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A Conjecture about Compact Quotients by Tori

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Let $T \times X \to X$ be a meromorphic action of $T = (\boldsymbol{C}^*)^k$ on a reduced compact normal complex analytic space X. The following is a basic problem of geometric invariant theory whose answer is not known even if the above is an algebraic action on a projective variety X.

Problem. *Let $T \times X \to X$ be as above. Classify all T invariant open sets $\mathcal{U} \subseteq X$ such that geometric quotient*

$$\mathcal{U} \longrightarrow \mathcal{U}/T$$

exists as a compact complex analytic space.

In this paper we define a finite regular k complex, $\mathscr{C}(X)$, that we call **the moment complex.** We give a conjectural answer to the above question in terms of this k complex and a proof of part of the conjecture. We also discuss the classification of semi-geometric quotients in terms of this complex. For simplicity we assume that X is irreducible and that X can be equivariantly embedded into a complex Kähler manifold on which T acts. This latter assumption, which is always true if X is a normal projective variety, allows us to use moment functions to simplify arguments (cf. § 0).

The previous work on this problem consists of 3 parts:

a) as a special case of his geometric invariant theory [11], Mumford gave a prescription for construction of some of those quotients with $\mathcal{U}/T$ projective,

b) for $k=1$ a complete and simple answer is given in [2] for very general X [see also 4, 6],

c) for $k=2$ a complete answer has recently been given [see 3] in the case that X is a compact Kähler manifold.

The answers in b) and c) are given in terms of special cases of the moment complex, $\mathscr{C}(X)$.

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The organization of this paper is as follows.

In Section 0 we give the definitions of meromorphic actions and the quotients by them. We also give those background facts that make these definitions easy to use. We also summarize the chief results about the main tools we use, i.e. the Douady space of closures of orbits (following Fujiki [5] and Lieberman [9]) and moment functions (following Atiyah [1]). More details about these tools can be found in [2, 3].

In Section 1 we construct the moment complex, make our conjecture, and give a proof of it in one direction.

For simplicity of exposition we leaned heavily on the moment mapping in the definition of the moment complex. In [3] for $C^* \times C^*$ we gave a definition of the abstract complex of which the moment complex is a realization that doesn't refer to the moment mapping. This definition is far more involved and we used the moment mapping to show existence. Such a definition will be given in general in a sequel.

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§ 0. Background material

In this section we fix notation and collect background material. More details can be found in [2, 3].

(0.1) T will always denote the k dimensional torus $(C^*)^k$. T has an Iwasawa decomposition

$$T \approx K \times A$$

where $K \approx (S^1)^k$ is the maximal compact subgroup of T and where $A \approx (R^+)^k$. The character group of T is denoted by $\chi(T)$ and the group of one parameter subgroups of T is denoted by $\chi^*(T)$. The map:

$$\chi(T) \times \chi^*(T) \xrightarrow{\Gamma} Z$$

defined by $\Gamma(\chi, \alpha) = \chi \circ \alpha \in \chi(C^*) = Z$ is a perfect pairing. Since $T \approx (C^*)^k$ we have $\chi(T) \approx Z^k$ and $\chi^*(T) \approx Z^k$. Then $\chi_R(T) = \chi(T) \otimes_Z R$ and $\chi_R^*(T) = \chi^*(T) \otimes_Z R$ are dual k dimensional vector spaces. Using the fixed isomorphism $T \approx (C^*)^k$ we have a canonical basis $\{e_1, \cdots, e_k\} \subseteq \chi(T)$ with $e_1 = (1, 0, \cdots, 0), \cdots, e_k = (0, \cdots, 0, 1)$.

Using this choice of basis we get orientations on $\chi_R(T)$ and $\chi_R^*(T)$.

(0.2) Let X be a compact irreducible normal complex analytic space. Let $T\times X\to X$ be a holomorphic action of T on X. Throughout this paper it will be assumed that X can be embedded T equivariantly into a compact Kähler manifold on which T acts. This Kählerian assumption rules out many pathologies, [e.g. 13].

Let $\bar{T}$ denote any equivariant compactification of T as an algebraic variety that induces the usual linear algebraic group structure, e.g. the obvious compactification of T as $(\boldsymbol{P}_{\boldsymbol{C}}^1)^k$. The above action is said to be meromorphic if the holomorphic map $T\times X\to X$ extends meromorphically to $\bar{T}\times X$ for one and hence any such compactification $\bar{T}$. A key result of Sommese [12] is that if X is a compact Kähler manifold then a holomorphic action by T is meromorphic if and only if T has at least one fixed point on X. We denote the connected components of the fixed point set by $\{F_i \mid i=1, \cdots, r\}$.

Throughout this paper it is assumed that $T\times X\to X$ is a meromorphic action of T on an X as above.

The next two results are immediate consequences of the work of A. Fujiki [5] and D. Lieberman [9] on compactness of components of the Douady space of Kähler manifolds.

(0.2.2) **Theorem.** *Let X and T be as above. There is for any $x \in X$ with* dim $Tx=k$ *a diagram*:

*)
$$\begin{array}{ccc} \mathscr{X}_x & \xrightarrow{\phi_x} & X \\ {\scriptstyle f_x}\downarrow & & \\ \mathscr{Q}_x & & \end{array}$$

with the following properties:

a) *f_x is a flat surjective morphism of connected compact complex spaces $\mathscr{X}_x$ and $\mathscr{Q}_x$,*

b) *the restriction of ϕ_x to each fibre $\mathscr{X}_x(q)=f_x^{-1}(q)$ is an embedding and there is a $q \in \mathscr{Q}_x$ such that $\phi_x(\mathscr{X}_x(q))=\overline{Tx}$,*

c) *there is a natural action of T on $\mathscr{X}_x$ making f_x and ϕ_x equivariant with respect to the trivial action of T on $\mathscr{Q}_x$ and the given action of T on X,*

d) *there is a dense Zariski open set $\mathscr{O}_x\subseteq\mathscr{Q}_x$ such that for each $q \in \mathscr{O}_x$, $\mathscr{X}_x(q)$, is reduced and $\phi_x(\mathscr{X}_x(q))$ is the closure of a T orbit,*

e) *the reduction of every fibre of f_x is pure k dimensional and for fibres $\{\mathscr{X}_x(q), \mathscr{X}_x(q')\}$ that are reduced, $\phi_x(\mathscr{X}_x(q))=\phi_x(\mathscr{X}_x(q'))$ only if $q=q'$,*

f) *given any diagram*:

$$\text{**)} \qquad \begin{array}{ccc} \mathscr{X}' & \xrightarrow{\phi'} & X \\ {\scriptstyle f'}\downarrow & & \\ \mathscr{Q}' & & \end{array}$$

that satisfies properties a) *through* e) *there is a holomorphic map*:

$$c\colon \mathscr{Q}' \longrightarrow \mathscr{Q}_x$$

such that **) *is the pullback of* *).

There is one diagram from (0.2.2):

$$\begin{array}{ccc} \mathscr{X}_x & \xrightarrow{\phi_x} & X \\ {\scriptstyle f_x}\downarrow & & \\ \mathscr{Q}_x & & \end{array}$$

with $\mathscr{X}_x$ and $\mathscr{Q}_x$ compact irreducible spaces and ϕ_x a bimeromorphic holomorphic map. We will drop here the subscripts and refer to the diagram:

$$\text{(0.2.3*)} \qquad \begin{array}{ccc} \mathscr{X} & \xrightarrow{\phi} & X \\ {\scriptstyle f}\downarrow & & \\ \mathscr{Q} & & \end{array}$$

One consequence of these assumptions is that the analytic form of Sumihiro's theorem [14] holds.

(0.3) **Theorem** (M. Koras [8]). *Let* X *and* T *be as above. Given any* $x \in X^T$ *there is a* T *invariant Stein neighborhood* $\mathscr{U} \in X$ *of* x *and a proper embedding* Ψ *of* $\mathscr{U}$ *into* $\boldsymbol{C}^N$ *such that*:

a) $\Psi(x) = \mathit{origin}$,

b) Ψ *is equivariant with respect to the usual action of* T *on* X *and a linear action of* T *on* $\boldsymbol{C}^N$.

This result has the following immediate consequence by pulling back the usual Luna slice theorem on $\boldsymbol{C}^N$ [10].

(0.3.1) **Corollary.** *Let* X *and* T *be as above. Let* x *be a point on* X *with finite isotropy subgroup* $I_x \subseteq T$. *Then there exist a neighborhood* $\mathscr{U}$ *of* Tx *and an analytic subset* D *of* $\mathscr{U}$ *containing* x *such that*:

$$\mathscr{U} \approx Tx_{I_x} D.$$

The above result makes geometric quotients easy to handle.

(0.3.2) **Theorem** (Holmann [7]). *Let $\mathcal{U}$ be an open T invariant subset of X. Assume that* $\dim Tx=k$ *for all $x \in \mathcal{U}$ and that the set $\mathcal{U}/T$ of T orbits on $\mathcal{U}$ is Hausdorff with respect to the induced topology. Then there is a unique complex structure on $\mathcal{U}/T$ such that the quotient map:*

$$A: \mathcal{U} \longrightarrow \mathcal{U}/T$$

is holomorphic and such that given any open set V on $\mathcal{U}/T$ the space of holomorphic functions on V pulls back to the space of holomorphic functions on $A^{-1}(V)$ constant on fibres. $\mathcal{U}/T$ is normal. We call $\mathcal{U}/T$ with this analytic structure, the ***geometric quotient*** *of $\mathcal{U}$ by T.*

Let $\mathcal{U}$ be a T invariant open subset of X. By a **semi-geometric quotient** of $\mathcal{U}$ by T, we mean a holomorphic surjection $A: \mathcal{U} \to \mathcal{U}/T$ onto a normal complex analytic space such that:

a) given any $x \in \mathcal{U}/T$, $A^{-1}(x)$ is a connected union of orbits of T containing exactly one orbit that is closed in $\mathcal{U}$,

b) $\mathcal{U}/T$ has the quotient topology and given any $x \in \mathcal{U}/T$ there is a Stein neighborhood V such that $A^{-1}(V)$ is Stein, and the space of holomorphic functions on V pulls back to the space of holomorphic functions on $A^{-1}(V)$ invariant under T.

We do not spend much time on semi-geometric quotients by tori. This is not because they are not important but because the experience with one dimensional tori [4, 6] strongly indicates that the classification of semi-geometric quotients by tori can be reduced by straightforward arguments to the classification of geometric quotients by tori.

By W/G, where G is a compact group acting continuously on a complex analytic space W, we mean the space of G orbits with the induced topology. W/G is Hausdorff and has the structure of a simplicial complex if $G \subseteq T$ and the action of G is the restriction of a holomorphic action of T.

The following criteria from [3, (0.4)] for existence and compactness of quotients are the analogues of [2, (1.2)]; we use the notation of (0.2.3).

(0.4) **Theorem.** *Let $\mathcal{U} \subseteq X$ be a T invariant open set. A geometric quotient $\mathcal{U}/T$ exists if and only if for each $q \in \mathcal{Q}$, $\phi(\mathcal{Z}(q)) \cap \mathcal{U}$ is either empty or of the form Tx for some $x \in X$ with* $\dim Tx=k$. *Assuming that $\mathcal{U}/T$ exists then $\mathcal{U}/T$ is compact if and only if $\phi(\mathcal{Z}(q)) \cap \mathcal{U}$ is non-empty for all $q \in \mathcal{Q}$.*

We need some results on moment functions [3, Section 1].

(0.5.1) **Theorem.** *Let X be a connected compact Kähler manifold on which T acts meromorphically. Then there is a map $m: X \to \chi_{\boldsymbol{R}}(T) \approx \boldsymbol{R}^k$ that is called the moment map and which is constant on K orbits and connected components of X^T. Further given any $\mathscr{X}_x$ from* (0.2.4), *we have that*:

a) *$m(\phi_x(\mathscr{X}_x)) = m(\overline{Tx})$ is the convex hull of $m((\overline{Tx}) \cap X^T)$, and the last set is the set of vertices,* (i.e.) *extreme points of $m(\overline{Tx})$,*

b) *m factors*:

$$\begin{array}{ccc} \phi(\mathscr{X}_x(q)) & \xrightarrow{\ m\ } & m(\overline{Tx}) \\ & {}_a\searrow \quad \nearrow_b & \\ & \phi(\mathscr{X}_x(q))/K & \end{array}$$

with b a homeomorphism for all $q \in \mathscr{Q}_x$.

(0.5.2) **Theorem.** *Let X and T be as above. Given a one parameter subgroup $\alpha \in \chi^*(T)$, there is a projection*:

$$\alpha^*: \chi_{\boldsymbol{R}}(T) \longrightarrow \boldsymbol{R}$$

defined by $\alpha^*(\chi) = \Gamma(\chi, \alpha) = \chi \circ \alpha \in \chi(\boldsymbol{C}^*) = \boldsymbol{Z}$ for $\chi \in \chi(T)$. Let $\alpha^* \circ m: X \to \chi_{\boldsymbol{R}}(\alpha)$ be the composition of α^* and the moment function m. Then for all $x \in X - X^\alpha$, $\alpha^* \circ m(\alpha(t) \cdot x)$ is a monotone increasing function of $|t|$.

§ 1. The moment complex

Throughout this section $T \times X \to X$ is a meromorphic action of $T = (\boldsymbol{C}^*)^k$ on X, a compact normal complex analytic space. It is further assumed that there is a T equivariant embedding of X into a compact Kähler manifold, Y, on which T acts. Let

$$m: X \longrightarrow \chi_{\boldsymbol{R}}(T)$$

be a moment map as defined in the last section.

Let CELLS$(X, T) = \{Tx/K: x \in X\}$. We say that $c_1 = Tx_1/K$ and $c_2 = Tx_2/K$ are equivalent if $\dim c_1 = \dim c_2$ and

$$\{F_i \mid \overline{Tx_1} \cap F_i \neq \phi\} = \{F_i \mid \overline{Tx_2} \cap F_i \neq \phi\}.$$

This gives an equivalence relation on CELLS (X, T) which breaks CELLS (X, T) up into finitely many classes. By (0.5.1a) the moment function m induces a homeomorphism of each $\bar{c}_i = \overline{Tx_i}/K$ onto a compact convex closed cell.

Note that if c_1 and c_2 are equivalent m induces a homeomorphism of $\bar{c}_1$ and $\bar{c}_2$ onto $m(\bar{c}_1)$ which preserves all incidence relations. Let

$$\{e_1, e_2, \cdots, e_N\}$$

be representatives of the equivalence relation on CELLS (X, T). $\mathscr{C}(X)$ is constructed out of the closed cells $\bar{e}_i$ where this denotes a closed cell homeomorphic to $\overline{m(e_i)}$; we identify e_i with the obvious subset of $\overline{m(e_i)}$. To describe $\mathscr{C}(X)$ it is only necessary to know when a closed cell $\bar{e}_a$ is on the boundary of $\bar{e}_b$, and what is the attaching map. If there is a representative Tx/K of e_b and a representative c of e_a such that $c \in \overline{Tx/K} - Tx/K$ then we say that $\bar{e}_a$ is on the boundary of $\bar{e}_b$ and use m to give the attaching map. This makes sense since $m(e_a) \subseteq \overline{m(e_b)} - m(e_b)$.

Note that distinct e_i with $\dim e_i = k$ correspond to disjoint sets. Note that because of the convexity of the images of the $m(e_i)$, $\mathscr{C}(X)$ inherits a piecewise linear structure, i.e. $\mathscr{C}(X)$ is regular.

Remark. In the above, the only place where the moment map m was indispensable was in showing that $\mathscr{C}(X)$ is regular, i.e. has a piecewise linear structure. $C(X)$ is a covariant functor on the category of normal compact complex spaces which T equivariantly embed in Kähler manifolds.

We call the cells e_i the *simple subcomplexes* or *simple cells* of $C(X)$. The map of CELLS (X, T) onto $C(X)$ induced by m gives rise to a map

$$M: X \longrightarrow \mathscr{C}(X)$$

which is usually discontinuous. We call M the *refined moment map.*

(1.1) **Lemma.** *The moment map* $m: X \rightarrow \chi_{\mathbf{R}}(T)$ *factors* $m = Mm'$ *where* m' *is the continuous map from* $\mathscr{C}(X)$ *to* $\chi_{\mathbf{R}}(T)$ *induced by* m *and* M *is the refined moment map.*

There is a unique e_i with $\bar{e}_i$ mapped onto $m(X)$ by m; this can be seen from (0.2.3) and (0.5.1). We call this $\bar{e}_i$ the *principal cell* of $\mathscr{C}(X)$, and we denote it by P (or P_x if there is any possibility of confusion). By a *spanning subcomplex of* $\mathscr{C}(X)$ we mean any subcomplex B such that

a) B is homeomorphic with a closed k ball under the map m',

and

b) the boundaries of B and P are equal.

(1.2) **Example.** Given a fibre $Z(q)$ of the family of closures of orbits in (0.2.3), it can be seen using (0.5.1) that $M(\phi(\mathscr{Z}(q)))$ is a spanning subcomplex.

(1.2.1) **Example.** This example can be generalized. Let F be any spanning complex and let e_a be a simple cell of dimension k that is a subcomplex of F. Let $x \in X$ be such that $e_a = Tx/K$. Choose any fibre $\mathscr{Z}_x(q)$ of the family of closures of orbits in (0.2.2). F with $\bar{e}_a$ replaced by $M(\phi_x(\mathscr{Z}_x(q)))$ is a spanning complex. This construction applied to the principle cell gives the example of the last paragraph.

By a geometric open subcomplex of $\mathscr{C}(X)$ we mean an open subcomplex which meets each spanning subcomplex in exactly one simple k cell.

(1.3) **Conjecture.** *There is a one to one correspondence between the geometric open subcomplexes of $\mathscr{C}(X)$ and the T invariant open sets $\mathscr{U} \subseteq X$ such that the geometric quotient $\mathscr{U} \to \mathscr{U}/T$ exists and $\mathscr{U}/T$ is a compact complex analytic space. The correspondence is gotten by sending a geometric open subcomplex, F, to its inverse image $M^{-1}(F) \subseteq X$.*

(1.3.1) **Theorem.** *Let F be a geometric open subcomplex of $\mathscr{C}(X)$. Then $M^{-1}(F)$ is a T invariant Zariski open set. The geometric quotient $M^{-1}(F) \to M^{-1}(F)/T$ exists with $M^{-1}(F)/T$ a compact complex analytic space.*

Proof. Using (0.5.1) it follows that the complement of $M^{-1}(F)$ is a union of:

a) the orbits of T which have dimension less than k, and
b) finitely many sets of the form $\phi_x(\mathscr{Z}_x)$.

To see b) let G be a spanning subcomplex of $\mathscr{C}(X)$. Let e_a be any k dimensional simple cell of $G-F$. Let $x \in X$ be such that $M(Tx/K) = M(e_a)$. Then $\phi_x(\mathscr{Z}_x)$ is in the complement of $M^{-1}(F)$. If this wasn't true, then using the construction of example (1.2.1) we can construct a spanning subcomplex of $\mathscr{C}(X)$ which meets F in at least two simple cells e_b of dimension k in contradiction of the assumption that F is a geometric open subcomplex. From this it follows that $M^{-1}(F)$ is a T invariant Zariski open subset of X.

As noted in (1.2) given any $q \in \mathscr{Q}$ it follows that $M(\phi(Z(q)))$ is a spanning subcomplex. Using this, the theorem is an immediate consequence of (0.4). □

(1.3) Most of the technical problems that we encounter in trying to prove the other half of the conjecture stem from two facts:

a) not every spanning subcomplex is of the form $M(\phi(\mathscr{Z}(q)))$, and
b) given $y \in \mathscr{C}(X)$, the set $M^{-1}(y)$ can be disconnected.

In [3] we dealt with these problems successfully for $(C^*)^2$ and a smooth X.

We close this section with a conjectural classification of the semi-geometric quotients by T.

By a **semi-geometric open subcomplex** of $\mathscr{C}(X)$ we mean an open subcomplex F of $\mathscr{C}(X)$ such that for each spanning subcomplex B of $\mathscr{C}(X)$:

a) $F \cap B$ is non-empty and equal to the interior of its closure in $\mathscr{C}(X)$, and

b) $F \cap B$ contains at most one subcomplex that is closed in F.

(1.4) **Conjecture.** *There is a one to one correspondence between the semi-geometric open subcomplexes of $\mathscr{C}(X)$ and the T invariant Zariski open $\mathscr{U} \subseteq X$ such that the semi-geometric quotient $\mathscr{U} \to \mathscr{U}/T$ exists with $\mathscr{U}/T$ a compact complex analytic space. The correspondence is gotten by sending a semi-geometric open subcomplex, F, of $\mathscr{C}(X)$ to its inverse image $M^{-1}(F) \subseteq X$.*

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