Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 165–198

Isolated Q-Gorenstein Singularities of Dimension Three

Shihoko Ishii

Introduction

Let (X, x) be a germ of an analytic space with isolated singularity at x. The symbol X will also denote a sufficiently small Stein neighbourhood of the germ (X, x).

Using the plurigenera $\{\delta_m\}_{m\in N}$, which where introduced by Watanabe [W1], we define an invariant κ_* for a singularity (X, x) as follows,

$$\kappa_*(X, x) = -\infty, \quad \text{if } \delta_m(X, x) = 0 \text{ for every } m \in N, \text{ and}$$

$$\kappa_*(X, x) = s, \quad \text{if } \sup_{k\leq m} \delta_k(X, x) \text{ grows in order } s \text{ as a function of } m.$$

For a normal isolated Q-Gorenstein singularity (X, x), $\kappa_*(X, x)$ turns out to be $-\infty$, 0 or dim X ([W1]). In some sense, we may think of singularities (X, x) satisfying $\kappa_*(X, x) = -\infty$ as almost non-singular points. These singularities include canonical singularities. So it is natural for us to study singularities (X, x) with $\kappa_*(X, x) = 0$. Two dimensional normal Q-Gorenstein singularities with $\kappa_* = 0$ were completely determined by Watanabe ([W1]) and Tsunoda-Miyanishi ([T-M]) independently.

Higher dimensional isolated Gorenstein singularities with $\kappa_* = 0$ were studied in [I]. Then, Watanabe extended this discussion to the case of isolated quasi-Gorenstein singularities ([W2]).

In this paper, we will investigate *higher-dimensional* (espesially, 3-*dimensional*, in detail) *normal isolated Q-Gorenstein singularities with* $\kappa_* = 0$.

In Section 1, we summarize the notation, definitions and basic facts which will be used in this paper.

In Section 2, we will show that, for a normal isolated Q-Gorenstein singularity (X, x) of dimension $n \geq 2$, it is log-canonical if and only if $\kappa_*(X, x) \leq 0$. Moreover, it will be shown that the singularity (X, x) with $\kappa_*(X, x) \leq 0$ satisfies the condition that the canonical map $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^i(E, \mathcal{O}_E)$ is an isomorphism for every $i > 0$, where $f: \tilde{X} \to X$ is a good resolution of the singularity (X, x); i.e. f is a resolution with the divisor

Received February 4, 1985.
Revised March 1, 1986.
Partially supported by JSPS fellowship.

$E=f^{-1}(x)_{\text{red}}$ simple normal crossings (the singularity with this condition was named a Du Bois singularity by Steenbrink [S]).

In Section 3, we introduce the concept "essential part" of the exceptional divisor of a good resolution and investigate the basic properties of the essential part.

In Section 4, we will study minimal resolutions of higher dimensional $\boldsymbol{Q}$-Gorenstein singularities. Of course, to define a minimal resolution, we have to allow some kinds of mild singularities on a resolved space ([R2]).

In and after Section 5, we restrict our consideration to the three dimensional case. In Section 5, under a certain assumption, we study the configuration of essential part of a 3-dimensional normal $\boldsymbol{Q}$-Gorenstein singularity with $\kappa_*=0$. The configurations which may occur in the essential part is listed in Table (1) at the last of Section 5.

In Section 6, we construct a three-dimensional normal isolated $\boldsymbol{Q}$-Gorenstein singularity with $\kappa_*=0$ in each class listed in Table (1).

§ Preliminaries

Let (X, x) be a germ of normal isolated singularity of an analytic space of dimension $n\geq 2$.

Definition 1.1. A resolution $f\colon \tilde{X}\to X$ of the singularity is called a good resolution if the support of the inverse image of the singular point is a divisor with simple normal crossings on $\tilde{X}$.

Definition 1.2. A singularity (X, x) is called a Gorenstein singularity if the local ring $\mathcal{O}_{X,x}$ is a Gorenstein ring.

The following is well known.

Proposition 1.3. *For a singularity (X, x) of dimension $n\geq 2$, the following conditions are equivalent:*

(1) *The singularity (X, x) is a Gorenstein singularity,*

(2) *the canonical sheaf ω_X is invertible at x and the local ring $\mathcal{O}_{X,x}$ is a Cohen-Macaulay ring.*

We generalize the concept of Gorenstein singularity.

Definition 1.4. A normal singularity (X, x) is called a $\boldsymbol{Q}$-Gorenstein singularity if there exists an integer $r>0$ such that $\omega_X^{[r]}=i_*(\omega_{X_{\text{reg}}}^{\otimes r})$ is invertible at x, where X_{reg} is the open subspace of X consisting of all non-singular points and $i\colon X_{\text{reg}}\to X$ is the inclusion. For a $\boldsymbol{Q}$-Gorenstein singularity (X, x), the minimal positive integer r such that $\omega_X^{[r]}$ is invertible is called the index of (X, x). We frequently use the terminology "r-Gorenstein singularity" for a $\boldsymbol{Q}$-Gorenstein singularity of index r.

Remark 1.5. A 1-Gorenstein singularity is not Gorenstein! A 1-Gorenstein singularity is called a quasi-Gorenstein singularity in [W2].

Definition 1.6. An r-Gorenstein singularity (X, x) is said to be a canonical (resp. terminal) singularity if the following condition is satisfied:

For a good resolution $f: \tilde{X} \to X$ of the singularity, we have in Div $(\tilde{X}) \otimes \boldsymbol{Q}$,

$$K_{\tilde{X}} = f^*(K_X) + \sum a_i E_i \tag{1}$$

with $a_i \geq 0$ (resp. $a_i > 0$) for all i, where E_i are the irreducible components of the exceptional divisor E.

The singularity (X, x) is called a log-canonical (resp. log-terminal) singularity, if $a_i \geq -1$ (resp. $a_i > -1$) for all i in the equality (1).

Proposition 1.7. *Suppose φ: $(Y, y) \to (X, x)$ is a proper morphism with X and Y normal varieties and φ is étale in codimension* 1 *on Y.*

Then

(I) *if (X, x) is canonical (resp. log-canonical, terminal, log-terminal), so is (Y, y) and*

(II) *if (Y, y) is log-canonical (resp. log-terminal), so is (X, x).*

Proof. This is an easy consequence of the logarithmic ramification formula (Lemma 1.6, [Ka2]). For later use, we present a proof here.

Take a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\varphi]{} & X \end{array}$$

with f and g good resolutions. Let $D \subset \tilde{Y}$ be an irreducible component of the exceptional divisor for g: $\tilde{Y} \to Y$.

First, assume $\tilde{\varphi}(D)$ is a divisor on $\tilde{X}$. If we write down canonical divisors $K_{\tilde{Y}}$ and $K_{\tilde{X}}$ as

$$K_{\tilde{Y}} = g^*K_Y + aD + (\text{other terms})$$
$$K_{\tilde{X}} = f^*K_X + b\tilde{\varphi}(D) + (\text{other terms}),$$

then we have $a = e(b+1) - 1$, where e is the ramification index of $\tilde{\varphi}$ at D. Now, we have the following implications: $a \geq -1 \leftrightarrow b \geq -1$, $a > -1 \leftrightarrow b > -1$, $b \geq 0 \to a \geq 0$ and $b > 0 \to a > 0$.

Next, for the divisor D mapped to a lower dimensional subvariety by $\tilde{\varphi}$, we have $a \geq -1$ (resp. $a > -1$), if X is log-canonical (resp. log-terminal) by log. ramification formula. This completes the proof.

For an r-Gorenstein singularity (X, x), by means of a local generator, we identify $\omega_X^{[r]}$ with $\mathcal{O}_X$, and define in the obvious way an algebraic structure on $A=\mathcal{O}_X \oplus \omega_X^{[1]} \oplus \cdots \oplus \omega_X^{[r-1]}$. The finite cyclic cover $Y=\mathrm{Spec}_X A \to X$ is called a canonical cover of X. Then, Y is a normal and 1-Gorenstein and the morphism is étale outside the singularity ([R1]). By Proposition 1.7, some arguments on an r-Gorenstein singularity will be reduced to the case of 1-Gorenstein.

In the following, we fix a good resolution $f\colon \tilde{X} \to X$ of the singularity (X, x). Denote $f^{-1}(x)_{\mathrm{red}}$ by E.

Definition 1.8. For a normal isolated singularity (X, x), we define the plurigenera $\{\delta_m\}_{m \in N}$ by

$$\delta_m(X, x)=\dim_C \Gamma(X-\{x\}, \mathcal{O}(mK))/L^{2/m}(X-\{x\}),$$

where $L^{2/m}(X-\{x\})$ denotes the set of all $L^{2/m}$-integrable m-ple holomorphic n-forms on $X-\{x\}$.

Proposition 1.9. (Watanabe [W1]). *The plurigenus $\delta_m(X, x)$ is represented as*

$$\delta_m(X, x)=\dim_C \Gamma(\tilde{X}-E, \mathcal{O}(mK))/\Gamma(\tilde{X}, \mathcal{O}(mK+(m-1)E)).$$

The following is proved in the same way as Theorem 1.13 an Example 1.14 of [W1].

Proposition 1.10 (Watanabe [W1]). *For an r-Gorenstein singularity* (X, x), either

(I) $\delta_m(X, x)=0$ *for every* $m \in N$,

(II) $\delta_m(X, x)=1$ *for* $m \equiv 0 \pmod r$ *and* $\delta_m(X, x)=0$ *for* $m \not\equiv 0 \pmod r$, *or*

(III) $\delta_{rm}(X, x)$ *grows in order n as a function of m.*

We will close this section by a proposition about a Du Bois singularity. A Du Bois singularity is characterized as a singularity such that the natural maps $(R^i f_* \mathcal{O}_{\tilde{X}})_x \to H^i(E, \mathcal{O}_E)$ are isomorphisms for all $i>0$.

Proposition 1.11. *Let (X, x) be a normal isolated singularity and let $\pi\colon Y \to X$ be a finite Galois cover étale on $X-\{x\}$ with Y normal.*

Then, Y is Du Bois, then so is (X, x).

Proof. Let S be the set of points of Y which correspond to x by the morphism π. Denote the Galois group by G. Take a commutative diagram,

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

where f and g are good resolutions, G acts on $\tilde{Y}$, and the morphisms g and $\tilde{\pi}$ are G-equivariant with the trivial action on $\tilde{X}$. Here, the existence of such a diagram follows from Theorem 5.3.1 of [H2]. Denote $f^{-1}(x)_{\text{red}}$, $g^{-1}(S)_{\text{red}}$ by E and F respectively. Since Y is Du Bois, $R^i g_* \mathcal{O}_{\tilde{Y}} \simeq H^i(F, \mathcal{O}_F)$ for $i>0$. So $R^i g_* I_F = 0$ for $i>0$, where I_F is the ideal sheaf of F in $\mathcal{O}_{\tilde{Y}}$. Then $R^i f_* I_E = (R^i g_* I_F)^G = 0$ for $i>0$. This means that $R^i f_* \mathcal{O}_{\tilde{X}} \simeq H^i(E, \mathcal{O}_E)$ for $i>0$.

§ 2. Log-canonical singularities

In the following sections, (X, x) will denote a normal isolated r-Gorenstein singularity of dimension $n \geq 2$. We fix a good resolution $f: \tilde{X} \to X$ of the singularity (X, x). We denote $f^{-1}(x)_{\text{red}}$ by E and decompose E into irreducible components E_i $(i=1, 2, \cdots, s)$.

Theorem 2.1. *For a normal isolated r-Gorenstein singularity (X, x), the following equivalences hold;*

(i) *the singularity (X, x) is log-canonical if and only if $\delta_m(X, x) \leq 1$ for every $m \in \mathbf{N}$, and*

(ii) *the singularity (X, x) is log-terminal if and only if $\delta_m(X, x) = 0$ for every $m \in \mathbf{N}$.*

Proof. Denote a canonical divisor by $K_{\tilde{X}} = f^* K_X + \sum_{i=1}^{s} m_i E_i$, in $\mathbf{Q} \otimes \text{Div}(\tilde{X})$.

First assume $m_i \geq -1$ for every i. Then, for a positive integer m which can be divided by r, we get

$$f^*(mK_X) - E \leq mK_{\tilde{X}} + (m-1)E.$$

Therefore,

$$\Gamma(\tilde{X}, \mathcal{O}(f^*(mK_X) - E)) \subset \Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}} + (m-1)E)). \tag{1}$$

On the other hand,

$$\Gamma(\tilde{X}, \mathcal{O}(f^*(mK_X))) = \Gamma(\tilde{X} - E, \mathcal{O}(mK_{\tilde{X}})). \tag{2}$$

The equality (2) and the relation (1) yield the surjective map:

$$\begin{aligned} &\Gamma(\tilde{X}, \mathcal{O}(f^*(mK_X)))/\Gamma(\tilde{X}, \mathcal{O}(f^*(mK_X) - E)) \\ &\qquad \longrightarrow \Gamma(\tilde{X} - E, \mathcal{O}(mK_X))/\Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}} + (m-1)E)). \end{aligned}$$

Here, note that the left hand side is contained in $\Gamma(E, \mathcal{O}_E) \simeq \boldsymbol{C}$ by the exact sequence

$$0 \longrightarrow \mathcal{O}(f^*(mK_X)-E) \longrightarrow \mathcal{O}(f^*(mK_{\tilde{X}})) \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

This means that $\delta_m(X, x) \leq 1$ for every m which can be divided by r. Hence $\delta_m(X, x) \leq 1$ for every $m \in \boldsymbol{N}$.

Next assume $m_i > -1$ for every i. Then,

$$f^*(mK_X) \leq mK_{\tilde{X}} + (m-1)E$$

for any m which can be devided by r. Therefore, $\Gamma(\tilde{X}-E, \mathcal{O}(mK_{\tilde{X}})) = \Gamma(\tilde{X}, \mathcal{O}(f^*(mK_X))) \subseteq \Gamma(\tilde{X}, \mathcal{O}(mK_{\tilde{X}}+(m-1)E))$, which means that $\delta_m(X, x) = 0$ for any m which can be divided by r and, obviously for every $m \in \boldsymbol{N}$.

Conversely, assume that $\delta_m(X, x) \leq 1$ for every $m \in \boldsymbol{N}$. If $\delta_r(X, x) = 0$, then $\delta_{rm}(X, x) = 0$ for every $m \in \boldsymbol{N}$ (Theorem 1.6 [W1]). In this case, we have $\Gamma(\tilde{X}-E, \mathcal{O}(rK_{\tilde{X}})) = \Gamma(\tilde{X}, \mathcal{O}(rK_{\tilde{X}}+(r-1)E))$. This means that any r-ple holomorphic n-form θ on X, $f^*\theta$ has poles on E of order at most $r-1$. Therefore, $m_i \geq -(r-1)/r > -1$.

If $\delta_r(X, x) = 1$, then

$$\Gamma(\tilde{X}-E, \mathcal{O}(mrK_{\tilde{X}})) = \langle \theta^m \rangle \oplus \Gamma(\tilde{X}, \mathcal{O}(mrK_{\tilde{X}}+(mr-1)E)),$$

where θ is a r-ple meromorphic n-form which is holomorphic on $\tilde{X}-E$. The r-ple meromorphic n-form θ must have poles of order r on E. In fact, for a holomorphic n-form g, $\theta^{m-1}g \in \Gamma(\tilde{X}, \mathcal{O}(mrK_{\tilde{X}}+(mr-1)E))$ for any $m \geq 1$. So g has zeros of order at least $s(m-1)-mr+1$, where s is the order of poles of θ at E. Therefore s must be r. Now we have $rm_i \geq -r$ for any i and the equality holds for some i. This completes the proof.

Definition 2.2. A $\boldsymbol{Q}$-Gorenstein singularity (X, x) with $\delta_m(X, x) \leq 1$ for every $m \in \boldsymbol{N}$ and $\delta_m(X, x) = 1$ for some m is called a periodically elliptic singularity. In particular, a singularity (X, x) with $\delta_m(X, x) = 1$ for every $m \in \boldsymbol{N}$ is called a purely elliptic singularity.

Note that a singularity is periodically elliptic if and only if it is log-canonical and not log-terminal, by Theorem 2.1. For a periodically elliptic singularity (X, x) of index r, $\delta_m(X, x) = 0$ for $m \not\equiv 0 \pmod r$ and $\delta_m(X, x) = 1$ for $m \equiv 0 \pmod r$.

Proposition 2.3 ([I]), [W2]). *A* 1-*Gorenstein singularity* (X, x) *is log-canonical if and only if it is a Du Bois singularity.*

Proof. If the singularity (X, x) is Gorenstein, the equivalence of "log-canonical" and "Du Bois" was proved in [I], and this was extended to the case of 1-Goresntein in [W2].

Theorem 2.4. *If a normal isolated singularity (X, x) is log-canonical, then it is a Du Bois singularity.*

Proof. Let $\pi: Y \to X$ be the canonical cover of (X, x). Then all singularities of Y are isolated 1-Gorenstein and log-canonical by Proposition 1.7. Then Y is Du Bois by Proposition 2.3. Consequently, we observe that (X, x) is Du Bois by Proposition 1.11.

Remark 2.5. For a general Q-Gorenstein singularity (X, x), the converse of Theorem 2.4 does not hold. In fact, a rational two dimensional singularity is always Q-Gorenstein ([A1]) and Du Bois ([S]), but there is a rational two-dimensional singularity which is not a log-canonical singularity.

§ 3. A good resolution of normal isolated Q-Gorenstein singularities of dimension $n \geq 2$

In this section, we consider the essential part of a good resolution of a normal isolated Q-Gorenstein singularity (X, x). First, we introduce the concept "essential part" of the exceptional divisor which, in fact, plays an essential role in the exceptional divisor.

Definition 3.1. Let $f: \tilde{X} \to X$ be a good resolution of a singularity (X, x). We denote $f^{-1}(x)_{\mathrm{red}}$ by E and decompose E into irreducible components E_i $(i=1, 2, \cdots, s)$. We write a canonical divisor of $\tilde{X}$ in $Q \otimes \mathrm{Div}(\tilde{X})$ as

$$K_{\tilde{X}} = f^* K_X + \sum_{i \in I} m_i E_i - \sum_{j \in J} m_j E_j,$$

where $m_i \geq 0$ for any $i \in I$ and $m_j > 0$ for any $j \in J$. We define the essential divisor of $K_{\tilde{X}}$ to be the divisor $\sum_{j \in J} [m_j] E_j$ and denote it by E_J. We call the reduced divisor $E_{J\,\mathrm{red}}$ the essential part of E. A component of $E_{J\,\mathrm{red}}$ is called an essential component of E.

Remark 3.2. If a singularity (X, x) is log-terminal, the essential divisor of $K_{\tilde{X}}$ is empty. If a singularity (X, x) is periodically elliptic (i.e., log-canonical but not log-terminal), then the essential divisor is reduced.

The following is proved as Proposition 3.7 in [I].

Proposition 3.3. *Let $f: \tilde{X} \to X$ be a good resolution of a singularity (X, x), E_J be the essential divisor of $K_{\tilde{X}}$. Then,*

(1) *$h^{n-1}(D, \mathcal{O}_D) = p_g(X, x)$ for any integral divisor D with $D \geq E_J$,*

(2) *$h^{n-1}(D, \mathcal{O}_D) = 0$ for any effective divisor D which does not have a common component of E_J.*

Lemma 3.4. *Let (X, x) be an r-Gorenstein singularity with $r>1$ and let $\pi: Y \to X$ be the canonical covering.*

Then the set $\pi^{-1}(x)_{\text{red}}$ consists of only one point.

Proof. Take a point $y \in \pi^{-1}(x)_{\text{red}}$ and the subgroup H of $\mathbf{Z}/(r)$ consisting of the elements which fix the point y. Then the singularity (Y, y) is 1-Gorenstein and $(Y, y)/H = (X, x)$. Now, denote the morphism $(Y, y) \to (X, x)$ by π'. Noting that π' is étale outside of x, we get $\pi'_*\mathcal{O}(K_Y) = \mathcal{O}(K_X) \otimes \pi'_*\mathcal{O}_Y$ on $X - \{x\}$, which is isomorphic to $\pi'_*\mathcal{O}_Y$ on $X - \{x\}$ since (Y, y) is 1-Gorenstein. Therefore, $\wedge^s(\mathcal{O}(K_X) \otimes \pi'_*\mathcal{O}_Y) \simeq \wedge^s \pi'_*\mathcal{O}_Y$ on $X - \{x\}$, where s is the order of H. By tensoring the invertible sheaf $(\wedge^s \pi'_*\mathcal{O}_Y)^{-1}$ on the both sides of the equality, we have $\mathcal{O}(sK_X) \simeq \mathcal{O}_X$ on $X - \{x\}$. The definition of the index of $\mathbf{Q}$-Gorenstein singularity yields that s must be r, which means $H = \mathbf{Z}/(r)$.

Lemma 3.5. *Let (X, x) be an r-Gorenstein singularity with $r>1$. Take a commutative diagram*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

where f and g are good resolutions, $\tilde{\pi}$ is the canonical covering.

Then the essential part of $g^{-1}\pi^{-1}(x)_{\text{red}}$ is mapped generically finitely onto the essential part of $f^{-1}(x)_{\text{red}}$ by the morphism $\tilde{\pi}$.

Proof. Let E_i be an irreducible component of the exceptional divisor for $f: \tilde{X} \to X$ and F_i be one of the irreducible components of the exceptional divisor for $g: \tilde{Y} \to Y$ which correspond to E_i by $\tilde{\pi}$.

If we write down canonical divisors $K_{\tilde{Y}}$ and $K_{\tilde{X}}$ as follows;

$$K_{\tilde{Y}} = g^*K_Y + aF_i + (\text{other terms})$$
$$K_{\tilde{X}} = f^*K_X + bE_i + (\text{other terms}),$$

then $a \leq -1$ if and only if $b \leq -1$ by the proof of Proposition 1.7. This completes the proof.

Proposition 3.6. *Assume a singularity (X, x) is periodically elliptic. For a good resolution $f: \tilde{X} \to X$ of the singularity (X, x), the essential part E_J of $E = f^{-1}(x)_{\text{red}}$ is connected.*

Proof. First, consider the case where (X, x) is of index one. Then $p_g(X, x) = h^{n-1}(E_J, \mathcal{O}_{E_J}) = 1$. Therefore, we can apply the argument for the proof of Corollary 3.9 of [I]. Consequently we find that E_J is connected.

For a general r-Gorenstein singularity (X, x), take the canonical cover $\pi\colon Y \to X$. Then, by the above argument, Lemma 3.4 and Lemma 3.5, we obtain that E_J is also connected.

Proposition 3.7. *Let (X, x) be a periodically elliptic singularity of index r. If $r>1$ (resp, $r=1$), a reduced effective divisor E' with $E' \leq E_J$ (resp. $E'<E_J$) satisfies $H^{n-1}(E', \mathcal{O}_{E'})=0$.*

Proof. For $r>0$, $p_g(X, x)=h^{n-1}(E_J, \mathcal{O}_{E_J})=0$. Therefore $h^{n-1}(E', \mathcal{O}_{E'})=0$ for $E' \leq E_J$. If $r=1$, then $1=p_g(X, x)>h^{n-1}(E', \mathcal{O}_{E'})$ for $E'<E_J$ by Proposition 3.8 of [I]. This completes the proof.

Lemma 3.8. *Let $\pi\colon (Y, y) \to (X, x)$ be a finite Galois covering of germs of periodically elliptic singularities with the Galois group G, which is étale outside the singularities. Let $g\colon \tilde{Y} \to Y$ be a G-equivariant good resolution of the singularity (Y, y) with the essential part F.*

Assume the index r of the singularity (X, x) is greater than one. Then $H^{n-1}(F, \mathcal{O}_F)^G=0$.

Proof. Since $H^{n-1}(F, \mathcal{O}_F)=0$ for a periodically elliptic singularity (Y, y) of index greater than one (cf. Proposition 3.7), it is sufficient to show the assertion of the lemma for a purely elliptic singularity (Y, y).

We may assume that the morphism g appears on the following commutative diagram;

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

where f and g are good resolutions, G acts on $\tilde{Y}$, and the morphisms g and $\tilde{\pi}$ are G-equivariant with the trivial action on $\tilde{X}$. Denote the essential part of $f^{-1}(x)_{\mathrm{red}}$ by E. Then the morphism $\pi|_F$ is decomposed as $F \xrightarrow{\varphi} F/G \xrightarrow{\psi} E$, where φ is a finite Galois covering and ψ induces an isomorphism between Zariski open subsets $F/G-D \simeq E-S$.

Then we have the exact sequence of Hodge structures;

$$H^{n-1}(E, \boldsymbol{C}) \longrightarrow H^{n-1}(F/G, \boldsymbol{C}) \oplus H^{n-1}(S, \boldsymbol{C}) \longrightarrow H^{n-1}(D, \boldsymbol{C}).$$

By taking Gr_F^0, we obtain

$$(1) \qquad \mathrm{Gr}_F^0 H^{n-1}(E) \longrightarrow \mathrm{Gr}_F^0 H^{n-1}(F/G) \oplus \mathrm{Gr}_F^0 H^{n-1}(S) \longrightarrow \mathrm{Gr}_F^0 H^{n-1}(D).$$

Here, noting that E is a Du Bois variety, we have $\mathrm{Gr}_F^0 H^{n-1}(E) \simeq H^{n-1}(E, \mathcal{O}_E)$, which is zero because the singularity (X, x) is periodically elliptic of index $r>1$. On the other hand, $\mathrm{Gr}_F^0 H^{n-1}(S)=0$, since S is a proper variety of dimension $<n-1$. Moreover, it is checked that $\mathrm{Gr}_F^0 H^{n-1}(D)$ also vanishes. In fact, $\mathrm{Gr}_F^0 H^{n-1}(D)$ is contained in $\mathrm{Gr}_F^0 H^{n-1}(\varphi^{-1}(D))$ which is isomorphic to $H^{n-1}(\tilde{D}, \mathcal{O}_{\tilde{D}})$, where $\tilde{D}$ is the maximal reduced divisor contained in $\varphi^{-1}(D)$. However, we know that $H^{n-1}(\tilde{D}, \mathcal{O}_{\tilde{D}})=0$ since $\tilde{D} \subsetneqq F$ (cf. Proposition 3.7).

Now, by the exact sequence (1), we get $\mathrm{Gr}_F^0 H^{n-1}(F/G)=0$.

Next, consider the commutative diagram;

$$\begin{array}{ccc} H^{n-1}(F/G, \mathcal{O}_{F/G}) & = & H^{n-1}(F, \mathcal{O}_F)^G \\ \rho\downarrow & & \downarrow\rho^G \\ 0 = \mathrm{Gr}_F^0 H^{n-1}(F/G) & \longrightarrow & \{\mathrm{Gr}_F^0 H^{n-1}(F)\}^G \end{array}$$

where ρ and ρ^G are induced from natural morphisms $\Omega^{\cdot} \to \underline{\Omega}^{\cdot}$ from the stupid filtered De Rham complex to the Du Bois' filtered complex on F/G and F respectively.

Here, we note that ρ^G is an isomorphism since F is a Du Bois variety. Therefore ρ is injective. Consequently, we obtain $H^{n-1}(F, \mathcal{O}_F)^G=0$ as desired.

In the rest of this section, we will define the types of periodically elliptic singularities.

Let $f: \tilde{X} \to X$ be a good resolution of a purely elliptic singularity (X, x), and let E_J be the essential divisor of f.

Then, since E_J is a complete variety with simple normal crossings,

$$H^{n-1}(E_J, \mathcal{O}_{E_J}) \simeq \mathrm{Gr}_F^0 H^{n-1}(E_J) = \bigoplus_{i=0}^{n-1} H_{n-1}^{0,i}(E_J),$$

where $H_m^{i,j}(*)$ is the (i, j) component of $\mathrm{Gr}_{i+j}^W H^m(*)$. Because the left hand side is a one-dimensional C-vector space, it must coincide with one of $H_{n-1}^{0,i}(E_J)$ $(i=0, 1, \cdots, n-1)$.

Definition 3.9. A purely elliptic singularity (X, x) is of type $(0, i)$ $(i= 0, 1, 2, \cdots, n-1)$ if $H^{n-1}(E_J, \mathcal{O}_{E_J})$ consists of the $(0, i)$-Hodge-component. A periodically elliptic singularity (X, x) is called of type $(0, i)$, if the purely elliptic singularity on the canonical cover of (X, x) is of type $(0, i)$.

Note that a purely elliptic singularity (X, x) is of type $(0, i)$ if and only if $H^{n-1}(f^{-1}(x)_{\mathrm{red}}, \mathcal{O})$ consists of the $(0, i)$-Hodge-component, since this is canonically isomorphic to $H^{n-1}(E_J, \mathcal{O}_{E_J})$.

Here, we note that the type of a purely elliptic singularity (X, x) defined above is independent of the choice of a good resolution $f: \tilde{X} \to X$

(Proposition 4.2, [I]).

Theorem 3.10. *For a periodically elliptic singularity* (X, x) *of type* $(0, 0)$, *the index of* (X, x) *is either one or two.*

Proof. Assuming the index $r>1$, we will show that r must be two. For a periodically elliptic singularity (X, x) of type $(0, 0)$ of index $r>1$, take the canonical cover $\pi: (Y, y)\to(X, x)$. Let $G=\langle\lambda\rangle$ be the Galois group of π. In what follows, we will use the same notation $\tilde{X}$, $\tilde{Y}$, f, g, $\tilde{\pi}$, F, E, etc. as in the proof of Lemma 3.8. Denote the dual graph of F by Γ. Then an automorphism λ on F induces an automorphism on $H^{n-1}(\Gamma, \boldsymbol{Z})$. Here, remember that $H^{n-1}(\Gamma, \boldsymbol{C})=W_0H^{n-1}(F)$ and the right hand side is a one-dimensional $\boldsymbol{C}$-vector space since the singularity (Y, y) is of type $(0, 0)$. Let σ be a free generator of $H^{n-1}(\Gamma, \boldsymbol{Z})$. Then, $\lambda(\sigma)$ is either σ or $-\sigma$ in $H^{n-1}(F, \mathcal{O}_F)$. Now, by Lemma 3.8, $\lambda(\sigma)$ must be $-\sigma$, which means that the order r of G can be devided by 2. Therefore, the subgroup $H=\langle\lambda^2\rangle$ is a proper subgroup of G. Here, if $H\neq\langle 1\rangle$, the quotient $(Y', y')=(Y, y)/H$ is again a purely elliptic singularity by Lemma 3.8. Now, we have a cyclic covering $h: (Y', y')\to(X, x)$ étale outside the singularity with (Y', y') a purely elliptic singularity. Note that the degree of the morphism h is less than r. However, this is a contradiction to the definition of the index.

§ 4. A minimal resolution of a Q-Gorenstein singularity

Definition 4.1. A projective birational morphism $g: Y\to X$ with $Y-g^{-1}(x)\simeq X-\{x\}$ from a normal variety Y is called a partial resolution of the singularity (X, x). A partial resolution $g: Y\to X$ is called a minimal resolution of the singularity (X, x), if Y has only terminal singularities and K_Y is relatively numerically effective with respect to g. For simplicity, we use "nef" instead of "numerically effective".

Noting that a terminal singularity on a surface is a non-singular point, we see that a minimal resolution defined in 4.1 coincides with the well known one for a surface singularity.

Proposition 4.2. *Let* $g: Y\to X$ *be a minimal resolution of a* Q*-Gorenstein singularity* (X, x) *of dimension* $n\geq 2$.

Then the inverse image $g^{-1}(x)$ *is of pure codimension one.*

Moreover, if we denote $g^{-1}(x)_{\mathrm{red}}$ *by* $\sum_{i=1}^{s} D_i$, *a canonical divisor on* Y *is presented as* $K_Y=g^*K_X-\sum_{i=1}^{s} n_iD_i$ *with* $n_i\geq 0$ *for every* i *as* Q*-Cartier divisors.*

Proof. As is well known, a projective birational morphism $g: Y\to X$

is obtained by a blowing up of some ideal sheaf on X. Therefore, there are positive numbers m_i ($i=1, 2, \cdots, t$) such that $L=-\sum_{i=1}^{t} m_i D_i$ is relatively ample Cartier divisor with respect to g, where all D_i ($i=1, \cdots, s$) are the Weil divisors contained in $g^{-1}(x)$ and D_i ($i=s+1, \cdots, t$) are not contained in $g^{-1}(x)$. Since K_Y is relatively nef, K_Y+aL ($a\geq 0, a \in \boldsymbol{Q}$) is relatively nef. Here, we denote a canonical divisor K_Y by $g^*K_X-\sum_{i=1}^{s} n_i D_i$ with $n_i \in \boldsymbol{Q}$.

If there exists a negative n_i, let a be a positive integer $-\min_{i=1,\dots,s}\{n_i/m_i\}$. Then, we have

$$K_Y+aL=g^*K_X-\sum_{i=s+1}^{t} am_i D_i-\sum_{i=1}^{s} \beta_i D_i$$

where $\beta_i=0$ for the i's such that n_i/m_i attain the minimal value, and $\beta_i>0$ for the other i's. Here, we may assume the existence of i ($1\leq i\leq s$) for which n_i/m_i does not attain $-a$, because, if not, we have $n_i=-am_i>0$ for all $i=1, \cdots, s$. Consequently K_Y+aL is not nef on some D_i with $\beta_i=0$, which is a contradiction.

Now we have only to show that $g^{-1}(x)_{\text{red}}=\sum_{i=1}^{s} D_i$. Let C be a curve contained in an irreducible component of $g^{-1}(x)_{\text{red}}$ of codimension ≥ 2. We can take a curve C snch that C is not contained in $\sum_{i=1}^{s} D_i$ and intersects it. Then, $K_Y C=(f^*K_X-\sum_{i=1}^{s} n_i D_i)C<0$, since $n_i>0$ for $i=1, 2, \cdots, s$. Therefore, $g^{-1}(x)_{\text{red}}$ must coincide with $\sum_{i=1}^{s} D_i$.

Proposition 4.3. *Let g: $Y\to X$ be a minimal resolution of purely elliptic singularity (X, x).*

Then $K_Y=g^*K_X-\sum_{i=1}^{s} D_i$ where D_i are Weil divisors with $g^{-1}(x)_{\text{red}}=\sum_{i=1}^{s} D_i$.

Proof. Since the index of the singularity (X, x) is one, all the coefficients n_i of $K_Y=g^*K_Y-\sum_{i=1}^{s} n_i D_i$ are integers. Noting that the singularity (X, x) is log-canonical, $n_i=0, 1$, for $i=1, \cdots, s$, by Proposition 4.2. However $n_i=0$ leads a contradiction.

Proposition 4.4. *For a 3-dimensional singularity (X, x), there exists a minimal resolution of the singularity, if and only if there is a resolution f: $\tilde{X}\to X$ such that the algebra $\bigoplus_{m\geq 0} f_*\mathcal{O}(mK_{\tilde{X}})$ is finitely generated over $\mathcal{O}_X$.*

Proof. First, assume that g: $Y\to X$ is a minimal resolution. Then, by a relative version of Kawamata's theorem ([Ka1]), the algebra $\bigoplus g_*\mathcal{O}(mK_Y)$ is finitely generated over $\mathcal{O}_X$. Since Y has only terminal singularities, the algebra is isomorphic to $\bigoplus f_*\mathcal{O}(mK_{\tilde{X}})$, where h: $\tilde{X}\to Y$ is a resolution and $f=gh$.

Next, conversely assume that the algebra $\bigoplus f_*\mathcal{O}(mK_{\tilde{X}})$ is finitely generated over $\mathcal{O}_X$ for a resolution f: $\tilde{X}\to X$. Denote the variety

Proj $(\oplus f_*\mathcal{O}(mK_{\tilde{X}}))$ by Y and the canonical morphism $Y \to X$ by g. Then Y has only canonical singularities and K_Y is relatively ample $\boldsymbol{Q}$-Cartier divisor with respect to g. Here, by M. Reid ([R1]), there exists a partial resolution h: $Y' \to Y$ such that Y' has only terminal singularities and $K_{Y'} = h^*K_Y$ as $\boldsymbol{Q}$-Cartier divisors. Thus, we have a minimal resolution gh: $Y' \to X$ of the singularity.

Definition 4.5. A singularity (X, x) which has a resolution f: $\tilde{X} \to X$ with finitely generated $\mathcal{O}_X$-algebra $\bigoplus_{m \geq 0} f_*\mathcal{O}(mK_{\tilde{X}})$ is called an f.g. singularity.

Remark 4.6. It is conjectured that every 3-dimensional $\boldsymbol{Q}$-Gorenstein singularity is an f.g. singularity. Shepherd-Barron has shown that a singularity defined by a non-degenerate polynomial is an f.g. singularity ([SB2]).

Proposition 4.7. *Let g: $Y \to X$ be a minimal resolution of a purely elliptic f.g. singularity (X, x) of dimension three.*

Then, the singularities of the divisor $D = g^{-1}(x)_{\text{red}}$ is only normal crossings except for finite points.

Proof. First, we note that every singularity of Y is isolated, since the singularities on Y are 3-dimensional and terminal ([R2]).

Let C be a one dimensional irreducible component of the singular locus of D. Then D is a Cartier divisor on a non-singular 3-fold at a general point of C. Denote the multiplicity of D at a general point by m. Next, take the blowing up σ: $Y' \to Y$ at the center C, and represent a canonical divisor $K_{Y'}$ as

$$K_{Y'} = \sigma^* g^* K_X - \sum_{i=1}^{s} [D_i] - dD_0,$$

where $[D_i]$ is the proper transform of an irreducible component D_i of D $(i = 1, \cdots, s)$ and $D_0 = \sigma^{-1}(C)_{\text{red}}$. Then, d must be $m-1$, because at a general point of C, σ is considered as a blowing-up of a non-singular 3-fold with a non-singular curve as the center. Therefore, we have a good resolution f: $\tilde{X} \to X$ factored through $g \cdot \sigma$ with $K_{\tilde{X}} = f^*K_X - (m-1)[D_0] - $(other terms). Here, m must be ≤ 2 by Theorem 2.1, since (X, x) is purely elliptic.

If D is not ordinary at a general point of C, then, by successive blowing-ups of Y with suitable curves as centers, we have a partial resolution g'': $Y'' \to X$ factored through g with $K_{Y''} = g''K_X - D_1' - D_2' - D_3' -$ (other terms), where D_1', D_2' and D_3' are components of $g''^{-1}(x)_{\text{red}}$ and intersect at a curve C'. By passing through the blowing-up of Y'' with center C', we get a good resolution f: $\tilde{X} \to X$ with $K_{\tilde{X}} = f^*K_X - dE_i -$(other terms) $(d \geq 2)$, which is a contradiction.

§ 5. Configurations of the essential parts of 3-dimensional periodically elliptic f.g. singularities

In this section, we study the configuration of the essential part of a good resolution of 3-dimensional periodically elliptic f.g. singularities. It turns out to be a distinguished figure according to the type of the singularity (cf. Definition 3.9). First, we mention the main results of this section.

Theorem 5.1. *Let f: $\tilde{X} \to X$ be a good resolution of a purely elliptic f.g. singularity (X, x) of dimension 3.*

Then the essential part E_J of $E = f^{-1}(x)_{\text{red}}$ is as follows.

(1) *If (X, x) is of type $(0, 2)$, E_J is an irreducible surface birationally equivalent to a $K3$ or Abelian surface.*

(2) *If (X, x) is of type $(0, 1)$, E_J is either*

(2a) *a chain of surfaces $E_1, E_2, \cdots, E_s$ $(s \geq 2)$ with rational surfaces E_1, E_s and elliptic ruled surfaces $E_2, \cdots, E_{s-1}$, where E_i and E_{i+1} intersect at an elliptic curve for $i = 1, \cdots, s-1$, or*

(2b) *a circle of elliptic ruled surfaces $E_1, E_2, \cdots, E_s, E_{s+1} = E_1$ $(s \geq 2)$, where E_i and E_{i+1} intersect at each section for $i = 1, \cdots, s$, and E_J has a ruling over an elliptic curve; i.e., there is a morphism p: $E_J \to C$ to an elliptic curve C whose restriction on each component is the canonical projection of the ruled surface to the base curve.*

(3) *If (X, x) is of type $(0, 0)$, then E_J consists of rational surfaces with rational intersection curves and the dual graph of E_J is a two dimensional simplicial complex which is a triangulation of an image of a compact orientable real surface by a continuous map.*

Theorem 5.2. *Let f: $\tilde{X} \to X$ be a good resolution of 3-dimensional periodically elliptic f.g. singularity (X, x) of index $r > 1$.*

Then the essential part E_J of $E = f^{-1}(x)_{\text{red}}$ is as follows.

(1) *If (X, x) is of type $(0, 2)$, then either*

(1a) *E_J is an irreducible surface birationally equivalent to an Enriques surface or a bielliptic surface and the index r is 2, or 2, 3, 4, 6 respectively, or*

(1b) *E_J is either a ratinal surface or an elliptic ruled surface.*

(2) *If (X, x) is of type $(0, 1)$, then either*

(2a) *the index r equals to 2, E_J is a chain of elliptic ruled surfaces (possibly with a rational surface at one end) and all double curves are elliptic,*

(2b) *E_J is a chain of rational surfaces and all double curves are rational, or*

(2c) *E_J is a circle of the rational surfaces and all double curves are rational.*

(3) *If* (X, x) *is of type* $(0, 0)$, *then the index* r *turns out to be* 2, E_J *consists of rational surfaces with rational intersection curves and the dual graph of* E_J *is a triangulation of an image of the quotient of an orientable real surface by a reflection which does not preserve the orientation.*

To prove our theorems, we prepare basic lemmas.

Lemma 5.3. *Let* S *be a normal surface and* $\phi: \tilde{S} \to S$ *be a resolution of the singularities of* S.

Then, for an effective Weil divisor C *on* X, *the inverse image* ϕ^*C *defined by Sakai* ([Sa]) *is again effective; i.e.* $\phi^*C = \sum_{i=1}^{s} a_i C_i + C_0$ *with* $a_i \geq 0$ $a_i \in \mathbf{Q}$, *where* $\{C_i\}_{i=1,\dots,s}$ *are the excpetional curves and* C_0 *is the proper transform of* C.

More precisely, if C *passes through the singularities on* S, *then* $a_i \gneq 0$ *for any* i.

Proof. We will use the same notation as in [Sa]. First, denote the inverse image ϕ^*C by $\sum_{i=1}^{s} a_i C_i + C_0$. Next, decompose the integral divisor $\sum_{i=1}^{s} [a_i] C_i$ as

$$\sum_{i=1}^{s} [a_i] C_i = C^+ - C^-,$$

where C^+ and C^- are effective and have no common components. Then we have a relation of two integral divisors

$$[\phi^*C] = C_0 + C^+ - C^- \leq C_0 + C^+.$$

Therefore, we have the inclusions

$$\Gamma(\tilde{S}, \mathcal{O}([\phi^*C])) \xrightarrow{\iota} \Gamma(\tilde{S}, \mathcal{O}(C_0 + C^+)) \longrightarrow \Gamma\left(\tilde{S} - \sum_{i=1}^{s} C_i, \mathcal{O}(C_0)\right).$$

Here, the first term is isomorphic to $\Gamma(S, \mathcal{O}_S(C))$ by Theorem 2.1 of [Sa]. Noting that $\Gamma(S, \mathcal{O}(C)) = \Gamma(S - \{\text{singular points}\}, \mathcal{O}(C))$ by the normality of S, we get that $\Gamma(S, \mathcal{O}(C))$ is isomorphic to $\Gamma(\tilde{S} - \sum C_i, \mathcal{O}(C_0))$. Therefore the inclusion ι is the equality. This means that C^- is a fixed component of $|C_0 + C^+|$. But $C_0 + C^+$ and C^- have no common components, which means $C^- = 0$. Thus, we have $[a_i] \geq 0$, so $a_i \geq 0$. However, $a_i = 0$ does not occur, if C passes the point $\phi(E_i)$.

Lemma 5.4. *Let* $g: Y \to X$ *be a minimal resolution of a* 3-*dimensional purely elliptic f.g. singularity* (X, x). *Let* $D = g^{-1}(x)_{\text{red}}$.

Then, D *is a Gorenstein variety and* $\mathcal{O}_D(K_D) \simeq \mathcal{O}_D$.

Proof. By Proposition 4.3, $K_Y = g^*K_X - D$. Therefore, by the adjunction formula, $\mathcal{O}_D(K_D)\simeq\mathcal{O}_D$ outside the singularities of Y. Here, we note that Y has only isolated singularities ([R2]). Then we have $\mathcal{O}_D(K_D)\simeq\mathcal{O}_D$ on D.

Proposition 5.5. *Let $g\colon Y\to X$ be a minimal resolution of a 3-dimensional purely elliptic f.g. singularity (X, x) and let D be $g^{-1}(x)_{\mathrm{red}}$. Let $D=\sum_{i=1}^{s} D_i$ be the decomposition into irreducible components.*

Then,

(i) *if (X, x) is of type $(0, 2)$, then D is an irreducible normal surface birational to either a $K3$- or Abelian surface,*

(ii) *if (X, x) is of type $(0, 1)$, then either*

(iia) *D is an irreducible normal surface birational to an elliptic ruled surface with two simple elliptic singularities,*

(iia′) *D is an irreducible normal rational surface with one simple elliptic singularity, or*

(iib) *for every 1-dimensional irreducible component C of the singular locus of D, a component of $\sigma^{-1}(C)$ is birational to an elliptic curve, where σ is the normalization of D, and every irreducible component of D is rational or elliptic ruled;*

More precisely, if an irreducible component D_i of D is rational, for the double curve C of D on D_i, $\sigma_i^{-1}(C)$ is irreducible, where σ_i is the normalization of D_i. If an irreducible component D_i of D is elliptic ruled, then for the double curve C of D on D_i, either $\sigma_i^{-1}(C)$ is a union of two sections or $\sigma_i^{-1}(C)$ is a section and D has one simple elliptic singularity on D_i,

(iii) *if (X, x) is of type $(0, 0)$, then either*

(iiia) *D is an irreducible normal rational surface with a cusp singularity, or*

(iiib) *every 1-dimensional irreducible component of the singular locus of D is a rational curve and every irreducible component of D is a rational surface.*

Besides, every isolated singular point of D other than the ones mentioned above is at worst a rational double singularity.

Proof. In the proof below, we use the notation; $f\colon \tilde{X}\to X$ a good resolution with the essential part E_J, which is factored as $\tilde{X}\xrightarrow{h}Y\xrightarrow{g}X$.

First of all, assume that D admits only isolated singularities. Then, D must be irreducible and, therefore, normal by Serre's criteria ([EGA]). In fact, if D is decomposed as D_1+D_2, then D_1 and D_2 intersect at points. Therefore, $\mathrm{Gr}_F^0H^2(D)\simeq\bigoplus_{i=1,2}\mathrm{Gr}_F^0H^2(D_i)$. So, one of $\mathrm{Gr}_F^0H^2(D_i)$ must be non-zero, which yields that one of $H^2(E_i, \mathcal{O}_{E_i})$ $(i=1, 2)$ must be non-zero, where E_i is the proper transform of D_i. This is a contradiction to Proposition 3.7.

For a normal Gorenstein surface D with trivial dualizing sheaf, we can apply a theorem of Umezu [U]. By her result, we get either that D is birationally equivalent to a $K3$- or Abelian surface with only rational double points as singularities or that D is birational to a ruled surface.

In the former two cases, the singularity (X, x) will be of type (0, 2). In fact, the essential part E_J contains a birational $K3$- or Abelian surface. Here, by Proposition 3.7, E_J coincides with the surface and so $H^2(E_J, \mathcal{O}_{E_J})$ consists of (0, 2)-Hodge-component.

Now, we consider the latter case. We assume that D is birationally equivalent to a ruled surface of genus g. Denote the proper transform of D on $\tilde{X}$ by E_0. Then the birational morphism $h|_{E_0}\colon E_0 \to D$ is factored through a minimal resolution $\pi\colon \tilde{D} \to D$. Then $K_{\tilde{D}} = -\Delta$, where $\Delta \geq 0$ is exceptional by π. Here, if $g \geq 2$, by Umezu [U], the member of $|-K_{\tilde{D}}|$ is $2C+$(rulings) where C is a section. Let C_1 be the proper transform of C on E_0. Then, since C_1 is contracted to a point by h, we may assume that there exists an exceptional divisor E_1 on $\tilde{X}$ such that $E_1 \cap E_0 = C_1$. Now, by the adjunction formula, we have

$$K_{\tilde{X}} = -E_0 - 2E_1 + (\text{other terms}). \tag{5.5.1}$$

This is a contradiction to the hypothesis of our proposition. Therefore g must be ≤ 1. By the result of [U], D is as follows;

(α) D is a rational surface with a cusp singularity and some (maybe no) rational double points, or

(β) D is a rational surface with a simple elliptic singularity and some (maybe no) rational double points, or

(γ) D is birationally equivalent to an elliptic ruled surface with two simple elliptic singularities and some (maybe no) rational double points.

Noting that $\mathrm{Gr}_F^0 H^2(D)$ consists of the (0, 0) or (0, 1)-Hodge-component in the case (α) or (β) (γ) respectively, we observe that the singularity (X, x) is of type (0, 0) or (0, 1) in the case (α) or (β) (γ) respectively. Now we get (iia), (iia′) and (iiia).

Next, assume that the singular locus of D has dimension one. Let C be a 1-dimensional singular locus of D on an irreducible component D_i of D. Take a normalization $\sigma\colon D'_i \to D_i$ and then take the minimal resolution $\pi\colon \tilde{D}_i \to D'_i$ of D'_i. Then $K_{D'_i} = -\sigma^{-1}(C)$. By Lemma 5.3 and the minimality of π, there is an effective member $[\sigma^{-1}(C)] + Z$ in $|-K_{\tilde{D}_i}|$, where Z is an effective divisor on $\tilde{D}_i$ supported on the exceptional sets of π and $[\sigma^{-1}(C)]$ is the proper transform of $\sigma^{-1}(C)$ by π. Therefore $\tilde{D}_i$ is a rational or elliptic ruled surface, because the genus g of $\tilde{D}_i \geq 2$ yields the representation (5.5.1) which is a contradiction. Now, by [U] again, $[\sigma^{-1}(C)] + Z$ is either two disjoint sections of an elliptic ruled surface, an elliptic curve on a rational surface or a circle of rational curves on a rational surface. Noting that an isolated singularity (D, p) is rational double if and only if $\pi^*K_D = K_{\tilde{D}}$ ([Sa]), we have the last assertion of the proposition. Finally, (iib) and (iiib) can be easily shown by calculating the Hodge component of $H^2(D, \boldsymbol{C})$.

Proof of Theorem 5.1, (1). By Proposition 5.5, the assertion (1) of Theorem 5.1 is now obvious.

Proof of Theorem 5.1, (2). First, we prepare a lemma.

Lemma 5.6. *For a purely elliptic f.g. singularity* (X, x) *of type* (0, 1), *every double curve on the essential part* E_J *has positive genus and there is no triple point on* E_J,

Proof. First, take any irreducible component E_j of E_J and put $E_j^\vee = E_J - E_j$. Then consider the exact sequence;

$$\cdots H^1(E_j, \mathcal{O}_{E_j}) \oplus H^1(E_j^\vee, \mathcal{O}_{E_j^\vee}) \longrightarrow H^1(E_j \cap E_j^\vee, \mathcal{O}) \longrightarrow H^2(E_J, \mathcal{O}_{E_J}) \longrightarrow 0.$$

Since $H^2(E_J, \mathcal{O}_{E_J})$ consists of the (0, 1)-component, there are (0, 1)-component on $H^1(E_j \cap E_j^\vee, 0)$. Therefore, $E_j \cap E_j^\vee$ contains at least one curve of positive genus. Note that this holds for any good resolution.

Here, if l is a rational double curve on E_J, take the blowing-up $\sigma: \tilde{X}' \rightarrow \tilde{X}$ with center l. Then the divisor $E_0 = \sigma^{-1}(l)$ is an essential component of $\sigma^{-1} \circ f^{-1}(x)_{\mathrm{red}}$ and the double curves of E'_J in E_0 are all rational, where E'_J is the essential part of $\sigma^{-1} \circ f^{-1}(x)_{\mathrm{red}}$. This is a contradiction to the fact mentioned above.

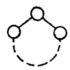
If there exists a triple point p on E_J, take the blowing-up at p. Then we also have an essential component with only rational double curves on it. Q.E.D. of Lemma 5.6.

Let (X, x) be a purely elliptic f.g. singularity of type (0, 1). To prove the theorem, it is sufficient to show the assertion for a good resolution $f: \tilde{X} \rightarrow X$ which factored through a minimal resolution $g: Y \rightarrow X$.

Now, we will study in detail the configuration of D on a minimal resolution of a singularity (X, x) of type (0, 1).

Since D is of normal crossings except for a set S of finite points, we can consider the dual graph Γ of $D - S$, which we call the generic dual graph of D. The generic dual graph Γ of D is connected, because, if not, there is a decomposition $D = D_1 + D_2$, where D_1 and D_2 intersect at points. This induces a contradiction in the same way as the proof of Proposition 5.5. Remarking that $K_{D_i} = (-D + D_i)|_{D_i}$, where D_i is a component of D, we can observe that Γ turns out to be either of followings;

(δ)  : an elliptic ruled surface which intersects with itself at a section,

(δ')  : a circle of elliptic ruled surfaces with a section as a double curve.

(ε) o: a surface described in (iia) or (iia$'$).

(ε') o—: a rational surface with a double curve C such that $\sigma^{-1}(C)$ is an irreducible elliptic curve, where σ is the normalization of D.

(ε'') $\underset{D_1}{\circ}$—$\underset{D_2}{\circ}$—$\cdots$—$\underset{D_s}{\circ}$ ($s \geq 2$): a chain of surfaces with elliptic intersection curves; each of D_1 and D_2 is either an elliptic ruled surface with one simple elliptic singularity or a rational surface and $D_2, \cdots, D_{s-1}$ are all elliptic ruled surfaces.

Again we note that every isolated singularity of D other than noted above is rational double.

For the cases (δ) and (δ'), we have that the essential part E_J is a circle of elliptic ruled surfaces with a section as a double curve, by Lemma 5.6 and the rational doubleness of isolated singularities in D. This provides us the first assertion of (2b) of Theorem 5.1. For the second assertion, we have only to remember that a circle of elliptic ruled surface E_J satisfies $H^2(E_J, \mathcal{O}_{E_J}) = \boldsymbol{C}$ if and only if E_J has a ruling over an elliptic curve. In what follows, we will show the cases (ε), (ε') and (ε'') provide us (2a) of Theorem 5.1.

Lemma 5.7. *In the cases* (ε), (ε') *and* (ε''), *for the essential part* E_J, *the vanishing* $H^1(E_J, \mathcal{O}_{E_J}) = 0$ *holds. And a fortiori, the Hodge component* $H_1^{01}(E_J) = 0$.

Proof. First, we will show the vanishing $\mathrm{Gr}_F^0 H^1(D) = 0$, where F is the Hodge filtration on $H^1(D, \boldsymbol{C})$. In fact, for case (ε), the singularities of D are all Du Bois, then $\mathrm{Gr}_F^0 H^1(D) \simeq H^1(D, \mathcal{O}_D) = 0$. The other cases are proved by Mayer-Vietoris exact sequence, which will be left to the reader.

By Proposition 4.7, we have a resolution $f'\colon \tilde{X}' \to Y$ with $E' = f'^{-1} \circ g^{-1}(x)_{\mathrm{red}}$ of normal crossings (not necessarily simple normal crossings) and isomorphic outside a set S of finite points on Y. Denote the essential part of E' by E'_J. Then, we have an exact sequence;

$$0 = \mathrm{Gr}_F^0 H^1(D) \longrightarrow H^1(E'_J, \mathcal{O}_{E'_J}) \oplus \underset{\substack{\| \\ 0}}{\mathrm{Gr}_F^0 H^1(S)} \longrightarrow H^1(f'^{-1}(S), \mathcal{O}).$$

Here the last term is zero, because each point of S is a terminal singularity on Y. This yields $H^1(E'_J, \mathcal{O}_{E'_J})=0$.

Now the vanishing $H^1(E_J, \mathcal{O}_{E_J})=0$ for the essential part of a good resolution factored through $gf'\colon \tilde{X}'\to X$ is not difficult, so we omit the proof.

Lemma 5.8. *For the case* (ε), (ε') *and* (ε''), *the essential part* E_J *contains the following configuration*:

(1) $\underset{E_0}{\cdots\!-\!\circ\!-\!\cdots}$: E_0 *an elliptic ruled surface with the disjoint sections as double curves, or*

(2) $\underset{E_0}{\circ\!-\!\cdots}$: E_0 *a rational surface with an elliptic curve as a double curve.*

Proof. Immediate consequence from (ε), (ε') and (ε'').

Lemma 5.9. *Assume* $H^{01}_1(E_J)=0$ *for a purely elliptic singularity* (X, x). *Denote a canonical divisor* $K_{\tilde{X}}$ *by* $\sum_{i\in I} m_iE_i - E_J$ $(m_i\geq 0)$. *If* E_i $(i\in I)$ *intersects to a component of* E_J *at a curve of gneus* $g>0$, *then*, m_i *must be zero.*

To prove the lemma we have to prepare two sublemmas.

Sublemma 5.10. *Assume* $H^{01}_1(E_J)=0$. *Put*

$$F_1=\sum_{\substack{i\in I\\ E_i\cap E_J \text{ contains}\\ \text{positive genus curves}}} E_i,\qquad F_2=\sum_{\substack{i\in I,\, E_i\not\subset F_1\\ E_i\cap F_1 \text{ contains}\\ \text{positive genus curves}}} E_i,\ \cdots\cdots,$$

$$F_t=\sum_{\substack{i\in I,\, E_i\not\subset F_1\cup\cdots\cup F_{t-1}\\ E_i\cap F_{t-1} \text{ contains}\\ \text{positive genus curves}}} E_i.$$

Then, for any component E_i *of* F_k $(k=1, 2, \cdots, t)$ *and an irreducible curve* C_{i0} *in* $C_i=E_i\cap F_{k-1}$ *of genus* >0, $q(E_i)=g(C_{i0})$ *and other components of* C_i *are all rational.*

Proof. First, take any component $E_1\leq F_1$. Then we have an exact sequence induced from Mayer-Vietoris sequence;

$$0\longrightarrow H^{01}_1(E_J+E_1)\longrightarrow H^{01}_1(E_J)\oplus H^{01}_1(E_1)\xrightarrow{\ \varphi\ } H^{01}_1(C_1)\longrightarrow H^{01}_2(E_J+E_1)$$
$$\simeq H^{01}_2(E_J).$$

By the surjectivity of φ, we get $q(E_1)=h^{01}_1(E_1)\geq h^{01}_1(C_1)=\sum_j g(C_{1j})$, where $\{C_{1j}\}$ are the components of C_1. Let C_{10} have positive genus. Then $q(E_1)\leq g(C_{10})$, because E_1 is a ruled surface which appears on a resolution of a terminal singularity. Therefore, $q(E_1)=g(C_{10})$ and $g(C_{1j})=0$ for $j\neq 0$. Now we get the assertion for any component of F_1. To prove the sublemma for $F_2, \cdots, F_t$, note that φ is an isomorphism, so $H_1^{01}(E_J+E_1)=0$ for any $E_1<F_1$. Take two components E_1, E_2 of F_1 with $E_1\cap E_2\neq\phi$, and put $C_2=E_2\cap E_J$. Then, on the exact sequence;

$$0\longrightarrow H_1^{01}(E_J+E_1+E_2)\longrightarrow \underset{\substack{\| \\ 0}}{H_1^{01}(E_J+E_1)}\oplus H_1^{01}(E_2)\xrightarrow{\varphi'} H_1^{01}(C_2+(E_1\cap E_2))\longrightarrow 0,$$

by the surjectivity of φ', $q(E_2)\geq g(C_{20})+h_1^{01}(E_1\cap E_2)$, where C_{20} is the irreducible curve of positive genus in C_2. However, $q(E_2)=g(C_{20})$ yields that any component of $E_1\cap E_2$ is rational and φ' is an isomorphism. Thus, we get $H_1^{01}(E_J+E_1+E_2)=0$. Inductively we have $H_1^{01}(E_J+F_1)=0$. Now, we stand on the same stage as we did on the first discussion about a component of F_1. Therefore, we shall prove the assertion for $F_2, F_3, \cdots, F_t$ successively.

Sublemma 5.11. *Under the same assumption as* 5.10. *Let E_1 be a component of F_1, C_1 a curve of positive genus in $E_1\cap E_J$ and E_0 the component of E_J with $E_1\cap E_0=C_1$. Put $G_1=E_1$,*

$$G_2=\sum_{\substack{i\in I, E_i\neq E_1\\ E_i\cap G_1 \text{ contains}\\ \text{positive genus curves}}} E_i,\qquad G_3=\sum_{\substack{i\in I, E_i\not\subset G_1\cup G_2\\ E_i\cap G_2 \text{ contains}\\ \text{positive genus curves}}} E_i,\ \cdots\cdots,$$

$$G_t=\sum_{\substack{i\in I, E_i\not\subset G_1\cup\cdots\cup G_{t-1}\\ E_i\cap G_{t-1} \text{ contains}\\ \text{positive genus curves}}} E_i.$$

Then, there exist a canonical surjective morphism π: $G=\sum_{k=1}^{t} G_k\rightarrow C_1$, an open subspace U of $\tilde{X}$ and a contraction;

$$\begin{array}{ccc} U & \xrightarrow{\ \Pi\ } & V \\ \uparrow & & \uparrow \\ G\cap U & \xrightarrow[\pi]{} & C_1\cap U \end{array}$$

that is an isomorphism outside $G\cap U$.

Proof. For any component E_i of G_k, the only one intersection curve of E_i and G_{k-1} with positive genus is a section of ruled surface E_i (Sublemma 5.10). Therefore, there is a surjective morphism π_i: $E_i\rightarrow G_{k-1}$ corresponding to the intersection curve of positive genus. Now we get the first assertion.

Next, take an open subspace U of $\tilde{X}$ such that $U\cap E_i\cap G_{k-1}$ is étale on C_1 for any $E_i<G_k$ and $G\cap U$ does not intersect a divisor $E-E_0-G$.

Since E is contracted to a point, there exist positive integers $\{n_i\}_{i\in I\cup J}$ such that $-\sum_{i\in I\cup J} n_iE_i|_G$ is ample on G. Therefore restricting it on $G\cap U$, $-(\sum_{E_i\le G} n_iE_i+n_0E_0)|_{G\cap U}$ is relatively ample with respect to π. Because $E_0|_{G\cap U}=C_1|_{G\cap U}$ is a section on a ruled surface E_1 in U on a Stein space, its linear system is base point free. Then $-Z=-\sum_{E_i\le G} n_iE_i|_{G\cap U}$ is relatively ample with respect to π. On the other hand, we have $\pi_*\mathcal{O}_Z=\mathcal{O}_C$, which is proved in the same way as Proposition 4.2 of [SB1]. Hence, by [A2] and [H1], we have a contraction

$$\begin{array}{ccc} U & \xrightarrow{\Pi} & V \\ \uparrow & & \uparrow \\ G\cap U & \longrightarrow & C_1\cap U, \end{array}$$

such that Π is an isomorphism outisde $G\cap U$. Q.E.D. of Sublemma 5.11.

Proof of Lemma 5.9. Assume a component $E_1<E_I$ of non-essential part intersects to a component $E_0<E_J$ at a curve of positive genus. By 5.10 and 5.11, we have a contraction morphism Π. We will use the same notation as in 5.11. Let v be a general point of $C_1\cap U$ in V and $H\cap V$ a general hyperplane through v. Then H is a normal surface with a singularity v and V is regarded as $H\times C_1$ in the neighbourhood of v. Let $\tilde{H}$ be the proper transform of H by Π. Then $\Pi|_{\tilde{H}}\colon \tilde{H}\to H$ is a resolution since U is regarded as $\tilde{H}\times C_1$ on the neighbourhood of v. Now, denote a canonical divisor $K_{\tilde{X}}$ by $\sum_{i\in I} m_iE_i-E_J$ with $m_i\ge 0$. Then $K_U=\sum_{E_i\le G} m_iE_i-E_0$ with $m_i\ge 0$, and therefore $K_V=-E_0$.

Cutting the two canonical divisors by hyperplanes H, $\tilde{H}$, we get $K_{\tilde{H}}=\sum_{E_i\le G} m_iE_i-E_0$ and $K_H=-E_0|_H$, because $\mathcal{O}(H)\otimes\mathcal{O}_H\simeq\mathcal{O}_H$ and $\mathcal{O}(\tilde{H})\otimes\mathcal{O}_{\tilde{H}}\simeq\mathcal{O}_{\tilde{H}}$. Denote $E_i|_{\tilde{H}}$ by e_i, then for $i\ge 1$, e_i is a configuration of rational curves $\{e_{i1}, e_{i2}, \cdots, e_{ij_i}\}$ and in particular e_1 is a rational curve. Then for a resolution $\Pi'=\Pi|_{\tilde{H}}\colon \tilde{H}\to H$ of a surface singularity (H, v), we have $K_H=-e_0$ and $K_{\tilde{H}}=\sum m_ie_i-[e_0]$ with $m_i\ge 0$. Therefore, if we write $K_{\tilde{H}}=\Pi'^*K_H+\Sigma_{i,j} n_{ij}e_{ij}$, the coefficient n_{ij} of every exceptional curve e_{ij} must be strictly positive by Lemma 5.3. This means that the singularity (H, v) is a non-singular point. So, the morphism Π' is obtained by a successive blowing up at suitable points. Thus, on $K_{\tilde{H}}=\sum m_ie_i-[e_0]$, we get $m_1=0$ since e_1 intersects $[e_0]$. This completes the proof of Lemma 5.9.

Now, we will turn to the proof of Theorem 5.1 (2a) in the case (ε), (ε') and (ε''). Consider the case that E_J contains a configuration (1) described in Lemma 5.8 (for the other case, we omit the proof, since it is shown similarly).

Since $H^1(E_J, \mathcal{O}_{E_J})=0$, we have $H^1(\Gamma_{E_J}, \boldsymbol{C})=0$, where Γ_{E_J} is the dual graph of E_J. Therefore Γ_{E_J} does not contain a circle. So $E_0^\vee$ is decomposed into two disjoint connected components E_- and E_+, where, for a divisor $D<E_J$, $D^\vee$ means E_J-D. Put $C_1=E_0\cap E_+$ and $C_{-1}=E_0\cap E_-$. Then we know that they are disjoint elliptic curves on E_0.

First, we study the configuration of E_+. Denote by E_1 the component of E_+ which intersects E_0 at C_1. Put $E_0^*=E_-+E_0$, $E_1^*=E_-+E_0+E_1$.

We claim that $H^1(E_+, \mathcal{O}_{E_+})=H^1(E_0^*, \mathcal{O}_{E_0^*})=0$. In fact, by the decomposition $E_J=E_0^*+E_+$, we have the Mayer-Vietoris exact sequence;

$$\begin{array}{l} 0=H^1(E_J, \mathcal{O}_{E_J})\longrightarrow H^1(E_0^*, \mathcal{O})\oplus H^1(E_+, \mathcal{O})\longrightarrow H^1(E_0\cap E_+, \mathcal{O}) \\ \longrightarrow H^2(E_J, \mathcal{O})\longrightarrow 0. \end{array}$$

where $H^1(E_0\cap E_+, \mathcal{O}) = \boldsymbol{C}$ and $H^2(E_J, \mathcal{O}) = \boldsymbol{C}$.

Then, $H^1(E_0^*, \mathcal{O})=H^1(E_+, \mathcal{O})=0$.

Since E_1 is either the proper transform of a component of D or a component which appears on a resolution of a terminal singularity, E_1 is a ruled surface. Here, noting that E_1 contains an elliptic curve $E_1\cap E_0$, we get that E_1 is either rational or elliptic ruled.

If E_1 is rational, $E_1^*=E_J$. In fact, we have an exact sequence;

$$\begin{array}{l} H^1(E_1^*, \mathcal{O}_{E_1^*})\longrightarrow H^1(E_0^*, \mathcal{O})\oplus H^1(E_1, \mathcal{O})\longrightarrow H^1(E_0^*\cap E_1, \mathcal{O}) \\ \longrightarrow H^2(E_1, \mathcal{O})\longrightarrow 0. \end{array}$$

where $H^1(E_0^*, \mathcal{O}) = 0$, $H^1(E_1, \mathcal{O}) = 0$ and $H^1(E_0^*\cap E_1, \mathcal{O}) = \boldsymbol{C}$.

Then, $H^2(E_1^*, \mathcal{O})\neq 0$, which means $E_1^*=E_J$ by Proposition 3.7.

If E_1 is an elliptic ruled surface, then E_1 intersects to only one component of E_J other than E_0 at a section. In fact, let E_1 intersects a divisor $E_J-E_0-E_1$ at C. Then by Lemma 5.6, C is a disjoint union of positive genus curves. If C is not a section, the curve $E_1^\vee|_{E_1}=C_1+C$ intersects a general ruling f at more than two points. Since $K_{E_1}=\sum_{i\in I} m_iE_i-E_1^\vee|_{E_1}$, there must exist a component E_i with $m_i>0$, which intersects E_1 and crosses the ruling f of E_1. However, this is a contradiction to Lemma 5.9. Therefore, C must be a section.

By the successive process we finally have a configuration of E_+; E_1, $E_2, \cdots, E_s$, where E_i $(i=1, 2, \cdots, s-1)$ are elliptic ruled, E_s is rational and E_i, E_{i+1} intersect at a section for $i=1, \cdots, s-1$. In the same way, we have the similar discription of the configuration of E_-.

Proof of Theorem 5.1, (3). Let (X, x) be a purely elliptic singularity of type (0, 0).

Lemma 5.12. *$H_1^{01}(E_J)=0$ for a purely elliptic singularity (X, x) of type $(0, 0)$.*

Proof. We can easily check that $H_1^{01}(D)=0$ by (iii) of Proposition 5.5. By the same argument of Lemma 5.7, we have the assertion.

Lemma 5.13. *For each component E_j of the essential part E_J, the double curves of E_J on E_j form some circles.*

Proof. For an irreducible component E_j of E_J, $E_j \cap E_j^\vee$ contains at least one circle of curves, where $E_j^\vee = E_J - E_j$. In fact, on the exact sequence

$$H^1(E_J, \mathcal{O}) \longrightarrow H^1(E_j, \mathcal{O}) \oplus H^1(E_j^\vee, \mathcal{O}) \longrightarrow H^1(E_j \cap E_j^\vee, \mathcal{O}) \xrightarrow{\varphi} H^2(E_J, \mathcal{O}) \longrightarrow 0,$$

the surjectivity of φ yields that $H^1(E_j \cap E_j^\vee, \mathcal{O})$ contains a $(0, 0)$-component, which means it contains at least one circle C of curves. If there is a component C_0 in $E_j \cap E_j^\vee$ which is not contained in any circle of $E_j \cap E_j^\vee$, take an irreducible component $E_0 < E_J$ such that $E_0 \cap E_j = C_0$. Consider the diagram

$$\begin{array}{ccccccc}
\to H^1(E_j, \mathcal{O}) \oplus H^1(E_j^\vee, \mathcal{O}) & \longrightarrow & H^1(E_j \cap E_j^\vee, \mathcal{O}) & \xrightarrow{\varphi} & H^2(E_J, \mathcal{O}) & \to & 0, \\
\downarrow & & \downarrow & & \downarrow & & \\
\to H^1(E_j, \mathcal{O}) \oplus H^1(E_j^\vee - E_0, \mathcal{O}) & \longrightarrow & H^1(E_j \cap (E_j^\vee - E_0), \mathcal{O}) & \xrightarrow{\varphi'} & H^2(E_J - E_0, \mathcal{O}) & \to & 0.
\end{array}$$

By taking the $(0, 0)$-components of the diagram, we have

$$\begin{array}{ccccccc}
\to H_1^{00}(E_j) \oplus H_1^{00}(E_j^\vee) & \longrightarrow & H_1^{00}(E_j \cap E_j^\vee) & \xrightarrow{\varphi} & H_2^{00}(E_J) & \to & 0, \\
\alpha \downarrow & & \beta \downarrow & & \downarrow \gamma & & \\
\to H_1^{00}(E_j) \oplus H_1^{00}(E_j^\vee - E_0) & \longrightarrow & H_1^{00}(E_j \cap E_j^\vee - E_0) & \xrightarrow{\varphi'} & H_2^{00}(E_J - E_0) & \to & 0.
\end{array}$$

Here, we may assume α is surjective. Because, if E_J has no triple point on C_0, then by blowing up $\tilde{X}$ at C_0 and taking the new exceptional component as E_0, we have $H_1^{00}(E_0 \cap (E_j^\vee - E_0))=0$, which induces {the cokernel of α}$=0$. If there is a triple point p of E_J on C_0, then by blowing up $\tilde{X}$ at p and taking the new exceptional component as E_0, we have also $H_1^{00}(E_0 \cap (E_j^\vee - E_0))=0$. On the other hand, by the definition of E_0, β is bijective. Therefore, by "snake lemma", γ is injective, which is a contradiction to Proposition 3.7. Q.E.D. of Lemma 5.13.

By the above lemma, the dual graph Γ of E_J is 2-dimensional simplicial complex without a boundary.

Here, since (X, x) is of type $(0, 0)$, $\boldsymbol{C} \simeq H^2(E_J, \mathcal{O}_{E_J}) \simeq H_2^{00}(E_J) \simeq H^2(\Gamma, \boldsymbol{C})$, which means that Γ contains a two dimensional simplicial complex which triangulates an image of a compact orientable real surface by a continuous map. Now by Proposition 3.7, Γ must coincide with the image complex. Therefore the assertion about the dual graph Γ in (3) of 5.1 follows.

Now, we will show that every component E_j of E_J is rational and all intersection curves are rational. Since an irreducible component E_j is either the proper transform of a component of D or a component of a resolution of a terminal singularity, E_j is ruled. By Lemma 5.13, the intersection curves form a circle of curves, so it contains a (multi-) section of a ruled surface E_j. Therefore, it is sufficient to show that any intersection curves of E_J are rational.

Suppose E_J has a double curve C_1 with genus $g(C_1)>0$ on a component E_1. By Lemma 5.12, we can apply Lemma 5.9 to our case. Then, by adjunction formula, either

(i) E_1 is a ruled surface of irregularity $g(C_1)$, $E_1 \cap E_1^\vee$ consists of a section C_1 another section C_1' and some rational curves, or

(ii) E_1 is a ruled surface of irregularity $q \leq g(C_1)$, $E_1 \cap E_1^\vee$ consists of a double section and some rational curves.

Therefore, E_J contains a configuration E' which is either,

(a) a chain of ruled surfaces $E_1, \cdots, E_s$ of positive irregularity, E_1 and E_2 (also E_{s-1} and E_s) intersect at a double section of E_1 (of E_s) E_i and E_{i+1} $(1<i<s-1)$ intersect at a section of E_i, or

(b) a circle of ruled surfaces of the positive common irregularity whose intersection curve is a section of each component.

Since E' cannot be contracted to a curve on Y by (iii) of Proposition 5.5, E' is contracted to a terminal singularity. But a good resolution of a terminal singularity does not contain such configurations.

Now we will turn to the consideration of the case index $r>1$.

Let (X, x) be a periodically elliptic f.g. singularity of index $r>0$ and (X_1, x_1) the canonical cover of (X, x). For the convenience, we fix a diagram;

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X_1 & \xrightarrow[\pi]{} & X \end{array}$$

where f and g are good resolutions with the essential parts E_J and E_J'

respectively, G acts on $\tilde{X}_1$ and g, $\tilde{\pi}$ are G-equivariant morphisms with the trivial action on $\tilde{X}$.

Proof of Theorem 5.2 (1). If (X, x) is of type (0, 2), the essential part E_J is an image of generically finite morphism $\tilde{\pi}|_{E'_J}: E'_J \to E_J$ from the essential part E'_J for a good resolution g of (X_1, x_1). Since E'_J is irreducible and birational to either a $K3$- or Abelian surface, E_J is also irreducible and birational to either an Enriques, a bielliptic, an elliptic ruled or a rational surface.

When E_J is either an Enriques surface or a bielliptic surface, the index r turns out to be 2 or 2, 3, 4, 6 respectively. In fact, consider the case of an Enriques surface (the other case is proved in a similar way, so it will be omitted). First, we note that morphism

$$Y_1 = \operatorname{Proj} \bigoplus_{m \geq 0} g_* \mathcal{O}(mK_{\tilde{X}_1}) \xrightarrow{h} X_1$$

admits an action of G which is compatible with the action of G on X_1. Denote $h^{-1}(x_1)_{\mathrm{red}}$ by D', and D'/G by D. In our case, D' has only rational double points as singularities and the minimal resolution is a minimal $K3$-surface. On the other hand, D is birational equivalent to an Enriques surface with only quotient singularities.

Claim that $\mathcal{O}(K_{D'}) = \mathcal{O}_{D'}$ and $\mathcal{O}(2K_D) = \mathcal{O}_D$. The first assertion is trivial. For the second claim, let $\alpha: \tilde{D} \to D$ be the minimal resolution of the singularities of D, then $K_{\tilde{D}} = \alpha^* K_D - \Delta$ in $\boldsymbol{Q} \otimes \operatorname{Div}(\tilde{D})$, where $\Delta \geq 0$. Since $\tilde{D}$ is obtained by blowing-ups from the Enriques surface, $|2K_{\tilde{D}}|$ has an effective member. Therefore, we may regard $\alpha^*(2K_D)$ as an effective $\boldsymbol{Q}$-divisor, and, by [Sa], $\mathcal{O}(2K_D) = \alpha_* \mathcal{O}_{\tilde{D}}([\alpha^* 2K_D])$. The right hand side contains $\mathcal{O}_D$, so $2K_D$ is trivial or $|2K_D|$ has an effective member. If $|2K_D|$ has an effective member, by the ramification formula for the restricted morphism $h|_{D'}: D' \to D$, $|2K_{D'}|$ has an effective member, which is a contradiction to $\mathcal{O}(K_{D'}) = \mathcal{O}_{D'}$. Thus we obtain $\mathcal{O}(2K_D) = \mathcal{O}_D$.

From this and the ramification formula, the morphism $h|_{D'}$ is étale. Since $(h|_D)^* K_D = \mathcal{O}_{D'}$, the morphism $h|_{D'}$ is factored through the étale double covering $h': D_0 \to D$ corresponding to the invertible sheaf $\mathcal{O}(K_D)$ of order 2. Now we have a subgroup H of G such that $D'/H = D_0$. Noting that $\mathcal{O}(K_{D_0}) = \mathcal{O}(h'^* K_D) = \mathcal{O}_D$, we have $H^2(D', \mathcal{O}_{D'})^H = H^2(D_0, \mathcal{O}_{D_0}) \neq 0$. Therefore the singularity of X_1/H is of index 1 by Lemma 3.8. Thus H must be $< 1 >$ by the definition of the index of (X, x). This yields that $r = 2$.

Proof of Theorem 5.2. (2). Let (X, x) be of type (0, 1) with index $r > 1$.

First, claim that E_J is a chain or circle of surfaces whose components are elliptic ruled or rational, and double curves are elliptic or rational.

By (2) of Theorem 5.1, any component of E_J is rational or elliptic ruled and any intersection curve is rational or elliptic. Since an element of G induces an automorphism of the dual graph $\Gamma_{E'_J}$ of E'_J, the quotient E'_J/G is either a circle or a chain of normal surfaces. On the other hand, E_J has no triple point. In fact, if E_J has components E_1, E_2, E_3 with $E_1 \cap E_2 \cap E_3 \neq \phi$, then there must be a connected configuration D in E'_J/G which intersects $[E_1]$, $[E_2]$ and $[E_3]$, where $[E_i]$ is the proper transform of E_i by the canonical morphism $\psi: E'_J/G \to E_J$. This is a contradiction to the configuration of E'_J/G. Considering that ψ is an isomorphism between an open subset of E'_J/G and an open dense subset on E_J, E_J is a chain or a circle of surfaces which are rational or elliptic ruled. Here, we note that E_J does not contain the configuration noted in (2) of Theorem 5.1. In fact, if E_J contains the configuration E' noted in (2) of Theorem 5.1, then $H^2(E', \mathcal{O}_{E'}) = \boldsymbol{C}$, which is a contradiction to Proposition 3.7.

Lemma 5.14. *Let E_j be a rational component of E_J which intersects to two other components of E_J. Assume that there exists an elliptic ruled component E'_j of E'_J which is mapped onto E_j.*

Then, E_j intersects the other components of E at rational curves.

Proof. Let C be a double curve of E_J which lies on E_j. Then, there is a double curve $\tilde{C}$ of E'_J which lies on E'_j and is mapped onto C. Let H be the subgroup of G which fixes a component E'_j. Then H fixes each double curve of E'_J on E'_j, by the assumption on E_j. Since $\tilde{C}$ is a section of an elliptic ruled surface E'_j, there exists an H-equivariant isomorphism $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \simeq H^1(E'_j, \mathcal{O}_{E'_j})$. By the definition of E'_j, E'_j/H is a rational surface. Therefore, $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^H = H^1(E'_j, \mathcal{O}_{E'_j})^H = 0$. Considering that $H^1(C, \mathcal{O}_C) = H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})^H$, we get that C is a rational curve.

Now we are going to establish the assertions (2a) (2b) and (2c) in Theorem 5.2.

First, assume that E_J is a circle of components. Then E'_J is a circle of elliptic ruled surfaces noted in (2b) of Theorem 5.1. If E_J contains an elliptic ruled component E_j, then E_j must intersect the other components of E_J at elliptic curves. Therefore, by Lemma 5.14, there is no rational component in E_J. Now E_J is a circle of elliptic ruled surfaces with elliptic curves as double curves. Moreover E_J has a ruling of an elliptic curve C/G where $p: E'_J \to C$ is a projection noted in (2b) of Theorem 5.1. In fact, any automorphism on E'_J maps a ruling of E'_J to a ruling of E'_J. Here, we have a compatible action of G on C, and induced morphism $E_J \to C/G$ determines a ruling on E_J. However, the ruling of E_J over an elliptic curve implies $H^2(E_J, \mathcal{O}_{E_J}) = \boldsymbol{C}$, which is a contradiction by Proposition 3.7. Consequently, in this case, the circle E_J consists of rational components and double curves are all rational.

Next, assume that E_J is a chain of components. To determine the configuration of E_J, we will divide it into three cases.

Case 1. E_J consists of rational components.

If E_J has an elliptic double curve, then E_J contains a configuration (2a) of Theorem 5.1. Therefore, double curves of E_J are all rational.

Case 2. E_J consists of elliptic ruled components. Obviously, double curves of E_J are all elliptic. The assertion about the index r of the singularity (X, x) is shown as follows.

Since the components of E_J are all elliptic ruled, E'_J is a circle of elliptic ruled surfaces noted in (2b) of Theorem 5.1. Then, a generator σ of G does not preserve the orientation of the dual graph $\Gamma_{E'_J}$. Let H be the subgroup of G generated by σ^2. Then, H preserve the orientation of $\Gamma_{E'_J}$. Considering that E_J consists of elliptic ruled component, E'_J/H is a circle of elliptic ruled surfaces. Therefore, $H^2(E'_J, \mathcal{O}_{E'_J})^H = \boldsymbol{C}$ which implies $H = \langle\sigma^2\rangle = 1$ by Lemma 3.8

Case 3. E_J contains both of rational and elliptic ruled components.

Noting that an elliptic ruled component intersects the other components at elliptic curves, E_J consists of elliptic ruled surfaces with rational surfaces at edges and all double curves are elliptic, by Lemma 5.14. However, the configuration with rational components at both edges is just the same as (2a) of Theorem 5.1. Therefore E_J contains only one rational component at an edge. Now, we will consider the index r of the singularity. In this case, E'_J is (2a) of Theorem 5.1. Two rational components of E'_J are mapped to the rational component of E_J. So a generator σ of G induces the reflection of $\Gamma_{E'_J}$ at a center of the chain. Then the subgroup $H = \langle\sigma^2\rangle$ fixes all components of E'_J. Considering that all components except for one edge are elliptic ruled surfaces, E'_J/H is a chain of elliptic surface with rational surfaces at both edges and all double curves are elliptic. Therefore, $H^2(E'_J, \mathcal{O}_{E'_J})^H = \boldsymbol{C}$ which implies $\langle\sigma^2\rangle = 1$ by Lemma 3.8. Hence the index r is 2.

Proof of Theorem 5.2, (3). Let (X, x) be of type (0, 0) of index $r > 1$. Then the equality $r = 2$ follows immediately from Theorem 3.10. By (3) of Theorem 5.1, all components and all intersection curves are rational. The shape of the dual graph Γ_{E_J} of E_J follows from the consideration of the quotient of a compact orientable real surface by an involution which does not preserve the orientation.

§ 6. Constructions of periodically elliptic singularities

The aim of this section is to construct examples of periodically elliptic singularities in each class (I) ~ (XIX) of Table (1).

Table of periodically elliptic singularities (Table (1))

Theorem	type	E_J	index	$\dim R^1 f_* \mathcal{O}_{\tilde{X}}$	p_g		
5.1	0, 2	a $K3$-surface	1	0	1	Gorenstein	I
		an Abelan surface	1	2	1	not Gorenstein	II
	0, 1	(2a) in 5.1	1	0	1	Gorenstein	III
		(2b) in 5.1	1	2	1	not Gorenstein	IV
	0, 0	triangulation of an image of S^2	1	0	1	Gorenstein	V
		.... of $S^1 \times S^1$	1	2	1	not Gorenstein	VI
		.... of compact orientable of genus $g>1$	1	2g	1	not Gorenstein	VII
5.2	0, 2	an Enriques suface	2	0	0	rational	VIII
		a bielliptic suface	2, 3, 4 or 6	1	0	not rational	IX
		a rational surface	>1	0	0	rational	X
		an elliptic ruled surface	>1	1	0	not rational	XI
	0, 1	a chain of elliptic ruled surfaces	2	1	0	not rational	XII
		a chain of elliptic ruled surfaces with rational surface at one end	2	0	0	rational	XIII
		(2b) in 5.2	>1	0	0	rational	XIV
		(2c) in 5.2	>1	1	0	not rational	XV
	0, 0	triangulation of an image of RP^2	2	0	0	rational	XVI
		.... of Klein bottle	2	1	0	not rational	XVII
		.... of compact non-orient able surface of genus $g>1$	2	g	0	not rational	XVIII
		the others in 5.2, (3)	2	≥ 0	0	rational or not	XIX

First, we quote the following theorem of Persson and Kulikov.

Theorem 6.1 (Persson [P], Kulikov [K] Friedman-Morrison [F-M]). *Let π: $Z \to \Delta$ be a semi-stable degeneration of projective surfaces with $mK_Z = 0$, such that the special fiber Z_0 is projective.* (*The least such m is necessarily* 1, 2, 3, 4, *or* 6) *Then either*

I) *Z_0 is smooth;*

II) *Z_0 is a circle of elliptic ruled components, or a chain of elliptic* *ruled components (possibly with rational surfaces at one or both ends of the chain), and all double curves are smooth elliptic curves; or*

III) Z_0 *consists of rational surfaces meeting along rational curves which form a circle on each component. If* Γ *is the dual graph of* Z_0, then Γ *is a triangulation of* S^2, $\boldsymbol{RP}^2$, $S^1 \times S^1$, *or the Klein bottle.*

Theorem 6.2. *Let* $\pi: Z \to \Delta$ *be a degeneration of surfaces in Theorem* 6.1 *with the special fiber* Z_0.

Then there exist a periodically elliptic singularity (X, x) *and a minimal resolution* $g: Y \to X$ *with* $g^{-1}(x)_{\text{red}} \simeq Z_0$.

Proof. The singularity and its resolution can be constructed as in Theorem 5.2 of [I].

Let C be a very ample divisor on Z_0 which is non-singular on each irreducible component of Z_0 and intersects to the double curves normally. Let $h: Y \to Z$ be the blowing up with the center C. Denote the proper transform of Z_0 by E. Then the restriction $h': E \to Z_0$ of h is an isomorphism, since h' is the blowing up by a Cartier divisor C on E. By the isomorphism h', $N_{E/Y}$ is isomorphic to $N_{Z_0/Z} \otimes \mathcal{O}_{Z_0}(-C) = \mathcal{O}_{Z_0}(-C)$ which is negative by the definition of C. So the divisor E is exceptional in Y. Let (X, x) be the singularity obtained by contracting E in Y. We claim that the singularity (X, x) is periodically elliptic. In fact, by the definition of C, Y has only isolated rational singularities locally defined by $x_1x_2 - x_3x_4$ in $\boldsymbol{C}^4$ which are terminal. On the other hand, $K_Y = -E$ in $\boldsymbol{Q} \otimes \mathrm{Div}(Y)$. This means that (X, x) is $\boldsymbol{Q}$-Gorenstein and log-canonical and not log-terminal; i.e. periodically elliptic. Since $K_Y \otimes \mathcal{O}_E = \mathcal{O}_E(-E)$ is ample on E, K_Y is relatively nef with respect to the contraction morphism $g: Y \to X$. Now, we find that g is a minimal resolution of a periodically elliptic singularity (X, x).

Remark 6.3. Now, by Theorem 6.2, we have periodically elliptic singularities I, II, III, IV, V, VI, VIII, IX, XII, XIII, XVI, XVII in Table (1).

Examples of VII and XVIII can be constructed by Tsuchihashi's method ([T]).

Next, we will construct examples of the other classes.

Example 6.4 (class X and XI). Let S be a rational or elliptic ruled surface with an exceptional set C such that the contraction $g: S \to S'$ of C is in the category of porjective varieties. Assume $K_S = ((1-r)/r)C$ $(r \geq 2)$. For example, let r be 2 (or 4) and C_1 be a union of 6-lines of general positions. (or 4, respectively) on P^2, and S be the blowing-up of P^2 at the double points. Let C be the proper transform of C_1 in S. Then S and C satisfy the above conditions.

For another example, let r be 3 and C_1 be a union of suitable 3-sections on a elliptic ruled surface R of degree 2, S be the blowing up of R at $C_1\cap C_2$ where C_2 is a ruling. Let C be the proper transform of C_1 on S. Then S and C satisfy our conditions too.

In the product $S\times P^1$, denote the divisor $S\times\{a\}$ ($a\in P^1$) by S_a. Then,

$$K_{S\times P^1}=-S_{a_0}-\frac{1}{r}(S_{a_1}+S_{a_2}+\cdots+S_{a_r})-\frac{r-1}{r}(C\times P^1)$$

for distinct $a_i\in P^1$ ($i=0, 1, 2, \cdots, r$).

On the other hand, take an ample divisor C' on S' which does not pass the singular points $g(C)$. Put $C''=g^*C'$ in S_{a_0}. Now, let Y be the blowing up of $S\times P^1$ at $C''\cup\{(\bigcup_{i=1}^r S_{a_i})\cap(C\times P^1)\}$ and E_0 and E_1 be the proper transform on Y of S_{a_0}, $C\times P^1$ respectively. Then the normal bundle of E_0+E_1 is negative and in a neighbourhood of E_0+E_1, we have

$$K_Y=-E_0-\frac{r-1}{r}E_1.$$

Therefore, the singularity (X, x) obtained by contracting E_0+E_1 is a periodically elliptic singularity with one rational (or elliptic ruled) essential component (X, XI in Table (1)).

Example 6.5 (class XIX). Let E_0 be the sum of two general hyperplanes in P^3, and H_i be general hyperplanes for $i=1, 2, \cdots, 4$ such that $\sum_{i=1}^4 H_i+E_0$ is of normal crossings. Then, in $Q\otimes\mathrm{Div}(P^3)$,

$$K_{P^3}=-E_0-1/2\sum_{i=1}^4 H_i.$$

Let p: $Y\to P^3$ be the successive blowing up at the intersections of two $[H_i]$'s, where $[H_i]$ means the proper transform of H_i on each stage. Here, on Y, the proper transforms $[H_i]$'s are disjoint each other and

$$K_Y=-[E_0]-1/2\sum_{i=1}^4 [H_i].$$

Let L be a very ample divisor on Y such that $L+[E_0]$ are simple normal crossings. Take the blowing-up q_1: $Y_1\to Y$ of Y at $L_0=[E_0]\cap L$. Next taking the blowing-up q_2: $Y_2\to Y_1$ of Y_1 at $\bigcup_{i=1}^4 [H_i]\cap[E_0]$, we get

$$K_{Y_2}=-[E_0]-1/2\sum_{i=1}^4 [H_i]-1/2\sum_{i=1}^4 E_i,$$

where E_i is the exceptional divisor on Y_2 mapped onto $[E_0]\cap[H_i]\subset Y_1$.

Last, take the blowing-up q_3: $Y_3\to Y_2$ of Y_2 at $\bigcup_{i=1}^4([H_i]\cap E_i)$. Consequently, by the similar argument to Theorem 6.2, Y_3 has at worst isolated cDV singularities, the normal bundle $N_{[E_0]/Y_3}$ are negative and

$$K_{Y_3}=-[E_0]-1/2\sum_{i=1}^4([H_i]+[E_i]) \qquad \text{in } Q\otimes\mathrm{Div}(Y_3).$$

Noting that $(\cup[H_i])\cap(\bigcup_{i=1}^4[E_i])=\phi$, we have

$$K_U=-[E_0]-1/2\sum_{i=1}^4[E_i] \tag{1}$$

on a neighbourhood $U\subset Y_3$ of $\bigcup_{i=0}^4[E_i]$.

Now, we will show that $(U, \sum_{i=0}^4[E_i])$ gives a partial good resolution of a periodically elliptic singularity (X, x) in the class XIV of Table (1). We can easily check that there is a morphism f': $U\to Z$ which contracts the divisor $\sum_{i=1}^4[E_i]$ to a curve in Z isomorphic to $\bigcup_{i=1}^4[E_i]\cap[E_0]$. Then,

$$\begin{aligned} N_{[E_0]/Z} &\simeq N_{[E_0]/U}+1/2\sum_{i=1}^4[E_i]|_{[E_0]} \\ &\simeq N_{[E_0]/Y}-1/2\sum_{i=1}^4[H_i]|_{[E_0]}, \end{aligned}$$

which is negative. Therefore $[E_0]$ is contractible in Z to a singular point (X, x). By (1) and the definition of E_0, (X, x) is a periodically elliptic singularity in the class XIV.

Example 6.6 (class XIV). Let C' be the sum of three general lines in P^2. Take a blowing-up b: $S\to P^2$ at sufficiently many general points on C' so that $N_{C/S}$ is negative, where C is the proper transform of C'. Let Y be the product $S\times P^1$. Denote the divisors $C\times P^1$ and $S\times\{a\}$ $(a\in P^1)$ on Y by E_0 and S_a respectively. Fix distinct 4-points $a_1, \cdots, a_4\in P^1$. Then,

$$K_Y=-E_0-1/2\sum_{i=1}^4 S_{a_i}.$$

Now, take the blowing-up, $Y_1\to Y$ at curves $\cup(S_{a_i}\cap E_0)$ and $\bigcup_{j=1}^m(S_{b_j}\cap E_0)$ for sufficiently many general points $b_j\in P^1$. Then,

$$K_{Y_1}=-[E_0]-1/2\sum_{i=1}^4([S_{a_i}]+E_i),$$

where E_i is the exceptional divisor which is mapped onto $S_{a_i}\cap E_0$. Next, again take the blowing up $Y_2\to Y_1$ at $([S_{a_i}]\cap E_i)$. Consequently, Y_2 has at worst isolated cDV points as singularities and

$$K_{Y_2} = -[E_0] - 1/2 \sum_{i=1}^{4} ([S_{a_i}] + [E_i]).$$

Noting that $(\bigcup [S_{a_i}]) \cap (\bigcup [E_i]) = \phi$, we have

(2) $$K_U = -[E_0] - 1/2 \sum_{i=1}^{4} [E_i],$$

on a neighbourhood U of $\bigcup_{i=0}^{4} [E_i]$. We can show that the divisor $\sum_{i=0}^{4} [E_i]$ is contractible to a singular point in the similar way to Example 6.5. By the above representation of a canonical divisor K_U and the definition of E_0, the singularity belongs to the class XV.

Note that the singularity constructed above has index 2. We remark that the similar construction give us a singularity in the class XV which has index 3.

Example 6.7 (class XIX). Let E_0 be the sum of three general hyperplanes in P^3, and H_1, H_2 be two general hyperplanes. Then, $K_{P^3} = -E_0 - 1/2(H_1 + H_2)$. Take a suitable blowing up $Y \to P^3$ and an open subset U of Y similarly to Example 6.5. Then U is a partial good resolution of a periodically elliptic singularity with the essential part $[E_0]$ which is isomorphic to E_0. This shows that the singularity belongs to the class XIX.

References

[A1] Artin, M., On isolated rational singularities of surfaces, Amer. J. Math., **88** (1966), 129–136.

[A2] ——, Algebraization of formal moduli, II: Existence of formal modifications, Ann. of Math., **91** (1979), 88–135.

[DB] Du Bois, P., Complexe de de Rham filtré d'une variété singulière, Bull. Soc. Math. France., **109** (1981), 41–81.

[EGA] Grothendieck, A., Éléments de géométrie algebriques IV, Publ. Math. IHES., **24** (1965).

[F-M] Friedman, R. and Morrison, R., The birational geometry of degenerations: an over view in The Birational Geometry of Degenerations, 1–32, ed. Friedman et al. Birkhäuser (1983).

[G] Grauert, H., Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann., **146** (1962), 331–368.

[H1] Hironaka, H., Formal line bundles along exceptional loci, Proc. of Bombay Colloquim. OUP, (1968).

[H2] ——, Desingularization Theorems, Memoiras de Mathematica del Instituto "Jorge Juan" **30**, (1977).

[I] Ishii, S., On isolated Gorenstein singularities, Math. Ann. **270** (1985), 541–554.

[Ka1] Kawamata, Y., Elementary contractions of algebraic 3-fold, Ann. of Math., **119** (1984), 95–110.

[Ka2] ——, The cone of curves of algebraic varieties, Ann. of Math., **119** (1984), 603–633.

[K] Kulikov, V., Degenerations of K3 surfaces and Enriques surfaces, Math. USSR Izv., **11** (1977), 957–989.

[P] Persson, U., On degenerations of alegbraic surfaces, Memoirs Amer. Math. Soc., **189** (1977).

[R1] Reid, M., Canonical 3 folds, Proc. Conf. Alg. Geom., Angers ed. A Beauville, Sijithoff and Nordhoff, (1979).

[R2] ——, Minimal models of canonical 3-folds, Symposia in Math. **1**, Kinokuniya-North Holland, (1981).

[Sa] Sakai, F., Weil divisors on normal surfaces, Duke Math. J., **51** (1984), 877–887.

[SB1] Shephered-Barron, N. I., Degenerations with numerically effective canonical divisor, in Birational Geometry of Degenerations, 33–84 ed. Friedman et al. Birkhäuser, (1983).

[SB2] ——, in this volume,

[S] Steenbrink, J., Mixed Hodge structures associated with isolated singularities, Proc. Sym. in Pure Math., **40** Part 2 (1983), 513–536.

[T] Tsuchihashi, H., in this volume,

[T-M] Tsunoda, S. and Miyanishi, M., The structure of open algebraic surfaces, II. in Classification of algebraic and analytic varieties, 499–544, Birkhäuser, (1983).

[U] Umezu, Y., On normal projective surfaces with trivial dualizing sheaf, Tokyo J. Math., **4** (1981), 343–354.

[W1] Watanabe, K., On plurigenera of normal isolated singularities I, Math. Ann., **250** (1980), 65–94.

[W2] ——, On plurigenera of normal isolated singularities II, in this volume,

Department of Mathematics
Tokyo Metropolitan University
Fukazawa-2, Setagaya Tokyo 158, Japan