{-1}(W)}).$$

Since $H^{-d}(N^{\bullet}_{X/T})$ is a sheaf with support in X_{sing} and $X_{\text{sing}}\subset T\times\{0\}\times U$ by the defining equation of the variety X in $T\times X_0\times U$, $\hat{u}^* H^{-d}(N^{\bullet}_{X/T})$ is zero when restricted to $\hat{\pi}^{-1}(W)$. So, the stalk of $(\hat{\pi}_*\hat{u}^*)(H^{-d}(N^{\bullet}_{X/T})$ at (t_0, x_0) must also be zero.

Denote this $\mathcal{O}_{T\times X_0}$ module by M. Then the module $v_t^* M$ is an $\mathcal{O}_{X_0,0}$ $=\boldsymbol{C}\{x_0\}$ module and is the result of the base extension u. Since M is $\mathcal{O}_{T\times X_0}$ coherent, it also follows that

$$\text{length}_{C\{x_0\}}(v_t^* M)\leq\text{length}_{C\{x_0\}}(v_0^* M).$$

We now only need to replace M by a coherent $\mathcal{O}_{T\times X_0}$ submodule associated to the section ϕ of $\mathcal{O}_U$.

This is accomplished by using the section $s(\phi)$. The base change morphism u lifts $s(\phi)$ to a section $\boldsymbol{s}(\phi)$ of M. As such, we obtain an $\mathcal{O}_{T\times X_0}$ submodule of M by setting $P_\phi=\mathcal{O}_{T\times X_0}\cdot(\boldsymbol{s}(\phi))$. It is clearly of finite type. Thus, P is a coherent $\mathcal{O}_{T\times X_0}$ submodule of M. It then follows that for t in T

$$\text{length}_{C\{x_0\}}(v_t^* P_\phi) \leq \text{length}_{C\{x_0\}}(v_0^* P_\phi).$$

To complete the proof of Theorem (3.1), we need only note that the base change u extends the $\boldsymbol{C}$-linear sheaf homomorphism H: $\boldsymbol{C} \to \pi_* H^{-d}(N^{\cdot}_{X/T})$) to *a* $\boldsymbol{C}\{x_0\}$ module homomorphism $\hat{H}$ so that $\hat{H}_t$: $N_\phi(t) \to v_t^* P_\phi$ is a $\boldsymbol{C}\{x_0\}$ module isomorphism for $t \neq 0$ and $\hat{H}_0$: $N_\phi(0) \to v_0^* P_\phi$ is a $\boldsymbol{C}\{x_0\}$ module epimorphism. This completes the proof of (3.1).
We state the desired corollary.

Corollary (3.8). *Let f_t be a* 1*-parameter family of germs of a analytic functions at the common isolated singular point $\bar{0}$. Let $U_t \Subset U_t'$ be a pair of Milnor neighborhoods for a representative of f_t. (Thus, each fiber $f_t^{-1}(w)$ is transverse to the boundary of U_t' and U_t for all w near to* 0 *and f_t is a C^∞ fibration off the singular fiber containing the unique singular point $\bar{0}$.) Let ρ_t be a C^∞ function which is* 1 *on U_t and* 0 *off U_t'. Define the generalized functions on $C^\infty(U_t', \boldsymbol{C})$*

$$I_t(s, \psi) = \int_{U_t'} |f_t|^{2s} |\psi|^2 \rho_t \, dx_1 d\bar{x}_1 \cdots dx_n d\bar{x}_n.$$

Let ϕ be an analytic function defined in an open set containing $\bigcup_t U_t'$. Let $\beta_\phi(t)$ be the largest pole of the analytic continuation of $I_t(s, \phi)$. Then if $\beta_\phi(t) > -1$ for all t, one has that for t sufficiently close to 0

$$\beta_\phi(t) \leq \beta_\phi(0).$$

Remark (3.9). Clearly, it suffices to require only $\beta_\phi(0)$ to be in $(-1, 0)$ in order to conclude (3.8) from (3.1).

Remark (3.10). Although the above proof applies to show an upper-semicontinuity of the $\kappa_\phi(\bar{0}_t)$ if f_t is a 1-parameter family of real-analytic germs, each of which having a singular point at $\bar{0} \in \boldsymbol{R}^n$, it does not imply that the corresponding $\beta_\phi(t)$ are upper semicontinuous. From Remark (2.9), one only knows that $\beta_\phi(t) + 1 \leq \kappa_\phi(\bar{0}_t)$ for each t. Indeed, Varcenko's example [26] indicates that the $\beta_\phi(t)$ need not have any type of semi-continuity behavior in general if one works only over $\boldsymbol{R}$.

Section 4.

There are two applications of Theorem (3.1) we wish to describe in this section. We will need to state first some definitions and results of Arnol'd and Varcenko. In the following, the open neighborhoods U of $\bar{0}$ in $\boldsymbol{C}^n$ and T of 0 in $\boldsymbol{C}$ should satisfy the properties of the neighborhoods U, T considered in Section 3. As there, f: $U \to T$ is a defining representative of an isolated hypersurface singularity.

Let $\omega \in \Gamma(U, \Omega_U^n)$ be a section of the sheaf of holomorphic n-differentials in U. Let

$$\mathscr{H}_{n-1} = \coprod_{t \in T-\{0\}} H_{n-1}(f^{-1}(t), \boldsymbol{C}) \text{ and } \mathscr{H}^{n-1} = \coprod_{t \in T-\{0\}} H^{n-1}(f^{-1}(t), \boldsymbol{C})$$

be the homology and cohomology bundles associated to the Milnor fibration $f\colon U - f^{-1}(0) \to T - \{0\}$. The Leray residue $\omega/df = \operatorname{Res}(\omega/(t-f))|_{\{f=t\}}$ determines a section of $\mathscr{H}^{n-1}$

$$\phi\colon t \longrightarrow [\omega/df|_{f^{-1}(t)}] = \text{cohomology class of } \omega/df|_{f^{-1}(t)}.$$

Let $\delta(t_0)$ be a fixed cycle representative in $H_{n-1}(f^{-1}(t), \boldsymbol{C})$. By means of the fibration f on $U - \{f^{-1}(0)\}$, $\delta(t_0)$ can be transported in a possibly multi-valued manner to a cycle in each smooth fiber $f^{-1}(t)$ in U. Denote this class by $\delta(t)$.

Define $I(t, \delta, \omega) = \int_{\delta(t)} \omega/df$. It is a classical theorem that in any angular sector $a < \arg(t) < b$, $0 < b - a < 2\pi$, one has a series expansion [16]

$$I(t, \delta, \omega) = \sum_{\lambda \in \Lambda} \sum_{\alpha \in L(\lambda)} \sum_{k=0}^{n-1} \frac{1}{k!} A_{k,\alpha}(\delta, \omega) t^{\alpha} (\ell n t)^k. \tag{4.1}$$

where i) Λ is the set of eigenvalues of monodromy action on H^{n-1}.

ii) $L(\lambda) = \{\alpha > -1 \colon \exp(-2\pi i \alpha) = \lambda\}$.

iii) The right hand side in (4.1) is single valued and converges in each sector, for $|t|$ sufficiently small, to the function I defined in that sector.

Following Varcenko, define now the index $\alpha(\omega)$ associated to ω.

Definition (4.2). Set $\alpha(\omega) = \min\{\alpha\colon$ for some k, $A_{k,\alpha}(\delta, \omega) \neq 0$ for some family $\{\delta(t)\}$ of cycles constructed as above$\}$.

Also define the Arnol'd exponent $\sigma(f)$ of the germ of f at $\bar{0}$ to be

Definition (4.3). $\sigma(f) = \min\{\alpha(\omega) + 1 \colon \omega \in \Gamma(U, \Omega_U^n)\}$.

Basic results proved by Varcenko [27] allow one to connect the leading poles β_ϕ of the generalized function $|f|^{2s}$ and the indices $\alpha(\omega)$ in a precise manner. We state this in the

Theorem (4.4). *Let* $(x_1, \cdots, x_n)$ *be a system of holomorphic coordinates in the open neighborhood* U *of* $\bar{0}$ *in* $\boldsymbol{C}^n$. *Let* $\omega = \phi(x_1, \cdots, x_n) dx_1 \cdots dx_n$ *be an element of* $\Gamma(U, \Omega_U^n)$. *Then, if* $\beta_\phi > -1$, *one has* $\alpha(\omega) + 1 = -\beta_\phi$.

Remark. This is independent of the coordinates since $\beta_\phi = \beta_{\phi \cdot G}$ if G is a local unit in U.

From (4.4), there is an immediate corollary

Corollary (4.5). *If* $\beta_1=\beta_\phi$ *then* $\sigma(f)=-\beta_1$ (*where* $\beta_1=\beta_\phi$ *for* ϕ *any local analytic unit in* U).

We now apply Corollary (3.8) and Remark (3.9) to obtain the first application of (3.1).

Theorem (4.6). 1) (*Steenbrink-Varcenko*) *If* $\{f_t\}$ *is any* 1-*parameter family of germs of analytic functions at the same isolated singular point* $\bar{0}$ *in* C^n, *then one has*

$$\sigma(f_t)\geq\sigma(f_0)$$

for t sufficiently close to 0, *if* $\sigma(f_0)<1$.

2) *In the situation of* (1), *let* U_t *be a neighborhood of* $\bar{0}$ *on which there is defined a representative of* f_t *satisfying the property that* $f_t\colon U_t\to T_t$ *is a Milnor fibration. Let U be an open neighborhood of* $\bar{0}$ *containing* $\bigcup_t U_t$ *and let* $\omega\in\Gamma(U,\Omega_U^n)$. *Assume* $\omega=\phi(x)dx$. *Let* $\omega(t)=\omega|_{U_t}$ *and define* $\alpha_t(\omega(t))$ *to be the initial exponent of* $\omega(t)$ *for* f_t. *Then, if* $\beta_\phi(0)>-1$, *one has the inequality*

$$\alpha_t(\omega(t))\geq\alpha_0(\omega(0)),\qquad \textit{for all t sufficiently near } 0.$$

Proof. This is immediate from (3.8), (3.9), (4.4). Note that 2) is a distinct extension of the lower semi-continuity theorem of Steenbrink, stated in 1), although the version proved by Steenbrink does not require that $\sigma(f_0)<1$. Note that in the non-rational singularity situation Varcenko also used Elkik's theorem to prove the semicontinuity of the Arnol'd index [29]. (2) is however a distinct refinement both of [22] as well as [29].

Theorem (4.6) can be used to give a proof of an extension of a conjecture of Teissier using the same idea as Loeser did to prove the conjecture [13]. A sketch of [13] follows first and then we present our extension of the conjecture.

Let $f\colon(C^n,\bar{0})\to(C,0)$ be a germ of an analytic function with isolated singularity at $\bar{0}$. Let U, T be as above and $f\colon U\to T$ a representative of the germ. Teissier associated [23] a finite set of numbers $\{(e_q,m_q)\}$ to f. These form the polar multiplicities for f in U. The ratios e_q/m_q are called the polar invariants.

Let H be a generic plane of codimension i through $\bar{0}$ in U. The definition of generic is given in [23]. One of its consequences is that the set of polar invariants of $f|_H$, a representative in $H\cap U$ of the germ at $\bar{0}$ of $f|_H$, is independent of H. Denote this set by $\{e_q^{(i)}/m_q^{(i)}\}$.

The original form of the conjecture was [25]

$$\sigma(f)\geq\sum_{i=0}^{n-1}\frac{1}{1+\sup\{e_q^{(i)}/m_q^{(i)}\}}. \tag{4.7}$$

For f a germ at $\bar{0} \in C^2$ of an analytically irreducible plane curve with characteristic sequence $(n, \beta_1, \cdots, \beta_g)$, one knows the following.

a) $\sigma(f)=\frac{1}{n}+\frac{1}{\beta_1}$ if $\sigma(f)\leq 1$

b) $1+\sup\{e_q/m_q\}=\frac{\bar{\beta}_g}{n_1\cdots n_{g-1}}$ where

$\bar{\beta}_1=\beta_1$, $\bar{\beta}_q=n_{q-1}\bar{\beta}_{q-1}+(\beta_q-\beta_{q-1})$ for $q=2, \cdots, g$, and if $e^{(q)}=\text{g.c.d.}(e^{(q-1)}, \beta_q)$, then $n_q=e^{(q-1)}/e^{(q)}$.

One can show [18] that $\bar{\beta}_g/(n_1\cdots n_{g-1})=(\mu+\beta_g-1)/n$, where μ is the Milnor number of f at $\bar{0}$. Thus, the lower bound in (4.7) for this f can be considerably smaller than the value for $\sigma(f)$.

On the other hand one also has that

$$\frac{1}{\beta_1}=\frac{1}{1+\inf\{e_q/m_q\}}.$$

This suggests an improvement of (4.7) can be made by conjecturing

$$\sum_{i=0}^{n-1}\frac{1}{1+\inf\{e^{(i)}/m_q^{(i)}\}}\geq\sigma(f)\geq\sum_{i=0}^{n-1}\frac{1}{1+\sup\{e_q^{(i)}/m_q^{(i)}\}}. \tag{4.8}$$

Loeser [13] has proved the following version of (4.8)

$$\begin{aligned}\sum_{i=0}^{n-1}\frac{1}{1+[\![\inf\{e_q^{(i)}/m_q^{(i)}\}]\!]}&\geq\sigma(f)\geq\sum_{i=0}^{n-1}\frac{1}{1+[\![\sup\{e_q^{(i)}/m_q^{(i)}\}]\!]}\\&\geq\sum_{i=0}^{n-1}\frac{1}{1+\sup\{e_q^{(i)}/m_q^{(i)}\}}.\end{aligned} \tag{4.9}$$

He does so by using Steenbrink's theorem (4.6) (1), valid without any restrictions $\sigma(f_0)$ [22].

Given $f\colon (C^n, \bar{0})\to(C, 0)$, choose coordinates $(x_1, \cdots, x_n)$ in a neighborhood of $\bar{0}$ so that $x_1=0$ defines a generic hyperplane for f in the sense of Teissier [23].

Define the two pairs of families of functions as follows.

I) Set $\alpha=\inf\{e_q/m_q\}$ and $A=[\![\alpha]\!]+1$.

Define $F_t(x_1, \cdots, x_n)=f(tx_1, x_2, \cdots, x_n)+x_1^A$ and $G_t(x_1, \cdots, x_n)=f(x_1, \cdots, x_n)+tx_1^A$.

II) Set $\beta=\sup\{e_q/m_q\}$ and $B=[\![\beta]\!]+1$.

Define $\tilde{F}_t(x_1, \cdots, x_n)=f(tx_1, x_2, \cdots, x_n)+x_1^B$ and $\tilde{G}_t(x_1, \cdots, x_n)=f(x_1, \cdots, x_n)+tx_1^B$.

(4.10) implies (cf. [13] for the proof of the equality part of (4.11.i))

(4.11) i) $\sigma(G_1)=\sigma(G_t)\geq\sigma(G_0)$ for $t\neq 0$
ii) $\sigma(\tilde{F}_1)=\sigma(\tilde{F}_t)\geq\sigma(\tilde{F}_0)$ for $t\neq 0$.

In addition, one also has by the invariance of the Arnol'd exponent in a μ-constant deformation [28] that

(4.12) i) $\sigma(F_1)=\sigma(F_t)=\sigma(F_0)$
ii) $\sigma(\tilde{G}_1)=\sigma(\tilde{G}_t)=\sigma(\tilde{G}_0)$,

as well as the equality in (4.11.ii). (The $\{\tilde{F}_t\}$, $t\neq 0$, form a μ-constant deformation [23])

Remark (4.13). Under the assumption that f does not define a rational singularity at $\bar{0}$, theorem (4.6) can be used to prove (4.9) in the same way.

To see this it suffices to observe the following.

Let H_1 be a generic plane of codimension 1 [23]. Set $f^{(1)}=f|_{H_1}$. Choose coordinates in U such that $x_1=0$ defines H_1 in U. Let B be the integer in II above. By the additivity of the Arnol'd exponent with respect to the Thom-Sebastiani operation one has

$$\sigma(f^{(1)}+x_1^B)=\sigma(f^{(1)})+\sigma(x_1^B)=\sigma(f^{(1)})+\frac{1}{B}.$$

(4.11) implies that

$$\sigma(f^{(1)})=\sigma(f^{(1)}+x_1^B)-\frac{1}{B}\leq\sigma(f^{(1)}+x_1^B)\leq\sigma(f).$$

Now let $0\subset H_{n-1}\subset H_{n-2}\subset\cdots\subset H_1$ be a generic flag in U of planes such that codim $H_i=i$. Here generic should mean that H_{i+1} is a generic hyperplane for the germ of $f^{(i)}=f|_{H_i}$ at the isolated singularity $\bar{0}_i$ in C^{n-1} in the sense of [23]. Let $x_1, \cdots, x_n$ be coordinates in U such that for each i, $U\cap H_i=\{x_1=\cdots=x_i=0\}$. Set $\bar{0}_i$ to be the origin in $U\cap H_i$.

Let $\alpha_i=\inf\{e_q^{(i)}/m_q^{(i)}\}$, $A_i=[\![\alpha_i]\!]+1$;
$\beta_i=\sup\{e_q^{(i)}/m_q^{(i)}\}$, $B_i=[\![\beta_i]\!]+1$.

Define for each i, the functions

$$F_t^{(i)}(x_i, \cdots, x_n)=f^{(i)}(tx_i, x_{i+1}, \cdots, x_n)+x_i^{A_i}$$
$$G_t^{(i)}(x_i, \cdots, x_n)=f^{(i)}(x_i, x_{i+1}, \cdots, x_n)+tx_i^{A_i}$$
$$\tilde{F}_t^{(i)}(x_i, \cdots, x_n)=f^{(i)}(tx_i, x_{i+1}, \cdots, x_n)+x_i^{B_i}$$
$$\tilde{G}_t^{(i)}(x_i, \cdots, x_n)=f^{(i)}(x_i, x_{i+1}, \cdots, x_n)+tx_i^{B_i}$$

Repeating the above reasoning shows that each $f^{(i)}$ defines a non-rational singular point at $\bar{0}_i$. Thus, one has the inequalities (4.11), (4.12) for each i. This gives (4.9) by a simple manipulation.

On the other hand the fact that (4.6) applies to exponents other than the smallest implies that (4.9) can be extended. Let $p(x)=x_1^{i_1}\cdots x_n^{i_n}$ be a monomial in the coordinates $(x_1, \cdots, x_n)$ defined in (4.13). Set $\omega=p(x)dx_1 \cdots dx_n$. If the exponent $\alpha(\omega)$ lies in $(-1, 0)$, one can use Varcenko's description of the Mixed Hodge Structure on the vanishing cohomology of Thom-Sebastiani polynomials [27] to estimate $\alpha(\omega)$ as follows.

Set $\omega_1=x_1^{i_1}dx_1$ and $\omega_2=x_2^{i_2}\cdots x_n^{i_n}dx_2\cdots dx_n$, so that $\omega=\omega_1\wedge\omega_2$. Then, when one considers ω to be a holomorphic section of the cohomology bundle for the function $f^{(1)}+x_1^{A_1}=F_0$ resp. of the cohomology bundle for the function $f^{(1)}+x_1^{B_1}=\tilde{F}_0$, one has that $\alpha(\omega)=\alpha(\omega_1)+\alpha(\omega_2)+1$ where $\alpha(\omega_1)$ equals $(i_1/A_1)-1$ resp. $(i_1/B_1)-1$, and $\alpha(\omega_2)$ is the exponent of the section $\omega_2/df^{(1)}$ for the cohomology bundle for the function $f^{(1)}$.

For $g\colon U\to C$ a Milnor fibration of a holomorphic function with isolated critical point at $\bar{0}$, denote by $\alpha_g(\omega)$ the value of the exponent associated to the section $[\omega/dg]$ of the cohomology bundle $\mathscr{H}_g^{n-1}$ for g. Then in the notation of I, II above, one deduces from (4.6) part 2 that

$$\alpha_{f^{(1)}+x_1^{B_1}}(\omega)+1\leq\alpha_f(\omega)+1\leq\alpha_{f^{(1)}+x_1^{A_1}}(\omega)+1, \tag{4.15.1}$$

that is,

$$\alpha_{f^{(1)}}(\omega_2)+1+\frac{i_1}{B_1}\leq\alpha_f(\omega)+1\leq\alpha_{f^{(1)}}(\omega_2)+1+\frac{i_1}{A_1}. \tag{4.15.2}$$

Thus, $\alpha_f(\omega)\leq 0$ implies $\alpha_{f^{(1)}}(\omega_2)\leq 0$.

Applying this $n-1$ times in the same way as (4.13), using [28] applied to the entire spectrum, not just the Arnol'd exponent, in the analogues of (4.11), (4.12), one has the

Theorem (4.16). *If* $\omega(x)=x_1^{i_1}\cdots x_n^{i_n}dx_1\cdots dx_n$ *is such that* $\alpha_f(\omega)\in(-1, 0)$, *then in the notation of* (4.13) *one has the pair of estimates*

$$\sum_{j=0}^{n-1}\frac{i_j}{B_j}-1\leq\alpha_f(\omega)\leq\sum_{j=0}^{n-1}\frac{i_j}{A_j}-1. \tag{4.17}$$

Remark (4.18). The basis of (4.6) and (4.16) lies in theorem (4.4) as well as (3.1). How can one extend (4.4) to exponents greater than zero, in the sense of identifying the value of an exponent from the resolution data of multiplicities? A similar question is the content of Remark (2.8). In that it is also not clear how to describe the poles β_ϕ from resolution data once β_ϕ becomes smaller than -1. Perhaps the best candidates for such an extension would be the "leading" exponents, studied by Loeser [14].

Section 5.

In this section we extend Igusa's theory of "Forms of Higher Degree" [6] to allow test functions which may vanish at the singularities of the form. We then apply (4.16) to obtain upper bound estimates for the leading poles in $(-1, 0)$ of the extended zeta function defined below. Such estimates as well as the precise formulae derivable from [11] in the case of two variables (cf. (5.11)), therefore have a number theoretic interest. Unfortunately, there is as yet no p-adic cohomological interpretation to these results.

We briefly recall the constructions at the base of [6].

Let K be a local field which for the purposes here is a finite algebraic extension of $\boldsymbol{Q}_p$. Let $f \in K[x_1 \cdots x_n]$. On the additive group K^n there is an "additive" Haar measure μ denoted $|dx_1| \cdots |dx_n| = |dx_1 \cdots dx_n|$ which is an n-fold product measure of the additive Haar measure. We normalize the measure by forcing $\mu(\boldsymbol{R}^n) = 1$ where $\boldsymbol{R}$ is the ring of integers in K. Let $\mathscr{S}(K^n)$ be the space of Schwartz-Bruhat functions on K^n. These are *complex*-valued functions which are locally constant with compact support. Compact refers to the metric topology on K^n defined via the sup norm on K^n. That is, $|(v_1, \cdots, v_n)|_K = \max |v_i|_K$ where $|v|_K$ is the unique extension to K of the standard p-adic norm on $\boldsymbol{Q}_p$

For $f \in K[x_1, \cdots, x_n]$, $\Phi \in \mathscr{S}(K^n)$ define

$$Z(s, \Phi) = \int_{K^n - \{f=0\}} |f(x_1, \cdots, x_n)|^s \Phi |dx|.$$

For $\mathrm{Re}(s) > 0$, $Z(s, \Phi)$ is analytic in s for each $\Phi \in \mathscr{S}(K^n)$, as can be seen easily. Adapting the idea of the proof when $K = \boldsymbol{R}$ or $\boldsymbol{C}$, Igusa showed that $Z(s, -)$ has an analytic continuation to $\boldsymbol{C}$ with poles at a finite set of negative rationals. Examples of the explicit continuation may be found in [12, 17].

The use of $\mathscr{S}(K^n)$ as the space of test functions is not absolutely necessary for Igusa's theory. One can extend the space to include norms of polynomials cut off by a function in $\mathscr{S}(K^n)$ to maintain the boundedness of the support of the test function.

Such a test function need no longer be locally constant in a neighborhood of a point on the zero locus of the polynomial. Its support in the metric topology, although bounded, need not be totally bounded. Thus, the support need not be compact, although it will be contained in a compact set.

So, define $\tilde{\mathscr{S}}(K^n)=\{\zeta\colon K^n\to\boldsymbol{C}\colon \zeta=\sum_{i=1}^{r}\phi_i\cdot|p_i(x)|$ where $\phi_i\in\mathscr{S}(K^n)$, $p_i\in K[x_1,\cdots,x_n]\}$.

Define a zeta function for $\zeta\in\tilde{\mathscr{S}}(K^n)$ by

$$\tilde{Z}(s,\zeta)=\int_{K^n-\{f\zeta=0\}}|f|^s\zeta|dx|.$$

Note that if ζ is in $\mathscr{S}(K^n)$, $K^n-\{f\zeta=0\}=K^n-\{f=0\}-\{\zeta=0\}=\operatorname{supp}(\zeta)-\{f=0\}$ is what one integrates over in any case and $\tilde{Z}(s,\zeta)=Z(s,\zeta)$, as defined by Igusa . In this way $\tilde{Z}$ is a generalization of the local zeta function of Igusa.

We show

Theorem (5.1). *The zeta function $\tilde{Z}(s,\zeta)$ admits an analytic Continuation to* $\boldsymbol{C}$ *with poles at finitely many negative rationals. In general, $\tilde{Z}(s,-)$ admits poles of arbitrarily large absolute value in Q_-. These poles lie in finitely many arithmetic progressions.*

Proof. Unlike the situation in [6], if $\pi\colon X\to K^n$ is a resolution of singularities of $\{f=0\}$, as embedded in K^n, the preimage π^{-1} (supp (ζ)) need not be compact. On the other hand, the analytic continuation of an integral of the form

$$\int_U|x_1^{M_1}\cdots x_n^{M_n}|^s|x_1|^{m_1}\cdots|x_n|^{m_n}|dx_1\cdots dx_n|,\tag{5.2}$$

where U is a compact open subset of $\mathscr{R}_K^{(n)}$ minus the locus $\{x_1\cdots x_n=0\}$, can always be explicitly determined by the evaluation of (5.2) as

$$\prod_{i=1}^n\int_{R-\{x_i=0\}}|x_i|^{M_is+m_i}|dx_i|.$$

One can use the resolution theorem to arrive at a local situation as in (5.2) by applying it not to the ideal sheaf $(f)\mathcal{O}_{K^n}$ but to $(fp)\mathcal{O}_{K^n}$ if $\zeta=\phi|p|$. Note that if $\zeta=\sum\phi_i|p_i|$, and if $\zeta_i=\phi_i|p_i|$, $\tilde{Z}(s,\zeta)=\sum\tilde{Z}(s,\zeta_i)$, so that the analytic continuation of $\tilde{Z}(s,\zeta)$ would be determined by that for the individual $\tilde{Z}(s,\zeta_i)$.

If $V=\operatorname{supp}(\phi)$, where $\zeta=\phi|p|$, V is compact. Let $\pi\colon X\to K^n$ be a proper birational morphism between K^n and a smooth K variety X which is an isomorphism off the singular locus of $\{fp=0\}$ and for which in a neighborhood (in the K-analytic topology) of a point z in $\pi^{-1}(\{fp=0\})$, there are K rational coordinates $y_1,\cdots,y_n$ such that (5.3) holds:

$$f\circ\pi(y_1,\cdots,y_n)=V_1(y)\Pi y_i^{M_i}\tag{5.3}$$

$$p\circ\pi(y_1,\cdots,y_n)=V_2(y)\Pi y_i^{r_i}$$
$$\det d\pi(y_1,\cdots,y_n)=V_3(y)\Pi y_i^{b_i}.$$

Since the V_i are units in the neighborhood of z, by shrinking the neighborhood we may assume that $|V_i(y)|=|V_i(z)|$ for all points $(y_1,\cdots,y_n)$ in the neighborhood and $i=1, 2, 3$.

Consider now $\pi^{-1}(V)$. Because it is compact, it can be covered by finitely many compact open affine discs each of the form $D_i=z_i+(\mathscr{P}^g)^{(n)}$ for some positive g. It is possible to find the z_i and g so that $D_i\cap D_j=\phi$ if $i\neq j$ [6]. In each D_i there are local K coordinates centered at z_i so that equations of the form given in (5.3) hold in D_i.

It is convenient to form the set of multiplicities of $f\circ\pi$, $p\circ\pi$, $\det d\pi$ along the divisors in $\pi^{-1}(fp^{-1}(0))$. Let $D_1,\cdots,D_M$ be the irreducible components of $\pi^{-1}(fp^{-1}(0))$.

Form the set

$$\text{(5.4)}\quad \mathscr{N}_\pi=\{(\text{mult}_{D_i}(f\circ\pi),\ \text{mult}_{D_i}(p\circ\pi),\ \text{mult}_{D_i}(\det d\pi)) : i=1,\cdots,M\}.$$

Evidently, (5.3) says that for each $i=1,\cdots,n$, the triples (M_i, r_i, b_i) are elements of $\mathscr{N}_\pi$.

Consider now $\tilde{Z}(s,\zeta)$: we then have

$$\text{(5.5)}\quad \tilde{Z}(s,\zeta)=\sum_{i=1}^{N} a_i|V_1(z_i)|^s|V_2(z_i)||V_3(z_i)| \cdot\int_{D_i}|y_1^{M_1}\cdots y_n^{M_n}|^s|y_1^{r_1}\cdots y_n^{r_n}||y_1^{b_1}\cdots y_n^{b_n}||dy_1\cdots dy_n|.$$

Thus the analytic continuation of the term

$$\int_{D_i}|y_1^{M_1}\cdots y_n^{M_n}|^s|y_1^{r_1}\cdots y_n^{r_n}||y_1^{b_1}\cdots y_n^{b_n}||dy_1\cdots dy_n|$$

can be determined very easily. Indeed, this term equals

$$\text{(5.6)}\quad \prod_{j=1}^{n}\int_{\mathscr{P}^g}|y_j^{M_j}|^s|y_j|^{b_j+r_j}|d^+y_j|=\sum_{j=1}^{n}(1-q)^n\cdot\frac{q^{-[(b_j+r_j+1)g+sgM_j]}}{1-q^{-[(b_j+r_j+1)+sM_j]}}$$

where $q=p^r$. Here, $p=\text{char}\ \mathscr{R}/\mathscr{P}$ and $r=[K;Q_p]$.

Each term admits poles at the n ratios

$$\text{(5.7)}\quad s_j=\frac{-(1+b_j+r_j)}{M_j},\qquad \text{if } M_j\neq 0.$$

If $M_j=0$, one understands the analytic continuation of the term to be possible to all of $\boldsymbol{C}$ in a trivial way.

Thus, $\tilde{Z}(s, \zeta)$ is a sum of terms each of which admits a pole at at most n negative rationals. This gives the desired analytic continuation for $\tilde{Z}(s, \zeta)$ to $\boldsymbol{C}$. Note too that one can force b_j to grow arbitrarily large simply by setting $p(x_1, \cdots, x_n)$ to equal f^v for $v=1, 2, \cdots$.

This proves theorem (5.1).

Remark (5.8). Consider one of the ratios s_0 in (5.7). Let $D_{i_1}, \cdots, D_{i_k}$ be those divisors in which the value s_0 arises as a fraction of the form $-(1+\tilde{b}_\ell+\tilde{r}_\ell)/\tilde{M}_\ell$ where $(\tilde{M}_\ell, \tilde{r}_\ell, \tilde{b}_\ell) \in \mathcal{N}_\pi$ (defined in (5.4)). Assume that s_0 satisfies the following property

(*) Let D_j be a divisor in $\pi^{-1}(fp^{-1}(0))$ and $D_j \cap D_{i_v} \neq \phi$ for some $v=1, \cdots, k$. Let (M_j, r_j, b_j) be the element of $\mathcal{N}_\pi$ corresponding to D_j. Then

$$s_0 > \frac{-(1+b_j+r_j)}{M_j}.$$

Such a ratio s_0 must be a pole of $\tilde{Z}(s, \zeta)$ if $\zeta > 0$ in $\operatorname{supp}(\zeta)$ and $\operatorname{supp}(\zeta)$ contains the singular locus of $\{fp=0\}$. For in this case the sign of the residue of any term of form (5.6), forming the contribution to $\operatorname{Res}_{s=s_0} \tilde{Z}(s, \zeta)$ in a neighborhood of a point which either lies in or intersects some D_{i_v}, must be positive. So, the total value of $\operatorname{Res}_{s=s_0} \tilde{Z}(s, \zeta)$ is just a finite sum of positive quantities and cannot be zero.

Let us agree to consider in the following only those ζ whose support satisfies the condition $\{fp=0\}_{sg} \subset \operatorname{supp}(\zeta)$.

Remark (5.9). For given $\zeta = \phi|p|$, $p \in K[x_1, \cdots, x_n]$, set $\beta_\zeta = \sup\{-(1+r_i+b_i)/M_i\colon (M_i, r_i, b_i)$ is an element of the set of multiplicity triples $\mathcal{N}_\pi$ obtained by constructing a resolution defined over K, $\pi\colon X \to K^n$, so that both $(f \circ \pi)$ and $(p \circ \pi)$ are locally in normal crossing form$\}$. We then have that if ϕ has constant sign on $V = \operatorname{supp}(\phi)$ and V contains the singular locus of $\{fp=0\}$ then β_ζ is the largest pole of $\tilde{Z}(s, \zeta)$ with the sign of the residue at $s=\beta_\zeta$ given by the sign of ϕ. The value β_ζ depends only on p then. We henceforth denote it by β_p.

The value of β_p depends, a priori on the field K. This is because the set of divisors, rational over K and used to determine the set of ratios, the maximum of which is β_p, clearly may change if one works over an extension of K. To emphasize the dependence on K, denote the ratio as $\beta_p(K)$.

On the other hand, there is a finite extension L of K with the property that if L' is any finite extension of L then for $p \in K[x_1, \cdots, x_n]$, $\beta_p(L') = \beta_p(L)$. One sees this by looking over $\bar{K}$.

Any resolution $\pi\colon X \to \bar{K}^n$ obtained as in Theorem (5.1) is determined as a composition of birational morphisms. Hence, π is defined over a finite extension, say L, of K obtained by adjoining to K all the coefficients used in the finitely many rational maps comprising π. Moreover, X itself is rational over L if we include the coefficients of all defining equations (polynomials over $\bar{K}$) used to define X in its various affine neighborhoods. Thus, π descends to a resolution $\pi_L: X_L \to L^n$ of $f \cdot p$ as discussed above. The multiplicity data forming $\mathcal{N}_L$ as determined by π_L is therefore the same as the set $\mathcal{N}_\pi$ as determined by π.

Clearly, one has $\beta_p(K) \leq \beta_p(L)$.

Now, over K or L one can also define the quasi-adjoint characters at a singular point ξ of $\{f=0\}$ exactly as was done in section (2). Let $\kappa_p(\xi, K)$ resp. $\kappa_p(\xi, L)$ be these characters for the polynomial p.

If $\bar{0}$ is an isolated singular point of $\{f=0\}$, then one has $\beta_p(K) \leq \beta_p(L) = \kappa_p(\bar{0}, L) - 1$ if $\kappa_p(\bar{0}, L) \in (0, 1)$. This is because adjointness is a property defined purely algebraically. Thus, to evaluate $\kappa_p(\bar{0}, L)$, one notes first that $\kappa_p(\bar{0}, L) = \kappa_p(\bar{0}, \bar{K})$. Then observe that $\kappa_p(\bar{0}, \bar{K}) = \kappa_p(\bar{0}, \boldsymbol{C})$ because one can work within $\bar{K}$ or within $\boldsymbol{C}$ by "choosing" an embedding $K \hookrightarrow \boldsymbol{C}$.

Because $\bar{0}$ is an isolated singular point in the complex hypersurface $\{f=0\}$, one now obtains an upper bound estimate for $\beta_p(K)$ in terms of the polar invariant data from section (4). Observe that this data is independent of the embedding $K \hookrightarrow \boldsymbol{C}$. Indeed, if $p(x) = x_1^{i_1} \cdots x_n^{i_n}$, then if $\kappa_p(\bar{0}, \boldsymbol{C}) \in (0, 1)$, one has $\kappa_p(\bar{0}, \boldsymbol{C}) = 1 + \beta_p(\boldsymbol{C})$, with $\beta_p(\boldsymbol{C})$ the largest pole of $I_p(s)$ (2.6). One also knows that when $\beta_p(\boldsymbol{C}) \in (-1, 0)$ then $\beta_p(\boldsymbol{C}) = -(\alpha_f(p dx) + 1)$. From (4.16), one has under the assumption imposed there the inequalities stated in (4.17). Thus, one concludes

$$\beta_p(K) \leq \beta_p(L) = \beta_p(\boldsymbol{C}) \leq -\sum_{j=0}^{n-1} \frac{i_j}{\boldsymbol{B}_j}. \tag{5.10}$$

Loeser has also observed (5.10) independently [15].

Remark (5.11). If f defines a germ of an irreducible plane curve at $\bar{0}$ in $\bar{K}^2$ with characteristic sequence $(n, \beta_1, \cdots, \beta_g)$, one can show that for a monomial $p(x, y)$ in coordinates centered at $\bar{0}$, the quantities β_p in $(-1, 0)$ are independent of the intermediate field L and depend only upon K. Moreover, the values of certain β_p are computed in [11]. Indeed, from the analysis in [11], some of the smaller poles of

$$\langle |f|^s, \phi |p| \rangle$$

can also be determined. In this way, the results of [17] are extended.

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