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Moonshine for $PSL_2(F_7)$

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0. In [1], Conway and Norton assigned a Thompson series of the form

$$q^{-1}+\sum_{n=1}^{\infty} H_n(m)q^n, \quad q=e^{2\pi i z}$$

to each element m of the Fischer-Griess group F_1 where H_n are characters of F_1, and they conjectured among others that Thompson series are generators of the modular function fields of genus zero for some modular groups which contain $\Gamma_0(N)$ for some N. In [6], Queen studied moonshine for other simple groups, for example, Thompson's group F_3.

In this paper, we consider these phenomena for $PSL_2(F_7)$ and its relation to Conway-Norton's monstrous moonshine.

Let $G=PSL_2(F_7)$. G acts on $F_7\cup\{\infty\}$ as linear fractional transformations, so G can be considered as the subgroup of S_8. Then, each element of G is written by products of cycles and these are of the following forms:

$$1^8,\ 1\cdot 7,\ 1^2\cdot 3^2,\ 2^4,\ 4^2.$$

For each product of cycles of length n_i, $m=(n_1)(n_2)\cdots(n_s)$, $n_1\geq\cdots\geq n_s\geq 1$ $\sum_{i=1}^{s} n_i=8$, in G, we associate following modular forms:

$$\eta_{1,m}(z)=\prod_{i=1}^{s}\eta(3n_i z),$$

$$\eta_{2,m}(z)=\prod_{i=1}^{s}\eta(n_i z)^3,$$

where $\eta(z)$ is the Dedekind η-function. Then $\eta_{1,m}(z)$ (resp. $\eta_{2,m}(z)$) is a cusp form of weight $s/2$ (resp. $3s/2$) on $\Gamma_0(9n_1n_s)$ (resp. $\Gamma_0(n_1n_s)$) with some character and is known to be a common eigenfunction of all Hecke operators (cf. [4]).

We shall prove

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Theorem 0.1. *For each m of G, there exists a modular form* $\vartheta_m(z) = \sum_{n=0}^{\infty} a_n(m)q^n$ *satisfying following properties:*

(0.1) $\vartheta_m(z)$ *is a* θ*-function of some quadratic lattice.*

(0.2) *There exist characters* H_n, $n \geq 0$, *of G such that*

$$H_n(m) = a_n(m) \quad \text{for all } m \in G.$$

(0.3) *Put* $j_{1,m}(z) = \vartheta_m(3z)/\eta_{1,m}(z)$. *Then* $j_{1,m}(z)$ *coincides with a Thompson series to some element of Thompson's group* F_3.

(0.4) *Put* $j_{2,m}(z) = \vartheta_m(z)^3/\eta_{2,m}(z)$. *Then* $j_{2,m}(z)$ *coincides with a Thompson series to some element of* F_1 *up to a constant term.*

(0.5) $j_{2,m}(z) = j_{1,m}(z/3)^3$ *for all* $m \in G$.

Similar theorems can be proved for $PSL_2(F_5)$ and $PSL_2(F_3)$. (see § 3).

1. We define $\vartheta_m(z)$ as follows:

m	$\vartheta_m(z)$	
1^8	$\theta_{E_8}(z)$	$1+240\sum_{n=1}^{\infty}(\sum_{d\mid n, d>0} d^3)q^n$
$1\cdot 7$	$\theta\left(z, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\right)$	$1+2\sum_{n=1}^{\infty} c_{1,\chi}(n)q^n$
$1^2\cdot 3^2$	$\theta\left[z, \left[\begin{array}{cc\|cc} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]\right]$	$1+12\sum_{n=1}^{\infty}(\sum_{d\mid n, d>0, (d,3)=1} d)q^n$
2^4	$\theta\left[z, \left[\begin{array}{cc\|cc} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right]\right]$	$1+8\sum_{n=1}^{\infty}(\sum_{d\mid n, d>0, 4\nmid d} d)q^n$
4^2	$\theta\left(z, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\right)$	$1+4\sum_{n=1}^{\infty} c_{1,\psi}(n)q^{2n}$

Here, in last column, we give explicit description of $\vartheta_m(z)$ as Eisenstein series. E_8 means E_8-lattice, namely, even integral unimodular 8-dimensional quadratic lattice. For any positive definite, even integral symmetric matrix $A = (a_{ij})_{1\leq i,j\leq n}$, we define $\theta(z, A) = \sum_{x=(x_1,\cdots,x_n)\in Z^n} e^{\pi i z x A {}^t x}$. χ and ψ are non-trivial real Dirichlet characters mod 7 and mod 4 respectively, and put $c_{1,\chi}(n)=\sum_{\substack{d|n\\d>0}}\chi(d)$ and $c_{1,\psi}(n)=\sum_{\substack{d|n\\d>0}}\psi(d)$. Hence (0.1) is true. With these data, we can prove following propositions by some calculations and comparing these with tables in [1] and [6].

Proposition 1.1. *Let F_3 be Thompson's simple group. For each m in G, there exists an element of F_3 whose Thompson series in Table* III *of* [6] *coincides with $j_{1,m}(z)$.*

G	1^8	$1\cdot 7$	$1^2\cdot 3^2$	2^4	4^2
F_3	1A	7A	3A	4A	8A

where elements of F_3 are denoted by the same symbol as in Table III *of* [6].

Proposition 1.2. *For each m in G, there exists an element of F_1 whose Thompson series in Table* 4 *of* [1] *coincides with $j_{2,m}(z)$ up to a constant term c_m.*

G	1^8	$1\cdot 7$	$1^2\cdot 3^2$	2^4	4^2
F_1	1A	7A	3A	4A	8B
c_m	720	6	36	16	0

Hence (0.3), (0.4) and (0.5) are proved to be true.

Remark 1.3. (0.5) for $m=1^8$ shows

$$\left(\frac{\theta_{E_8}(z)}{\eta(z)^8}\right)^3=j(z),\quad j(z) \text{ the modular invariant,}$$

which is a well-known identity.

To prove (0.2), we need the character table of $PSL_2(F_7)$;

	1^8	$1\cdot 7$	$1^2\cdot 3^2$	2^4	4^2
1_G	1	1	1	1	1
ψ	7	0	1	-1	-1
x_2	8	1	-1	0	0
θ_2	6	-1	0	2	0
$\eta_1+\eta_2$	6	-1	0	-2	2

For this table, we refer Dornhoff's textbook [3]. For any fixed integer $n\geq 1$, we consider a class function χ on G by $\chi(m)=a_n(m)$. Then χ is written by

$$\chi=x1_G+y\psi+z\chi_2+w\theta_2+u(\eta_1+\eta_2)$$

with some constants x, y, z, w and u. Solving linear equations, we get

$$(1.1)\qquad \begin{cases} z=x+y-a_n(1^2\cdot 3^2), \\ u=\frac{1}{2}(y-x)+\frac{1}{2}a_n(4^2), \\ w=y-x+\frac{1}{2}a_n(4^2)+\frac{1}{2}a_n(2^4), \\ 24y=a_n(1^8)+8a_n(1^2\cdot 3^2)-6a_n(4^2)-3a_n(2^4), \\ 168x=a_n(1^8)+48a_n(1\cdot 7)+56a_n(1^2\cdot 3^2)+42a_n(4^2)+21a_n(2^4). \end{cases}$$

Therefore, it is easily seen that x, y, z, w and u are integers if and only if the following congruences hold:

$$(1.2)\qquad a_n(1^8)-a_n(1^2\cdot 3^2)\equiv 0 \pmod 3,$$

$$(1.3)\qquad a_n(1^8)-a_n(1\cdot 7)\equiv 0 \pmod 7,$$

$$(1.4)\qquad a_n(1^8)\equiv a_n(2^4) \pmod 4,$$

$$(1.5)\qquad a_n(1^8)-3a_n(2^4)+2a_n(4^2)\equiv 0 \pmod 8.$$

From the explicit description of $a_n(m)$ for $n\geq 1$, we get

$$(1.6)\qquad \begin{cases} a_n(1^8)\equiv 0 \pmod{48}, & a_n(1^2\cdot 3^2)\equiv 0 \pmod 3, \\ a_n(2^4)\equiv 0 \pmod 8, & a_n(4^2)\equiv 0 \pmod 4. \end{cases}$$

Hence (1.2), (1.4) and (1.5) are proved to be true. We see that

$$a_n(1^8)=240\sum_{\substack{d|n\\ d>0}} d^3\equiv 2\sum_{\substack{d|n\\ d>0}}\left(\frac{d}{7}\right)=a_n(1\cdot 7) \pmod 7,$$

so (1.3) is proved to be true.

To show that x, y, z, w and u are positive, it is sufficient to prove that

$$(1.7)\qquad \begin{cases} a_n(1^8)-8a_n(1\cdot 7)-14a_n(4^2)-7a_n(2^4)>0, \\ a_n(1^8)+6a_n(1\cdot 7)-7a(1^2\cdot 3^2)>0. \end{cases}$$

From the explicit description of $a_n(m)$ for $n\geq 1$, we get

$$(1.8)\qquad \begin{cases} a_n(1^8)\geq 120a_n(1\cdot 7), & a_n(1^8)\geq 20a_n(1^2\cdot 3^2), \\ a_n(1^8)\geq 30a_n(2^4), & a_n(1^8)\geq 60a_n(4^2). \end{cases}$$

Hence (1.7) is proved.

2. We give several remarks.

Remark 2.1. $\vartheta_{1^8}(z)$ is the θ-function of E_8-lattice Λ. Hence, there arises a following question; does $PSL_2(\boldsymbol{F}_7)$ act on Λ as isometries such that

$$\text{(2.1)} \qquad \vartheta_m(z)=\theta(z, \Lambda_m) \qquad \text{with } \Lambda_m=\{x \in \Lambda \mid mx=x\}?$$

If this is true, we get another proof of (0.2).

There is an action of S_8 on Λ which arises in Lie group theory; let $e_1, \cdots, e_8$ be an orthonormal basis of $\boldsymbol{R}^8$, i.e. $\langle e_i, e_j\rangle=\delta_{ij}$. Put $\tilde{e}=\frac{1}{2}\sum_{i=1}^8 e_i$, $L=\sum_{i=1}^8 \boldsymbol{Z}e_i+\boldsymbol{Z}\tilde{e}$. Then $\Lambda=\{a=\sum_{i=1}^8 a_ie_i \in L \mid \sum_{i=1}^8 a_i$ even integer$\}$. S_8 acts on Λ by natural permutations of $e_1, \cdots, e_8$. Therefore $PSL_2(\boldsymbol{F}_7)$ which is a subgroup of S_8 acts on Λ as isometries. However, in this situation, one knows that

$$\text{(2.2)} \qquad \begin{cases} \vartheta_m(z)=\theta(z, \Lambda_m) & \text{for } m=1^8,\ 1\cdot 7,\ 1^2\cdot 3^2, \\ \vartheta_m(z)\neq\theta(z, \Lambda_m) & \text{for } m=2^4,\ 4^2. \end{cases}$$

In fact, we can answer the above question affirmatively in constructing a good lattice by using code theory. The detail will appear in elsewhere.

Remark 2.2. In Table III of L. Queen's paper [6], she gave Thompson series $j_m(z)$ to each element m of Thompson's group F_3. One sees that

$$\text{(2.3)} \qquad j_m(z)=qf_m(3z) \qquad \text{for some } f_m(z)=\sum_{n=0}^{\infty} a_n e^{2\pi i n z},\ a_n \in \boldsymbol{Z}.$$

Put

$$\text{(2.4)} \qquad \tilde{j}_m(z)=j_m\left(\frac{z}{3}\right)^3=qf_m(z)^3.$$

Then, comparing these with Table 4 in Conway-Norton's paper [1], one knows that $\tilde{j}_m(z)$ coincides with Thompson series to some element of F_1 up to a constant term, except for 18B, 27A and 27C. For 18B of F_3, $\tilde{j}_m(z)$ coincides with Thompson series to 3A of $3\cdot 2$ Suzuki. I wonder that this coincidence can be interpretated via E_8-lattice. See Remark 2.3.

Remark 2.3. In [2], it is known that $PSL_2(\boldsymbol{F}_7)$ is a maximal subgroup of the Mathieu group M_{24}. There exists an element Π of type 3^8 in S_{24}, but not in M_{24} such that the centralizer C_Π of Π in M_{24} is isomorphic to $PSL_2(\boldsymbol{F}_7)$. This isomorphism maps an element of type $(n_1)\cdots(n_s)$ to $(n_1)^3\cdots(n_s)^3$.

M_{24} acts on the Leech lattice L as isometries. Then, for the element g of type 3^8 in M_{24}, it is supposed to be true that

$$\theta(z, L_g)=E_4(3z)=\theta_{E_8}(3z) \tag{2.5}$$

with $L_g=\{x \in L \mid gx=x\}$. Does $PSL_2(F_7)$ act on L_m as isometries such that

$$\vartheta_m(3z)=\theta(z, (L_g)_m) \qquad \text{for all } m \in PSL(F_7)? \tag{2.6}$$

Thompson's group F_3 appears as the subgroup of the centralizer of 3C in F_1 and it is known that Thompson series to 3C in F_1 coincides with

$$\frac{E_4(3z)}{\eta(3z)^8}=\frac{\theta_{E_8}(z)}{\eta(z)^8}\bigg|_{z\to 3z}.$$

For the results mentioned here, we refer [5].

Remark 2.4. For any product m of cycles of length n_i, $m=(n_1)\cdots(n_s)$, $n_1\geq n_2\geq\cdots\geq n_s$, m is called symmetric if

$$\{n_1, n_2, \cdots, n_s\}=\left\{\frac{N}{n_1}, \cdots, \frac{N}{n_s}\right\} \quad \text{with } N=n_1n_s \tag{2.7}$$

and

$$n_i \text{ divides } n_1 \qquad \text{for all } i. \tag{2.8}$$

We assume that $\sum_{i=1}^{s} n_i=8$ and s is even. Then all solutions of symmetric products are given by

$$1^8,\ 1\cdot 7,\ 1^2\cdot 3^2,\ 2^4,\ 4^2 \ \text{ and } \ 2\cdot 6. \tag{2.9}$$

One knows that, except for $2\cdot 6$, they appear in $PSL_2(F_7)$. For $2\cdot 6$, we define

$$\vartheta_{2\cdot 6}(z)=\theta\left(z, \begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix}\right). \tag{2.10}$$

Then we can prove that the analogous statement to (0.3), (0.4) and (0.5) for $2\cdot 6$ is true.

3. In cases of $PSL_2(F_5)$ and $PSL(F_3)$, moonshines become more elementary than that of $PSL_2(F_7)$.

Let $p=3$ or 5. Let $\Omega=F_p\cup\{\infty\}$ and $L=\sum_{i\in\Omega} Ze_i$ such that $(e_i, e_j)=2\delta_{ij}$ be the even integral, quadratic lattice of rank $p+1$. $PSL_2(F_p)$ acts on Ω by linear fractional transformations, so we can define action of $PSL_2(F_p)$ on L by

$$m\cdot e_i=e_{m(i)} \qquad \text{for } m\in PSL_2(\boldsymbol{F}_p),\ i\in\Omega,$$

which gives an isometry on L.

We put, for any $m\in PSL_2(\boldsymbol{F}_p)$,

$$\vartheta_m(z)=\theta(z, L_m) \qquad \text{where } L_m=\{x\in L\,|\,m\cdot x=x\}.$$

The action of m in $PSL_2(\boldsymbol{F}_p)$ on Ω induces a permutation of Ω, so m can be written by a product of cycles of length n_i:

$$m=(n_1)\cdots(n_s), \qquad n_1\geq n_2\geq\cdots\geq n_s\geq 1.$$

For such m, we put

$$\eta_m(z)=\eta(n_1z)\cdots\eta(n_sz)$$

and

$$j_{1,m}(z)=\begin{cases}\dfrac{\vartheta_m(2z)}{\eta_m(4z)} & \text{when } p=5,\\[2mm] \dfrac{\vartheta_m(3z)}{\eta_m(6z)} & \text{when } p=3,\end{cases}$$

$$j_{2,m}(z)=\begin{cases}\dfrac{\vartheta_m(z)^2}{\eta_m(2z)^2} & \text{when } p=5,\\[2mm] \dfrac{\vartheta_m(z)^3}{\eta_m(2z)^3} & \text{when } p=3.\end{cases}$$

Then we can prove

Theorem 3.1. *The notation being as above, the following statements are true;*

(3.1) *For any $m\in PSL_2(\boldsymbol{F}_p)$, $j_{1,m}(z)$ and $j_{2,m}(z)$ coincide with Thompson series to some elements of F_1 up to constant terms, except for $j_{1,1\cdot 3}(z)$.*

(3.2) *For any $m\in PSL_2(\boldsymbol{F}_3)$, $j_{1,m}(z)$ coincides with Thompson series to some element of F_3.*

(3.3) *Let* $j_{1,m}(z)=\sum_{n=-1}^{\infty} H_n(m)q^n, \quad H_n(m)\in\boldsymbol{Z}.$

Then $H_n(m)$, for all $n\geq 1$, are characters of $PSL_2(\boldsymbol{F}_p)$.

(3.4) $$j_{2,m}(z)=\begin{cases} j_{1,m}\!\left(\dfrac{z}{2}\right)^2 & \textit{when } p=5,\\[2mm] j_{1,m}\!\left(\dfrac{z}{3}\right)^3 & \textit{when } p=3.\end{cases}$$

Case of $PSL_2(F_5)$:

m	1^6	$1\cdot 5$	$1^2\cdot 2^2$	3^2
$j_{1,m}(z)$	8B	40B	16A	24E
$j_{2,m}(z)$	4A	20A	8A	12D

Case of $PSL_2(F_3)$:

m		1^4	$1\cdot 3$	2^2
$j_{1,m}(z)$	F_1	12D		24E
	F_3	4A	12A	8A
$j_{2,m}(z)$		4A	12A	8B

Proof. (3.4) is obvious. The proofs of (3.1) and (3.2) can be done by some computations of $\vartheta_m(z)$ and $\eta_m(z)$. The similar results to (3.3) are already mentioned in [1] (see (2) in page 317), [7] and [8], so we omit the proof.

Remark 3.1. Analogous result to Remark 2.4 can be proved in case of $PSL_2(F_5)$; for $2\cdot 4$, we define

$$\vartheta_{2\cdot 4}(z)=\theta\left(z, \begin{pmatrix} 4 & \\ & 8 \end{pmatrix}\right).$$

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