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Du Bois Singularities on a Normal Surface

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Introduction

The purpose of this paper is to investigate Du Bois singularities on normal analytic surfaces. Du Bois introduced the concept of what we call Du Bois singularity by means of his differential complex ([2]). A normal isolated singularity (X, x) is Du Bois if and only if the canonical maps $R^i f_* \mathcal{O}_{\tilde{X}} \to H^i(E, \mathcal{O}_E)$ are bijective for all $i > 0$, where $f: \tilde{X} \to X$ is a good resolution meaning that the divisor $E = f^{-1}(x)_{\text{red}}$ is of normal crossings.

In this paper, a Du Bois singularity is characterized by the property that any holomorphic 2-form on $\tilde{X} - E$ has poles on E of order at most one (Theorem 1.8). Then, we show that any resolution of a Du Bois singularity is a good resolution. We also show that any connected subdivisor of the fibre $E = f^{-1}(x)_{\text{red}}$ of a Du Bois singularity (X, x) is contracted to a Du Bois singularity.

In Theorem 3.2, we get a numerical sufficient condition for a connected configuration of curves with normal crossings on a non-singular surface to be contracted to a Du Bois singularity, which gives examples of Du Bois singularities with arbitrary geometric genus. However, Du Bois' condition is not completely determined by numerical conditions on E. In Section 4, we have examples of Du Bois and non-Du Bois singularities with the same numerical conditions on E (Proposition 4.2).

In this paper, we work only on surface singularities, so, "a singularity" always means a normal singularity on an analytic surface.

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§ 1. General 2-forms around a Du Bois singularity

In this section, we introduce the concept of a general 2-form around

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a singularity on a normal surface and characterize a Du Bois singularity by means of a general 2-form.

Definition 1.1. Let (X, x) be a singurality. For a holomorphic 2-form θ on $X-\{x\}$, denote the associated Weil divisor to θ by D_θ. In other words, D_θ is the closure of the set {zeros of θ on $X-\{x\}$} in X.

(1.2) Fix a resolution $f: \tilde{X}\to X$ of the singularity (X, x). Let E be the reduced divisor $f^{-1}(x)_{\text{red}}$ and decompose E into irreducible components $E_1, \cdots, E_s$ $(s\geq 1)$. We can regard a holomorphic 2-form θ on $X-\{x\}$ as a meromorphic 2-form on $\tilde{X}$ with the poles on E by the isomorphism $\Gamma(X-\{x\}, \mathcal{O}(K_X))\simeq\Gamma(\tilde{X}-E, \mathcal{O}(K_{\tilde{X}}))$. Since $\dim \Gamma(\tilde{X}-E, \mathcal{O}(K_{\tilde{X}}))/\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))$ is finite, the order of the pole of $\theta\in\Gamma(\tilde{X}-E, \mathcal{O}(K_{\tilde{X}}))$ at each E_i is bounded. Let m_i be the minimal value of $v_{E_i}(\theta)$ for all $\theta\in\Gamma(X-\{x\}, \mathcal{O}(K_X))$, where v_{E_i} is the valuation associated to the divisor E_i.

Definition 1.3. A holomorphic 2-form θ on $X-\{x\}$ is called a general holomorphic 2-form if $v_{E_i}(\theta)=m_i$ for every i. We can easily check that the subset Δ consisting of all general holomorphic 2-forms is a dense subset of $\Gamma(X-\{x\}, \mathcal{O}(K_X))$.

Proposition 1.4. *Let us use the notation of* (1.2) *and* (1.3). *Then, for a holomorphic 2-form θ on $\tilde{X}-E$, a canonical divisor is represented as*

$$K_{\tilde{X}}=\sum v_{E_i}(\theta)E_i+[D_\theta],$$

where $[D_\theta]$ is the proper transform of the Weil divisor D_θ.

Proof. The 2-form θ induces an isomorphism

$$\mathcal{O}_{\tilde{X}}\simeq\mathcal{O}(K_{\tilde{X}}-\sum v_{E_i}(\theta)E_i-[D_\theta]).$$

Definition 1.5. For a resolution $f: \tilde{X}\to X$ with $E=f^{-1}(x)_{\text{red}}$, we define the index sets I, J as $I=\{i\,|\,m_i\geq 0\}$, $J=\{j\,|\,m_i<0\}$ and denote the reduced divisors $\sum_{i\in I}E_i$, $\sum_{j\in J}E_j$ by E_I, E_J respectively. We call E_J the essential part of E.

The configuration of E_I and E_J of a singularity will be discussed in Section 3.

Now we are going to investigate Du Bois singularities. To this end, we define a good resolution of a singularity first.

Definition 1.6. A resolution $f: \tilde{X}\to X$ of a singularity (X, x) is called a good resolution if the reduced divisor $E=f^{-1}(x)_{\mathrm{red}}$ crosses normally with itself.

We take the following characterization of Du Bois singularity as its definition.

Proposition 1.7 (Steenbrink [4]). *A normal singularity (X, x) of dimension two is Du Bois if and only if the canonical map $R^1f_*\mathcal{O}_{\tilde{X}}\to H^1(E, \mathcal{O}_E)$ is an isomorphism, where $f\colon \tilde{X}\to X$ is a good resolution.*

Theorem 1.8. *Let $f\colon \tilde{X}\to X$ be a good resolution of a normal singularity (X, x) of dimension two, E the reduced divisor $f^{-1}(x)_{\mathrm{red}}$ and $E=\sum E_i$ the irreducible decomposition.*

Then, the singularity (X, x) is Du Bois if and only if a general holomorphic 2-form on $\tilde{X}-E$ has poles of order at most one.

Proof. By taking X sufficiently small, we may assume that the singularity (X, x) is Du Bois if and only if

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})\simeq H^1(E, \mathcal{O}_E).$$

The left hand side $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is the dual of $\Gamma(\tilde{X}-E, K_{\tilde{X}})/\Gamma(\tilde{X}, K_{\tilde{X}})$ ([5]). For the right hand side, consider the exact sequence;

$$0\to\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))\to\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))\to\Gamma(E, \mathcal{O}(K_E))\to H^1(\tilde{X}, \mathcal{O}(K_{\tilde{X}})).$$

Here, $H^1(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))=0$, by the Grauert-Riemenschneider vanishing theorem. By the duality for a compact Gorenstein variety E, $H^1(E, \mathcal{O}_E)$ is the dual of $\Gamma(E, \mathcal{O}(K_E))=\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))/\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))$. Combining the relation $\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))\subset\Gamma(\tilde{X}-E, \mathcal{O}(K_{\tilde{X}}))$ and the isomorphism $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})\simeq H^1(E, \mathcal{O}_E)$, we see that $\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))=\Gamma(\tilde{X}-E, \mathcal{O}(K_{\tilde{X}}))$. Now the singularity (X, x) is Du Bois if and only if, for every holomorphic 2-form θ on $\tilde{X}-E$ has poles on E with order at most one.

Corollary 1.9. *Under the same notation as in Theorem* 1.8, *take a canonical divisor $K_{\tilde{X}}$ as*

$$K_{\tilde{X}}=\sum r_iE_i+D$$

with an effective divisor D which does not contain any components of E.

Then the singularity (X, x) is Du Bois if and only if, for any canonical divisor $K_{\tilde{X}}$ as above, the inequality $r_i\geq -1$ holds for any i.

Proof. This follows immediately from Theorem 1.8.

Proposition 1.10. *If a singularity (X, x) satisfies $\delta_m(X, x)\leq 1$ for* *every $m \in N$, then (X, x) is Du Bois. In particular, for a Gorenstein singularity (X, x), the converse holds.*

Proof. Take a good resolution $f: \tilde{X} \to X$ of the singularity (X, x). Assume that the singularity (X, x) is not Du Bois. Then there is a meromorphic 2-form θ on $\tilde{X}$ which is holomorphic on $\tilde{X}-E$ and has a pole on E of order ≥ 2. Hence, for a holomorphic 2-form ω on $\tilde{X}$, the m-ple 2-form $\omega\theta^{m-1}$ has a pole on E of order $>m-1$, if we take m sufficiently large. Now, we obtain two elements θ^m, $\omega\theta^{m-1}$ of $\Gamma(\tilde{X}-E, \mathcal{O}(mK_{\tilde{X}}))/\Gamma(\tilde{X}, \mathcal{O}(mK+(m-1)E))$ which are linearly independent over $\boldsymbol{C}$. This implies that $\delta_m(X, x) \geq 2$.

For the second assertion, see [3].

Note that the converse is not true, in general. Indeed, there is a Du Bois singularity with arbitrarily large geometric genus $p_g(=\delta_1)$ (c.f. Corollary 3.3).

§ 2. Resolutions of Du Bois singularities

Lemma 2.1. *Let $f: \tilde{X} \to X$ be a good resolution of a singularity (X, x). Let $g: \tilde{X} \to Y$ be a birational morphism contracting a connected divisor E' where $E' < E = f^{-1}(x)_{\text{red}}$. Denote $E-E'$ by E^*, the reduced Weil divisor $g(E^*)_{\text{red}}$ by $\bar{E}^*$ and the number of points of E^* corresponding to the point $y=g(E')$ by p.*

Then,

$$h^1(E, \mathcal{O}_E) \leq h^1(E', \mathcal{O}_{E'}) + h^1(\bar{E}^*, \mathcal{O}_{\bar{E}^*}), \tag{1}$$

where the equality holds if and only if $h^0(\bar{E}^*, g_*\mathcal{O}_{E^*}/\mathcal{O}_{\bar{E}^*}) = p-1$.

Proof. Because E is of normal crossings, we have the exact sequence of Mayer-Vietois type;

$$\begin{aligned} 0 &\to \Gamma(E, \mathcal{O}_E) \to \Gamma(E', \mathcal{O}_{E'}) \oplus \Gamma(E^*, \mathcal{O}_{E^*}) \to \Gamma(E' \cap E^*, \mathcal{O}) \\ &\to H^1(E, \mathcal{O}_E) \to H^1(E', \mathcal{O}_{E'}) \oplus H^1(E^*, \mathcal{O}_{E^*}) \to 0. \end{aligned}$$

Then we have the equality

$$h^1(E, \mathcal{O}_E) = h^1(E', \mathcal{O}_{E'}) + h^1(E^*, \mathcal{O}_{E^*}) + p - q, \tag{2}$$

where q is the number of the connected components of E^*.

Next consider the restricted morphism $g' = g|_{E^*}: E^* \to \bar{E}^*$. Then we have an exact sequence;

$$\begin{aligned} 0 &\to \Gamma(\bar{E}^*, \mathcal{O}_{\bar{E}^*}) \to \Gamma(E^*, \mathcal{O}_{E^*}) \to \Gamma(\bar{E}^*, g'_*\mathcal{O}_{E^*}/\mathcal{O}_{\bar{E}^*}) \\ &\to H^1(\bar{E}^*, \mathcal{O}_{\bar{E}^*}) \to H^1(E^*, \mathcal{O}_{E^*}) \to 0. \end{aligned}$$

Since

$$(3) \qquad h^0(\bar{E}^*, g'_*\mathcal{O}_{E^*}/\mathcal{O}_{\bar{E}^*}) \geq p-1,$$

the exact sequence yields

$$(4) \qquad h^1(E^*, \mathcal{O}_{E^*}) + p - q \leq h^1(\bar{E}^*, \mathcal{O}_{\bar{E}^*}).$$

Combining (2) and (4), we show that

$$h^1(E, \mathcal{O}_E) \leq h^1(E', \mathcal{O}_{E'}) + h^1(\bar{E}^*, \mathcal{O}_{\bar{E}^*})$$

as desired. Clearly the equality of (1) holds if and only if the equality in (3) holds.

Theorem 2.2. *Let $f\colon \tilde{X} \to X$ be a good resolution of a Du Bois singularity (X, x). Let E' be a connected divisor in $\tilde{X}$ with $E' < E = f^{-1}(x)_{\mathrm{red}}$.*

Then the singularity (Y, y) obtained by contracting the divisor E' in $\tilde{X}$ is also Du Bois and the equality of (1) *in Lemma* 2.1 *holds.*

Proof. First, consider the spectral sequence;

$$E_2^{p,q} = H^p(Y, R^q g_* \mathcal{O}_{\tilde{X}}) \Rightarrow E^{p+q} = H^{p+q}(\tilde{X}, \mathcal{O}_{\tilde{X}}).$$

Take the trivial edge sequence;

$$0 \to H^1(Y, \mathcal{O}_Y) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to \Gamma(Y, R^1 g_* \mathcal{O}_{\tilde{X}}) \to 0.$$

Then we have that $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = h^1(Y, \mathcal{O}_Y) + h^0(Y, R^1 g_* \mathcal{O}_{\tilde{X}})$, where the left hand side is equal to $h^1(E, \mathcal{O}_E)$, since the singularity (X, x) is Du Bois.

Since Y and $\tilde{X}$ are of two-dimension,

$$(6) \qquad h^1(Y, \mathcal{O}_Y) \geq h^1(\bar{E}^*, \mathcal{O}_{\bar{E}^*}),$$

$$(7) \qquad h^0(Y, R^1 g_* \mathcal{O}_{\tilde{X}}) \geq h^1(E', \mathcal{O}_{E'}).$$

Therefore, by Lemma 2.1, all the equalities in (1), (6) and (7) hold. Particularly, $h^0(Y, R^1 g_* \mathcal{O}_{\tilde{X}}) = h^1(E', \mathcal{O}_{E'})$ which means the isomorphism $\Gamma(Y, R^1 g_* \mathcal{O}_{\tilde{X}}) = H^1(E', \mathcal{O}_{E'})$ because the map is always surjective.

Theorem 2.3. *Every resolution of a Du Bois singularity is a good resolution.*

Proof. Let $h\colon Y \to X$ be a resolution of a Du Bois singularity (X, x). Then Y is obtained by contracting some exceptional divisors of $\tilde{X}$, where $f\colon \tilde{X} \to X$ is a suitable good resolution. By Theorem 2.2, the equality $h^0(\bar{E}^*, g_* \mathcal{O}_{E^*}/\mathcal{O}_{\bar{E}^*}) = p-1$ holds. It is known that the singularity on a curve of embedded dimension two is ordinary double if and only if the above equality holds.

§ 3. Configuration of the exceptional curves of a Du Bois singularity

Proposition 3.1. *For a resolution* $f\colon \tilde{X}\to X$ *of a Du Bois singularity* (X, x), *let* E_J *be the essential part of* $E=f^{-1}(x)_{\mathrm{red}}$ *and let* E_I *be* $E-E_J$. *Then*

(i) $H^1(E, \mathcal{O}_E)\simeq H^1(E_J, \mathcal{O}_{E_J})$, *and*

(ii) *the singular points obtained by contracting* E_I *in* $\tilde{X}$ *are all rational.*

Proof. First, we note that for a divisor D on $\tilde{X}$,

$$\Gamma(D, \mathcal{O}(K_D))\simeq\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+D))/\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}})). \tag{1}$$

Because, in the exact sequence;

$$0\to\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))\to\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+D))\to\Gamma(D, \mathcal{O}(K_D))\to H^1(\tilde{X}, \mathcal{O}(K_{\tilde{X}})),$$

the last term $H^1(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))$ vanishes by the Grauert-Riemenschneider vanishing theorem.

For the assertion (i), apply (1) to the divisors E and E_J. Then, by the duality of a compact Gorenstein curve, we find that $H^1(E, \mathcal{O}_E)$ and $H^1(E_J, \mathcal{O}_{E_J})$ are dual to

$$\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))/\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}})) \quad \text{and} \quad \Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E_J))/\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))$$

respectively. By the definition of E_J, we have

$$\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E))=\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E_J)).$$

Therefore the assertion (i) follows.

Next, to show the assertion (ii), we have to remember the definition of E_I. Since $\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}))=\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E_I))$, we get $H^1(E_I, \mathcal{O}_{E_I})=\Gamma(E_I, \mathcal{O}(K_{E_I}))^*=0$. By Theorem 2.2, the morphism $g\colon \tilde{X}\to Y$ which contracts E_I satisfies $\Gamma(Y, R^1g_*\mathcal{O}_{\tilde{X}})\simeq H^1(E_I, \mathcal{O}_{E_I})=0$. This completes the proof of (ii).

Theorem 3.2. *Let* $f\colon \tilde{X}\to X$ *be a good resolution of a singularity* (X, x), *E the reduced exceptional divisor* $f^{-1}(x)_{\mathrm{red}}$ *and* $E=\sum E_i$ *the irreducible decomposition.*

Assume $E_i^2<-2(E_i \sum_{j\neq i} E_j+\max\{g(E_i)-1, 0\})$ *for every i.*

Then, the singularity (X, x) *is Du Bois.*

Proof. Take a canonical divisor $K_{\tilde{X}}$ as

$$K_{\tilde{X}}=\sum a_i E_i + D,$$

where $a_i \in \boldsymbol{Z}$ and D is an effective divisor which does not contain any component E_i. We may assume that $a_1=\min_i \{a_i\}$. Then,

$$K_{\tilde{X}}=a_1E_1+a_1\sum_{j\neq 1} E_j+\sum_{j\neq 1}(a_j-a_1)E_j+D.$$

By the adjunction formula,

$$2g(E_1)-2=K_{\tilde{X}}E_1+E_1^2=(a_1+1)E_1^2+a_1\sum_{j\neq 1} E_jE_1+\sum_{j\neq 1}(a_j-a_1)E_jE_1+DE_1.$$

Since $\sum_{j\neq 1}(a_j-a_1)E_jE_1\geq 0$ and $DE_1\geq 0$, we have

$$2g(E_1)-2\geq(a_1+1)E_1^2+a_1\sum_{j\neq 1} E_jE_1.$$

If $a_1\leq -2$, then

$$\begin{aligned} E_1^2 &\geq 2\{g(E_1)-1\}/(a_1+1)-\{a_1/(a_1+1)\}\sum_{j\neq 1} E_jE_1 \\ &\geq -2\{\sum_{j\neq 1} E_jE_1+\max\{g(E_1)-1, 0\}\}. \end{aligned}$$

This contradicts assumption of the theorem. So all a_i must be ≥ -1. Consequently by Corollary 1.9, the singularity (X, x) is Du Bois.

Brenton ([6]) obtained a results similar to Theorem 3.2. Although the detail is unavailable, his proof seems a little different from ours.

Corollary 3.3. *Let Y be a non-singular surface and E be a reduced divisor on Y with normal crossings.*

Then, there exists a Du Bois singularity (X, x) with a good resolution $f\colon \tilde{X}\to X$ with $f^{-1}(x)_{\mathrm{red}}\simeq E$.

In particular, if E is a tree of non-singular rational curves, then the singularity (X, x) is rational.

Proof. By taking a blowing up $g\colon \tilde{X}\to Y$ of Y at sufficiently many general points on E, the proper transform $[E]$ of E turns out to have the properties noted in Theorem 3.2. Then the singularity (X, x) obtained by contracting the proper transform $[E]$ of E in $\tilde{X}$ is Du Bois. For the second assertion, we have only to note that a Du Bois singularity with $H^1(E, \mathcal{O}_E)=0$ is rational.

§ 4. Elliptic Du Bois singularities

It is well known that any rational singularity (i.e. a normal surface singularity (X, x) with $p_g(X, x)=0$) is Du Bois (Steenbrink [4]).

In this section, we investigate Du Bois singularities (X, x) with $P_g(X, x)=1$, which we call elliptic Du Bois singularity.

Let $f: \tilde{X}\to X$ be a resolution of an elliptic Du Bois singularity (X, x). Then $E=f^{-1}(x)_{\mathrm{red}}$ is of normal crossings and $h^1(E, \mathcal{O}_E)=1$. Therefore the divisor E is decomposed as $E=A_1+A_2$, where A_1 is either a non-singular elliptic curve or a circle of r-rational curves with $r\geq 1$ (a circle of 1-rational curve means a rational curve with an ordinary double point), and A_2 is void or a disjoint union of trees of non-singular rational curves. We note that each connected component of A_2 intersects A_1 at only one point.

However, the singularity which has such a configuration is not necessarily Du Bois. In fact, we have the following proposition.

Proposition 4.1. *Let $f: \tilde{X}\to X$ be a resolution of an elliptic singularity (X, x). Assume the divisor $E=f^{-1}(x)_{\mathrm{red}}$ has a decomposition $E=A_1+A_2$ as noted above.*

Then the following three conditions are equivalent:

(i) *(X, x) is an elliptic singularity.*

(ii) *(X, x) is a Du Bois singularity.*

(iii) *A_1 is the essential part of E.*

Proof. The equivalence between (i) and (ii) follows immediately from $H^1(E, \mathcal{O}_E)=\boldsymbol{C}$. To show the implications (ii)$\leftrightarrow$(iii), first we assume that the singularity (X, x) is Du Bois, and E_J is the essential part of E. Then, by Proposition 3.1, we see that $A_1\leq E_J$. Assume that $A_1<E_J$. By the definition of E_J, a general holomorphic 2-form on $\tilde{X}-E$ has poles on each component of E_J. Therefore, $\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+A_1))\subsetneqq\Gamma(\tilde{X}, \mathcal{O}(K_{\tilde{X}}+E_J))$. By an argument similar to (1) of the proof of Proposition 3.1, we can observe that $\Gamma(A_1, \mathcal{O}(K_{A_1}))\subsetneqq\Gamma(E_J, \mathcal{O}(K_{E_J}))=\boldsymbol{C}$. However, this contradicts the definition of A_1.

Next, assume that $A_1=E_J$. For a general holomorphic 2-form θ on $X-\{x\}$, take a canonical divisor on $\tilde{X}$;

$$K_{\tilde{X}}=\sum_{j\in J} a_jE_j+\sum_{i\in I} b_iE_i+[D_\theta],$$

where $\sum_{j\in J} E_j$ and $\sum_{i\in I} E_i$ are the irreducible decompositions of A_1 and A_2 respectively. Then by the assumption, $b_i\geq 0$ for each $i\in I$.

If A is irreducible, by the adjunction formula,

$$(a_1+1)A_1^2+\sum b_iE_iA_1+[D_\theta]A_1=0.$$

Since $[D_\theta]A_1\geq 0$, $\sum b_iE_iA_1\geq 0$, a_1 must be ≥ -1.

If A_1 is not irreducible, put $a=\min_j\{a_j\}$ and suppose that a_i $(i=1,$

$\cdots, s)$ attain the minimal value. Denote $\sum_{j=1}^{s} E_j$ by B. Then, again by the adjunction formula,

$$\begin{aligned}\deg K_B &= \sum_{j\in J} a_j E_j B + B^2 + \sum b_i E_i B + B[D_\theta] \\ &= (a+1)A_1 B + \sum_{\substack{j\geq s+1\\ j\in J}} (a_j - a - 1)E_j B + \sum b_i E_i B + [D_\theta]B \\ &\geq (a+1)A_1^2 - (a+1)(A_1 - B)B + \sum_{\substack{j\geq s+1\\ j\in J}} (a_j - a - 1)E_j B \\ &\quad + \sum b_i E_i B + [D_\theta]B.\end{aligned}$$

Note that $(A_1 - B)B \geq 0$, $\sum (a_j - a - 1)E_j B \geq 0$, $\sum b_i E_i B \geq 0$, and $[D_\theta]B \geq 0$. If $a \leq -2$, then $\deg K_B$ must be positive. This contradicts the fact that $\deg K_B \leq 0$ which followed from $B \leq A_1$. Therefore, $a \geq -1$. Now, by Corollary 1.9, we conclude that the singularity (X, x) is Du Bois.

Next, we investigate the special case where both A_1 and A_2 are irreducible.

Proposition 4.2. *Let* $f\colon \tilde{X} \to X$ *be a good resolution of a singularity* (X, x). *Assume that* $E = f^{-1}(x)_{\text{red}}$ *has a decomposition* $E = A_1 + A_2$, *where* A_1 *is either a non-singular elliptic curve or a rational curve with a node and* A_2 *is a non-singular rational curve.*

Then the singularity (X, x) *is not Du Bois if and only if the normal bundle* $N_{A_1/\tilde{X}}$ *is isomorphic to* $\mathcal{O}_{A_1}(-A_2)$.

Proof. Suppose that the singularity (X, x) is not Du Bois. Then for a general holomorphic 2-form θ on $X - \{x\}$, we can represent a canonical divisor as

$$K_{\tilde{X}} = aA_1 + bA_2 + [D_\theta],$$

with $b < 0$ by Proposition 4.1. By the adjunction formula on A_1, A_2 respectively, we get

$$(1) \qquad (a+1)A_1^2 + b + [D_\theta]A_1 = 0,$$

$$(2) \qquad a + (b+1)A_2^2 + [D_\theta]A_2 = -2.$$

Since $[D_\theta]A_1$, $[D_\theta]A_2 \geq 0$, from (1) and (2) we see that

$$(1)' \qquad b \leq -(a+1)A_1^2,$$

$$(2)' \qquad a \leq -(b+1)A_2^2 - 2.$$

Substituting (2)′ into (1)′, we obtain

$$b(A_1^2 A_2^2 - 1) \geq -A_1^2 A_2^2 - A_1^2.$$

Since the intersection matrix of E is negative definite, $A_1^2A_2^2-1>0$. Therefore,

$$b \geq \frac{-A_1^2A_2^2-A_1^2}{A_1^2A_2^2-1}=-1+\frac{-A_1^2-1}{A_1^2A_2^2-1}\geq -1.$$

Since b is a negative integer, we have $b=-1$ and $a=-2$. Then the equality must hold. So, $A_1^2=-1$, $[D_\theta]A_1=[D_\theta]A_2=0$. Therefore, the singularity (X, x) is Gorenstein and $\mathcal{O}_{A_1}\simeq\mathcal{O}(K_{A_1})\simeq\mathcal{O}_{A_1}(K_{\tilde{X}}+A_1)\simeq N^*_{A_1/\tilde{X}}\otimes\mathcal{O}_{A_1}(-A_2)$.

Conversely, assume $N_{A_1/\tilde{X}}\simeq\mathcal{O}_{A_1}(-A_2)$ and A_1 is a non-singular elliptic curve or a rational curve with an ordinary double point. If the singularity (X, x) is Du Bois, then the canonical map $\operatorname{Pic}(\tilde{X})\to\operatorname{Pic}(E)$ is bijective, by the commutative diagram;

$$\begin{array}{ccccccccc}
H^1(\tilde{X}, Z) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}^*_{\tilde{X}}) & \longrightarrow & H^2(\tilde{X}, Z) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \| & & \\
H^1(E, Z) & \longrightarrow & H^1(E, \mathcal{O}_E) & \longrightarrow & H^1(E, \mathcal{O}^*_E) & \longrightarrow & H^2(E, Z) & \longrightarrow & 0.
\end{array}$$

On the other hand, put $F=-2A_1-A_2$. Then $\mathcal{O}_{A_1}(F)\simeq N^*_{A_1/\tilde{X}}\simeq\mathcal{O}_{A_1}(K_{\tilde{X}})$ and $\mathcal{O}_{A_2}(F)\simeq\mathcal{O}_{P^1}(-2)\otimes N^*_{A_2/\tilde{X}}\simeq\mathcal{O}_{A_2}(K_{\tilde{X}})$. Therefore, $\mathcal{O}_E(K_{\tilde{X}})\simeq\mathcal{O}_E(F)$, which implies $\mathcal{O}_{\tilde{X}}(K_{\tilde{X}})=\mathcal{O}_{\tilde{X}}(F)$ by $\operatorname{Pic}(\tilde{X})\simeq\operatorname{Pic}(E)$. This contradicts the hypothesis that the singularity (X, x) is Du Bois.

Remark 4.3. Under the notation of Proposition 4.2, if A_1 is a rational curve with a node, the isomorphism $N_{A_1/\tilde{X}}\simeq\mathcal{O}_{A_1}(-A_2)$ is equivalent to $A_1^2=-1$.

On the other hand, for an elliptic A_1, the isomorphism $N_{A_1/\tilde{X}}\simeq\mathcal{O}_{A_1}(-A_2)$ induces $A_1^2=-1$, however, the converse does not hold. Therefore, the configurations A_1+A_2 with $A_1^2=-1$ and $A_2^2=-m$ $(m>1)$ is contracted to both of Du Bois and non-Du Bois singularities according to the choice of intersection point $A_1\cap A_2$ on an elliptic curve A_1.

Corollary 4.4. *Let $f\colon \tilde{X}\to X$ be a good resolution of a singularity (X, x). Assume the fiber $E=f^{-1}(x)_{\text{red}}$ contains a configuration A_1+A_2, where A_1 is either a non-singular elliptic curve or a rational curve with an ordinary double point with $N_{A_1/\tilde{X}}\simeq\mathcal{O}_{A_1}(-A_2)$ and $A_1\simeq P^1$.*

Then the singularity (X, x) is not Du Bois.

Proof. This is an immediate consequence of Theorem 2.2 and Proposition 4.2.

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