Advanced Studies in Pure Mathematics 7, 1985
Automorphic Forms and Number Theory
pp. 149–174

On the Discriminant of Transformation Equations of Modular Forms

Yoshitaka Maeda

Introduction

In our previous paper [3], we have proved that the transformation equations of certain modular forms can be expressed by special values of the zeta functions of those forms. At the symposium, we talked about this result. Here we give some results obtained after that.

Let f be a modular form of weight k on the congruence subgroup $\Gamma_0(p)$ of $SL_2(\boldsymbol{Z})$. We assume that *p is an odd prime throughout the paper.* Then the transformation equation of f is defined by

$$\Phi(X;f)=\prod_{\alpha\in\Gamma_0(p)\backslash SL_2(\boldsymbol{Z})}(X-f|_k\alpha)=0,$$

where $(f|_k\gamma)(z)=\det(\gamma)^{k/2}f((az+b)/(cz+d))(cz+d)^{-k}$ for

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})=\{\gamma\in GL_2(\boldsymbol{R})\,|\,\det(\gamma)>0\}.$$

Obviously all coefficients σ_μ of $\Phi(X;f)$ are modular forms on $SL_2(\boldsymbol{Z})$, and therefore, the discriminant D of $\Phi(X;f)$ is also a modular form on $SL_2(\boldsymbol{Z})$ of weight $p(p+1)k$. We call that the transformation equation $\Phi(X;f)=0$ is *$\boldsymbol{Z}$-rational* if all coefficients σ_μ have $\boldsymbol{Z}$-rational Fourier expansions as modular forms (see § 1, for the $\boldsymbol{Z}$-rationality of Fourier expansions). Then one of our results is

Theorem 1. *If the transformation equation $\Phi(X;f)=0$ of f is $\boldsymbol{Z}$-rational and if p is an odd prime, then the discriminant D of $\Phi(X;f)$ is expressed as*

$$D=\begin{cases}(-1)^{(p-1)/2}p^p\Delta^{p+1}h^2, & \text{if } f \text{ is a cusp form,}\\ (-1)^{(p-1)/2}p^p\Delta^{p-1}h^2, & \text{otherwise,}\end{cases}$$

where h is a modular form on $SL_2(\boldsymbol{Z})$ with a $\boldsymbol{Z}$-rational Fourier expansion and Δ is Ramanujan's function $\exp(2\pi iz)\prod_{n=1}^{\infty}(1-\exp(2\pi inz))^{24}$.

Received February 1, 1984.

Especially, when f is the special cusp form discussed in [3], we even know the divisibility of the above form h by $\Delta^{(p+1)/2}$ (Proposition 8). We will prove this theorem in Section 2.

In Proposition 3, we will also show under the assumption in Theorem 1 the following congruence relation:

$$\Phi(X;f)\equiv(X-\sigma_1)(X^p-\sigma_p) \mod p.$$

Here both σ_1 and σ_p are certain modular forms on $SL_2(\boldsymbol{Z})$. Though this result follows from the Eichler-Shimura congruence relation [6, Theorem 7.9], we will give an elementary proof without using their result.

In [3], we considered the transformation equation $\Phi(X;f)=0$ for $f=gE^*_{\lambda,p}$, where g is a cusp form on $\Gamma_0(p)$ and $E^*_{\lambda,p}$ is a certain Eisenstein series. In Section 3, we will show that for a certain choice of g, the transformation equation $\Phi(X; gE^*_{\lambda,p})=0$ of $gE^*_{\lambda,p}$ is $\boldsymbol{Z}$-rational. In Section 4, we will give numerical examples of the transformation equations for the above $gE^*_{\lambda,p}$ and the specialized equations at several elliptic curves. (See [3, § 3], for the definition of the specialized equation at an elliptic curve.)

§ 1. Congruence relation of transformation equations

Let Γ be a subgroup of $SL_2(\boldsymbol{Z})$ containing

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, a\equiv d\equiv 1,\ b\equiv c\equiv 0 \bmod N\right\}$$

for some positive integer N. We denote by $\mathscr{M}_k(\Gamma)$ the vector space consisting of functions f on the upper half complex plane $\mathfrak{H}$ with the following three properties:

(i) f is holomorphic on $\mathfrak{H}$;

(ii) $f|_k\gamma=f$ for all $\gamma \in \Gamma$;

(iii) The Fourier expansion of $f|_k\gamma$ has the form $\sum_{n=0}^{\infty} a_\gamma(n)e(nz/N)$ at $i\infty$ for all $\gamma \in SL_2(\boldsymbol{Z})$ ($e(z)=\exp(2\pi iz)$).

Moreover, if $a_\gamma(0)=0$ for all $\gamma \in SL_2(\boldsymbol{Z})$, then f is called a cusp form. The subspace of $\mathscr{M}_k(\Gamma)$ consisting of all cusp forms is denoted by $\mathscr{S}_k(\Gamma)$.

Let Λ be a subring of $\boldsymbol{C}$ and f be a function on $\mathfrak{H}$ with a Fourier expansion of the form $\sum_{n=0}^{\infty} a(n)e(nz/N)$ for some positive integer N. Then we say that f is Λ-rational if $a(n)$ belongs to Λ for any n. Let $g(z)=\sum_{n=0}^{\infty} b(n)e(nz/N)$ be another Λ-rational function and $\mathfrak{m}$ be an ideal of Λ. Then we write $f\equiv g \bmod \mathfrak{m}$ if $a(n)\equiv b(n) \bmod \mathfrak{m}$ for all n. Further, for any field-automorphism ρ of $\boldsymbol{C}$, we define an action of ρ on f by

$$f^\rho(z)=\sum_{n=0}^{\infty} a(n)^\rho e(nz/N).$$

Let p be an *odd* prime and $\Gamma_0(p)$ be the subgroup of $SL_2(\boldsymbol{Z})$ defined by

$$\Gamma_0(p)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod p\right\}.$$

Lemma 2. *Let f be an element of $\mathcal{M}_k(\Gamma_0(p))$. Then the transformation equation $\Phi(X; f)=0$ is $\boldsymbol{Z}$-rational if and only if both f and $f|_k\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are $\boldsymbol{Z}$-rational.*

Proof. Put $\tau=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let us write the Fourier expansions of f and $f|_k\tau$ as

$$f(z)=\sum_{n=0}^{\infty} a(n)e(nz), \tag{1.1$_a$}$$

and

$$(f|_k\tau)(z)=\sum_{n=0}^{\infty} b(n)e(nz/p). \tag{1.1$_b$}$$

Further let us write the transformation equation of f as

$$\Phi(X; f)=X^{p+1}+\sum_{\mu=1}^{p+1}(-1)^{\mu}\sigma_{\mu}X^{p+1-\mu}. \tag{1.2}$$

First let us show that if both f and $f|_k\tau$ are $\boldsymbol{Z}$-rational, then all coefficients σ_μ are $\boldsymbol{Z}$-rational. Let us define a set $\mathcal{R}$ by

$$\mathcal{R}=\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_u=\begin{pmatrix} 0 & -0 \\ 1 & u \end{pmatrix} \,\middle|\, u=0, 1, \cdots, p-1\right\}. \tag{1.3}$$

Then, since p is a prime, the set $\mathcal{R}$ gives a complete set of representatives for $\Gamma_0(p)\backslash SL_2(\boldsymbol{Z})$ (e.g., [4, Lemma 2.2]). Since $\tau_u=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, it follows from (1.1$_b$) that for any integer u,

$$(f|_k\tau_u)(z)=\sum_{n=0}^{\infty} \zeta^{nu}b(n)e(nz/p), \tag{1.4}$$

where $\zeta=e(1/p)$. Since all Fourier coefficients $b(n)$ belong to $\boldsymbol{Z}$, the modular forms $f|_k\tau_u$ are $\boldsymbol{Z}[\zeta]$-rational. Thus the coefficients σ_μ are $\boldsymbol{Z}[\zeta]$-rational. In fact, the coefficients σ_μ are symmetric functions in $\{f|_k\alpha\}_{\alpha\in\mathcal{R}}$. On the other hand, (1.4) shows that for any field-automorphism ρ of $\boldsymbol{C}$, $\{(f|_k\alpha)^\rho\}_{\alpha\in\mathcal{R}}=\{f|_k\alpha\}_{\alpha\in\mathcal{R}}$. Thus we have $\sigma_\mu^\rho=\sigma_\mu$. This shows that the modular forms σ_μ are $\boldsymbol{Z}$-rational. Conversely assume that all coefficients σ_μ are $\boldsymbol{Z}$-rational. Then we have, for any field-automorphism ρ of $\boldsymbol{C}$,

$$(1.5) \qquad \prod_{\alpha\in\mathscr{R}}(X-f|_k\alpha)=\prod_{\alpha\in\mathscr{R}}(X-(f|_k\alpha)^\rho).$$

Especially, we have $f^\rho=f|_k\alpha$ for some $\alpha\in\mathscr{R}$. Assume $\alpha\neq\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Then f^ρ belongs to both $\mathscr{M}_k(\Gamma_0(p))$ and $\mathscr{M}_k(\alpha^{-1}\Gamma_0(p)\alpha)$; therefore, f^ρ is an element of $\mathscr{M}_k(\Gamma)$, where Γ is the subgroup of $SL_2(\boldsymbol{Z})$ generated by $\Gamma_0(p)$ and $\alpha^{-1}\Gamma_0(p)\alpha$. Since p is a prime, $\Gamma_0(p)$ is maximal in $SL_2(\boldsymbol{Z})$; thus Γ is different from $\Gamma_0(p)$, and therefore, Γ is equal to $SL_2(\boldsymbol{Z})$. This shows that f^ρ, and therefore, f belongs to $\mathscr{M}_k(SL_2(\boldsymbol{Z}))$. Hence we have $f^\rho=f$ for any field-automorphism ρ. Namely, f is $\boldsymbol{Q}$-rational. On the other hand, the vector space $\mathscr{M}_k(\Gamma(p))$ has a $\boldsymbol{Z}$-rational basis (e.g., [8, (9)]). Thus we can write f as

$$f(z)=c\sum_{n=0}^{\infty}d(n)e(nz)$$

with a rational number c and rational integers $d(n)$. Let us consider the equation $\Phi(X;f)=0$ over the ring $\boldsymbol{Q}[[q]]$ of formal power series in $q=e(z)$. Then f is integral over $\boldsymbol{Z}[[q]]$ and is contained in its quotient field. In fact, all coefficients of $\Phi(X;f)$ are $\boldsymbol{Z}$-rational from the assumption, and therefore, they belong to $\boldsymbol{Z}[[q]]$ for $q=e(z)$. Since $\boldsymbol{Z}$ is principal, $\boldsymbol{Z}[[q]]$ is a normal ring (e.g., [2, VII. 3, Exercise 9c]), and this shows that f is $\boldsymbol{Z}$-rational. Next we shall show that $f|_k\tau$ is also $\boldsymbol{Z}$-rational. Put $g=f|_k\tau$. It follows from (1.5) that for any field-automorphism ρ of $\boldsymbol{C}$, $g^\rho=g|_k\tau^{-1}\alpha$ for some $\alpha\in\mathscr{R}$. The set $\tau^{-1}\mathscr{R}$ gives a complete set of representatives for $\tau^{-1}\Gamma_0(p)\tau\backslash SL_2(\boldsymbol{Z})$. Therefore, if the space $\mathscr{M}_k(\tau^{-1}\Gamma_0(p)\tau)$ is stable under the action of ρ, the above argument for f can be applied to g, and therefore, g is $\boldsymbol{Q}$-rational. In fact, the stableness of $\mathscr{M}_k(\tau^{-1}\Gamma_0(p)\tau)$ under the action of ρ follows from [7, Theorem 6]. Let us verify an assumption of [7, Theorem 6] in our case. Let G be the group GL_2, viewed as an algebraic group defined over $\boldsymbol{Q}$, and $G_{\boldsymbol{A}}$ be the adelization of G. Put $U'_\ell=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\boldsymbol{Z}_\ell)\,|\,c\equiv0 \bmod p\boldsymbol{Z}_\ell\right\}$,

$$U'=\{x=(x_\ell)\in\prod_\ell GL_2(\boldsymbol{Z}_\ell)\times GL_2^+(\boldsymbol{R})\,|\,x_\ell\in U'_\ell \text{ for all primes } \ell\},$$

and

$$S'=\boldsymbol{Q}^\times U',$$

where $\boldsymbol{Z}_\ell$ is the ring of ℓ-adic integers. Furthermore put $S=\tau^{-1}S'\tau$, and $\Gamma_S=S\cap G_{\boldsymbol{Q}+}$, where $G_{\boldsymbol{Q}+}=GL_2(\boldsymbol{Q})\cap GL_2^+(\boldsymbol{R})$. Then we have $\Gamma_S=$

$\boldsymbol{Q}^{\times}(\tau^{-1}\Gamma_0(p)\tau)$ and $\boldsymbol{Q}^{\times}\det(S)=\boldsymbol{Q}_A^{\times}$, where $\boldsymbol{Q}_A^{\times}$ is the idele group of $\boldsymbol{Q}$. Moreover S contains the set Δ_S defined by

$$\Delta_S=\left\{\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\in G_A \,\middle|\, t\in\prod_{\ell}\boldsymbol{Z}_{\ell}^{\times}\right\}.$$

Therefore, it follows from [7, Theorem 6] that the space $\mathcal{M}_k(\tau^{-1}\Gamma_0(p)\tau)=\mathcal{M}_k(\Gamma_S)$ has a $\boldsymbol{Q}$-rational basis. Consequently g is $\boldsymbol{Q}$-rational. The argument which shows the $\boldsymbol{Z}$-rationality of f can be also applied to g. However note that we need consider the transformation equation over the ring $\boldsymbol{Z}[[q^{1/p}]]$ in place of the ring $\boldsymbol{Z}[[q]]$.

Remark. A similar argument as in the proof of Lemma 2 shows that *the discriminant D of $\Phi(X;f)$ is equal to 0 if and only if f belongs to $\mathcal{M}_k(SL_2(\boldsymbol{Z}))$.*

Proposition 3. *Let f be an element of $\mathcal{M}_k(\Gamma_0(p))$ and let us write the transformation equation of f as*

$$\Phi(X;f)=X^{p+1}+\sum_{\mu=1}^{p+1}(-1)^{\mu}\sigma_{\mu}X^{p+1-\mu}.$$

If $\Phi(X;f)=0$ is $\boldsymbol{Z}$-rational, then we have a congruence

$$\Phi(X;f)\equiv(X-\sigma_1)(X^p-\sigma_p) \bmod p.$$

Proof. Let us write $\zeta=e(1/p)$ and $\mathfrak{p}$ be the unique prime ideal of $\boldsymbol{Z}[\zeta]$ which divides p. Then we know that $\zeta\equiv 1 \bmod \mathfrak{p}$. Since $f|_k\tau$ is $\boldsymbol{Z}$-rational by Lemma 2, the Fourier expansion (1.4) of $f|_k\tau_u$ shows

$$f|_k\tau_u\equiv f|_k\tau \bmod \mathfrak{p}$$

for any τ_u. Thus we have

$$\begin{aligned}\Phi(X;f)&\equiv(X-f)(X-f|_k\tau)^p && \bmod \mathfrak{p},\\ &\equiv(X-f)(X^p-(f|_k\tau)^p) && \bmod \mathfrak{p}.\end{aligned}$$

Especially, we have

$$\sigma_1\equiv f \quad \bmod \mathfrak{p},$$

and

$$\sigma_p\equiv(f|_k\tau)^p \quad \bmod \mathfrak{p}.$$

Thus we have

$$\Phi(X;f)\equiv(X-\sigma_1)(X^p-\sigma_p) \bmod \mathfrak{p}.$$

Here all coefficients of $\Phi(X;f)$ are $\boldsymbol{Z}$-rational, and especially both σ_1 and σ_p are $\boldsymbol{Z}$-rational. Therefore we have

$$\Phi(X;f)\equiv(X-\sigma_1)(X^p-\sigma_p) \bmod p.$$

§ 2. Proof of Theorem 1

By the definition of the discriminant, we have

$$D=\prod (f|_k\alpha-f|_k\beta)^2,$$

where the product is taken over all non-ordered pairs (α, β) with $\alpha\neq\beta$ in the representative set $\mathscr{R}$ as in (1.3). Then we see

$$D=\prod_{u=0}^{p-1}(f-f|_k\tau_u)^2\cdot\prod_{0\leqq u<v\leqq p-1}(f|_k\tau_u-f|_k\tau_v)^2.$$

Obviously D is a modular form on $SL_2(\boldsymbol{Z})$ of weight $kp(p+1)$. Let us put

$$\delta=\prod_{u=0}^{p-1}(f-f|_k\tau_u)\cdot\prod_{0\leqq u<v\leqq p-1}(f|_k\tau_u-f|_k\tau_v).$$

Then we see $D=\delta^2$. First let us show

Lemma 4. *δ is a modular form on $SL_2(\boldsymbol{Z})$.*

Proof. It is sufficient to prove that $\delta|_m\begin{pmatrix}1&1\\0&1\end{pmatrix}=\delta$ and $\delta|_m\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\delta$ for $m=kp(p+1)/2$. Put $\sigma=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\tau=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. The right multiplication of σ on the coset space $\Gamma_0(p)\backslash SL_2(\boldsymbol{Z})$ induces a permutation on the representative set $\mathscr{R}$, and $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is transformed to $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, τ_{p-1} to τ_0, and τ_u to τ_{u+1} for $u=0, 1, \cdots, p-2$. Thus we observe that the first factor $\prod_{u=0}^{p-1}(f-f|_k\tau_u)$ of δ is invariant under σ, and that

$$\{\prod_{0\leqq u<v\leqq p-1}(f|_k\tau_u-f|_k\tau_v)\}|_{kp(p-1)/2}\sigma=(-1)^{p-1}\prod_{0\leqq u<v\leqq p-1}(f|_k\tau_u-f|_k\tau_v).$$

Since p is odd, this shows that $\delta|_m\sigma=\delta$. Also, τ induces a permutation on $\mathscr{R}$, and $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ is transformed to τ_0, τ_0 to $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, and τ_u to $\tau_{\nu(u)}$ for $u=1, 2, \cdots, p-1$. Here $\nu(u)$ is an integer in $\{1, 2, \cdots, p-1\}$ satisfying $\nu(u)u\equiv-1 \bmod p$. Rewriting δ as

$$\prod_{u=0}^{p-1}(f-f|_k\tau_u)\cdot\prod_{v=1}^{p-1}(f|_k\tau_0-f|_k\tau_v)\cdot\prod_{1\leqq u<v\leqq p-1}(f|_k\tau_u-f|_k\tau_v),$$

we observe that

$$\delta|_m\tau = -\prod_{u=0}^{p-1}(f - f|_k\tau_u)\cdot\prod_{v=1}^{p-1}(f|_k\tau_0 - f|_k\tau_v)\cdot\prod_{1\leqq u<v\leqq p-1}(f|_k\tau_{\nu(u)} - f|_k\tau_{\nu(v)}).$$

Thus, in order to prove that $\delta|_m\tau=\delta$, we have to show that the last factor $\prod_{1\leqq u<v\leqq p-1}(f|_k\tau_u - f|_k\tau_v)$ of δ is alternating under the permutation ν; namely, it is sufficient to prove that the permutation ψ on $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ defined by $\psi(a)=-1/a$ for $a\in(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ is an odd permutation. Since $\psi^2=\mathrm{id}$, we see that

$$\psi=\prod(a, b),$$

where the product is taken over all the transpositions (a, b) between a and b with $ab=-1$ and $a\neq b$. The number ℓ of the elements in $(\boldsymbol{Z}/p\boldsymbol{Z})^\times$ with $a^2=-1$ is 2 or 0 according as $p\equiv 1 \bmod 4$ or not. Thus $(p-1-\ell)/2$ is odd. Since the number of the transpositions in ψ is $(p-1-\ell)/2$, ψ is an odd permutation. This concludes the proof of Lemma 4.

We claim that

(2.1) *$\delta=cg$, where c is a constant with $c^2=(-1)^{(p-1)/2}p^p$ and g is a $\boldsymbol{Z}$-rational modular form on $SL_2(\boldsymbol{Z})$.*

In fact, both f and $f|_k\tau$ are $\boldsymbol{Z}$-rational by Lemma 2. Thus, using the Fourier expansion (1.4) of $f|_k\tau_u$, we can find polynomials $\beta_n(x, y)$ in $\boldsymbol{Z}[x, y]$ so that

$$f|_k\tau_u - f|_k\tau_v = (\zeta^u-\zeta^v)\sum_{n=1}^{\infty}\beta_n(\zeta^u, \zeta^v)e(nz/p) \tag{2.2}$$

for all τ_u and τ_v. Put $c=\prod_{0\leqq u<v\leqq p-1}(\zeta^u-\zeta^v)$. Then we see that

$$\begin{aligned} c^2 &= (-1)^{p(p-1)/2}\prod_{0\leqq u<v\leqq p-1}(\zeta^u-\zeta^v)(\zeta^v-\zeta^u) \\ &= (-1)^{(p-1)/2}\prod_{\substack{u,v=0\\ u\neq v}}^{p-1}(\zeta^u-\zeta^v) \\ &= (-1)^{(p-1)/2}\prod_{u=0}^{p-1}\left\{\zeta^u\prod_{v=1}^{p-1}(1-\zeta^v)\right\} \\ &= (-1)^{(p-1)/2}p^p. \end{aligned}$$

Further put $g=\prod_{u=0}^{p-1}(f-f|_k\tau_u)\cdot\prod_{0\leqq u<v\leqq p-1}\{\sum_{n=1}^{\infty}\beta_n(\zeta^u, \zeta^v)e(nz/p)\}$. Then $\prod_{u=0}^{p-1}(f-f|_k\tau_u)$ is $\boldsymbol{Z}$-rational as shown below. For any field-automorphism ρ of $\boldsymbol{C}$, we defined the action of ρ on a Fourier series $\varphi=\sum_{n=0}^{\infty}c(n)e(nz/p)$ by $\varphi^\rho=\sum_{n=0}^{\infty}c(n)^\rho(nz/p)$. Then the Fourier expansions

(1.1$_a$) and (1.4) of f and $f|_k\tau_u$ show that any automorphism ρ induces a bijection of the set $\{f-f|_k\tau_u \mid u=0, 1, 2, \cdots, p-1\}$. Thus $\prod_{u=0}^{p-1}(f-f|_k\tau_u)$ is $\boldsymbol{Q}$-rational. Moreover, all Fourier coefficients of $f-f|_k\tau_u$ belong to $\boldsymbol{Z}[\zeta]$. Thus $\prod_{u=0}^{p-1}(f-f|_k\tau_u)$ is $\boldsymbol{Z}$-rational. A similar argument combined with the symmetricity in x and y of the polynomial $\beta_n(x, y)$ shows that $\prod_{0\leqq u<v\leqq p-1}\{\sum_{n=1}^{\infty}\beta_n(\zeta^u, \zeta^v)e(nz/p)\}$ is also $\boldsymbol{Z}$-rational. Therefore g is $\boldsymbol{Z}$-rational. Note that $\delta=cg$. Then Lemma 4 shows that g is a $\boldsymbol{Z}$-rational modular form on $SL_2(\boldsymbol{Z})$. Thus (2.1) is established.

Now we prove the divisibility of g by the power of Δ as indicated in the theorem. Let us put $h=g/\Delta^{(p-1)/2}$, where $\Delta(z)=e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24}$. Since the Fourier expansion of g starts from $e((p-1)/2\cdot z)$, h is still a modular form. In fact, the Fourier expansion (2.2) of $f|_k\tau_u-f|_k\tau_v$ starts from $e(z/p)$, and therefore, by counting the number of such factors in the product for g, we know that the Fourier expansion of g starts from $e((p-1)/2\cdot z)$. Further the cusp form Δ is nowhere vanishing on $\mathfrak{H}$ and its Fourier expansion starts from $e(z)$. Thus h is holomorphic on $\mathfrak{H}$ and so even at $i\infty$. Since the first coefficient of the Fourier expansion of Δ is equal to 1, h is again $\boldsymbol{Z}$-rational. Next we assume that f is a cusp form. Then, since the Fourier expansion of the first factor $\prod_{u=0}^{p-1}(f-f|_k\tau_u)$ of g starts from $e(z)$ or higher term, that of g starts from $e\left(\frac{p+1}{2}z\right)$ or higher term. A similar argument as above is still valid. This completes the proof of Theorem 1.

We remark the following direct consequence of Theorem 1, which may be well known:

Corollary 5. *Let us consider the specialized equation of f at an elliptic curve $\mathscr{E}$ defined over $\boldsymbol{Q}$. For the definition of the specialized equation and the details, see* [3, § 3]. *Under the same notation as in* [3, §3], *if the specialized equation $\Phi(X; f, \mathscr{E})=0$ is irreducible, then the prime p always ramifies in the splitting field of the equation.*

Remark. The above proof of Theorem 1 shows that *without assuming the $\boldsymbol{Z}$-rationality of transformation equation, we may express D as*

$$D=\Delta^{p-1}h^2$$

with a modular form h on $SL_2(\boldsymbol{Z})$.

§ 3. The transformation equation of $gE^*_{\lambda,p}$

In this section, let us consider in more detail the transformation equation of $gE^*_{\lambda,p}$ as in [3]. Here g is a cusp form in $\mathscr{S}_\ell(\Gamma_0(p))$ and $E^*_{\lambda,p}$

is the Eisenstein series defined by

$$E^*_{\lambda,p}(z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)}(cz+d)^{-\lambda},\quad \left(\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\right),$$

where λ is an even integer >2 and $\Gamma_\infty=\left\{\pm\begin{pmatrix}1&n\\0&1\end{pmatrix}\middle|\, n\in \boldsymbol{Z}\right\}$. The Eisenstein series $E^*_{\lambda,p}$ is a modular form on $\Gamma_0(p)$ of weight λ, and is expressed as follows (see [9, p. 794]):

$$E^*_{\lambda,p}(z)=\frac{2\lambda}{(p^\lambda-1)B_\lambda}\{G_\lambda(z)-p^\lambda G_\lambda(pz)\}, \tag{3.1}$$

where B_λ is the λ-th Bernoulli number and

$$G_\lambda(z)=-B_\lambda/2\lambda+\sum_{n=1}^{\infty}\{\sum_{0<d|n}d^{\lambda-1}\}e(nz). \tag{3.2}$$

Let us take a cusp form $\varphi(z)=\sum_{n=1}^\infty a(n)e(nz)$ in $\mathscr{S}_\ell(\Gamma_0(p))$ with the following three properties:

$$\varphi|_\ell\begin{pmatrix}0&-1\\p&0\end{pmatrix}=\gamma\varphi \quad \text{for } \gamma=\pm1; \tag{3.3$_a$}$$

$$a(1)=1; \tag{3.3$_b$}$$

$$\varphi \text{ is } \boldsymbol{Z}\text{-rational}. \tag{3.3$_c$}$$

For example, any $\boldsymbol{Q}$-rational primitive form in $\mathscr{S}_\ell(\Gamma_0(p))$ satisfies these conditions. Moreover, we can construct another example of such cusp forms. We will give this example at the end of this section.

Let us put

$$g=\frac{p^{\ell/2}}{d}N_\lambda\varphi, \tag{3.4}$$

where we write

$$(p^\lambda-1)B_\lambda/2\lambda=N_\lambda/D_\lambda \tag{3.5}$$

with mutually prime integers N_λ and D_λ, and d is the greatest common divisor of $p^{\ell/2}$ and D_λ. Then we have

Proposition 6. *The transformation equation of $gE^*_{\lambda,p}$ is $\boldsymbol{Z}$-rational.*

Proof. It is sufficient to show that $gE^*_{\lambda,p}$ and $gE^*_{\lambda,p}|_k\tau$ are $\boldsymbol{Z}$-rational (see Lemma 2). Here $k=\ell+\lambda$ and $\tau=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. We see from (3.1), (3.2) and (3.5) that

$$E^*_{\lambda,p}(z)=\frac{D_\lambda}{N_\lambda}\left\{\frac{N_\lambda}{D_\lambda}+\sum_{n=1}^{\infty}b(n)e(nz)\right\} \tag{3.6}$$

with $b(n)\in \boldsymbol{Z}$. Thus $N_\lambda E^*_{\lambda,p}$ is $\boldsymbol{Z}$-rational, and therefore, $gE^*_{\lambda,p}$ is $\boldsymbol{Z}$-rational. Since $G_\lambda(z)$ is a modular form in $\mathcal{M}_\lambda(SL_2(\boldsymbol{Z}))$, we have

$$G_\lambda(pz)|_\lambda\tau=p^{-\lambda}G_\lambda(z/p), \tag{3.7}$$

and hence, (3.1) shows that

$$(N_\lambda E^*_{\lambda,p}|_\lambda\tau)(z)=D_\lambda\{G_\lambda(z)-G_\lambda(z/p)\}. \tag{3.8}$$

On the other hand, it follows from (3.3_a) that

$$(\varphi|_\ell\tau)(z)=\gamma p^{-\ell/2}\varphi(z/p). \tag{3.9}$$

Thus $gE^*_{\lambda,p}|_k\tau$ is again $\boldsymbol{Z}$-rational. In fact, both the modular forms $\varphi(z/p)$ and $G_\lambda(z)-G_\lambda(z/p)$ are $\boldsymbol{Z}$-rational; therefore, the modular form:

$$(gE^*_{\lambda,p}|_k\tau)(z)=\gamma\frac{D_\lambda}{d}\varphi(z/p)\{G_\lambda(z)-G_\lambda(z/p)\} \tag{3.10}$$

is $\boldsymbol{Z}$-rational. This is what we wanted to show.

Proposition 7. *Let us write the transformation equation of $gE^*_{\lambda,p}$ as*

$$\Phi(X;gE^*_{\lambda,p})=X^{p+1}+\sum_{\mu=1}^{p+1}(-1)^\mu\sigma_\mu X^{p+1-\mu}.$$

Then the modular form σ_p has a Fourier expansion of the form

$$\sigma_p(z)=-\gamma\left(\frac{D_\lambda}{d}\right)^p e(2z)+\sum_{n=3}^{\infty}c(n)e(nz) \tag{3.11}$$

with rational integers $c(n)$.

Proof. Since σ_p is the p-th elementary symmetric function in $\{gE^*_{\lambda,p}|_k\alpha\}_{\alpha\in R}$, we have by (1.3) that

$$\sigma_p=gE^*_{\lambda,p}\left\{\sum_{u=0}^{p-1}\prod_{\substack{v=0\\ v\neq u}}^{p-1}(gE^*_{\lambda,p}|_k\tau_v)\right\}+\prod_{u=0}^{p-1}(gE^*_{\lambda,p}|_k\tau_u).$$

Here $\tau_u=\begin{pmatrix}0&-1\\1&u\end{pmatrix}$. Since $\tau_u=\tau\begin{pmatrix}1&u\\0&1\end{pmatrix}$, it follows from (3.10) that

$$(gE^*_{\lambda,p}|_k\tau_u)(z)=\gamma\frac{D_\lambda}{d}\varphi((z+u)/p)\{G_\lambda(z)-G_\lambda((z+u)/p)\}. \tag{3.12}$$

Thus we can find polynomials $\beta_n(x)$ in $\boldsymbol{Z}[x]$ so that (3.12) is rewritten as

$$(3.13) \qquad (gE^*_{\lambda,p}|_k\tau_u)(z) = -\gamma\frac{D_\lambda}{d}\sum_{n=2}^{\infty}\beta_n(\zeta^u)e(nz/p)$$

for all τ_u. Here $\zeta=e(1/p)$. Especially we have the second polynomial $\beta_2(x)=x^2$. Therefore we have

$$(3.14) \qquad \prod_{u=0}^{p-1}(gE^*_{\lambda,p}|_k\tau_u)(z) = -\gamma\left(\frac{D_\lambda}{d}\right)^p\sum_{n=2p}^{\infty}w(n)e(nz/p)$$

with rational integers $w(n)$. Note $w(2p)=1$. On the other hand, since the Fourier expansion of $gE^*_{\lambda,p}$ starts from $e(z)$, (3.13) shows that

$$(3.15) \qquad gE^*_{\lambda,p}\left\{\sum_{u=0}^{p-1}\prod_{\substack{v=0\\ v\neq u}}^{p-1}(gE^*_{\lambda,p}|_k\tau_v)\right\}(z) = \sum_{n=3p-2}^{\infty}w'(n)e(nz/p)$$

with rational integers $w'(n)$. Thus, considering $w(2p)=1$ and $3p-2>2p$, we see that the Fourier expansion of σ_p starts from $e(2z)$ with the coefficient $-\gamma(D_\lambda/d)^p$. Moreover, it follows from Proposition 6 that σ_p is a $\boldsymbol{Z}$-rational modular form. This concludes the proof of Proposition 7.

Remark. Our modification of φ as in (3.4) is best possible. In fact, analyzing carefully the above proof of Proposition 6, one sees that if both $c\varphi E^*_{\lambda,p}$ and $c\varphi E^*_{\lambda,p}|_k\tau$ are $\boldsymbol{Z}$-rational for a constant c, then c is a rational integer and a multiple of $p^{\ell/2}N_\lambda/d$.

Proposition 8. *The discriminant D of $\Phi(X; gE^*_{\lambda,p})$ is expressed as*

$$D=(-1)^{(p-1)/2}p^p\Delta^{2(p+1)}h^2,$$

where h is a $\boldsymbol{Z}$-rational modular form on $SL_2(\boldsymbol{Z})$.

Proof. The Fourier expansion (3.13) of the modular form $gE^*_{\lambda,p}|_k\tau_u$ shows that both the Fourier expansions of the modular forms $gE^*_{\lambda,p}-gE^*_{\lambda,p}|_k\tau_u$ and $gE^*_{\lambda,p}|_k\tau_u-gE^*_{\lambda,p}|_k\tau_v$ start from $e(2z/p)$ for any τ_u and τ_v. Thus that of the modular form

$$\prod_{u=0}^{p-1}(gE^*_{\lambda,p}-gE^*_{\lambda,p}|_k\tau_u)\cdot\prod_{0\leq u<v\leq p-1}(gE^*_{\lambda,p}|_k\tau_u-gE^*_{\lambda,p}|_k\tau_v)$$

starts from $e((p+1)z)$. Then a similar argument as in the proof of Theorem 1 shows our assertion.

Now let us give examples of the cusp forms φ satisfying the conditions ($3.3_{a,b,c}$) when there exists a primitive form of conductor p. Let us take a primitive form f in $\mathscr{S}_\ell(\Gamma_0(p))$ of conductor p and write the Fourier expansion of f as

$$f(z)=\sum_{n=1}^{\infty} b(n)e(nz).$$

(For the primitiveness of cusp forms, see, for example, [9, p. 789].) We denote by M the module generated over $\boldsymbol{Z}$ by all $b(n)$ in $\boldsymbol{C}$ and by K the field generated by M. For any isomorphism σ of K into $\boldsymbol{C}$, we define the conjugate f^σ of f by

$$f^\sigma(z)=\sum_{n=1}^{\infty} b(n)^\sigma e(nz).$$

As is well known, f^σ is again a primitive form in $\mathscr{S}_\ell(\Gamma_0(p))$. We define a cusp form $\mathrm{Tr}(\alpha f)$ in $\mathscr{S}_\ell(\Gamma_0(p))$ for any element α in K by

$$\mathrm{Tr}(\alpha f)=\sum \alpha^\sigma f^\sigma,$$

where σ runs over all isomorphisms of K into $\boldsymbol{C}$. Since f is primitive, it follows from [1, Lemma 3] that

$$f|_\ell\begin{pmatrix}0 & -1\\ p & 0\end{pmatrix}=\gamma f, \quad \gamma=\pm 1. \tag{3.16$_\mathrm{a}$}$$

Moreover we have

$$f^\sigma|_\ell\begin{pmatrix}0 & -1\\ p & 0\end{pmatrix}=\gamma f^\sigma \tag{3.16$_\mathrm{b}$}$$

for any f^σ, because γ is expressed as

$$\gamma=-b(p)p^{1-\ell/2}. \tag{3.17}$$

Thus we see

$$\mathrm{Tr}(\alpha f)|_\ell\begin{pmatrix}0 & -1\\ p & 0\end{pmatrix}=\gamma\,\mathrm{Tr}(\alpha f) \tag{3.18}$$

for any element α in K. We see easily that

(3.19) $\mathrm{Tr}(\alpha f)$ is $\boldsymbol{Z}$-rational if and only if α belongs to

$$\mathscr{D}=\{\beta\in K \mid \mathrm{Tr}_{K/\boldsymbol{Q}}(\beta x)\in \boldsymbol{Z} \text{ for all } x\in M\}.$$

Proposition 9. *Let us put*

$U=\{\alpha\in\mathscr{D}\mid \mathrm{Tr}(\alpha f)$ *has a Fourier expansion of the form* $e(z)+\sum_{n=2}^{\infty} c(n)e(nz)$ *with rational integers* $c(n)\}$, *and*

$$V=\{\alpha \in \mathscr{D} \mid \mathrm{Tr}_{K/Q}(\alpha)=0\}.$$

Then we have

(1) *U is not empty*;

(2) *V is isomorphic to $\boldsymbol{Z}^{d-1}$ for $d=[K:\boldsymbol{Q}]$*;

(3) *$U=\alpha_0+V$ for any element α_0 of U.*

Proof. Let α be an element of $\mathscr{D}$. Since $b(1)=1$ and $\mathrm{Tr}(\alpha f)=\sum_{n=1}^{\infty}\mathrm{Tr}_{K/Q}(\alpha b(n))e(nz)$, we see that

$$\alpha \text{ belongs to } U \text{ if and only if } \mathrm{Tr}_{K/Q}\alpha=1. \tag{3.20}$$

Since M generates K and since M is a $\boldsymbol{Z}$-free module, M is isomorphic to $\boldsymbol{Z}^d$. Let $\{\omega_i\}_{i=1}^d$ be a $\boldsymbol{Z}$-basis of M and $\{\eta_i\}_{i=1}^d$ be the dual basis of $\{\omega_i\}$ with respect to $\mathrm{Tr}_{K/Q}$; hence we have

$$\mathrm{Tr}_{K/Q}(\omega_i\eta_j)=\delta_{ij}. \tag{3.21}$$

Then we know

$$\mathscr{D}=\sum_{i=1}^{d}\boldsymbol{Z}\eta_i \quad \text{(direct sum)}. \tag{3.22}$$

Since 1 $(=b(1))$ belongs to M, we may write as $1=\sum_{i=1}^d m_i\omega_i$ for some rational integers m_i. Then (3.21) shows that $m_i=\mathrm{Tr}_{K/Q}\eta_i$ for any i. Namely, we have

$$1=\sum_{i=1}^{d}(\mathrm{Tr}_{K/Q}\eta_i)\omega_i. \tag{3.23}$$

Let c be the greatest common divisor of $\{\mathrm{Tr}_{K/Q}\eta_i\}_{i=1}^d$. Since f is primitive, all Fourier coefficients $b(n)$ of f, and therefore, all ω_i are algebraic integers. This combined with (3.23) shows that c is equal to 1. Thus considering (3.22), we know

$$\mathrm{Tr}_{K/Q}\mathscr{D}=\boldsymbol{Z}. \tag{3.24}$$

Especially there exists an element α_0 of $\mathscr{D}$ such that $\mathrm{Tr}(\alpha_0 f)$ belongs to U. Since $V\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is isomorphic to $\boldsymbol{Q}^{d-1}$, we see the assertion (2). The third assertion is clear from (3.20).

Now let us put $\varphi=\mathrm{Tr}(\alpha f)$ for any element α of U. Then (3.18), (3.19) and (3.20) show that φ satisfies the conditions $(3.3_{\mathrm{a,b,c}})$.

§ 4. Numerical examples

In this section, we are going to give several numerical examples of the transformation equations $\Phi(x; gE^*_{\lambda,p})=0$ and the specialized equations $\Phi(X; gE^*_{\lambda,p}, \mathscr{E})=0$ at various elliptic curves $\mathscr{E}$ defined over $\boldsymbol{Q}$. See [3, § 3] for the definition of the specialized equation at an elliptic curve. For simplicity, we consider only the case $\dim \mathscr{S}_\ell(\Gamma_0(p))=1$. Thus we may take as φ in (3.4) with ($3.3_{a,b,c}$) the unique primitive form in $\mathscr{S}_\ell(\Gamma_0(p))$. Let us modify φ as in (3.4) and write the modified modular form as g.

Let us explain how to read the table given below by taking the following case I as an example. This case is the restatement of the example given in our previous paper [3, § 5]. We will add several new examples here. We use the same notation in Section 3 and write simply G, H, and D for $12g_2$, $216g_3$, and Δ, respectively. Here we put $g_2=20G_4$ and $g_3=-\frac{7}{3}G_6$, where G_4 and G_6 are the Eisenstein series defined by (3.2). Thus both $12g_2$ and $216g_3$ are $\boldsymbol{Z}$-rational and their constant terms are equal to 1.

Case I. $p=5,\ \ell=4,\ \lambda=4,\ k=8.$
$g=-5\cdot 13\varphi.$

X^6	1
X^5	0
X^4	$-25GD$
X^3	$-1440D^2$
X^2	$155G^2D^2$
X	$GH^2D^2+18096GD^3$
1	$65H^2D^3+538240D^4$

The above table can be read that the transformation equation $\Phi(X; gE^*_{4,5})$ is given by the polynomial

$$(4.1)\quad X^6-25GDX^4-1440D^2X^3+155G^2D^2X^2+(GH^2D^2+18096GD^3)X \\ +(65H^2D^3+538240D^4).$$

Thus, for example, the monomial $-25GD$ given at the right-hand side of X^4 is the isobaric polynomial of the coefficient of X^4.

$\mathrm{Tr}(X)$	0
$\mathrm{Tr}(X^2)$	$50GD=2^{-30}\cdot 3\cdot 14!\cdot\dfrac{2^{20}}{3^6\cdot 7^2\cdot 11\cdot 13}f_{16}$
$\mathrm{Tr}(X^3)$	$4320D^2=2^{-46}\cdot 3\cdot 22!\sum_\sigma\left(\dfrac{2^{29}}{3^8\cdot 5^3\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19\sqrt{144169}}f_{24}\right)^\sigma$

$\mathrm{Tr}\,(X^4)$ $\quad 630G^2D^2=2^{-62}\cdot 3\cdot 30!$

$$\cdot\sum_\sigma\left(\frac{2^{34}}{3^{14}\cdot 5^6\cdot 7^3\cdot 11^2\cdot 13^2\cdot 17\cdot 19\cdot 23\cdot 29\sqrt{18295489}}f_{32}\right)^\sigma$$

$\mathrm{Tr}\,(X^5)$ $\quad -5GH^2D^2+89520GD^3=2^{-78}\cdot 3\cdot 38!$

$$\cdot\sum_\sigma\left(\frac{-2^{43}(\alpha-3537792)}{3^{18}\cdot 5^7\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)}f_{40}\right)^\sigma$$

$$N(\alpha-3537792)=2^{18}\cdot 3^7\cdot 7^2\cdot 11\cdot 23\cdot 31\cdot 73\cdot 2161$$

$\mathrm{Tr}\,(X^6)$ $\quad 7610H^2D^3+16815360D^4=2^{-94}\cdot 3\cdot 46!$

$$\cdot\sum_\sigma\left(\frac{2^{56}(5117\alpha+17457217536)}{3^{21}\cdot 5^9\cdot 7^5\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\psi'(\alpha)}f_{48}\right)^\sigma$$

$$N(5117\alpha+17457217536)$$
$$=-2^{37}\cdot 3^8\cdot 11\cdot 383^2\cdot 3129512851870124265857.$$

Here $\mathrm{Tr}(X^\mu)$ indicates the μ-th power sum $\mathrm{Tr}\,(gE_{4,5}^*)^\mu$ of all the roots of the transformation equation given in (4.1), and the corresponding isobaric polynomial is given at the right-hand side of $\mathrm{Tr}\,(X^\mu)$. As we have seen in [3], the power sum $\mathrm{Tr}\,(gE_{4,5}^*)^\mu$ can be expressed as

$$\mathrm{Tr}\,(gE_{4,5}^*)^\mu=2^{-2(8\mu-1)}\cdot 3\cdot(8\mu-2)!\sum_{f\in P(8\mu)}\frac{D(8\mu-1,f,g^\mu E_{4,5}^{*\mu-1}}{\pi^{8\mu}\langle f,f\rangle}f.$$

(See [3, Theorem] for the notation.) After the isobaric polynomial in the table, we have given this expression of the power sum. (This expression is not given in [3, § 5].) Thus, for example, in the expression corresponding to $\mathrm{Tr}\,(X^5)$, the value

$$-2^{43}(\alpha-3537792)/\{3^{18}\cdot 5^7\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)\}$$

gives the special value $D(39,f_{40},g^5E_{4,5}^{*4})/\pi^{40}\langle f_{40},f_{40}\rangle$. Here f_{40} indicates a primitive form in $\mathscr{S}_{40}(SL_2(\boldsymbol{Z}))$; α is a generator of the field $K(f_{40})$ generated over $\boldsymbol{Q}$ by all Fourier coefficients of f_{40}; ψ indicates the characteristic polynomial of α and $\psi'(x)=d\psi/dx$. Further, in the above expression of $\mathrm{Tr}(X^5)$, the summation is over all isomorphisms σ of $K(f_{40})$ into $\boldsymbol{C}$. Note that in the limit of the calculation we have done, all the primitive forms in $\mathscr{S}_m(SL_2(\boldsymbol{Z}))$ are conjugate under the automorphisms of $\boldsymbol{C}$. We denote by $N(\gamma)$ the norm of an algebraic number γ, for example, $N(\alpha-3537792)$ indicates the norm of the number $\alpha-3537792$. If the factors in the listed numbers are less than 10^{10}, then they are primes; otherwise, we do not know whether they are prime or not.

Let us now list the characteristic polynomials $\psi(x)$ and their discriminants $D(\psi)$ of a generator α of the fields $K(f_m)$:

m	$\psi(x)$ and $D(\psi)$
40	$\psi(x)=x^3-548856x^2-810051757056x+213542160549543936$ $D(\psi)=2^{26}\cdot 3^{12}\cdot 5^2\cdot 7^2\cdot 13^2\cdot 73\cdot 59077\cdot 92419245301$
48	$\psi(x)=x^4-5785560x^3-467142374034432x^2$ $+14268305621832538521 60x$ $+32979138288402143208076738 56$ $D(\psi)=2^{70}\cdot 3^{22}\cdot 5^6\cdot 7^6\cdot 31\cdot 383^2$ $\cdot 102107536163441411992455248734239414 99439$
50	$\psi(x)=x^3+24225168x^2-566746931810304x$ $-1363488322874273641 2672$ $D(\psi)=2^{32}\cdot 3^{12}\cdot 5^4\cdot 7^4\cdot 1228462869413174261 9401$
60	$\psi(x)=x^5+449691864x^4-2209450184054433792x^3$ $-73601006039351369787034828 8x^2$ $+8106347633348129724162336484396892 16x$ $+26322221615706082411520309890223724856501 8624$ $D(\psi)=2^{148}\cdot 3^{38}\cdot 5^8\cdot 7^8\cdot 17^4\cdot 23\cdot 1019$ $\cdot 65191632047210387890272707448050309485567043235713$ $20700882988973280588502206945747301717487795597$*

*This number 65191 $\cdots$ 5597 is a number of 97-figures.

Case II. $p=5,\ \ell=4,\ \lambda=6,\ k=10.$

$g=5^2\cdot 31\varphi$

X^6	1
X^5	0
X^4	$-145G^2D$
X^3	$587520HD^2$
X^2	$3635GH^2D^2-377403840GD^3$
X	$G^2H^3D^2+6290064G^2HD^3$
1	$-775H^4D^3-7058849600H^2D^4$

$\mathrm{Tr}(X)\quad 0$

$$\mathrm{Tr}(X^2)\quad 290G^2D=2^{-38}\cdot 3\cdot 18!\cdot\frac{2^{23}\cdot 29}{3^9\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17}f_{20}$$

$\mathrm{Tr}(X^3)$ $\quad -1762560HD^2=2^{-58}\cdot 3\cdot 28!$

$$\cdot\sum_{\sigma}\left(\frac{-2^{35}}{3^{11}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 19\cdot 23\sqrt{51349}}f_{30}\right)^{\sigma}$$

$\mathrm{Tr}(X^4)$ $\quad 27510GH^2D^2+1582277760GD^3=2^{-78}\cdot 3\cdot 38!$

$$\cdot\sum_{\sigma}\left(\frac{2^{44}(131\alpha+1196402688)}{3^{17}\cdot 5^7\cdot 7^4\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)}f_{40}\right)^{\sigma}$$

$$N(131\alpha+1196402688)=-2^{27}\cdot 3^7\cdot 7^2\cdot 3833\cdot 32619042931$$

$\mathrm{Tr}(X^5)$ $\quad -5G^2H^3D^2-457402320G^2HD^3=2^{-98}\cdot 3\cdot 48!$

$$\cdot\sum_{\sigma}\left(\frac{-2^{52}(\alpha-8757800448)}{3^{23}\cdot 5^9\cdot 7^6\cdot 11^4\cdot 13^3\cdot 17^7\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\psi'(\alpha)}f_{50}\right)^{\sigma}$$

$$N(\alpha-8757800448)=2^{21}\cdot 3^7\cdot 5^2\cdot 19\cdot 73\cdot 4235321855794559$$

$\mathrm{Tr}(X^6)$ $\quad 2939450H^4D^3+1421841072000H^2D^4+585580127846400D^5$
$=2^{-118}\cdot 3\cdot 58!$

$$\cdot\sum_{\sigma}\left(\frac{2^{68}(168536131\alpha^2+47995636461477888\alpha+999350356402218729052569 6)}{3^{27}\cdot 5^{11}\cdot 7^8\cdot 11^5\cdot 13^4\cdot 17^3\cdot 19^3\cdot 23^2\cdot 29^2\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\psi'(\alpha)}f_{60}\right)^{\sigma}$$

$$N(168536131\alpha^2+47995636461477888\alpha+9993503564022187290525696)$$
$$=-2^{105}\cdot 3^{23}\cdot 5^3\cdot 11^2\cdot 13\cdot 17^4$$
$$\cdot 116584253158876173092059456339030406653826382655372507793225580762957007062391$$

Case III. $p=5,\ \ell=4,\ \lambda=8,\ k=12.$
$g=-5\cdot 13\cdot 313\varphi$

X^6	1
X^5	$60480D$
X^4	$-625H^2D+1301832000D^2$
X^3	$117113760H^2D^2+11768083937280D^3$
X^2	$69755H^4D^2+1728323786880H^2D^3+393094371172147 20D^4$
X	$H^6D^2+17889611952H^4D^3-18249030627747840H^2D^4+15417626668505432064D^5$
1	$20345H^6D^3+335091233981440H^4D^4+6660452326511923200H^2D^5+32175921734973802414080D^6$

$\mathrm{Tr}(X)$	$-60480D$
$\mathrm{Tr}(X^2)$	$1250H^2D+1054166400D^2$
$\mathrm{Tr}(X^3)$	$-464741280H^2D^2-20325436323840D^3$
$\mathrm{Tr}(X^4)$	$502230H^4D^2+27308861671680H^2D^3+411430804078878720D^4$

$\mathrm{Tr}(X^5)$ $\quad -5H^6D^2-552459647760H^4D^3-1001721601502668800H^2D^4$
$\qquad -8528203665906974392320D^5$
$\mathrm{Tr}(X^6)$ $\quad 226940810H^6D^3+10109205590 0113920H^4D^4$
$\qquad +301244776201771812864 00H^2D^5$
$\qquad +1786641266178486720685670 40D^6$

Case IV. $p=5,\ \ell=4,\ \lambda=10,\ k=14.$
$g=5^2\cdot 71\cdot 521\varphi$

X^6 $\quad 1$
X^5 $\quad 0$
X^4 $\quad -2545GH^2D-604109741760GD^2$
X^3 $\quad 25344112320H^3D^2-21193152057391 1040HD^3$
X^2 $\quad 1207235G^2H^4D^2-27110066987928960G^2H^2D^3$
$\qquad -1839342399957117683 7120G^2D^4$
X $\quad GH^7D^2+22434273283920GH^5D^3$
$\qquad +5557901335458375149568GH^3D^4$
$\qquad -306714023877649287994343424GHD^5$
1 $\quad -924775H^8D^3-217790930732661680 00H^6D^4$
$\qquad -1377333793706508373865472 00H^4D^5$
$\qquad -13353978977429466150344392704 00H^2D^6$

$\mathrm{Tr}(X)$ $\quad 0$
$\mathrm{Tr}(X^2)$ $\quad 5090GH^2D+1208219483520GD^2$
$\mathrm{Tr}(X^3)$ $\quad -76032336960H^3D^2+635794561721733120HD^3$
$\mathrm{Tr}(X^4)$ $\quad 8125110G^2H^4D^2+114590105122832640G^2H^2D^3$
$\qquad +80347085617695248314 3680G^2D^4$
$\mathrm{Tr}(X^5)$ $\quad -5GH^7D^2-434675195691600GH^5D^3$
$\qquad -10164580382184703017 9840GH^3D^4$
$\qquad +641683050942935873278036869120GHD^5$
$\mathrm{Tr}(X^6)$ $\quad 14539127450H^8D^3+249071858318180032 3200H^6D^4$
$\qquad +7271821686052784666299576 3200H^4D^5$
$\qquad +6425407018434796912604359434829824 00H^2D^6$
$\qquad +8771463901699277047522726890361061376 00D^7$

Case V. $p=5,\ \ell=4,\ \lambda=12,\ k=16.$
$g=-5\cdot 31\cdot 601\cdot 691\varphi$

X^6 $\quad 1$
X^5 $\quad 81829440GD$
X^4 $\quad -10225G^2H^2D+1216590866568000G^2D^2$

X^3 $\quad 2910441252960H^4D^2+6127470158334076661760H^2D^3$
$\quad -3228232763504972929499 1360D^4$

X^2 $\quad 20065355GH^6D^2+20426028073872433 6320GH^4D^3$
$\quad +1025675627148717142617040 8960GH^2D^4$
$\quad +927548601074608807540446894 48960GD^5$

X $\quad G^2H^8D^2+301942307434744 80G^2H^6D^3$
$\quad -16160752402141715384813475 84G^2H^4D^4$
$\quad +42117422128340913862098973248 06144G^2H^2D^5$
$\quad -1322361548952497965184180353573 3751808G^2D^6$

1 $\quad 64370105H^{10}D^3+18604647958742074084 99840H^8D^4$
$\quad +14358613619629720425298218824 90880H^6D^5$
$\quad +2723246756592052120558494661460 99036160H^4D^6$
$\quad +1088901780807827047786927036936723 05131520H^2D^7$
$\quad +87186102896322601284933986395964551 5656724480D^8$

$\mathrm{Tr}(X)$ $\quad -81829440GD$

$\mathrm{Tr}(X^2)$ $\quad 20450G^2H^2D+4262875517577600G^2D^2$

$\mathrm{Tr}(X^3)$ $\quad -1124144183088 0H^4D^2-26765818188519726102 5280H^2D^3$
$\quad -333901542596495424649297920D^4$

$\mathrm{Tr}(X^4)$ $\quad 128839830GH^6D^2+35970816404844522 4320GH^4D^3$
$\quad +171765241739983259380327621 01760GH^2D^4$
$\quad +15348600887342729858716479124 930560GD^5$

$\mathrm{Tr}(X^5)$ $\quad -5G^2H^8D^2-3343343744366964 00G^2H^6D^3$
$\quad -62325620179500179366623180 80G^2H^4D^4$
$\quad -11052172542267893126003439704 2851151 8720G^2H^2D^5$
$\quad -704474107875704575852741574977576 721448960G^2D^6$

$\mathrm{Tr}(X^6)$ $\quad 907155508010H^{10}D^3+50642550664820076143 642880H^8D^4$
$\quad +84100815515218630048984550520668160H^6D^5$
$\quad +7113919327875099523376002049578430238 5949573120H^4D^6$
$\quad +152077104881047937683933892892491027290971 0$
$\quad 61616640H^2D^7$
$\quad +558823976148858965042637471021663302378110$
$\quad 42088386560D^8$

Case VI. $\quad p=5,\ \ell=6,\ \lambda=4,\ k=10.$
$\quad g=-5^2\cdot 13\varphi.$

X^6 $\quad 1$

X^5 $\quad 0$

X^4 $\quad -55G^2D$

X^3 $\quad -41040HD^2$

X^2 $\quad 395GH^2D^2-7266240GD^3$

X $\quad -G^2H^3D^2+121104G^2HD^3$

1 $\quad -325H^4D^3-2691200H^2D^4$

$\mathrm{Tr}(X)$ $\quad 0$

$$\mathrm{Tr}(X^2) \quad 110G^2D=2^{-38}\cdot 3\cdot 18!\cdot\frac{2^{23}}{3^9\cdot 5^2\cdot 7^2\cdot 13\cdot 17}f_{20}$$

$$\mathrm{Tr}(X^3) \quad 123120HD^2=2^{-58}\cdot 3\cdot 28!\sum_\sigma\left(\frac{2^{31}}{3^{11}\cdot 5^5\cdot 7^4\cdot 11^2\cdot 13^2\cdot 17\cdot 23\sqrt{51349}}f_{30}\right)^\sigma$$

$$\mathrm{Tr}(X^4) \quad 4470GH^2D^2+39519360GD^3=2^{-78}\cdot 3\cdot 38!$$
$$\cdot\sum_\sigma\left(\frac{2^{44}(149\alpha+142350336)}{3^{17}\cdot 5^7\cdot 7^5\cdot 11^3\cdot 13^2\cdot 17^2\cdot 19^2\cdot 23\cdot 29\cdot 31\cdot 37\psi'(\alpha)}f_{40}\right)^\sigma$$

$$N(149\alpha+142350336)=-2^{25}\cdot 3^7\cdot 5^2\cdot 7^2\cdot 19\cdot 8389\cdot 89003$$

$$\mathrm{Tr}(X^5) \quad 5G^2H^3D^2+10680480G^2HD^3=2^{-98}\cdot 3\cdot 48!$$
$$\cdot\sum_\sigma\frac{2^{52}(\alpha-180741120)}{3^{23}\cdot 5^9\cdot 7^6\cdot 11^4\cdot 13^3\cdot 17^2\cdot 19^2\cdot 23^2\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\psi'(\alpha)}f_{50}\Big)^\sigma$$

$$N(\alpha-180741120)=2^{22}\cdot 3^7\cdot 31\cdot 1223\cdot 18919300277$$

$$\mathrm{Tr}(X^6) \quad 204350H^4D^3+8391590400H^2D^4+5137086873600D^5$$
$$=2^{-118}\cdot 3\cdot 58!$$
$$\cdot\sum_\sigma\left(\frac{2^{68}(6637423\alpha^2+2121380494196736\alpha-675434433410334814371840)}{3^{27}\cdot 5^{11}\cdot 7^9\cdot 11^5\cdot 13^4\cdot 17^3\cdot 19^3\cdot 23^2\cdot 29^2\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47\cdot 53\psi'(\alpha)}f_{60}\right)^\sigma$$

$$N(6637423\alpha^2+2121380494196736\alpha-675434433410334814371840)$$
$$=2^{107}\cdot 3^{23}\cdot 5^3\cdot 7^5\cdot 17^4\cdot 181\cdot 233$$
$$\cdot 450456472421115659925683391938601142706911756985491297538017303$$

Case VII. $p=5,\ \ell=6,\ \lambda=6,\ k=12.$
$g=5^3\cdot 31\varphi.$

X^6 $\quad 1$

X^5 $\quad -10800D$

X^4 $\quad -175H^2D+39994560D^2$

X^3 $\quad -954000H^2D^2-58506624000D^3$

X^2 $\quad 4595H^4D^2+5976610560H^2D^3+29346922598400D^4$

X $\quad -H^6D^2-26913792H^4D^3+1347784593408H^2D^4$

1 $\quad 3875H^6D^3+35294248000H^4D^4$

$\mathrm{Tr}(X)$	$10800D$
$\mathrm{Tr}(X^2)$	$350H^2D+36650880D^2$
$\mathrm{Tr}(X^3)$	$8532000H^2D^2+139408128000D^3$
$\mathrm{Tr}(X^4)$	$42870H^4D^2+70958165760H^2D^3+554255811993600D^4$
$\mathrm{Tr}(X^5)$	$5H^6D^2+2374938960H^4D^3+433666967592960H^2D^4$ $+223776852111360000D^5$
$\mathrm{Tr}(X^6)$	$5935550H^6D^3+42310659964800H^4D^4$ $+2330955736245043200H^2D^5+908139108881675059200D^6$

Case VIII. $p=11,\ \ell=2,\ \lambda=4,\ k=6.$
$g=-11\cdot 61\varphi.$

X^{12}	1
X^{11}	0
X^{10}	$11088D$
X^9	$-9075HD$
X^8	$-5962H^2D+24952224D^2$
X^7	$-77H^3D-67215456HD^2$
X^6	$37678773H^2D^2+25829299584D^3$
X^5	$-17237913H^3D^2-119108926464HD^3$
X^4	$2011493H^4D^2+104087609758H^2D^3-68766745458048D^4$
X^3	$-55913H^5D^2-44737025102H^3D^3+98746847977536HD^4$
X^2	$440H^6D^2+6582378638H^4D^3-52178539740844H^2D^4$ $+21138255578398464D^5$
X	$-H^7D^2-308633685H^5D^3+6499878090033H^3D^4$ $-20914887319687488HD^5$
1	$-671H^6D^3-207290985242H^4D^4+1480882485474007H^2D^5$ $-3777866437306791104D^6$

$\mathrm{Tr}(X)$	0
$\mathrm{Tr}(X^2)$	$-22176D$
$\mathrm{Tr}(X^3)$	$27225HD$
$\mathrm{Tr}(X^4)$	$23848H^2D+146078592D^2$
$\mathrm{Tr}(X^5)$	$385H^3D-167040720HD^2$
$\mathrm{Tr}(X^6)$	$-375645699H^2D^2-1221354706176D^3$
$\mathrm{Tr}(X^7)$	$493425009H^3D^2+1841678960352HD^3$
$\mathrm{Tr}(X^8)$	$131680032H^4D^2+4757929285648H^2D^3+11020319744130048D^4$
$\mathrm{Tr}(X^9)$	$4634883H^5D^2-9276821678520H^3D^3-21750837501488832HD^4$
$\mathrm{Tr}(X^{10})$	$25245H^6D^2+340701804850H^4D^3-45226189141798280H^2D^4$ $-97227250729594799616D^5$

$$\begin{aligned}\operatorname{Tr}(X^{11}) \quad & 11H^7D^2+4400455461072H^5D^3+127109577108789495H^3D^4 \\ & +2391406460047937523 84HD^5\end{aligned}$$

$$\begin{aligned}\operatorname{Tr}(X^{12}) \quad & 825050231500H^6D^3-196655983317642 75H^4D^4 \\ & +393248139239476286700H^2D^5 \\ & +84518045159262798708 5568D^6\end{aligned}$$

In what follows, we are going to give the specialized equations $\Phi(X; gE^*_{\lambda,p}, \mathscr{E})=0$ at several elliptic curves $\mathscr{E}$ defined over $\boldsymbol{Q}$. Again let us explain how to read the table given below for specialized equations. We first list the curves where we specialize the transformation equations in Case I–V:

Case A:	$y^2=4x^3-2^2\cdot 3^{-1}x+3^{-3}\cdot 19$	(11A);
Case B:	$y^2= 4x^3-2^2x+1$	(37A);
Case C:	$y^2=4x^3-2^3\cdot 3^{-1}\cdot 5x+3^{-3}\cdot 251$	(37B);
Case D:	$y^2=4x^3+2^3\cdot 3x-2^3$	
Case E:	$y^2=4x^3+1$	(27A).

The curve in Case A is isogeneous to the modular curve $X_0(11)_{/\boldsymbol{Q}}$ $(\cong\mathfrak{H}/\Gamma_0(11))$. This curve is referred in [10] as 11A. The example of Case A is the restatement of [3, § 5]. The curves in Case B and Case C correspond the distinct non-isogeneous factors of the jacobian variety of $X_0(37)_{/\boldsymbol{Q}}$. The curve in Case D is found in Serre [5, 5.9.2], which has potential everywhere good reduction. The curve in Case E has complex multiplication under $\boldsymbol{Q}(\sqrt{-3})$. In the following table, we list the specialized equations of the transformation equations already listed above at these elliptic curves. In Case A, as is well known, all the specialized equations of level 5 are reducible; so, we here list only one of them which corresponds to that in Case I. All the factors of the equations listed below are irreducible over $\boldsymbol{Q}$.

Case A. $G=2^4,\ H=-2^3\cdot 19,\ D=-11.$

(I) $X^6+4400X^4-174240X^3+4801280X^2-340643072X+5881529280$
$=(X-22)(X^5+22X^4+4884X^3-66792X^2+3331856X$
$-267342240)$

Case B. $G=2^4\cdot 3,\ H=-2^3\cdot 3^3,\ D=37.$

(I) $X^6-44400X^4-1971360X^3+488897280X^2+47063460096X$
1162360730560
Discriminant$=2^{36}\cdot 3^{12}\cdot 5^5\cdot 11^6\cdot 37^{12}\cdot 42044237^2$
Constant term$=2^6\cdot 5\cdot 37^3\cdot 71711$

(II) $X^6-12360960X^4-173732014080X^3-906454164234240X^2$
$-1585928183337123 84X-617317300619300044800$

Discriminant$=2^{90}\cdot 3^{30}\cdot 5^5\cdot 37^{12}\cdot 431^2\cdot 1751588674548053514 8167^2$

Constant term$=-2^{19}\cdot 3^6\cdot 5^2\cdot 37^3\cdot 1275457$

(III) $X^6+2237760X^5+1781129088000X^4+603569053249044480X^3$
$+77756911326531739975680X^2$
$-5246190928164651604346142 72X$
$+1054703030814562065989248430 89920$

Discriminant$=2^{156}\cdot 3^{60}\cdot 5^5\cdot 37^{12}\cdot 5387^2\cdot 11719^2$
$\cdot 19132098094132357350596121 53^2$

Constant term$=2^{33}\cdot 3^{12}\cdot 5\cdot 37^3\cdot 71711\cdot 1272109$

(IV) $X^6-396974702319206 40X^4+2318403282667096971018240X^3$
$-795716495364315743888001623654 40X^2$
$+21547580043025511396710363748451011 7888X$
$-180650394914609884769327204746079333724979200$

Discriminant$=2^{156}\cdot 3^{90}\cdot 5^5\cdot 11^2\cdot 37^{12}$
$\times\begin{pmatrix}1692620601527228431550100005983156558085509852 48\\190802837328703\end{pmatrix}^2$

Constant term$=-2^{30}\cdot 3^{18}\cdot 5^2\cdot 37^3\cdot 2777\cdot 6469\cdot 19089662430217$

(V) $X^6+145329085440X^5+383734167247978168 3200X^4$
$-460214266235592346411892893286 40X^3$
$+351785812651605713387572533809327772794880X^2$
$-467907236561224841504789599481108148857636609064 96X$
$+507573706310871139843866968593054587223929974 4266$
893393920

Discriminant$=2^{210}\cdot 3^{102}\cdot 5^5\cdot 37^{12}\cdot 73^2\cdot 39521^2$
$\times\begin{pmatrix}16388230918081261220040555436226618726893298260\\185323381603186396722399746932636921\end{pmatrix}^2$

Constant term$=2^{43}\cdot 3^{18}\cdot 5\cdot 37^3\cdot 661$
$\cdot 88971329820430423822802081 29$

Case C. $G=2^5\cdot 5,\ H=-2^3\cdot 251,\ D=37.$

(I) $X^6-148000X^4-1971360X^3+5432192000X^2$
$+1029841968640X+14284097373120$

Discriminant$=2^{36}\cdot 5^5\cdot 37^{12}\cdot 97^2\cdot 251^4\cdot 158512865466953^2$

Constant term$=2^6\cdot 3\cdot 5\cdot 37^3\cdot 293749$

(II) $X^6-137344000X^4-1615064279040X^3+151709417062400X^2$
$-1666186390720675 8400X-539800778571532271616 00$

Discriminant $= 2^{96}\cdot 5^{11}\cdot 37^{12}\cdot 251^{2}\cdot 84649^{2}$
$\cdot 80524545706391590943442857^{2}$

Constant term $= -2^{18}\cdot 3\cdot 5^{2}\cdot 37^{3}\cdot 103\cdot 251^{2}\cdot 8353$

(III) $X^{6}+2237760X^{5}+1688966528000X^{4}+1242544486072320000X^{3}$
$+428210896271006105600000X^{2}$
$-1221019443602348105007104000000X$
$+121548575717669223512628264960000000$

Discriminant $= 2^{186}\cdot 5^{35}\cdot 37^{12}\cdot 149^{2}\cdot 251^{4}\cdot 613^{2}\cdot 7681^{2}\cdot 85999^{2}$
$\cdot 282871143533352749904 7^{2}$

Constant term $= 2^{30}\cdot 3\cdot 5^{6}\cdot 11\cdot 37^{3}\cdot 433421914559869$

(IV) $X^{6}-132384946524160000X^{4}+21274901477944622530560000X^{3}$
$-10242321004657701912901058560000000X^{2}$
$-66663381877744811933411598925824000000 0X$
$-1717659704516737715044221514686362812416000000000$

Discriminant $= 2^{156}\cdot 5^{41}\cdot 37^{12}\cdot 223^{2}\cdot 251^{2}$
$$\times\begin{pmatrix}210413886361011152471697101677904852736456762 34\\ 37015564475299986747399133\end{pmatrix}^{2}$$

Constant term $= -2^{30}\cdot 3\cdot 5^{8}\cdot 37^{3}\cdot 251^{2}\cdot 4277631339834905360883 1$

(V) $X^{6}+484430284800X^{5}+42637091095065067520000X^{4}$
$+1191013336577507913643327488000000X^{3}$
$+134572490568813512164803429231755264000000000X^{2}$
$+28017443414101677967629098569361263396126720000000000X$
$+179318260454300488579895539058291784993371546714$
11200000000000

Discriminant $= 2^{216}\cdot 5^{55}\cdot 37^{12}\cdot 251^{4}\cdot 3001^{2}$
$$\times\begin{pmatrix}59569294896991541925368974827809306186927780234\\ 9144890245644003939481473021094394736831441 03\end{pmatrix}^{2}$$

Constant term $= 2^{42}\cdot 3\cdot 5^{11}\cdot 37^{3}\cdot 809$
$\cdot 67923444740355187523117805785144 9$

Case D. $G=-2^{5}\cdot 3^{2}$, $H=2^{6}\cdot 3^{3}$, $D=-2^{6}\cdot 3^{5}$.

(I) $X^{6}-111974400X^{4}-348285173760X^{3}$
$+3109490031329280X^{2}+19395514284707414016X$
$+30756189783160164188160$

Discriminant $= 2^{156}\cdot 3^{102}\cdot 5^{5}\cdot 523^{2}\cdot 1993^{2}$

Constant term $= 2^{30}\cdot 3^{20}\cdot 5\cdot 31\cdot 53$

(II) $X^{6}+187042037760X^{4}+2455494063449702 40X^{3}$
$-4095999825264194774630 40X^{2}$
$-33910115106396916746100408 32X$

$-1232983430314568952093894456115200$

Discriminant $=2^{228}\cdot 3^{132}\cdot 5^{5}\cdot 13^{2}\cdot 176660195838663136987^{2}$

Constant term $=-2^{45}\cdot 3^{26}\cdot 5^{2}\cdot 551461$

(III) $X^{6}-940584960X^{5}+3148962351022080000X^{4}$

$-44180830874421892617338880X^{3}$

$+2280132588424701078050853649121280X^{2}$

$-17214744129018591819208640165526254911488X$

$+43732973945707100175825194219369249263344156672\,0$

Discriminant $=2^{306}\cdot 3^{162}\cdot 5^{5}\cdot 2857^{2}$

$\cdot 128103862043615822809259620839427^{2}$

Constant term $=2^{60}\cdot 3^{32}\cdot 5\cdot 31\cdot 53\cdot 509\cdot 48955757$

(IV) $X^{6}+4208045923858460442624 0X^{4}$

$+137755201561421637187333471076352 0X^{3}$

$-892215225577255581337202823712315120445030 40X^{2}$

$-1393508005060190232849573134375438826070233330 2$

$1057024X$

$-553005321475495000766649906481955867310524936332$

65100180684800

Discriminant $=2^{368}\cdot 3^{192}\cdot 5^{5}\cdot 47^{2}\cdot 109^{2}$

$\cdot 44275612064289958736301204301726556694592007012$

18607287629^{2}

Constant term $=-2^{72}\cdot 3^{38}\cdot 5^{2}\cdot 107\cdot 3240694984266366049$

(V) $X^{6}+366512097853440X^{5}+2440630437357417453972357120 0X^{4}$

$-1957290889419754283886842288845509560893440X^{3}$

$+23787048189330037674267146417212680864094364958 2$

$79598080X^{2}$

$-1646767309836256491740571252143341940013589479 47$

$98454628675502473216X$

$+294635125712074867303286507823764150716792024944 1$

$16288482193800643302215647232 0$

Discriminant $=2^{438}\cdot 3^{222}\cdot 5^{5}$

$$\times\begin{pmatrix}17055481350680990459643415602664324740824343889\\6355667551086293800093349973130262537 07\end{pmatrix}^{2}$$

Constant term $=2^{87}\cdot 3^{44}\cdot 5\cdot 19\cdot 9787\cdot 20795362588083644126474341$

Case E. $G=0,\ H=-2^{3}\cdot 3^{3},\ D=-3^{3}.$

(I) $X^{6}-1049760X^{3}+226351350720$

Discriminant $=2^{36}\cdot 3^{66}\cdot 5^{5}\cdot 11^{6}\cdot 17^{6}$

Constant term $=2^{6}\cdot 3^{12}\cdot 5\cdot 11^{3}$

(II) $X^6-92513249280X^3-1749903443385974784 00$

$\text{Discriminant}=2^{96}\cdot 3^{96}\cdot 5^{13}\cdot 7^6\cdot 41^6\cdot 61^6$

$\text{Constant term}=-2^{18}\cdot 3^{18}\cdot 5^2\cdot 41^3$

(III) $X^6-1632960X^5+949822848000X^4-227647896698880000X^3$
$+19303585597263052800000X^2-674475678084121598361600000X$
$+839433158209838194989465600000 0$

$$=(X^2-544320X+20323353600)^3$$

Discriminant of the irreducible factor $=2^{18}\cdot 3^8\cdot 5^3$

Constant term of the irreducible factor $=2^{10}\cdot 3^8\cdot 5^2\cdot 11^2$

(IV) $X^6-9012189878816256000000 00X^3$
$-1983712130025693212836833656832000000000000 00$

$\text{Discriminant}=2^{156}\cdot 3^{156}\cdot 5^{65}\cdot 17^6\cdot 23^6\cdot 59^6\cdot 71^6\cdot 70157^6$

$\text{Constant term}=-2^{30}\cdot 3^{30}\cdot 5^{13}\cdot 59^3\cdot 71^3$

(V) $X^6-227831878268154706479022080000000X^3$
$+420663885954424404794201383021715969885287219200$
000000000

$\text{Discriminant}=2^{216}\cdot 3^{186}\cdot 5^{55}\cdot 7^6\cdot 13^6\cdot 173^6\cdot 521^6\cdot 4519^6$
$\cdot 20703622163768078697283 67^2$

$\text{Constant term}=2^{42}\cdot 3^{36}\cdot 5^{11}\cdot 521^3\cdot 4519^3$

References

[1] T. Asai, On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution, J. Math. Soc. Japan, **28** (1976), 48–61.

[2] N. Bourbaki, Algèbre Commutative, Hermann, Paris, 1965.

[3] K. Doi, H. Hida and Y. Maeda, Transformation equations and the special values of Shimura's zeta functions, Hokkaido Math. J., **13** (1984).

[4] H. Hida, On the values of Hecke's L-functions at non-positive integers, J. Math. Soc. Japan, **30** (1978), 249–278.

[5] J. P. Serre, propriétés galoisiennes des points d'ordre fini des courbes elliptiques, Invent. math., **15** (1972), 259–331.

[6] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, No. 11, Iwanami Shoten and Princeton Univ. Press, Tokyo, 1971.

[7] ——, On some arithmetic properties of modular forms of one and several variables, Ann. of Math., **102** (1975), 491–515.

[8] ——, On the Fourier coefficients of modular forms of several variables, Nachr. Akad. Wiss. Göttingen, (1975), 261–268.

[9] ——, The special values of the zeta functions associated with cusp forms, Comm. Pure Appl. Math., **29** (1976), 783–804.

[10] Table 1, In "Modular functions of one variable IV", Lecture Notes in Math. **476**, Springer, (1975), 81–113.

Department of Mathematics
Hokkaido University
Sapporo, 060, Japan