Advanced Studies in Pure Mathematics 4, 1984
Group Representations and Systems of Differential Equations
pp. 291–330

Geometric Approach to the Completely Integrable Hamiltonian Systems Attached to the Root Systems with Signature

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Introduction

Because of their symmetry, completely integrable Hamiltonian systems are intimately related to the geometry of Lie groups and homogeneous spaces. Conversely it is probable to find new completely integrable Hamiltonian systems among the Hamiltonian systems naturally constructed in connection with Lie groups and homogeneous spaces. From the view point described above, we shall consider in this article the Hamiltonian systems attached to certain root systems with signature. In more detail, we shall treat the following ones. Throughout the paper we retain the following notations. Let n be an integer such that $n \geqq 2$, and let m be an integer satisfying $1 \leqq m \leqq n$. For a notational convenience we write

$$\sum_{(1)} = \sum_{1 \leqq j<k \leqq m,\ m<j<k \leqq n}, \qquad \sum_{(2)} = \sum_{1 \leqq j \leqq m,\ m<k \leqq n}$$

and

$$\mathrm{sh}\,(x) = \sinh\,(x), \qquad \mathrm{ch}\,(x) = \cosh\,(x)$$

and moreover for $q = (q_1, \cdots, q_n) \in \boldsymbol{R}^n$

$$q_{jk} = q_j - q_k, \qquad \hat{q}_{jk} = q_j + q_k.$$

(I) The Hamiltonian system attached to the root system with signature (A_{n-1}, ε_m).

This is the Hamiltonian system on the phase space $D_{(A_{n-1}, \varepsilon_m)} \times \boldsymbol{R}^n$ with the Hamiltonian $H_{(A_{n-1}, \varepsilon_m)}$ where

$$D_{(A_{n-1}, \varepsilon_m)} = \{q = (q_1, \cdots, q_n) \in \boldsymbol{R}^n;\ q_1 > \cdots > q_m,\ q_{m+1} > \cdots > q_n\} \tag{0.1}$$

and

Received August 10, 1983.

$$(0.2)\qquad H_{(A_{n-1},\,\varepsilon_m)}(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+c^2\Big(\sum\nolimits_{(1)}\mathrm{sh}^{-2}(q_{jk})-\sum\nolimits_{(2)}\mathrm{ch}^{-2}(q_{jk})\Big).$$

Here c is a nonzero real constant. This Hamiltonian system describes the motions of n particles with unit mass interacting each other on a line; they are devided into two types of particles, say, $q_1, \cdots, q_m$ and $q_{m+1}, \cdots, q_n$. The particles of the same type interact pairwise among themselves with the repulsive potential $c^2\,\mathrm{sh}^{-2}(x)$, while the particles of the different type interact pairwise with the attractive potential $-c^2\,\mathrm{ch}^{-2}(x)$.

(II) The Hamiltonian systems attached to the root systems with signature (C_n, ε_m), (D_n, ε_m), (C_n, ε'_m) and (D_n, ε'_m).

The phase spaces of the first and the second Hamiltonian systems are the same and given by

$$(0.3)\qquad \begin{aligned} &D_{(C_n,\,\varepsilon_m)}\times \boldsymbol{R}^n=D_{(D_n,\,\varepsilon_m)}\times\boldsymbol{R}^n\\ &\quad=\{q=(q_1,\cdots,q_n)\in\boldsymbol{R}^n;\,q_1>\cdots>q_m>0,\ q_{m+1}>\cdots>q_n>0\}.\end{aligned}$$

The Hamiltonians are given respectively by

$$(0.4)\qquad \begin{aligned} H_{(C_n,\,\varepsilon_m)}(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+c_1^2\{&\textstyle\sum_{(1)}(\mathrm{sh}^{-2}(q_{jk})+\mathrm{sh}^{-2}(\hat{q}_{jk}))\\ &-\textstyle\sum_{(2)}(\mathrm{ch}^{-2}(q_{jk})+\mathrm{ch}^{-2}(\hat{q}_{jk}))\}+c_2^2/2\displaystyle\sum_{j=1}^{n}\mathrm{sh}^{-2}(2q_j)\end{aligned}$$

and

$$(0.5)\qquad \begin{aligned} H_{(D_n,\,\varepsilon_m)}(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+c_1^2\{&\textstyle\sum_{(1)}(\mathrm{sh}^{-2}(q_{jk})+\mathrm{sh}^{-2}(\hat{q}_{jk}))\\ &-\textstyle\sum_{(2)}(\mathrm{ch}^{-2}(q_{jk})+\mathrm{ch}^{-2}(\hat{q}_{jk}))\}\end{aligned}$$

where c_1 and c_2 are nonzero real constants.

The phase spaces of the third and the fourth systems are the same and given by

$$(0.6)\qquad \begin{aligned} &D_{(C_n,\,\varepsilon'_m)}\times\boldsymbol{R}^n=D_{(D_n,\,\varepsilon'_m)}\times\boldsymbol{R}^n\\ &\quad=\{q=(q_1,\cdots,q_n)\in\boldsymbol{R}^n;\ q_1>\cdots>q_m,\\ &\qquad\qquad q_{m+1}>\cdots>q_n,\ q_m+q_n>0\}.\end{aligned}$$

The Hamiltonians are given respectively by

$$(0.7)\qquad \begin{aligned} H_{(D_n,\,\varepsilon'_m)}(q,p)=\frac{1}{2}\sum_{j=1}^{n}p_j^2+c_1^2\{&\textstyle\sum_{(1)}(\mathrm{sh}^{-2}(q_{jk})-\mathrm{ch}^{-2}(\hat{q}_{jk}))\\ &+\textstyle\sum_{(2)}(\mathrm{sh}^{-2}(\hat{q}_{jk})-\mathrm{ch}^{-2}(q_{jk}))\}\end{aligned}$$

and

$$H_{(C_n,\varepsilon'_m)}(q,p)=H_{(D_n,\varepsilon'_m)}(q,p)-c_2^2/2\sum_{j=1}^{n}\mathrm{ch}^{-2}(2q_j). \tag{0.8}$$

Here c_1 and c_2 are nonzero real constants. These systems are interpreted as the Hamiltonian systems describing the motions of $2n$ particles with unit mass on a line with the coordinates and momenta satisfying the restriction

$$q_{n+j}=-q_j,\quad p_{n+j}=-p_j\quad (1\leqq j\leqq n). \tag{0.9}$$

The systems are consisting of two types of particles and the particles are divided into 4 groups, that is, $\{q_1,\cdots,q_m\}$, $\{q_{m+1},\cdots,q_n\}$, $\{-q_1,\cdots,-q_m\}$ and $\{-q_{m+1},\cdots,-q_n\}$. In the case (C_n,ε_m) and (D_n,ε_m), the particles belonging to $\{q_1,\cdots,q_m\}$ and $\{-q_1,\cdots,-q_m\}$ are of the same type and the particles belonging to $\{q_{m+1},\cdots,q_n\}$ and $\{-q_{m+1},\cdots,-q_n\}$ are of the same type, but distinct from the former. On the contrary, in the case (C_n,ε'_m) and (D_n,ε'_m) the particles in $\{q_1,\cdots,q_m\}$ and $\{-q_{m+1},\cdots,-q_n\}$ are of the same type and the particles in $\{q_{m+1},\cdots,q_n\}$ and $\{-q_1,\cdots,-q_m\}$ are of the same type, but distinct from the former. The particles of the same type interact pairwise with the repulsive potential $c_1^2\,\mathrm{sh}^{-2}(x)$ except the pairs lying in the symmetric position arround the origin. While the particles of the different type interact pairwise with the attractive potential $-c_1^2\,\mathrm{ch}^{-2}(x)$ except the pairs lying in the symmetric position arround the origin. As for the particles in the symmetric position with respect to the origin, the following two cases occur; the first case, which corresponds to either (C_n,ε_m) or (C_n,ε'_m), is that they interact each other with the repulsive potential $2^{-1}c_2^2\,\mathrm{sh}^{-2}(x)$ if they are of the same type and with the attractive potential $-2^{-1}c_2^2\,\mathrm{ch}^{-2}(x)$ if they are of distinct type. The second case, which corresponds to (D_n,ε_m) and (D_n,ε'_m) is that they do not interact.

(III) The Hamiltonian systems attached to the root systems with signature (B_n,ε_m) and (BC_n,ε_m).

The phase spaces of these systems are identical and given by the same one as in (0.3). The Hamiltonians are given respectively by

$$\begin{aligned}H_{(B_n,\varepsilon_m)}(q,p)=&\frac{1}{2}\sum_{j=1}^{n}p_j^2+c_1^2\{\textstyle\sum_{(1)}(\mathrm{sh}^{-2}(q_{jk})+\mathrm{sh}^{-2}(\hat{q}_{jk}))\\&-\textstyle\sum_{(2)}(\mathrm{ch}^{-2}(q_{jk})+\mathrm{ch}^{-2}(\hat{q}_{jk}))\}+c_0^2\left(\sum_{j=1}^{m}\mathrm{sh}^{-2}(q_j)-\sum_{j=m+1}^{n}\mathrm{ch}^{-2}(q_j)\right)\end{aligned} \tag{0.10}$$

and

$$(0.11)\qquad H_{(BC_n,\varepsilon_m)}(q,p)=H_{(B_n,\varepsilon_m)}(q,p)+c_2^2/2\sum_{j=1}^{n}\mathrm{sh}^{-2}(2q_j).$$

These systems are interpreted as the Hamiltonian systems of $2n+1$ particles with unit mass on a line satisfying the condition that one particle is fixed at the origin and the rest $2n$ particles are constrained just as in (0.9). They are consisting of two types of particles; the particles in $\{q_1, \cdots, q_m, -q_1, \cdots, -q_m\}$ and the one at the origin are of the same type and the particles in $\{q_{m+1}, \cdots, q_n, -q_{m+1}, \cdots, -q_n\}$ are of the same type, but distinct from the former. The law of interaction is essentially the same as in (II).

We note that since the potential acting between the particles of distinct type is attractive and non-singular those particles can go through each other and may form bounded states. The more detailed behavior of the particles including the scattering process will appear in [12]. We further remark that if $m=n$, then the above systems have only one type of particles and the terms of attractive potential in the Hamiltonians disappear. Such systems were already considered in [18] and [20]. Therefore our Hamiltonian systems include them as special cases. One reason why we call the above systems the systems attached to root systems with signature is that the linear forms $q\mapsto q_{jk}$, $q\mapsto \hat{q}_{jk}$, $q\mapsto q_j$ and $q\mapsto 2q_j$ on $\boldsymbol{R}^n$ constitute root systems in $\boldsymbol{R}^n$ and the way of grouping the particles corresponds to a signature of roots (cf. § 1). Another reasons will be explained below.

The main tool of our study of the above systems is the reduction procedure of Hamiltonian systems with symmetry developed in [1], [15] and [17]. Here we employ the reverse of the reduction procedure, which was used in [15] for the Calogero system of n particles on a line moving under the inverse square potential. In fact, we shall show that the above systems are realized as the reduced Hamiltonian systems, which are obtained by reducing the Hamiltonian systems of the geodesic flow of various affine symmetric spaces under the action of certain isometry groups. Those affine symmetric spaces are constructed from the original Hamiltonian systems by using the corresponding root systems with signature (see § 4, § 5, and § 6). The realization of our systems as the reduced Hamiltonian systems is established in Theorems 4.3, 5.3 and 6.3. The fact that the Hamiltonian system of the geodesic flow of an affine symmetric space has sufficiently many involutive integrals of motion (cf. Proposition 3.5 and Corollary 3.6) implies the complete integrability of our systems (cf. Corollaries 4.4, 5.4 and 6.4). As a by-product of the above mentioned realization, the quadrature of the motion of our systems is reduced to linear algebra and moreover the Lax pairs of our systems are naturally obtained and the Hamiltonian flow of our systems are the isospectral ones. (cf. Corollaries 4.5, 5.5 and 6.5). Summarizing, we state the main results of this paper.

Theorem. *The Hamiltonian systems attached to the root systems with signature* (A_{n-1}, ε_m), (C_n, ε_m), (C_n, ε'_m), (D_n, ε_m) *and* (D_n, ε'_m) *are completely integrable. The Hamiltonian systems attached to the root systems with signature* (B_n, ε_m) *and* (BC_n, ε_m) *are completely integrable under the condition*

$$(c_0/c_1)^2 = (2c_1 - c_2)/c_1.$$

These systems have functionally independent, involutive integrals of motion, which are rational functions of $p_1, \cdots, p_n, \exp(q_1), \cdots, \exp(q_n)$.

The study of certain non-linear partial differential equations such as the Korteweg-de Vries equation and the discovery of "soliton" solutions have revived interest in the study of integrable Hamiltonian systems, and the various integrable systems have been found. In particular among one-dimensional many-body problems characterized by the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{1 \leq j < k \leq n} V(q_{jk}),$$

J. Moser showed that when $V(x) = x^{-2}$ and $\sin^{-2}(x)$, the systems are completely integrable, which are now called Calogero and Sutherland systems after the discoverers. After that, it was proved that when $V(x) = \sinh^{-2}(x)$ and $\wp(x)$ (Weierstrass $\wp$-function), the corresponding systems are also completely integrable (cf. [11] and [21]). Furthermore M. Olshanetsky and A. Perelomov ([20]) extended the systems of Calogero type described above in connection with root systems of classical type, and showed their complete integrability. The systems of Calogero type having two types of particles were suggested in [9] and studied in [22]. That corresponds to the system attached to the root system with signature (A_{n-1}, ε_m) in our terminology.

We need one more notation. We denote by I_+(resp. I_-) the set of indices given by $\{(j, k); 1 \leqq j \neq k \leqq m, m < j \neq k \leqq n\}$ (resp. $\{(j, k); 1 \leqq j \leqq m, m < k \leqq n$ or $1 \leqq k \leqq m, m < j \leqq n\}$).

§ 1. Hamiltonian dynamical systems attached to the root systems with signature

Let R be a root system. *A* singnature of R is a mapping ε of R into $\{1, -1\}$ satisfying

(i) $\varepsilon(-\alpha) = \varepsilon(\alpha)$ for $\alpha \in R$ and

(ii) $\varepsilon(\alpha+\beta)=\varepsilon(\alpha)\varepsilon(\beta)$ if α, β and $\alpha+\beta \in R$.

The pair (R, ε) is called a root system with signature. For such (R, ε), we set $R_{\varepsilon}=\{\alpha \in R;\ \varepsilon(\alpha)=1\}$. Then it is a subroot system of R. Let R^+ be a system of positive roots of R. We put $R_{\varepsilon}^+=R_{\varepsilon}\cap R^+$, which is a positive system of R_{ε}. We say that two signatures ε and ε' are equivalent if there exists an element s of the Weyl group W of R such that $\varepsilon'(\alpha)=\varepsilon(s\alpha)$ for all $\alpha \in R$. Now we list up the typical examples of the root systems with signature in $\boldsymbol{R}^n$, which will appear in this paper. Let $\{e_1, \cdots, e_n\}$ be the canonical basis of $\boldsymbol{R}^n$. The root system (BC_n) in $\boldsymbol{R}^n$ is given by the following set;

$$(BC_n)=\{\pm(e_j\pm e_k);\ 1\leqq j<k\leqq n\}\cup\{\pm e_j,\ \pm 2e_j;\ 1\leqq j\leqq n\}.$$

We remark that this root system contains the root systems

$$(A_{n-1})=\{\pm(e_j-e_k);\ 1\leqq j<k\leqq n\},$$

$$(B_n)=\{\pm(e_j\pm e_k);\ 1\leqq j<k\leqq n\}\cup\{\pm e_j;\ 1\leqq j\leqq n\},$$

$$(C_n)=\{\pm(e_j\pm e_k);\ 1\leqq j<k\leqq n\}\cup\{\pm 2e_j;\ 1\leqq j\leqq n\} \text{ and}$$

$$(D_n)=\{\pm(e_j\pm e_k);\ 1\leqq j<k\leqq n\}.$$

For each integer m $(1\leqq m\leqq n)$, we can define the signature ε_m of (BC_n) by

$$\varepsilon_m(e_j-e_k)=\varepsilon_m(e_j+e_k)=\begin{cases} 1 & (1\leqq j<k\leqq m \text{ or } m<j<k\leqq n), \\ -1 & (1\leqq j\leqq m,\ m<k\leqq n) \end{cases}$$

and

$$\varepsilon_m(e_j)=\begin{cases} 1 & (1\leqq j\leqq m), \\ -1 & (m<j\leqq n), \end{cases} \qquad \varepsilon_m(2e_j)=1 \quad (1\leqq j\leqq n).$$

The restriction of ε_m to each one of the subroot systems (A_{n-1}), (B_n), (C_n) and (D_n) is again a signature, so we denote it by the same letter. We note that ε_n means the trivial signature. Besides the above signatures, it is known (cf. [23]) that the root systems (C_n) and (D_n) have the following signatures ε'_m $(1\leqq m\leqq n)$, which are inequivalent to ε_m; we set

$$\varepsilon'_m(e_j-e_k)=\begin{cases} 1 & (1\leqq j<k\leqq m \text{ or } m<j<k\leqq n), \\ -1 & (1\leqq j\leqq m,\ m<k\leqq n), \end{cases}$$

$$\varepsilon'_m(e_j+e_k)=\begin{cases} -1 & (1\leqq j<k\leqq m \text{ or } m<j<k\leqq n), \\ 1 & (1\leqq j\leqq m,\ m<k\leqq n) \end{cases}$$

and in addition for (C_n) we set

$$\varepsilon'_m(2e_j)=-1 \quad (1\leqq j\leqq n).$$

One can easily check that any signature of the above root systems is equivalent to either ε_m or ε'_m for some m $(1\leqq m\leqq n)$ (cf. [23]).

Now we introduce the notion of the Hamiltonian dynamical system attached to the root system with signature. Let (R, ε) be a root system with signature in $\boldsymbol{R}^n$. We define the phase space by

$$\left(\Omega_{(R,\varepsilon)}\times \boldsymbol{R}^n, \sum_{i=1}^{n} dq_i\wedge dp_i\right),$$

where the (unlabelled) configuration space $\Omega_{(R,\varepsilon)}$ is an open subset of $\boldsymbol{R}^n$ given by

$$\Omega_{(R,\varepsilon)}=\{q\in \boldsymbol{R}^n; \langle\alpha, q\rangle\neq 0 \text{ for all } \alpha\in R_\varepsilon\}.$$

Here, $\langle\ ,\ \rangle$ means the canonical inner product of $\boldsymbol{R}^n$. We choose nonzero real constants c_α $(\alpha\in R)$ so that

$$c_{s\alpha}^2=c_\alpha^2 \text{ for all } \alpha\in R \text{ and } s\in W. \tag{1.1}$$

We define the Hamiltonian $H_{(R,\varepsilon)}$ on $\Omega_{(R,\varepsilon)}\times\boldsymbol{R}^n$ by

$$H_{(R,\varepsilon)}(q,p)=\frac{1}{2}\langle p,p\rangle+\sum_{\alpha\in R_\varepsilon^+} c_\alpha^2 \operatorname{sh}^{-2}\langle\alpha,q\rangle-\sum_{\alpha\in R^+\setminus R_\varepsilon} c_\alpha^2\operatorname{ch}^{-2}\langle\alpha,q\rangle. \tag{1.2}$$

We remark that the Hamiltonian $H_{(R,\varepsilon)}$ can be written as

$$H_{(R,\varepsilon)}=\frac{1}{2}\langle p,p\rangle+\frac{1}{2}\Big(\sum_{\alpha\in R_\varepsilon} c_\alpha^2 \operatorname{sh}^{-2}\langle\alpha,q\rangle-\sum_{\alpha\in R\setminus R_\varepsilon} c_\alpha^2\operatorname{ch}^{-2}\langle\alpha,q\rangle\Big) \tag{1.3}$$

and hence it does not depend on the choice of a positive system of roots. We call this Hamiltonian system the Hamiltonian system attached to the root system with signature (R, ε).

Lemma 1.1. *Let ε and ε' be equivalent signatures of the root system R. Then the Hamiltonian systems attached to (R, ε) and (R, ε') are isomorphic.*

Proof. Since ε and ε' are equivalent, there exists $s\in W$ such that $\varepsilon'(\alpha)=\varepsilon(s\alpha)$ for all $\alpha\in R$, and hence $\alpha\mapsto s\alpha$ gives rise to a bijective mapping of $R_{\varepsilon'}$ onto R_ε. If we consider the map of $\Omega_{(R,\varepsilon')}\times\boldsymbol{R}^n$ into $\boldsymbol{R}^n\times\boldsymbol{R}^n$ given by $(q,p)\mapsto(sq,sp)$, then we obtain immediately that it defines a symplectic diffeomorphism of $\Omega_{(R,\varepsilon')}\times\boldsymbol{R}^n$ onto $\Omega_{(R,\varepsilon)}\times\boldsymbol{R}^n$. Furthermore we can deduce from (1.1) and (1.3) that $H_{(R,\varepsilon)}(sq, sp)=H_{(R,\varepsilon')}(q, p)$ for all $(q, p) \in \Omega_{(R,\varepsilon')}\times \boldsymbol{R}^n$.

In the remainder of the paper, we restrict ourselves to considering the above Hamiltonian system on a connected component $D_{(R,\varepsilon)}\times \boldsymbol{R}^n$ of $\Omega_{(R,\varepsilon)}\times \boldsymbol{R}^n$, where $D_{(R,\varepsilon)}$ is given by

$$D_{(R,\varepsilon)}=\{q \in \boldsymbol{R}^n; \langle \alpha, q\rangle>0 \text{ for all } \alpha \in R_\varepsilon^+\}.$$

This amount to considering a labelled configuration space. We notice that all of the Hamiltonian systems given in the introduction are the special cases of the Hamiltonian systems attached to the root systems with signature described above. We shall afford one example for a convenience of the reader. Take (BC_n, ε_m). Choose

$$\{e_j\pm e_k;\ 1\leqq j<k\leqq n\}\cup\{e_j, 2e_j;\ 1\leqq j\leqq n\}$$

for its positive system. Then from the definition of ε_m, we can deduce

$$(BC_n)_{\varepsilon_m}^+=\{e_j\pm e_k;\ 1\leqq j<k\leqq m,\ m<j<k\leqq n\}$$
$$\cup\{e_j;\ 1\leqq j\leqq m\}\cup\{2e_j;\ 1\leqq j\leqq n\}.$$

Moreover it can be easily seen that $c^2_{e_j\pm e_k}$ are all equal, so we can put them c_1^2. Similarly $c^2_{e_j}$ (resp. $c^2_{2e_j}$) are all equal, so that we can put them c_0^2 (resp. $c_2^2/2$). Then we can conclude that $H_{(BC_n,\varepsilon_m)}$ can be written as (0.11).

§ 2. The structure of affine symmetric spaces

Let G be a connected reductive linear Lie group with Lie algebra $\mathfrak{g}$. We fix a nondegenerate invariant symmetric bilinear form $\langle X, Y\rangle$ $(X, Y \in \mathfrak{g})$ on $\mathfrak{g}$, and we identify $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$ under $\langle\ ,\ \rangle$. Let σ be an involutive automorphism of G, and let θ be a Cartan involution of G commuting with σ. Let H be a subgroup of G which lies between the fixed point group G_σ and its identity component G_σ°. The homogeneous space G/H is called an affine symmetric space. Put $K=G_\theta$. Then K is a maximal compact subgroup of G. We denote the involution of $\mathfrak{g}$ corresponding to σ (resp. θ) of G by the same letter σ (resp. θ). Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces for σ (resp. θ). The restriction of $\langle\ ,\ \rangle$ to each one of $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{k}$ and $\mathfrak{p}$ is nondegenerate, and hence we can identify the dual space of each one of $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{k}$ and $\mathfrak{p}$ with itself. We note that the above decompositions of $\mathfrak{g}$ are the orthogonal ones with respect to $\langle\ ,\ \rangle$.

Now we recall the several facts about the certain class of affine symmetric spaces (cf. [23]). Those affine symmetric spaces are constructed by using the signatures of roots, which we shall consider later in the sections 4, 5 and 6. Let G be as above. We first take a Cartan involution θ of G, and denote the corresponding Cartan decomposition of $\mathfrak{g}$ by $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote the set of non-zero roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ by R. Then $\mathfrak{g}$ is decomposed into the direct sum $\mathfrak{g}=\mathfrak{g}^0+\sum_{\alpha\in R}\mathfrak{g}^\alpha$ where $\mathfrak{g}^0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and each $\mathfrak{g}^\alpha$ is the root space corresponding to the root $\alpha\in R$. Let ε be a signature of R. Then we can construct an involution σ on $\mathfrak{g}$ commuting with θ in the following way; $\sigma=\varepsilon(\alpha)\theta$ on $\mathfrak{g}^\alpha$ ($\alpha\in R$) and $\sigma=\theta$ on $\mathfrak{g}^0$. Put

$$\mathfrak{h}=\{X\in\mathfrak{g};\sigma(X)=X\} \quad\text{and}\quad \mathfrak{q}=\{X\in\mathfrak{g};\sigma(X)=-X\}.$$

Let H^0 be the analytic subgroup of G with Lie algebra $\mathfrak{h}$, and let M be the centralizer of $\mathfrak{a}$ in K. Define $H=H^0M$. Then it is a closed subgroup of G and the homogeneous space G/H is an affine symmetric space. Set

$$\mathfrak{a}_+=\{X\in\mathfrak{a};\alpha(X)>0 \text{ for every } \alpha\in R_\varepsilon^+\}$$

and denote its closure by $\bar{\mathfrak{a}}_+$. Put $A_+=\exp(\mathfrak{a}_+)$ and $\bar{A}_+=\exp(\bar{\mathfrak{a}}_+)$. Then it is known (cf. [23]) that $G=K\bar{A}_+H$ and moreover the map of $K/M\times A_+$ into G/H defined by $(kM, X)\mapsto k\exp(X)H$ is an analytic diffeomorphism onto an open dense submanifold of G/H.

§ 3. Hamiltonian systems on the cotangent bundle over G/H

Let G/H be an affine symmetric space where G is a connected real reductive linear Lie group. We consider certain Hamiltonian dynamical system on the cotangent bundle over G/H. The notations are the same as in the previous sections.

We define the action of the product group $G\times H$ on $G\times\mathfrak{q}$ by

$$(g,h)\cdot(x,X)=(gxh^{-1},\operatorname{Ad}(h)X) \tag{3.1}$$

where $(g,h)\in G\times H$ and $(x,X)\in G\times\mathfrak{q}$. We denote by $\mathcal{M}$ the orbit space for the H-action in $G\times\mathfrak{q}$, and denote the canonical projection of $G\times\mathfrak{q}$ onto $\mathcal{M}$ by π. Since the H-action is proper and free, it follows that $\mathcal{M}$ is a smooth manifold and π is a submersion. The G-action on $G\times\mathfrak{q}$ induces the G-action on $\mathcal{M}$ by

$$g\pi(x,X)=\pi(gx,X) \qquad (g\in G,\ \pi(x,X)\in\mathcal{M}). \tag{3.2}$$

If we define a map $\tilde{\omega}$ of $\mathcal{M}$ onto G/H by $\tilde{\omega}(\pi(x,X))=xH$, then it is obvious that $\mathcal{M}$ is a G-homogeneous vector bundle over G/H with projection $\tilde{\omega}$.

Since the dual space $\mathfrak{q}^*$ of $\mathfrak{q}$ is canonically isomorphic to $\mathfrak{q}$ as H-modules, M can be regarded as the cotangent bundle over G/H. For $x \in G$ and $X \in \mathfrak{g}$, we write $X_x = (dL_x)_e(X)$, where L_x is the left translation by $x \in G$ on G. Then we know that the tangent space T_xG at x is given by $T_xG = \{X_x; X \in \mathfrak{g}\}$. We note that the tangent space $T_{(x,X)}(G \times \mathfrak{q})$ is isomorphic to $T_xG \times \mathfrak{q}$. The differential $d\pi_{(x,X)}$ of π at $(x, X) \in G \times \mathfrak{q}$ is a surjective linear map of $T_{(x,X)}(G \times \mathfrak{q})$ onto $T_{\pi(x,X)}M$, whose kernel is

$$\text{Ker}\, d\pi_{(x,X)} = \{(U_x, -[U, X]) \in T_xG \times \mathfrak{q}; U \in \mathfrak{h}\}. \tag{3.3}$$

Let Θ be the canonical 1-form and $\omega = -d\Theta$ the symplectic form on M. Then it can be easily seen that

$$\Theta_{\pi(x,X)}(d\pi_{(x,X)}(Z_x, Y)) = \langle X, Z \rangle \tag{3.4}$$

for $Z \in \mathfrak{g}$, $Y \in \mathfrak{q}$ and

$$\begin{aligned} &\omega_{\pi(x,X)}(d\pi_{(x,X)}(Z_x, Y), d\pi_{(x,X)}(Z'_x, Y')) \\ &= \langle Y', Z \rangle - \langle Y, Z' \rangle + \langle X, [Z, Z'] \rangle \\ &= \langle Y', Z \rangle - \langle [Z, X] + Y, Z' \rangle \end{aligned} \tag{3.5}$$

for $Z, Z' \in \mathfrak{g}$ and $Y, Y' \in \mathfrak{q}$. We observe from (3.4) and (3.5) that Θ and ω are G-invariant and hence the G-action on M is symplectic.

Proposition 3.1. (i) *The cotangent bundle (M, ω) over G/H is a Hamiltonian G-space whose moment map Ψ of M into $\mathfrak{g}$ is given by*

$$\Psi(\pi(x, X)) = \text{Ad}(x)X \qquad (\pi(x, X) \in M). \tag{3.6}$$

(ii) *If we restrict the G-action to that of the maximal compact subgroup K, then (M, ω) is a Hamiltonian K-space and the corresponding moment map Φ of M into $\mathfrak{k}$ is given by*

$$\Phi(\pi(x, X)) = \frac{1}{2}(\text{Ad}(x)X + \theta(\text{Ad}(x)X)). \tag{3.7}$$

Proof. (i) For $W \in \mathfrak{g}$, let W^M be the infinitesimal generator of the action corresponding to W. Then by (3.2) we have

$$(W^M)_{\pi(x,X)} = d\pi_{(x,X)}((\text{Ad}(x^{-1})W)_x, 0). \tag{3.8}$$

On the other hand we define a smooth function f_W on M for each $W \in \mathfrak{g}$ by

$$f_W(\pi(x, X)) = \langle \text{Ad}(x)X, W \rangle. \tag{3.9}$$

Then the differential of f_W is given by

$$(3.10)\qquad \begin{aligned}(df_W)_{\pi(x,X)}(d\pi_{(x,X)}(Z'_x, Y')) &= \langle \mathrm{Ad}(x)([Z', X]+Y'), W\rangle \\ &= \langle Y', \mathrm{Ad}(x^{-1})W)\rangle - \langle [\mathrm{Ad}(x^{-1})W, X], Z'\rangle\end{aligned}$$

for $Z' \in \mathfrak{g}$, $Y' \in \mathfrak{q}$. Let ξ_{f_W} be the Hamiltonian vector field corresponding to f_W. Then by definition

$$\omega_{\pi(x,X)}(\xi_{f_W}, d\pi_{(x,X)}(Z'_x, Y')) = (df_W)_{\pi(x,X)}(d\pi_{(x,X)}(Z'_x, Y'))$$

for all $Z' \in \mathfrak{g}$ and $Y' \in \mathfrak{q}$. If we write $(\xi_{f_W})_{\pi(x,X)} = d\pi_{(x,X)}(Z_x, Y)$ with $Z \in \mathfrak{g}$, $Y \in \mathfrak{q}$, then by (3.5) and (3.10) we have

$$\langle Y', Z\rangle - \langle [Z, X]+Y, Z'\rangle = \langle Y' \,\mathrm{Ad}(x^{-1})W\rangle - \langle [\mathrm{Ad}(x^{-1})W, X], Z'\rangle$$

for all $Z' \in \mathfrak{g}$ and $Y' \in \mathfrak{q}$. This implies that there exists $U \in \mathfrak{h}$ such that $Z = \mathrm{Ad}(x^{-1})W + U$ and $Y = [\mathrm{Ad}(x^{-1})W, X] - [Z, X] = -[U, X]$. In view of (3.3), we obtain

$$(3.11)\qquad (\xi_{f_W})_{\pi(x,X)} = d\pi_{(x,X)}((\mathrm{Ad}(x^{-1})W)_x, 0).$$

Comparing this with (3.8), we find $\xi_{f_W} = W^M$ for all $W \in \mathfrak{g}$. Let W and W' be elements of $\mathfrak{g}$. Then the Poisson bracket $\{f_W, f_{W'}\}$ is, by definition, equal to $\omega(\xi_{f_W}, \xi_{f_{W'}})$. Applying (3.11) to (3.5), we can easily deduce that $\{f_W, f_{W'}\} = f_{[W,W']}$. This implies that the map $W \mapsto f_W$ gives a Lie algebra homomorphism of $\mathfrak{g}$ into $C^\infty(M)$. The G-equivariance of the above map is clear from the definition. Consequently, (M, ω) is a Hamiltonian G-space. If we recall the moment map for this action is given by the relation

$$\langle \Psi(\pi(x, X)), W\rangle = f_W(\pi(x, X)) \quad \text{for all } W \in \mathfrak{g},$$

we have (3.6) immediately.

(ii) For $W \in \mathfrak{k}$, we set

$$f_W(\pi(x, X)) = \left\langle \frac{1}{2}(\mathrm{Ad}(x)X + \theta(\mathrm{Ad}(x)X)), W\right\rangle.$$

Since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$, it follows that $f_W(\pi(x, X)) = \langle \mathrm{Ad}(x)X, W\rangle$. Hence the assertion of (ii) follows from that of (i).

Now we recall briefly the reduction procedure for Hamiltonian systems with symmetry (cf. [1], [17]) in our setting. Let $C \in \mathfrak{k}$ and denote by K_C the subgroup of K such that $\mathrm{Ad}(k)C = C$. Let $\Phi^{-1}(C)$ be the inverse image of C in our Hamiltonian K-space (M, ω) for the moment map Φ.

Since Φ is K-equivariant, $\Phi^{-1}(C)$ is invariant under the action of K_C. We denote by $M(C)$ the set of all K_C-orbits in $\Phi^{-1}(C)$, and denote by π_C the canonical projection of $\Phi^{-1}(C)$ onto $M(C)$.

Theorem 3.2. (Marsden: Weinstein [17]). *Assume that*

(3.12) *$\Phi^{-1}(C)$ is a submanifold of M and*

(3.13) *$M(C)$ is a smooth manifold and π_C is a submersion.*

Then $M(C)$ has the unique symplectic structure ω_C satisfying $\pi_C^*\omega_C = i_C^*\omega$ where i_C is the natural inclusion of $\Phi^{-1}(C)$ into M. The resulting symplectic manifold $(M(C), \omega_C)$ is called a reduced phase space.

We consider a Hamiltonian system on the Hamiltonian K-space (M, ω) with a K-invariant Hamiltonian f. Let ϕ_t^f be the Hamiltonian flow on M corresponding to f. Then it is known (cf. [1]) that $\Phi(\phi_t^f(\pi(x, X))) = \Phi(\pi(x, X))$ and hence the flow ϕ_t^f leaves $\Phi^{-1}(C)$ invariant. Since f is K-invariant, ϕ_t^f commutes with the K-action. Therefore it induces a flow ψ_t on $M(C)$ by

$$\psi_t \circ \pi_C = \pi_C \circ \phi_t^f. \tag{3.14}$$

On the other hand it is evident that for each f in the space $C^\infty(M)^K$ of all K-invariant smooth functions on M there exists a unique $f^C \in C^\infty(M(C))$ such that

$$f^C \circ \pi_C = f \circ i_C. \tag{3.15}$$

Theorem 3.3 (Marsden-Weinstein [17]). *Keeping the notations described above, we obtain*

(i) *the flow ψ_t is the Hamiltonian flow on $M(C)$ corresponding to the Hamiltonian f^C, that is, $\psi_t = \phi_t^{f^C}$.*

(ii) *If $f, g \in C^\infty(M)^K$, then the Poisson bracket $\{f, g\}$ is again in $C^\infty(M)^K$ and moreover it holds that $\{f, g\}^C = \{f^C, g^C\}$ where the Poisson bracket in the right side is the one on $M(C)$.*

Hence the map $f \mapsto f^c$ is a Lie algebra homomorphism of $C^\infty(M)^K$ into $C^\infty(M(C))$.

For our Hamiltonian K-space (M, ω) we can say more. We define $\nabla\varphi(X) \in \mathfrak{q}$ for $\varphi \in C^\infty(\mathfrak{q})$ and $X \in \mathfrak{q}$ by

$$\langle Y, \nabla\varphi(X)\rangle = d\varphi_X(Y) = \frac{d}{dt}\varphi(X+tY)\Big|_{t=0} \quad \text{for } Y \in \mathfrak{q}. \tag{3.16}$$

Let $C^\infty(\mathfrak{q})^H$ be the space of smooth functions on $\mathfrak{q}$ invariant under the adjoint action of H.

Lemma 3.4. *For $\varphi \in C^\infty(\mathfrak{q})^H$ and $X \in \mathfrak{q}$,*

$$[X, \nabla\varphi(X)]=0. \tag{3.17}$$

Proof. Since $\varphi \in C^\infty(\mathfrak{q})^H$, it follows that $\varphi(\mathrm{Ad}(\exp(tU))X)=\varphi(X)$ for $t \in \boldsymbol{R}$, $U \in \mathfrak{h}$ and $X \in \mathfrak{q}$. If we notice that $\mathrm{Ad}(\exp(tU))X=X+t[U, X]+O(t^2)$ for small t, we have, by differentiation, $\langle [U, X], \nabla\varphi(X)\rangle=0$ for all $U \in \mathfrak{h}$. Since $\mathfrak{q}$ is the orthogonal complement of $\mathfrak{h}$, we conclude $[X, \nabla\varphi(X)] \in \mathfrak{q}$. But since $[\mathfrak{q}, \mathfrak{q}]\subset\mathfrak{h}$ and $X, \nabla\varphi(X) \in \mathfrak{q}$, we have $[X, \nabla\varphi(X)] \in \mathfrak{q}\cap\mathfrak{h}=(0)$.

Let $C^\infty(M)^G$ be the space of smooth functions on M invariant under the action of G. Of course, $C^\infty(M)^G\subset C^\infty(M)^K$. For $f \in C^\infty(M)^G$, we set

$$\bar{f}(X)=f(\pi(e, X)) \qquad (X \in \mathfrak{q}). \tag{3.18}$$

Then it is clear that $\bar{f} \in C^\infty(\mathfrak{q})^H$ and the map $f\mapsto\bar{f}$ yields a linear isomorphism of $C^\infty(M)^G$ onto $C^\infty(\mathfrak{q})^H$.

Proposition 3.5. (i) *If $f \in C^\infty(M)^G$, then the Hamiltonian vector field ξ_f of f is given by*

$$(\xi_f)_{\pi(x,X)}=d\pi_{(x,X)}((\nabla\bar{f}(X))_x, 0). \tag{3.19}$$

(ii) *For $f_1, f_2 \in C^\infty(M)^G$, the Poisson bracket $\{f_1, f_2\}$ vanishes and hence $C^\infty(M)^G$ is a commutative Lie subalgebra of $C^\infty(M)$.*

Proof. (i) From the G-invariance of f, we find that

$$df_{\pi(x,X)}(d\pi_{(x,X)}(Z'_x, Y'))=\langle Y', \nabla\bar{f}(X)\rangle$$

for all $Z' \in \mathfrak{g}$ and $Y' \in \mathfrak{q}$. If we write $(\xi_f)_{\pi(x,X)}=d\pi_{(x,X)}(Z_x, Y)$ with $Z \in \mathfrak{g}$ and $Y \in \mathfrak{q}$, then from the definition of ξ_f and (3.5) it holds that

$$\langle Y', Z\rangle-\langle [Z, X]+Y, Z'\rangle=\langle Y', \nabla\bar{f}(X)\rangle$$

for all $Z' \in \mathfrak{g}$ and $Y' \in \mathfrak{q}$. This implies that there exists $U \in \mathfrak{h}$ such that $Z=\nabla\bar{f}(X)+U$ and $Y=[X, Z]$. But since $\bar{f} \in C^\infty(\mathfrak{q})^H$ it follows from Lemma 3.4 that $[X, \nabla\bar{f}(X)]=0$ and hence $Y=-[U, X]$. Therefore by using (3.3) we have

$$(\xi_f)_{\pi(x,X)}=d\pi_{(x,X)}((\nabla\bar{f}(X)+U)_x, -[U, X])=d\pi_{(x,X)}((\nabla\bar{f}(X))_x, 0).$$

(ii) Let $f_1, f_2 \in C^\infty(M)^G$ and recall that $\{f_1, f_2\}=\omega(\xi_{f_1}, \xi_{f_2})$. Applying (3.19) to the right side, we can easily obtain

$$\{f_1, f_2\}(\pi(x, X)) = \langle X, [\nabla \bar{f}_1(X), \nabla \bar{f}_2(X)]\rangle = \langle [X, \nabla \bar{f}_1(X)], \nabla \bar{f}_2(X)\rangle$$

which is equal to zero from Lemma 3.4.

Combining this proposition with Theorem 3.3, we obtain the following corollary.

Corollary 3.6. *Let (M, ω) be the cotangent bundle over an affine symmetric space G/H, and let $(M(C), \omega_C)$ be a reduced phase space. Then the subspace of $C^\infty(M(C))$ consisting of f^C with $f \in C^\infty(M)^G$ becomes a commutative Lie subalgebra of $C^\infty(M(C))$.*

For the remainder of this paper we consider the Hamiltonian system on (M, ω) with the G-invariant Hamiltonian F which is given by

$$F(\pi(x, X)) = \bar{F}(X) = \frac{1}{2}\langle X, X\rangle \qquad (\pi(x, X) \in M). \tag{3.20}$$

Since $\nabla \bar{F}(X) = X$ and hence $(\xi_F)_{\pi(x,X)} = d\pi_{(x,X)}(X_x, 0)$, the Hamiltonian flow ϕ_t^F is given by

$$\phi_t^F(\pi(x, X)) = \pi(x \exp(tX), X). \tag{3.21}$$

Our aim in the remainder sections is to show that for suitable choice of affine symmetric spaces and $C \in \mathfrak{k}$ the Hamiltonian systems attached to the root systems with signature are realized as reduced Hamiltonian systems $(M(C), \omega_C, F^C)$. The above corollary plays an essential role in proving complete integrability of such systems.

§ 4. The Hamiltonian system attached to the root system with signature (A_{n-1}, ε_m)

We study here the Hamiltonian system in the phase space $D_{(A_{n-1}, \varepsilon_m)} \times \boldsymbol{R}^n$ with the Hamiltonian $H_{(A_{n-1}, \varepsilon_m)}$ attached to the root system with signature (A_{n-1}, ε_m). We shall show that it can be realized as a reduced Hamiltonian system mentioned in the previous section and prove its complete integrability. We keep the notations in the preceding sections.

Let $G = GL(n, \boldsymbol{C})$ and hence $\mathfrak{g} = M_n(\boldsymbol{C})$. We give a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ by

$$\langle X, Y\rangle = \mathrm{Re}(\mathrm{tr}(XY)) \qquad (X, Y \in \mathfrak{g}). \tag{4.1}$$

Fix an integer m such that $1 \leq m \leq n$. Define $J_m \in G$ by

$$J_m = \begin{bmatrix} 1_m & 0 \\ 0 & -1_{n-m} \end{bmatrix} \tag{4.2}$$

where 1_m (resp. 1_{n-m}) is the identity matrix of size m (resp. $n-m$). In the following we denote the hermitian conjugate of a complex matrix X by X^*. Define the involutive automorphisms σ and θ on G by

$$(4.3)\qquad \sigma(g)=J_m(g^*)^{-1}J_m,\quad \theta(g)=(g^*)^{-1}\quad (g\in G).$$

Put $H=G_\sigma$ and $K=G_\theta$. Then $H=U(m,n-m)$ and $K=U(n)$ and hence G/H is diffeomorphic to the manifold of hermitian matrices of signature $(m,n-m)$. The involution on $\mathfrak{g}$ induced by σ and θ are given by

$$(4.4)\qquad \sigma(X)=-J_mX^*J_m,\quad \theta(X)=-X^*\quad (X\in\mathfrak{g}).$$

Therefore we have

$$\mathfrak{k}=\{X\in\mathfrak{g};X^*=-X\},\quad \mathfrak{p}=\{X\in\mathfrak{g};X^*=X\},$$

$$\mathfrak{h}=\left\{\begin{bmatrix}X_1 & X_2\\ X_2^* & X_3\end{bmatrix};X_1^*=-X_1\in M_m(C),\ X_2\in M_{m,n-m}(C),\ X_3^*=-X_3\in M_{n-m}(C)\right\},$$

$$\mathfrak{q}=\left\{\begin{bmatrix}X_1 & X_2\\ -X_2^* & X_3\end{bmatrix};X_1^*=X_1\in M_m(C),\ X_2\in M_{m,n-m}(C),\ X_3^*=X_3\in M_{n-m}(C)\right\}.$$

Put $\mathfrak{a}=\{D(q)=\mathrm{diag}(q_1,\cdots,q_n);q=(q_1,\cdots,q_n)\in \boldsymbol{R}^n\}$. Then $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{q}\cap\mathfrak{p}$, and moreover it is maximal abelian both in $\mathfrak{q}$ and $\mathfrak{p}$. The root system R of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is of type A_{n-1} and the involution σ is nothing but the one canonically induced by the signature ε_m of R. We can take as a Weyl chamber $\mathfrak{a}_+$ the following one;

$$\mathfrak{a}_+=\{D(q)\in\mathfrak{a};q\in D\}.$$

Here we denote the configuration space $D_{(A_{n-1},\varepsilon_m)}$ simply by

$$D=\{q=(q_1,\cdots,q_n)\in\boldsymbol{R}^n;q_1>\cdots>q_m,\ q_{m+1}>\cdots>q_n\}.$$

Let Z_K be the center of K. Then $Z_K=\{u1_n;u\in U(1)\}$. If we put $\mathfrak{k}^1=[\mathfrak{k},\mathfrak{k}]$, then it holds that $\mathfrak{k}=\mathfrak{z}_{\mathfrak{k}}+\mathfrak{k}^1$ where $\mathfrak{z}_{\mathfrak{k}}$ is the Lie algebra of Z_K. Set

$$T=\{\mathrm{diag}(u_1,\cdots,u_n);u_j\in U(1)\ (1\leqq j\leqq n)\}.$$

Then it is a maximal torus of K and its Lie algebra is $\sqrt{-1}\,\mathfrak{a}$. We remark that T is contained in H. Define a column vector $\boldsymbol{e}\in\boldsymbol{C}^n$ by

$$\boldsymbol{e}={}^t(1,\cdots,1).$$

With a nonzero real constant c, we define a matrix C by

$$C=\sqrt{-1}\,c(\boldsymbol{e}\boldsymbol{e}^*-1_n) \tag{4.5}$$

Then it is clear that $C \in \mathfrak{k}^1$. The following elementary observation is useful.

Lemma 4.1. (cf. [15]). *Set*

$$K_C=\{k \in K;\ \mathrm{Ad}(k)C=C\} \quad \text{and} \quad K_{\boldsymbol{e}}=\{k \in K;\ k\boldsymbol{e}=\boldsymbol{e}\}.$$

Then $K_{\boldsymbol{e}}$ is a closed subgroup of K_C and it holds that $K_C=K_{\boldsymbol{e}}Z_K$ and $K_{\boldsymbol{e}}\cap T=\{1_n\}$.

Proof. For $k \in K$, put $k\boldsymbol{e}=\boldsymbol{v}={}^t(v_1, \cdots, v_n) \in \boldsymbol{C}^n$. Then $\mathrm{Ad}(k)C=kCk^{-1}=kCk^*=\sqrt{-1}\,c(\boldsymbol{v}\boldsymbol{v}^*-1_n)$. Thus we conclude that $k \in K_C$ if and only if $v_i\bar{v}_j=1$ for all i, j, which is equivalent to $v_i=v$ for some $v \in U(1)$ $(1\leqq i\leqq n)$. Moreover it is clear that $k \in K_{\boldsymbol{e}}$ if and only if $v_i=1$ $(1\leqq i\leqq n)$. Consequently $K_{\boldsymbol{e}}\subset K_C$ and $K_C=K_{\boldsymbol{e}}Z_K$. The assertion $K_{\boldsymbol{e}}\cap T=\{1_n\}$ is also obvious.

Let M be the cotangent bundle over G/H. Then we have already seen in Proposition 3.1 that it is a Hamiltonian K-space and the moment map Φ is given by (3.7). In the case at hand since $\theta(g)=(g^*)^{-1}$ and $\theta(X)=-J_mXJ_m$ for $X \in \mathfrak{q}$, we can write

$$\Phi(\pi(x, X))=\frac{1}{2}(\mathrm{Ad}(x)X-\mathrm{Ad}((x^*)^{-1})J_mXJ_m). \tag{4.6}$$

We note that the center Z_K of K acts trivially on M and hence Φ is in fact a mapping of M into $\mathfrak{k}^1$. For $q \in D$, define $Z(q)=(Z(q)_{jk})$ by $Z(q)_{jj}=0$ $(1\leqq j\leqq n)$ and

$$Z(q)_{jk}=\begin{cases}\sqrt{-1}\,c\,\mathrm{sh}^{-1}(q_{jk}) & ((j, k) \in I_+),\\ \sqrt{-1}\,c\,\mathrm{ch}^{-1}(q_{jk}) & ((j, k) \in I_-).\end{cases} \tag{4.7}$$

Furthermore for $(q, p) \in D\times\boldsymbol{R}^n$, we put

$$Z(q, p)=D(p)+Z(q). \tag{4.8}$$

Then we can check $Z(q), Z(q, p) \in \mathfrak{q}$.

Proposition 4.2. (i) *Let $C \in \mathfrak{k}^1$ given by* (4.5). *For each $\pi(x, X) \in \Phi^{-1}(C)$, there exist $k \in K_{\boldsymbol{e}}$ and $(q, p) \in D\times\boldsymbol{R}^n$ uniquely such that*

$$k\pi(x, X)=\pi(\exp(D(q)), Z(q, p)). \tag{4.9}$$

(ii) *$\Phi^{-1}(C)$ is a submanifold of M, which is diffeomorphic to $K_e \times D \times \boldsymbol{R}^n$.*

Proof. (i) Let $\pi(x, X) \in \Phi^{-1}(C)$. Since $G=K\bar{A}_+H$ (cf. [23]), we can write $x=k^{-1}ah$ with $k \in K$, $a \in \bar{A}_+$ and $h \in H$. If we put $Y=\mathrm{Ad}(h)X$, then $Y \in \mathfrak{q}$ and $k\pi(x, X)=\pi(a, Y)$. Since Φ is K-equivariant it follows that

$$(4.10) \qquad \Phi(\pi(a, Y))=\Phi(k\pi(x, X))=\mathrm{Ad}(k)C=\sqrt{-1}\,c(\boldsymbol{v}\boldsymbol{v}^*-1_n)$$

where we put $k\boldsymbol{e}=\boldsymbol{v}={}^t(v_1, \cdots, v_n)$. From (4.6) and the fact that a is a diagonal matrix, we can deduce that the diagonal entries of $\Phi(\pi(a, Y))$ are all zero. Hence by (4.10) we have $v_i \in U(1)$ $(1\leqq i\leqq n)$. Put $t=\mathrm{diag}(v_1^{-1}, \cdots, v_n^{-1})$. Then $t \in T$. Since $t\boldsymbol{v}=\boldsymbol{e}$, we have $tk \in K_e$. Put $Z=\mathrm{Ad}(t)Y$. Then $Z \in \mathfrak{q}$ because $T\subset H$. Notice that $ta=at$. Then we have $tk\pi(x, X)=\pi(ta, Y)=\pi(a, Z)$. On the other hand since $tk \in K_e\subset K_C$, we obtain that $\Phi(\pi(a, Z))=C$. Consequently the above observation shows that for each $\pi(x, X) \in \Phi^{-1}(C)$ there exist $k \in K_e$, $a \in \bar{A}_+$ and $Z \in \mathfrak{q}$ such that $k\pi(x, X)=\pi(a, Z) \in \Phi^{-1}(C)$. Now we write $a=\exp(D(q))$ with $D(q) \in \bar{\mathfrak{a}}_+$ and $Z=(Z_{jk})$. Then by (4.6) we have

$$\frac{1}{2}(\mathrm{Ad}(\exp(D(q))Z-\mathrm{Ad}(\exp(-D(q))J_mZJ_m)=C.$$

Comparing each entry of the matrices in both sides, we have

$$\mathrm{sh}(q_{jk})Z_{jk}=\sqrt{-1}\,c \quad ((j, k) \in I_+),\ \mathrm{ch}(q_{jk})Z_{jk}=\sqrt{-1}\,c \quad ((j, k) \in I_-).$$

Hence we conclude that $q_{jk}\neq 0$ for $(j, k) \in I_+$, which implies $q \in D$ and consequently $D(q) \in \mathfrak{a}_+$. Furthermore we have $Z_{jk}=\sqrt{-1}\,c\,\mathrm{sh}^{-1}(q_{jk})$ for $(j, k) \in I_+$ and $Z_{jk}=\sqrt{-1}\,c\,\mathrm{ch}^{-1}(q_{jk})$ for $(j, k) \in I_-$. If we put $p_j=Z_{jj}$ $(1\leqq j\leqq n)$ then $p_j \in \boldsymbol{R}$ because $Z \in \mathfrak{q}$ and from (4.7) and (4.8) we have $Z=Z(q, p)$. The uniqueness of $k \in K_e$ and $(q, p) \in D\times \boldsymbol{R}^n$ is proved as follows. Assume that there exist k_1, $k_2 \in K_e$ and (q_1, p_1), $(q_2, p_2) \in D\times\boldsymbol{R}^n$ such that

$$(4.11) \qquad \pi(x, X)=k_1\pi(\exp(D(q_1)), Z(q_1, p_1))=k_2\pi(\exp(D(q_2)), Z(q_2, p_2)).$$

Then $k_1\exp(D(q_1))H=k_2\exp(D(q_2))H$ in G/H. Notice that the centralizer of $\mathfrak{a}$ in K is equal to T and it is known (cf. [23]) that the map of $K/T\times\mathfrak{a}_+$ into G/H defined by $(kT, D(q))\mapsto k\exp(D(q))H$ is an injective diffeomorphism. Then we conclude $q_1=q_2$ and $k_2^{-1}k_1 \in T\cap K_e$. But by Lemma 4.1 we have $k_1=k_2$. Hence it follows from (4.11) that $\pi(\exp(D(q_1)), Z(q_1, p_1)) =\pi(\exp(D(q_1)), Z(q_1, p_2))$, which implies that there exists $h \in H$ such that

$\exp(D(q_1))h^{-1}=\exp(D(q_1))$ and $Z(q_1, p_2)=\mathrm{Ad}(h)Z(q_1, p_1)$. Thus $h=1_n$ and $Z(q_1, p_1)=Z(q_1, p_2)$. This clearly yields $p_1=p_2$.

(ii) For any $\pi(x, X)\in\Phi^{-1}(C)$, we shall show that the differential $d\Phi_{\pi(x,X)}: T_{\pi(x,X)}M\mapsto\mathfrak{k}^1$ is surjective. Since $\pi(x, X)=k\pi(\exp(D(q)), Z(q, p))$, it suffices to show the surjectivity of $d\Phi_{\pi(a,Z)}$ where we put for simplicity $a=\exp(D(q))$ and $Z=Z(q, p)$. The direct computation shows that

$$\begin{aligned}(4.12)\qquad & d\Phi_{\pi(a,Z)}(d\pi_{(a,Z)}(W_a, Y))\\ &=\frac{1}{2}(\mathrm{Ad}(a)([W, Z]+Y)+\mathrm{Ad}(a^{-1})\theta([W, Z]+Y))\end{aligned}$$

where $W\in\mathfrak{g}$ and $Y\in\mathfrak{q}$. Define the map ϕ of $\mathfrak{q}$ into $\mathfrak{k}^1$ by

$$(4.13)\qquad \phi(Y)=\frac{1}{2}(\mathrm{Ad}(a)(Y)+\mathrm{Ad}(a^{-1})\theta(Y)).$$

Note that $\mathfrak{q}=\mathfrak{q}\cap\mathfrak{k}+\mathfrak{q}\cap\mathfrak{p}$ and hence each $Y\in\mathfrak{q}$ can be written as $Y=Y_{\mathfrak{k}}+Y_{\mathfrak{p}}$ with $Y_{\mathfrak{k}}\in\mathfrak{q}\cap\mathfrak{k}$ and $Y_{\mathfrak{p}}\in\mathfrak{q}\cap\mathfrak{p}$. Then we can write

$$(4.14)\qquad \phi(Y)=\frac{1}{2}(\mathrm{Ad}(a)+\mathrm{Ad}(a^{-1}))(Y_{\mathfrak{k}})+\frac{1}{2}(\mathrm{Ad}(a)-\mathrm{Ad}(a^{-1}))(Y_{\mathfrak{p}}).$$

From this it follows that ϕ induces an isomorphism of $\mathfrak{q}\cap\mathfrak{k}$ onto itself and ϕ induces a linear map of $\mathfrak{q}\cap\mathfrak{p}$ into $\mathfrak{h}\cap\mathfrak{k}^1$. Since $a\in A_+$, we can check directly that $\phi(\mathfrak{q}\cap\mathfrak{p})$ coincides with the orthogonal complement of $\sqrt{-1}\,\mathfrak{a}$ in $\mathfrak{h}\cap\mathfrak{k}^1$. Therefore the image of ϕ is the orthogonal complement of $\sqrt{-1}\,\mathfrak{a}\cap\mathfrak{k}^1$ in $\mathfrak{k}^1$. Now consider the map ψ of $\mathfrak{g}$ into $\mathfrak{k}^1$ defined by

$$(4.15)\qquad \psi(W)=-\frac{1}{2}(\mathrm{Ad}(a)+\mathrm{Ad}(a^{-1})\circ\theta)\circ\mathrm{ad}(Z)(W).$$

We note that

$$(4.16)\qquad d\Phi_{\pi(a,Z)}(d\pi_{(a,Z)}(W_a, Y))=\psi(W)+\phi(Y).$$

Hence we have only to show that $\psi(\mathfrak{g})$ contains $\sqrt{-1}\,\mathfrak{a}\cap\mathfrak{k}^1$. From (4.15) it suffices to see that the image of $\mathrm{ad}(Z)$ contains $\sqrt{-1}\,\mathfrak{a}\cap\mathfrak{k}^1$. From the explicit form of $Z=Z(q, p)$ given in (4.8) it can be easily checked by direct matrix computations. Consequently $\Phi^{-1}(C)$ is a submanifold of M. Furthermore we can conclude from (i) that $\Phi^{-1}(C)$ is diffeomorphic to $K_e\times D\times \boldsymbol{R}^n$.

Theorem 4.3. *Let $M(C)$ be the set of all K_C-orbits in $\Phi^{-1}(C)$, and let π_C be the canonical projection of $\Phi^{-1}(C)$ onto $M(C)$.*

(i) *Define a mapping* φ *of* $D\times\boldsymbol{R}^n$ *into* $M(C)$ *by*

$$\varphi(q,p)=\pi_C\circ\pi(\exp(D(q)),\,Z(q,p)). \tag{4.17}$$

Then φ *is bijective and hence* $M(C)$ *has a smooth manifold structure under which* φ *is a diffeomorphism and moreover* π_C *is a submersion. Consequently* $M(C)$ *is a reduced phase space with the symplectic structure* ω_C *in the sense of Section* 3.

(ii) *It holds that* $\varphi^*\omega_C=\sum_{i=1}^n dq_i\wedge dp_i$ *and thus* φ *is a symplectic diffeomorphism.*

(iii) *Define a G-invariant Hamiltonian F on M by*

$$F(\pi(x,X))=\frac{1}{2}\langle X,X\rangle \tag{4.18}$$

and denote the corresponding reduced Hamiltonian by F^C. *Then we have*

$$F^C(\varphi(q,p))=H_{(A_{n-1},\varepsilon_m)}(q,p). \tag{4.19}$$

Hence the Hamiltonian system $(D_{(A_{n-1},\varepsilon_m)}\times\boldsymbol{R}^n,\sum dq_i\wedge dp_i,H_{(A_{n-1},\varepsilon_m)})$ *is isomorphic to the reduced Hamiltonian system* $(M(C),\omega_C,F^C)$.

Proof. (i) It is clear from Proposition 4.2 that φ is a bijection. So one can define a C^∞-structure on $M(C)$ under which φ is a diffeomorphism. Now put

$$\tilde{\varphi}(q,p)=(\exp(D(q)),\,Z(q,p))\qquad((q,p)\in D\times\boldsymbol{R}^n). \tag{4.20}$$

Then $\tilde{\varphi}$ is a smooth map of $D\times\boldsymbol{R}^n$ into $G\times\mathfrak{q}$, which satisfies

$$i_C\circ\pi\circ\tilde{\varphi}=\pi\circ\tilde{\varphi}\quad\text{and}\quad\pi_C\circ\pi\circ\tilde{\varphi}=\varphi. \tag{4.21}$$

Since φ is a diffeomorphism, π_C is clearly a submersion. Hence the assumptions (3.12) and (3.13) hold. Therefore $M(C)$ is a reduced phase space with the symplectic structure ω_C.

(ii) Notice that the assertion $\varphi^*\omega_C=\sum dq_i\wedge dp_i$ is equivalent to the assertion

$$(\varphi^*\omega_C)_{(q,p)}((\xi,\eta),(\xi',\eta'))=\langle\xi,\eta'\rangle-\langle\xi',\eta\rangle \tag{4.22}$$

for all $(q,p)\in D\times\boldsymbol{R}^n$ and $(\xi,\eta),(\xi',\eta')\in T_{(q,p)}D\times\boldsymbol{R}^n=\boldsymbol{R}^n\times\boldsymbol{R}^n$. Here $\langle\xi,\eta'\rangle$ means the canonical inner product of $\boldsymbol{R}^n$. Since $\varphi=\pi_C\circ\pi\circ\tilde{\varphi}$ and $\pi_C^*\omega_C=i_C^*\omega$, it follows that $\varphi^*\omega_C=(\pi\circ\tilde{\varphi})^*(i_C^*\omega)$. Moreover since $\pi\circ\tilde{\varphi}=i_C\circ\pi\circ\tilde{\varphi}$, we conclude $\varphi^*\omega_C=(\pi\circ\tilde{\varphi})^*\omega$. Hence we have

$$(4.23)\qquad (\varphi^*\omega_C)_{(q,p)}((\xi,\eta),(\xi',\eta'))=\omega_{\pi(\tilde{\varphi}(q,p))}(d\pi(d\tilde{\varphi}(\xi,\eta)),\ d\pi(d\tilde{\varphi}(\xi',\eta'))).$$

On the other hand we can easily obtain that

$$(4.24)\qquad (d\tilde{\varphi})_{(q,p)}(\xi,\eta)=(D(\xi)_{\exp(D(q))},\ D(\eta)+[D(\xi),\ V(q)])$$

where $V(q)=(V(q)_{jk})$ is given by $V(q)_{jj}=0$ $(1\leqq j\leqq n)$ and

$$(4.25)\qquad V(q)_{jk}=\begin{cases}-\sqrt{-1}\,c\,\mathrm{ch}(q_{jk})\,\mathrm{sh}^{-2}(q_{jk}) & ((j,k)\in I_+),\\ -\sqrt{-1}\,c\,\mathrm{sh}(q_{jk})\,\mathrm{ch}^{-2}(q_{jk}) & ((j,k)\in I_-).\end{cases}$$

We remark $V(q)\in\mathfrak{h}$. Applying (4.24) to (4.23) and using (3.5), we obtain

$$\begin{aligned}(\varphi^*\omega_C)_{(q,p)}((\xi,\eta),(\xi',\eta'))&=\langle D(\xi),\ D(\eta')+[D(\xi'),\ V(q)]\rangle\\ &-\langle D(\xi'),\ D(\eta)+[D(\xi),\ V(q)]\rangle-\langle Z(q,p),\ [D(\xi),\ D(\xi')]\rangle.\end{aligned}$$

But since $[D(\xi), D(\xi')]=0$, $\langle D(\xi), D(\eta')\rangle=\langle\xi,\eta'\rangle$ and $\langle D(\xi'), D(\eta)\rangle=\langle\xi',\eta\rangle$, we have (4.22).

(iii) Since $F^C\circ\pi_C=F\circ i_C$, $\varphi=\pi_C\circ\pi\circ\tilde{\varphi}$ and $i_C\circ\pi\circ\tilde{\varphi}=\pi\circ\tilde{\varphi}$, it follows that $F^C\circ\varphi=F^C\circ\pi_C\circ\pi\circ\tilde{\varphi}=F\circ i_C\circ\pi\circ\tilde{\varphi}=F\circ\pi\circ\tilde{\varphi}$ and hence

$$(4.26)\qquad F^C(\varphi(q,p))=F(\pi(\exp(D(q)),\ Z(q,p))=\frac{1}{2}\langle Z(q,p),\ Z(q,p)\rangle.$$

It is an easy task to show that $\frac{1}{2}\langle Z(q,p), Z(q,p)\rangle=H_{(A_{n-1},\varepsilon_m)}(q,p)$.

We define G-invariant smooth functions $F_1,\cdots,F_n$ on M by

$$(4.27)\qquad F_k(\pi(x,X))=\bar{F}_k(X)=k^{-1}\,\mathrm{tr}\,(X^k).$$

Note that they are real valued since $X\in\mathfrak{q}$ and moreover $F_2=F$. The functions $\bar{F}_k$ $(1\leqq k\leqq n)$ are homogeneous polynomials on $\mathfrak{q}$ of degree k, which are invariant under $\mathrm{Ad}(H)$. They are algebraically independent homogeneous generators of $S(\mathfrak{q})^H$. We notice that the restrictions of $\bar{F}_k$ $(1\leqq k\leqq n)$ to $\mathfrak{a}$ are given by

$$\bar{F}_k(D(p))=k^{-1}\sum_{j=1}^{n}p_j^k$$

and hence if we set

$$\mathfrak{a}'=\{D(p)\in\mathfrak{a};\ p_i\neq p_j\ (1\leqq i\neq j\leqq n)\},$$

then they are functionally independent on $\mathfrak{a}'$. Let F_k^C $(1\leqq k\leqq n)$ be the reduced Hamiltonians on $M(C)$ corresponding to F_k. Set

$$(4.28)\qquad I_k(q,p)=F_k^C(\varphi(q,p))\qquad ((q,p)\in D\times\boldsymbol{R}^n).$$

Then it follows that $I_k = F_k \circ \pi \circ \tilde{\varphi}$ and hence

(4.29) $$I_k(q, p) = k^{-1} \operatorname{tr}(Z(q, p)^k) = k^{-1} \operatorname{tr}((D(p) + Z(q))^k).$$

This implies that $I_2 = H_{(A_{n-1}, \varepsilon m)}$ and I_k $(1 \leqq k \leqq n)$ are rational functions of $p_1, \cdots, p_n, \exp(q_1), \cdots, \exp(q_n)$.

Corollary 4.4. *The Hamiltonian system*

$$(D_{(A_{n-1}, \varepsilon m)} \times \boldsymbol{R}^n, \sum dq_i \wedge dp_i, H_{(A_{n-1}, \varepsilon m)})$$

is completely integrable. More precisely the above rational functions I_1, $\cdots$, I_n are mutually involutive integrals of motion, which are generically functionally independent.

Proof. Since $F_1, \cdots, F_n$ are G-invariant, it follows from Proposition 3.5 that they are in involution and hence from Corollary 3.6 that $F_1^C, \cdots, F_n^C$ are in involution. On the other hand, we have shown that φ is a symplectic diffeomorphism, so that $I_1, \cdots, I_n$ are in involution. Since $I_2 = H_{(A_{n-1}, \varepsilon m)}$, they are integrals of motion. Now we shall show the generically functional independence of $I_1, \cdots, I_n$. Put

$$\Omega = \{(q, p) \in D \times \boldsymbol{R}^n ; \partial(I_1, \cdots, I_n)/\partial(p_1, \cdots, p_n) \neq 0\}.$$

It is clear that $I_1, \cdots, I_n$ are functionally independent in Ω. So we have only to show that Ω is a non-empty open subset of $D \times \boldsymbol{R}^n$. Note that whenever $D(q)$ tends to infinity in $\mathfrak{a}' \cap \mathfrak{a}_+$ it follows that $Z(q)$ tends to zero from (4.7) and hence $I_k(q, p)$ tends to $\bar{F}_k(D(p))$ from (4.29). But we know that $\bar{F}_k$ $(1 \leqq k \leqq n)$ are functionally independent on $\mathfrak{a}'$. This yields that Ω is non-empty and open.

Define $B(q) = (B(q)_{jk})$ for $q \in D$ by

(4.30) $$B(q)_{jk} = \begin{cases} -\sqrt{-1}\, c\, \mathrm{sh}^{-2}(q_{jk}) & ((j, k) \in I_+), \\ \sqrt{-1}\, c\, \mathrm{ch}^{-2}(q_{jk}) & ((j, k) \in I_-) \end{cases}$$

and

$$B(q)_{jj} = -\sum_{i \neq j} B(q)_{ij} \qquad (1 \leqq j \leqq n).$$

Then it can be easily seen that $B(q) \in \mathfrak{k}_e$ where $\mathfrak{k}_e$ is the Lie algebra of K_e. Define $U(q) = (U(q)_{jk})$ for $q \in D$ by

(4.31) $$U(q) = \frac{1}{2}(\mathrm{Ad}(\exp(-D(q))B(q) + \sigma(\mathrm{Ad}(\exp(-D(q)))B(q))).$$

Then one can easily check that $U(q)_{jj}=B(q)_{jj}$ $(1\leqq j\leqq n)$ and $U(q)_{jk}=V(q)_{jk}$ $(1\leqq j\neq k\leqq n)$, where $V(q)$ is already given by (4.25). We remark that

$$(4.32)\qquad Z(q)=\frac{1}{2}(\mathrm{Ad}(\exp(-D(q)))B(q)-\sigma(\mathrm{Ad}(\exp(-D(q)))B(q)))$$

and hence

$$(4.33)\qquad \mathrm{Ad}(\exp(-D(q)))B(q)=U(q)+Z(q).$$

Corollary 4.5. *Let $(q(t), p(t))=\phi_t(q, p)$ be the trajectory of the Hamiltonian flow starting from $(q, p)\in D\times \boldsymbol{R}^n$. Define the curve $k(t)$ in K_e by*

$$(4.34)\qquad \frac{d}{dt}k(t)=k(t)B(q(t)),\qquad k(0)=1_n.$$

Then we have

$$(4.35)\qquad \begin{aligned}&\exp(D(q))\exp(2tZ(q,p))\exp(D(q))J_m\\ &\qquad =k(t)\exp(2D(q(t)))J_m k(t)^{-1}.\end{aligned}$$

Moreover $Z(q(t), p(t))$ satisfies the following Lax's isospectral deformation equation;

$$(4.36)\qquad \frac{d}{dt}Z(q(t), p(t))+[U(q(t)), Z(q(t), p(t))]=0.$$

Remark. The left side of (4.35) is a hermitian matrix of signature $(m, n-m)$. So the relation (4.35) implies that the left side of (4.35) can be diagonalized by the unitary matrix $k(t)\in K_e$ and its eigenvalues are $\exp(q_1(t)), \cdots, \exp(2q_m(t)), -\exp(2q_{m+1}(t)), \cdots, -\exp(2q_n(t))$ where $q(t)=(q_1(t), \cdots, q_n(t))\in D$ is the trajectory of the motion. Hence the determination of the trajectory of our Hamiltonian flow is reduced to finding the eigenvalues of the matrix in the left side of (4.35), which depends only on the initial value (q, p).

Proof. We have already seen in (3.21) that the flow ϕ_t^F on M is given by $\phi_t^F(\pi(x, X))=\pi(x\exp(tX), X)$. Since $\phi_t^{FC}\circ\pi_C=\pi_C\circ\phi_t^F$ (cf. Theorem 3.3), we have

$$(4.37)\qquad \phi_t^{FC}(\pi_C\circ\pi(x, X))=\pi_C\circ\pi(x\exp(tX), X).$$

But Theorem 4.3 implies $\phi_t^{FC}\circ\varphi=\varphi\circ\phi_t$. Since $\varphi=\pi_C\circ\pi\circ\tilde{\varphi}$, we have

$$\phi_t^{FC} \circ \pi_C \circ \pi \circ \tilde{\varphi}(q, p) = \pi_C \circ \pi \circ \tilde{\varphi}(q(t), p(t)),$$

that is,

$$\phi_t^{FC}(\pi_C \circ \pi(\exp(D(q)), Z(q, p))) = \pi_C \circ \pi(\exp(D(q(t))), Z(q(t), p(t))).$$

Using (4.37), we have

$$\pi_C \circ \pi(\exp(D(q)) \exp(tZ(q, p)), Z(q, p)) = \pi_C \circ \pi(\exp(D(q(t))), Z(q(t), p(t))).$$

This means that there exist $k(t) \in K_e$ and $h(t) \in H$ such that

$$\text{(4.38)} \qquad \exp(D(q)) \exp(tZ(q, p)) = k(t) \exp(D(q(t))h(t)^{-1}$$

and

$$\text{(4.39)} \qquad Z(q, p) = \mathrm{Ad}(h(t))Z(q(t), p(t)).$$

Put for simplicity $a(t) = \exp(D(q(t)))$, $a = a(0) = \exp(D(q))$, $Z(t) = Z(q(t), p(t))$ and $Z = Z(0) = Z(q, p)$. Then by (4.38) we have $a \exp(2tZ)a = k(t)a(t)^2\sigma(k(t)^{-1})$. Using the definition of σ, we can write

$$\text{(4.40)} \qquad a \exp(2tZ)J_m a = k(t)a(t)^2 J_m k(t)^{-1}.$$

Therefore to prove (4.35) we have only to show that $k(t)$ satisfies (4.34). It is clear from (4.38) that $k(0) = 1_n$. On the other hand since $h(t) = (a \exp(tZ))^{-1}k(t)a(t)$, it follows from (4.39) that

$$\text{(4.41)} \qquad \mathrm{Ad}(k(t))(\mathrm{Ad}(a(t))Z(t)) = \mathrm{Ad}(a \exp(tZ))Z = \mathrm{Ad}(a)Z.$$

Set $W(t) = \mathrm{Ad}(a(t))Z(t)$ and $W = W(0) = \mathrm{Ad}(a)Z$. Then we have by (4.40)

$$\text{(4.42)} \qquad \exp(2tW)a^2 = k(t)a(t)^2\sigma(k(t)^{-1})$$

and by (4.41)

$$\text{(4.43)} \qquad \mathrm{Ad}(k(t))W(t) = W.$$

Put $B(t) = k^{-1}(t)\dot{k}(t)$, where $\dot{k}(t) = (d/dt)k(t)$. Then $B(t) \in \mathfrak{k}_e$. Using the fact $\dot{q}(t) = p(t)$, we have

$$\begin{aligned}
&\frac{d}{dt} k(t)a(t)^2\sigma(k(t)^{-1}) \\
&\qquad = \mathrm{Ad}(k(t))(2D(p(t)) + B(t) - \mathrm{Ad}(a(t)^2)\sigma(B(t)))k(t)a(t)^2\sigma(k(t)^{-1}) \\
&\qquad = \mathrm{Ad}(k(t))(2D(p(t)) + B(t) - \mathrm{Ad}(a(t)^2)\sigma(B(t))) \exp(2tW)a^2.
\end{aligned}$$

On the other hand we have

$$\frac{d}{dt}\exp(2tW)a^2=2W\exp(2tW)a^2=2\mathrm{Ad}(k(t))W(t)\exp(2tW)a^2.$$

Hence we conclude

$$W(t)=D(p(t))+\frac{1}{2}(B(t)-\mathrm{Ad}(a(t)^2)\sigma(B(t))).$$

Remembering $W(t)=\mathrm{Ad}(a(t))Z(t)$, we obtain

$$Z(t)=D(p(t))+\frac{1}{2}(\mathrm{Ad}(a(t)^{-1})B(t)-\mathrm{Ad}(a(t))\sigma(B(t))).$$

Since $Z(t)=Z(q(t),p(t))=D(p(t))+Z(q(t))$, it follows that

$$Z(q(t))=\frac{1}{2}(\mathrm{Ad}(a(t)^{-1})B(t)-\sigma(\mathrm{Ad}(a(t)^{-1})B(t))).$$

Comparing the entries of the matrices in both sides and using the fact $B(t)\in\mathfrak{k}_e$, we conclude that $B(t)=B(q(t))$ where $B(q)$ is given by (4.30). Hence $\dot{k}(t)=k(t)B(q(t))$. Since $k(0)=1_n$, $k(t)$ satisfies (4.34) and consequently we have (4.35). Differentiating both sides in (4.43), we have

$$\frac{d}{dt}W(t)+[B(t),W(t)]=0.$$

Replacing $W(t)$ by $\mathrm{Ad}(a(t))Z(t)$, we obtain

$$\frac{d}{dt}Z(t)+[D(p(t))+\mathrm{Ad}(a(t)^{-1})B(t),Z(t)]=0.$$

Using (4.8) and (4.33) we can deduce (4.36).

§ 5. The Hamiltonian systems attached to (C_n,ε_m), (D_n,ε_m), (C_n,ε'_m) and (D_n,ε'_m).

In this section we consider the Hamiltonian systems attached to the root systems with signature (C_n,ε_m), (D_n,ε_m), (C_n,ε'_m) and (D_n,ε'_m) simultaneously. We recall that the configuration spaces of the above systems are given respectively by

$$D_1=D_{(C_n,\varepsilon_m)}=D_{(D_n,\varepsilon_m)}=\{q\in \boldsymbol{R}^n;\,q_1>\cdots>q_m>0,\,q_{m+1}>\cdots>q_n>0\}$$

and

$$D_{-1}=D_{(C_n,\varepsilon'_m)}=D_{(D_n,\varepsilon'_m)}$$
$$=\{q\in \boldsymbol{R}^n;\, q_1>\cdots>q_m,\, q_{m+1}>\cdots>q_n,\, q_{m+1}+q_n>0\}.$$

Throughout the section we write an element $X\in M_{2n}(\boldsymbol{C})$ as a block form

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \qquad \text{where } X_{11},\, X_{12},\, X_{21} \text{ and } X_{22}\in M_n(\boldsymbol{C}).$$

Let G be the closed subgroup of $GL(2n, \boldsymbol{C})$ defined by

$$G=\{g\in GL(2n, \boldsymbol{C});\, gQg^*=Q\}$$

where Q is the matrix given by

$$Q=\begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

Thus G is isomorphic to $U(n, n)$. We define a nondegenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G by

$$(5.1) \qquad \langle X, Y\rangle=\frac{1}{2}\operatorname{tr}(XY) \qquad (X, Y\in\mathfrak{g}).$$

Define the matrix J by

$$(5.2) \qquad J=\begin{bmatrix} J_m & 0 \\ 0 & \delta J_m \end{bmatrix} \qquad \text{where } J_m=\begin{bmatrix} 1_m & 0 \\ 0 & -1_{n-m} \end{bmatrix}$$

and δ is either 1 or -1. We define the involutive automorphisms σ and θ of G by

$$(5.3) \qquad \sigma(g)=J(g^*)^{-1}J \quad \text{and} \quad \theta(g)=(g^*)^{-1}.$$

Put $H=G_\sigma$ and $K=G_\theta$. Then K is a maximal compact subgroup of G and $K=G\cap U(2n)$. Hence each $k\in K$ is a unitary matrix of the form

$$k=\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{11} \end{bmatrix}.$$

In the case at hand we have

$$\mathfrak{h}=\left\{\begin{bmatrix} X_1 & X_2 \\ \delta J_m X_2 J_m & J_m X_1 J_m \end{bmatrix};\, X_1^*=-J_m X_1 J_m,\, X_2^*=-X_2\right\}$$

and

$$\mathfrak{q}=\left\{\begin{bmatrix} X_1 & X_2 \\ -\delta J_m X_2 J_m & -J_m X_1 J_m \end{bmatrix};\ X_1^*=J_m X_1 J_m,\ X_2^*=-X_2\right\}.$$

Put $\mathfrak{a}=\{D(q)=\mathrm{diag}(q_1, \cdots, q_n, -q_1, \cdots, -q_n);\ q=(q_1, \cdots, q_n)\in \boldsymbol{R}^n\}$. Then $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{q}\cap\mathfrak{p}$, which is maximal abelian both in $\mathfrak{q}$ and $\mathfrak{p}$. The root system R of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is of type C_n. We remark that it contains the root system of type D_n as a subroot system. The involution σ is equal to the one corresponding either to the signature ε_m if $\delta=1$ or to the signature ε'_m if $\delta=-1$. Put

$$\mathfrak{a}_+=\{D(q)\in\mathfrak{a};\ q\in D_\delta\}.$$

Then it is a Weyl chamber either for R_{ε_m} if $\delta=1$ or $R_{\varepsilon'_m}$ if $\delta=-1$. Set

$$T=\{\mathrm{diag}(u_1, \cdots, u_n, u_1, \cdots, u_n);\ u_j\in U(1)\ (1\leqq j\leqq n)\}.$$

Then it is a maximal torus of K, which is contained in H. The center Z_K of K is $\{u1_{2n};\ u\in U(1)\}$.

Let c_1, c_2 be real constants such that $c_1\neq 0$. We remark that in the following discussion the case $c_2\neq 0$ corresponds to (C_n, ε_m) and (C_n, ε'_m) and the case $c_2=0$ corresponds to (D_n, ε_m) and (D_n, ε'_m). Put

$$\tilde{\boldsymbol{e}}={}^t(\boldsymbol{e}, \boldsymbol{e})\in \boldsymbol{C}^{2n} \qquad \text{where } \boldsymbol{e}={}^t(1, \cdots, 1)\in\boldsymbol{C}^n.$$

Define $C\in\mathfrak{k}$ by

$$C=\sqrt{-1}\begin{bmatrix} c_1(\boldsymbol{e}\boldsymbol{e}^*-1_n) & c_1(\boldsymbol{e}\boldsymbol{e}^*-1_n)+c_2 1_n \\ c_1(\boldsymbol{e}\boldsymbol{e}^*-1_n)+c_2 1_n & c_1(\boldsymbol{e}\boldsymbol{e}^*-1_n) \end{bmatrix}. \tag{5.4}$$

Then $C\in\mathfrak{k}^1=[\mathfrak{k}, \mathfrak{k}]$. The proof of the following lemma is quite analogous to Lemma 4.1. So we shall omit it.

Lemma 5.1. (i) *For each $k\in K$, there exists $\boldsymbol{v}\in\boldsymbol{C}^n$ such that $k\tilde{\boldsymbol{e}}={}^t(\boldsymbol{v}, \boldsymbol{v})$.*

(ii) *For each $k\in K$, we have*

$$\mathrm{Ad}(k)C=\sqrt{-1}\begin{bmatrix} c_1(\boldsymbol{v}\boldsymbol{v}^*-1_n) & c_1(\boldsymbol{v}\boldsymbol{v}^*-1_n)+c_2 1_n \\ c_1(\boldsymbol{v}\boldsymbol{v}^*-1_n)+c_2 1_n & c_1(\boldsymbol{v}\boldsymbol{v}^*-1_n) \end{bmatrix}$$

where $\boldsymbol{v}$ is given in (i).

(iii) *Set $K_C=\{k\in K;\ \mathrm{Ad}(k)C=C\}$ and $K_{\tilde{e}}=\{k\in K;\ k\tilde{\boldsymbol{e}}=\tilde{\boldsymbol{e}}\}$. Then $K_C=Z_K K_{\tilde{e}}$ and $K_{\tilde{e}}\cap T=(1_{2n})$.*

Let M be the cotangent bundle over G/H and Φ be the moment map for the K-action given by (3.7). In the case at hand since $\theta(X)=-JXJ$ for $X\in\mathfrak{q}$, if follows that

$$\Phi(\pi(x, X))=\frac{1}{2}(\mathrm{Ad}(x)X-\mathrm{Ad}((x^*)^{-1})JXJ). \tag{5.5}$$

To describe the structure of $\Phi^{-1}(C)$ we need the following. For $q\in D_\delta$, we define $Z(q)\in\mathfrak{q}$ by

$$Z(q)=\begin{bmatrix} Z_1(q) & Z_2(q) \\ -\delta J_m Z_2(q) J_m & -J_m Z_1(q) J_m\end{bmatrix} \tag{5.6}$$

where $Z_1(q)=(Z_1(q)_{jk})$ and $Z_2(q)=(Z_2(q)_{jk})$ are given as follows;

$$Z_1(q)_{jk}=\begin{cases} 0 & (1\leqq j=k\leqq n), \\ \sqrt{-1}\,c_1\,\mathrm{sh}^{-1}(q_{jk}) & ((j,k)\in I_+), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-1}(q_{jk}) & ((j,k)\in I_-)\end{cases} \tag{5.7}$$

and either

$$Z_2(q)_{jk}=\begin{cases} \sqrt{-1}\,c_2\,\mathrm{sh}^{-1}(2q_j) & (1\leqq j=k\leqq n), \\ \sqrt{-1}\,c_1\,\mathrm{sh}^{-1}(\hat{q}_{jk}) & ((j,k)\in I_+), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-1}(\hat{q}_{jk}) & ((j,k)\in I_-)\end{cases} \tag{5.8}$$

if $\delta=1$ or

$$Z_2(q)_{jk}=\begin{cases} \sqrt{-1}\,c_2\,\mathrm{ch}^{-1}(2q_j) & (1\leqq j=k\leqq n), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-1}(\hat{q}_{jk}) & ((j,k)\in I_+), \\ \sqrt{-1}\,c_1\,\mathrm{sh}^{-1}(\hat{q}_{jk}) & ((j,k)\in I_-)\end{cases} \tag{5.9}$$

if $\delta=-1$. Furthermore we define $Z(q,p)\in\mathfrak{q}$ by

$$Z(q,p)=D(p)+Z(q) \qquad ((q,p)\in D_\delta\times\boldsymbol{R}^n). \tag{5.10}$$

Proposition 5.2. (i) *For each $\pi(x, X)\in\Phi^{-1}(C)$ there exist unique $k\in K_{\tilde{\varepsilon}}$ and $(q,p)\in D_\delta\times\boldsymbol{R}^n$ such that*

$$k\pi(x, X)=\pi(\exp(D(q)), Z(q,p)).$$

(ii) *$\Phi^{-1}(C)$ ia a submanifold of M, which is diffeomorphic to $K_{\tilde{\varepsilon}}\times D_\delta\times\boldsymbol{R}^n$.*

Proof. (i) Let $\pi(x, X)\in\Phi^{-1}(C)$. Since $G=K\bar{A}_+H$, we can write $x=k^{-1}ah$ ($k\in K$, $a\in\bar{A}_+$, $h\in H$). Hence $k\pi(x, X)=\pi(a, \mathrm{Ad}(h)X)$ and

$\Phi(\pi(a, \mathrm{Ad}(h)X))=\mathrm{Ad}(k)C$. Since a is a diagonal matrix, we can deduce from (5.5) that the diagonal entries of $\Phi(\pi(a, \mathrm{Ad}(h)X))$ vanish and so do the diagonal entries of $\mathrm{Ad}(k)C$. Therefore it follows from Lemma 5.1 (ii) that $v_j \in U(1)$ $(1 \leqq j \leqq n)$. Set $t=\mathrm{diag}(v_1, \cdots, v_n, v_1, \cdots, v_n)$. Then $t \in T$ and $t^{-1}k \in K_{\mathfrak{z}}$. Hence we have $t^{-1}k\pi(x, X)=\pi(a, Z)$ where $Z=\mathrm{Ad}(t^{-1}h)X \in \mathfrak{q}$ and $\Phi(\pi(a, Z))=C$. But as in Proposition 4.2, the last identity implies that $a=\exp(D(q)) \in A_+$ and Z is of the form $Z(q, p)$. As the centralizer of a in K is also equal to T in this case, so the proof of the uniqueness of $k \in K_{\mathfrak{z}}$ and $(q, p) \in D_\delta \times \boldsymbol{R}^n$ is quite similar to that of Proposition 4.2.

(ii) The differential $d\Phi_{\pi(a,Z)}$ where $a=\exp(D(q)) \in A_+$ and $Z=Z(q, p)$ is given by the same formula as in (4.12). Thus the proof of the assertion (ii) is quite analogous to that of Proposition 4.2. So we shall omit it.

Theorem 5.3. (i) *Let $M(C)$ be the set of K_C-orbits in $\Phi^{-1}(C)$ and let π_C be the canonical projection. Define a map φ of $D_\delta \times \boldsymbol{R}^n$ into $M(C)$ by*

$$\varphi(q, p)=\pi_C \circ \pi(\exp(D(q)), Z(q, p)). \tag{5.11}$$

Then φ is bijective and hence $M(C)$ has a smooth manifold structure under which φ is a diffeomorphism and π_C is a submersion. Thus $M(C)$ is a reduced phase space with the symplectic structure ω_C.

(ii) *It holds that $\varphi^*\omega_C=\sum_{i=1}^n dq_i \wedge dp_i$ and hence φ is a symplectic diffeomorphism.*

(iii) *Define a G-invariant Hamiltonian F on M by*

$$F(\pi(x, X))=\frac{1}{2}\langle X, X\rangle \qquad (\pi(x, X) \in M)$$

and denote reduced Hamiltonian on $M(C)$ by F^C. Then

$$F^C \circ \varphi=H_{(C_n, \varepsilon_m)} \quad \textit{if } \delta=1 \textit{ and } c_2 \neq 0,$$

$$F^C \circ \varphi=H_{(D_n, \varepsilon_m)} \quad \textit{if } \delta=1 \textit{ and } c_2=0.$$

$$F^C \circ \varphi=H_{(C_n, \varepsilon'_m)} \quad \textit{if } \delta=-1 \textit{ and } c_2 \neq 0 \quad \textit{and}$$

$$F^C \circ \varphi=H_{(D_n, \varepsilon'_m)} \quad \textit{if } \delta=-1 \textit{ and } c_2=0.$$

Proof. (i) It is a direct consequence of Proposition 5.2 that φ is bijective and hence $M(C)$ has a C^∞-structure under which φ is a diffeomorphism. If we define a map $\tilde{\varphi}$ of $D_\delta \times \boldsymbol{R}^n$ into $G \times \mathfrak{q}$ by $\tilde{\varphi}(q, p)=$

$(\exp(D(q)), Z(q, p))$, then we can check that $i_C \circ \pi \circ \tilde{\varphi} = \pi \circ \tilde{\varphi}$ and $\pi_C \circ \pi \circ \tilde{\varphi} = \varphi$. Thus π_C is a submersion.

(ii) We have only to show the same formula as in (4.22) for $(q, p) \in D_\delta \times \boldsymbol{R}^n$ and (ξ, η), $(\xi', \eta') \in \boldsymbol{R}^n \times \boldsymbol{R}^n$. But the proof is quite similar to that of (ii) in Theorem 4.3. The only difference is the definition of $V(q)$. In the case at hand one may take $V(q) \in \mathfrak{h}$ as

$$V(q) = \begin{bmatrix} V_1(q) & V_2(q) \\ \delta J_m V_2(q) J_m & J_m V_1(q) J_m \end{bmatrix} \tag{5.12}$$

where $V_1(q) = (V_1(q)_{jk})$ and $V_2(q) = (V_2(q)_{jk})$ are given respectively by

$$V_1(q)_{jk} = \begin{cases} 0 & (1 \leqq j = k \leqq n), \\ -\sqrt{-1}\, c_1 \operatorname{ch}(q_{jk}) \operatorname{sh}^{-2}(q_{jk}) & ((j, k) \in I_+), \\ -\sqrt{-1}\, c_1 \operatorname{sh}(q_{jk}) \operatorname{ch}^{-2}(q_{jk}) & ((j, k) \in I_-) \end{cases} \tag{5.13}$$

and either

$$V_2(q)_{jk} = \begin{cases} -\sqrt{-1}\, c_2 \operatorname{ch}(2q_j) \operatorname{sh}^{-2}(2q_j) & (1 \leqq j = k \leqq n), \\ -\sqrt{-1}\, c_1 \operatorname{ch}(\hat{q}_{jk}) \operatorname{sh}^{-2}(\hat{q}_{jk}) & ((j, k) \in I_+), \\ -\sqrt{-1}\, c_1 \operatorname{sh}(\hat{q}_{jk}) \operatorname{ch}^{-2}(\hat{q}_{jk}) & ((j, k) \in I_-) \end{cases} \tag{5.14}$$

if $\delta = 1$ or

$$V_2(q)_{jk} = \begin{cases} -\sqrt{-1}\, c_2 \operatorname{sh}(2q_j) \operatorname{ch}^{-2}(2q_j) & (1 \leqq j = k \leqq n), \\ -\sqrt{-1}\, c_1 \operatorname{sh}(\hat{q}_{jk}) \operatorname{ch}^{-2}(\hat{q}_{jk}) & ((j, k) \in I_+), \\ -\sqrt{-1}\, c_1 \operatorname{ch}(\hat{q}_{jk}) \operatorname{sh}^{-2}(\hat{q}_{jk}) & ((j, k) \in I_-) \end{cases} \tag{5.15}$$

if $\delta = -1$.

(iii) As in Theorem 4.3 we can deduce

$$F^C(\varphi(q, p)) = 2^{-1}\langle Z(q, p), Z(q, p)\rangle = 4^{-1}\operatorname{tr}(Z(q, p)^2).$$

Since $Z(q, p) = D(p) + Z(q)$, we have

$$F^C(\varphi(q, p)) = 4^{-1}\operatorname{tr}(D(p)^2) + 4^{-1}\operatorname{tr}(Z(q)^2) + 4^{-1}\operatorname{tr}(D(p)Z(q) + Z(q)D(p))$$

But since the diagonal part of $Z(q)$ is zero, the last term vanishes. It is clear that $4^{-1}\operatorname{tr}(D(p)^2) = 2^{-1}\sum p_j^2$. Since $Z(q)$ is given by (5.6), we have

$$4^{-1}\operatorname{tr}(Z(q)^2) = 2^{-1}\operatorname{tr}(Z_1(q)^2) - \delta 2^{-1}\operatorname{tr}((J_m Z_2(q))^2).$$

By direct matrix computations we obtain

$$2^{-1}\mathrm{tr}(Z_1(q)^2)=c_1^2(\textstyle\sum_{(1)}\mathrm{sh}^{-2}(q_{jk})-\sum_{(2)}\mathrm{ch}^{-2}(q_{jk}))$$

and if $\delta=1$

$$2^{-1}\mathrm{tr}((J_m Z_2(q))^2)=-c_1^2(\textstyle\sum_{(1)}\mathrm{sh}^{-2}(\hat{q}_{jk})-\sum_{(2)}\mathrm{ch}^{-2}(\hat{q}_{jk}))-c_2^2/2\sum_{j=1}^{n}\mathrm{sh}^{-2}(2q_j)$$

and if $\delta=-1$

$$2^{-1}\mathrm{tr}((J_m Z_2(q))^2)=-c_1^2(\textstyle\sum_{(1)}\mathrm{ch}^{-2}(\hat{q}_{jk})-\sum_{(2)}\mathrm{sh}^{-2}(\hat{q}_{jk}))-c_2^2/2\sum_{j=1}^{n}\mathrm{ch}^{-2}(2q_j).$$

These formulas clearly yeild (iii).

We define G-invariant smooth functions $F_1, \cdots, F_n$ on M by

$$(5.16)\qquad F_k(\pi(x, X))=\bar{F}_k(X)=(2k)^{-1}\mathrm{tr}(X^{2k}).$$

Since $X\in\mathfrak{q}$, they are real valued and moreover $F_1=2F$. The functions $\bar{F}_k$ $(1\leqq k\leqq n)$ are homogeneous polynomials on $\mathfrak{q}$ of degree $2k$, which are invariant under $\mathrm{Ad}(H)$. They are algebraically independent homogeneous generators of $S(\mathfrak{q})^H$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ such that

$$\mathfrak{a}'=\{D(p)\in\mathfrak{a}; p_i\pm p_j\neq 0\ (1\leqq i<j\leqq n), p_i\neq 0\ (1\leqq i\leqq n)\}.$$

Then the restrictions of $\bar{F}_k$ $(1\leqq k\leqq n)$ to $\mathfrak{a}'$ are known to be functionally independent (cf. [7]). Let F_k^C $(1\leqq k\leqq n)$ be the reduced Hamiltonians on $M(C)$ corresponding to F_k. Put

$$(5.17)\qquad I_k(q, p)=F_k^C(\varphi(q, p))\qquad ((q, p)\in D_\delta\times\boldsymbol{R}^n, 1\leqq k\leqq n).$$

Then we have

$$(5.18)\qquad I_k(q, p)=(2k)^{-1}\mathrm{tr}(Z(q, p)^{2k})=(2k)^{-1}\mathrm{tr}((D(p)+Z(q))^{2k}).$$

This yields that $2^{-1}I_1$ is the Hamiltonian of our system and I_k $(1\leqq k\leqq n)$ are rational functions of $p_1, \cdots, p_n, \exp(q_1), \cdots, \exp(q_n)$. The proof of the following corollary is similar to that of Corollary 4.4. So we shall leave it to the reader.

Corollary 5.4. *The Hamiltonian systems attached to the root systems with signature* (C_n, ε_m), (D_n, ε_m), (C_n, ε'_m) *and* (D_n, ε'_m) *are completely integrable. The above rational functions* $I_1, \cdots, I_n$ *are mutually involutive integrals of motion, which are generically functionally independent.*

For $q\in D_\delta$ we set

$$B(q)=\begin{bmatrix} B_1(q) & B_2(q) \\ B_2(q) & B_1(q) \end{bmatrix} \tag{5.19}$$

where $B_1(q)=(B_1(q)_{jk})$ and $B_2(q)=(B_2(q)_{jk})$ are given in the following manner; we put

$$B_1(q)_{jk}=\begin{cases} -\sqrt{-1}\,c_1\,\mathrm{sh}^{-2}(q_{jk}) & ((j,k)\in I_+), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-2}(q_{jk}) & ((j,k)\in I_-). \end{cases} \tag{5.20}$$

Moreover we put

$$B_2(q)_{jk}=\begin{cases} -\sqrt{-1}\,c_2\,\mathrm{sh}^{-2}(2q_j) & (1\leqq j=k\leqq n), \\ -\sqrt{-1}\,c_1\,\mathrm{sh}^{-2}(\hat{q}_{jk}) & ((j,k)\in I_+), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-2}(\hat{q}_{jk}) & ((j,k)\in I_-) \end{cases} \tag{5.21}$$

if $\delta=1$ and put

$$B_2(q)_{jk}=\begin{cases} \sqrt{-1}\,c_2\,\mathrm{ch}^{-2}(2q_j) & (1\leqq j=k\leqq n), \\ \sqrt{-1}\,c_1\,\mathrm{ch}^{-2}(\hat{q}_{jk}) & ((j,k)\in I_+), \\ -\sqrt{-1}\,c_1\,\mathrm{sh}^{-2}(\hat{q}_{jk}) & ((j,k)\in I_-) \end{cases} \tag{5.22}$$

if $\delta=-1$. Finally we define

$$B_1(q)_{jj}=-\Big(\sum_{i\neq j} B_1(q)_{ij}+\sum_{i} B_2(q)_{ij}\Big) \qquad (1\leqq j\leqq n). \tag{5.23}$$

Then one can see that $B(q)\in \mathfrak{k}_{\varepsilon}$ where $\mathfrak{k}_{\varepsilon}$ is the Lie algebra of K_{ε}. Define $U(q)\in \mathfrak{h}$ for $q\in D_{\delta}$ by

$$U(q)=\frac{1}{2}(\mathrm{Ad}(\exp(-D(q)))B(q)+\sigma(\mathrm{Ad}(\exp(-D(q)))B(q))). \tag{5.24}$$

Then we have $U(q)_{jj}=B(q)_{jj}$ $(1\leqq j\leqq 2n)$ and $U(q)_{jk}=V(q)_{jk}$ $(1\leqq j\neq k\leqq 2n)$, where $V(q)$ is given by (5.12). We remark that the following relations hold;

$$Z(q)=\frac{1}{2}(\mathrm{Ad}(\exp(-D(q)))B(q)-\sigma(\mathrm{Ad}(\exp(-D(q)))B(q))) \tag{5.24}$$

and hence

$$\mathrm{Ad}(\exp(-D(q)))B(q)=U(q)+Z(q). \tag{5.26}$$

Corollary 5.5. *Let $(q(t),p(t))=\phi_t(q,p)$ be the trajectory of the Hamiltonian flow starting from $(q,p)\in D_{\delta}\times \mathbf{R}^n$. Then we have*

(5.27) $\exp(D(q))\exp(2tZ(q,p))\exp(D(q))J=k(t)\exp(2D(q(t)))Jk(t)^{-1}$

where $k(t)$ *is a curve in* $K_{\bar{\varepsilon}}$ *given by*

(5.28) $$\frac{d}{dt}k(t)=k(t)B(q(t)), \qquad k(0)=1_{2n}.$$

Moreover $Z(q(t),p(t))$ *satisfies the following Lax's isospectral deformation equation*;

(5.29) $$\frac{d}{dt}Z(q(t),p(t))+[U(q(t)),Z(q(t),p(t))]=0.$$

Proof. The proof is parallel to that of Corollary 4.5. So we shall omit it.

§ 6. The Hamiltonian systems attached to (B_n, ε_m) and (BC_n, ε_m)

In this section we treat the Hamiltonian systems attached to the root systems with signature (B_n, ε_m) and (BC_n, ε_m) simultaneously. We recall that the configuration spaces $D_{(B_n,\varepsilon_m)}$ and $D_{(BC_n,\varepsilon_m)}$ are identical, which we denote simply by D;

$$D=\{q=(q_1,\cdots,q_n)\in \boldsymbol{R}^n; q_1>\cdots>q_m>0, q_{m+1}>\cdots>q_n>0\}.$$

Throughout the section we write an element X of $M_{2n+1}(\boldsymbol{C})$ as a block form

$$X=\begin{bmatrix} X_{00} & X_{01} & X_{02} \\ X_{10} & X_{11} & X_{12} \\ X_{20} & X_{21} & X_{22} \end{bmatrix}$$

where $X_{00}\in \boldsymbol{C}$, $X_{01}, X_{02}\in M_{1n}(\boldsymbol{C})$, $X_{10}, X_{20}\in M_{n1}(\boldsymbol{C})$ and $X_{11}, X_{12}, X_{21}, X_{22}\in M_n(\boldsymbol{C})$. Let G be the closed subgroup of $GL(2n+1,\boldsymbol{C})$ given by

$$G=\{g\in GL(2n+1,\boldsymbol{C}); gQg^*=Q\}$$

where Q is given by

$$Q=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{bmatrix}$$

Thus G is isomorphic to $U(n+1,n)$. Define J by

$$J=\begin{bmatrix}1 & 0 & 0\\ 0 & J_m & 0\\ 0 & 0 & J_m\end{bmatrix}, \qquad \text{where } J_m=\begin{bmatrix}1_m & 0\\ 0 & -1_{n-m}\end{bmatrix}.$$

We define a nondegenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G by

$$\langle X, Y\rangle=\frac{1}{2}\operatorname{tr}(XY). \tag{6.1}$$

We introduce involutive automorphisms σ and θ of G by

$$\sigma(g)=J(g^*)^{-1}J \quad \text{and} \quad \theta(g)=(g^*)^{-1}.$$

Then the corresponding involutions on $\mathfrak{g}$ are given by

$$\sigma(X)=-JX^*J \quad \text{and} \quad \theta(X)=-X^*.$$

We set $H=G_\sigma$ and $K=G_\theta$. Then we have

$$\mathfrak{k}=\left\{\begin{bmatrix}X_{00} & -X_{10}^* & -X_{10}^*\\ X_{10} & X_{11} & X_{12}\\ X_{10} & X_{12} & X_{11}\end{bmatrix};\ \overline{X}_{00}=-X_{00},\ X_{11}^*=-X_{11},\ X_{12}^*=-X_{12}\right\},$$

$$\mathfrak{p}=\left\{\begin{bmatrix}0 & X_{10}^* & -X_{10}^*\\ X_{10} & X_{11} & X_{12}\\ -X_{10} & -X_{12} & -X_{11}\end{bmatrix};\ X_{11}^*=X_{11},\ X_{12}^*=-X_{12}\right\},$$

$$\mathfrak{h}=\left\{\begin{bmatrix}X_{00} & -X_{10}^*J_m & -X_{10}^*\\ X_{10} & X_{11} & X_{12}\\ -J_mX_{10} & J_mX_{12}J_m & J_mX_{11}J_m\end{bmatrix};\right.$$

$$\left.\overline{X}_{00}=-X_{00},\ X_{11}^*=-J_mX_{11}J_m,\ X_{12}^*=-X_{12}\right\},$$

$$\mathfrak{q}=\left\{\begin{bmatrix}0 & X_{10}^*J_m & -X_{10}^*\\ X_{10} & X_{11} & X_{12}\\ -J_mX_{10} & -J_mX_{12}J_m & -J_mX_{11}J_m\end{bmatrix};\ X_{11}^*=J_mX_{11}J_m,\ X_{12}^*=-X_{12}\right\}.$$

Put

$$\mathfrak{a}=\{D(q)=\operatorname{diag}(0, q_1, \cdots, q_n, -q_1, \cdots, -q_n);\ q=(q_1, \cdots, q_n)\in \boldsymbol{R}^n\}.$$

Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{q}\cap\mathfrak{p}$, which is maximal abelian both in $\mathfrak{q}$ and $\mathfrak{p}$. The root system R of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is of type BC_n, which contains the root system of type B_n as a subroot system. The involution σ is easily seen to be the same as the one corresponding to the signature ε_m. Set $\mathfrak{a}_+=\{D(q)\in\mathfrak{a}; q\in D\}$. Then it is a Weyl chamber for R_{ε_m}. Since $K=G\cap U(2n+1)$, each $k\in K$ is a unitary matrix of the form

$$k=\begin{bmatrix} k_{00} & k_{01} & k_{01} \\ k_{10} & k_{11} & k_{12} \\ k_{10} & k_{12} & k_{11} \end{bmatrix}. \tag{6.2}$$

Puf $T=\{\mathrm{diag}(u_0, u_1, \cdots, u_n, u_1, \cdots, u_n); u_j\in U(1)\ (0\leqq j\leqq n)\}$ and $Z_K=\{u1_{2n+1}; u\in U(1)\}$. Then T is a maximal torus of K contained in H and Z_K is the center of K.

Let c_0, c_1, c_2 be real constants such that c_0 and c_1 are nonzero. In the remainder of the section, we assume that these constants satisfy the following relation;

$$(c_0/c_1)^2=(2c_1-c_2)/c_1. \tag{6.3}$$

Put $\boldsymbol{e}={}^t(1, \cdots, 1)\in \boldsymbol{C}^n$ and define $\tilde{\boldsymbol{e}}\in\boldsymbol{C}^{2n+1}$ by

$$\tilde{\boldsymbol{e}}={}^t(c_0/c_1, \boldsymbol{e}, \boldsymbol{e}).$$

Furthermore we define $C\in M_{2n+1}(C)$ by

$$C=\sqrt{-1}\begin{bmatrix} 0 & c_0\boldsymbol{e}^* & c_0\boldsymbol{e}^* \\ c_0\boldsymbol{e} & c_1(\boldsymbol{ee}^*-1_n) & c_1(\boldsymbol{ee}^*-1_n)+c_21_n \\ c_0\boldsymbol{e} & c_1(\boldsymbol{ee}^*-1_n)+c_21_n & c_1(\boldsymbol{ee}^*-1_n) \end{bmatrix} \tag{6.4}$$

Then we have $C\in\mathfrak{k}^1=[\mathfrak{k},\mathfrak{k}]$.

Lemma 6.1. *Assume* (6.3). *Then we obtain that*

(i) *for every $k\in K$, there exist $\boldsymbol{v}_0\in \boldsymbol{C}$ and $\boldsymbol{v}\in\boldsymbol{C}^n$ such that* $k\tilde{\boldsymbol{e}}={}^t(v_0, \boldsymbol{v}, \boldsymbol{v})$.

(ii) *For each $k\in K$, we have*

$$\mathrm{Ad}(k)C=\sqrt{-1}\begin{bmatrix} c_1(|v_0|^2-(c_0/c_1)^2) & c_1v_0\boldsymbol{v}^* & c_1v_0\boldsymbol{v}^* \\ c_1\bar{v}_0\boldsymbol{v} & c_1(\boldsymbol{vv}^*-1_n) & c_1(\boldsymbol{vv}^*-1_n)+c_21_n \\ c_1\bar{v}_0\boldsymbol{v} & c_1(\boldsymbol{vv}^*-1_n)+c_21_n & c_1(\boldsymbol{vv}^*-1_n) \end{bmatrix}$$

where v_0 and $\boldsymbol{v}$ are given in (i).

(iii) *If we set* $K_C=\{k\in K;\ \mathrm{Ad}(k)C=C\}$ *and* $K_{\tilde{e}}=\{k\in K;\ k\tilde{e}=\tilde{e}\}$, *then we have* $K_C=Z_K K_{\tilde{e}}$ *and* $K_{\tilde{e}}\cap T=(1_{2n+1})$.

Proof. (i) Each $k\in K$ can be written in the form (6.2). Hence if we put

$$(6.5)\qquad v_0=(c_0/c_1)k_{00}+2k_{01}\boldsymbol{e}\quad\text{and}\quad \boldsymbol{v}=(c_0/c_1)k_{10}+(k_{11}+k_{12})\boldsymbol{e}$$

then we have $k\tilde{e}={}^t(v_0,\boldsymbol{v},\boldsymbol{v})$.

(ii) Put $\mathrm{Ad}(k)C=\sqrt{-1}\,C'$ and denote its block expression by $C'=(C'_{rs})_{0\leqq r,s\leqq 2}$. First we shall show $C'_{00}=c_1(|v_0|^2-(c_0/c_1)^2)$. By direct matrix computation, we have

$$C'_{00}=c_1(2c_1^{-1}c_0(\bar{k}_{00}k_{01}\boldsymbol{e}+k_{00}\boldsymbol{e}^*k^*_{01})+4k_{01}\boldsymbol{e}\boldsymbol{e}^*k^*_{01}-2c_1^{-1}(2c_1-c_2)k_{01}k^*_{01}.$$

On the other hand by (6.5) we have

$$|v_0|^2=2c_1^{-1}c_0(\bar{k}_{00}k_{01}\boldsymbol{e}+k_{00}\boldsymbol{e}^*k^*_{01})+4k_{01}\boldsymbol{e}\boldsymbol{e}^*k^*_{01}+(c_0/c_1)^2|k_{00}|^2.$$

Hence we can write

$$C'_{00}=c_1(|v_0|^2-(c_0/c_1)^2|k_{00}|^2-2c_1^{-1}(2c_1-c_2)k_{01}k^*_{01}).$$

Since k is a unitary matrix and hence $|k_{00}|^2+2k_{01}k^*_{01}=1$, it follows that

$$C'_{00}=c_1(|v_0|^2-c_1^{-1}(2c_1-c_2)-((c_0/c_1)^2-c_1^{-1}(2c_1-c_2))|k_{00}|^2).$$

Using (6.3), we have $C'_{00}=c_1(|v_0|^2-(c_0/c_1)^2)$. Similarly by direct calculation, we have $C'_{10}=C'_{20}=(C'_{01})^*=(C'_{02})^*$ and

$$C'_{10}=c_0\bar{k}_{00}(k_{11}+k_{12})\boldsymbol{e}+2c_0k_{10}\boldsymbol{e}^*k^*_{01}+2c_1(k_{11}+k_{12})(\boldsymbol{e}\boldsymbol{e}^*-1_n)k^*_{01}$$
$$+c_2(k_{11}+k_{12})k^*_{01}.$$

By (6.5) this can be written as

$$C'_{10}=c_1\bar{v}_0\boldsymbol{v}-c_1^{-1}c_0^2\bar{k}_{00}k_{10}-(2c_1-c_2)(k_{11}+k_{12})k^*_{01}.$$

Since $\bar{k}_{00}k_{10}+(k_{11}+k_{12})k^*_{01}=0$, we have

$$C'_{10}=c_1\bar{v}_0\boldsymbol{v}-c_1((c_0/c_1)^2-c_1^{-1}(2c_1-c_2))\bar{k}_{00}k_{10}.$$

Again by using (6.3), we have $C'_{10}=c_1\bar{v}_0\boldsymbol{v}$. The cases for C'_{11}, C'_{12}, C'_{21} and C'_{22} are treated quite analogously. So we shall omit the proof.

(iii) From (ii) it follows that $k\in K_C$ if and only if a) $|v_0|^2=(c_0/c_1)^2$, b) $c_1\bar{v}_0\boldsymbol{v}=c_0\boldsymbol{e}$ and c) $\boldsymbol{v}\boldsymbol{v}^*=\boldsymbol{e}\boldsymbol{e}^*$. From these we can easily deduce that $k\in K_C$ if and only if $k\tilde{e}=u\tilde{e}$ for some $u\in U(1)$. This yields the assertion.

Let M be the cotangent bundle over G/H and Φ the moment map for the K-action given by (3.7). We consider the reduced phase space $M(C)$ where C is given by (6.4). For $q \in D$, we define $Z(q) \in \mathfrak{q}$ by

$$Z(q)=\begin{bmatrix} 0 & Z_0(q)^*J_m & -Z_0(q)^* \\ Z_0(q) & Z_1(q) & Z_2(q) \\ -J_mZ_0(q) & -J_mZ_2(q)J_m & -J_mZ_1(q)J_m \end{bmatrix} \tag{6.6}$$

where

$$Z_0(q)=\sqrt{-1}\,c_0^t(\mathrm{sh}^{-1}(q_1), \cdots, \mathrm{sh}^{-1}(q_m), \mathrm{ch}^{-1}(q_{m+1}), \cdots, \mathrm{ch}^{-1}(q_n)) \tag{6.7}$$

and $Z_1(q)$ and $Z_2(q)$ are given respectively by the same formulas (5.7) and (5.8). Moreover we define $Z(q, p) \in \mathfrak{q}$ by

$$Z(q, p)=D(p)+Z(q). \tag{6.8}$$

Proposition 6.2. *Under the assumption* (6.3), *we have*

(i) *for each $\pi(x, X) \in \Phi^{-1}(C)$ there exist unique $k \in K_{\tilde{e}}$ and $(q, p) \in D\times \boldsymbol{R}^n$ such that $k\pi(x, X)=\pi(\exp(D(q)), Z(q, p))$.*

(ii) *$\Phi^{-1}(C)$ is a submanifold of M diffeomorphic to $K_{\tilde{e}}\times D\times \boldsymbol{R}^n$.*

Proof. Let $\pi(x, X) \in \Phi^{-1}(C)$. Since $G=K\bar{A}_+H$, we can write $x=k^{-1}ah$ ($k \in K$, $a \in \bar{A}_+$, $h \in H$). Then $k\pi(x, X)=\pi(a, \mathrm{Ad}(h)X)$ and $\Phi(\pi(a, \mathrm{Ad}(h)X))=\mathrm{Ad}(k)C$. Since a is a diagonal matrix, we can easily obtain that the diagonal part of $\Phi(\pi(a, \mathrm{Ad}(h)X))$ vanishes and so does the diagonal part of $\mathrm{Ad}(k)C$. Thus we conclude from Lemma 6.1 (ii) that $|v_0|^2=(c_0/c_1)^2$ and $v_i \in U(1)$ $(1\leqq i\leqq n)$. Put $t=\mathrm{diag}(c_0^{-1}c_1v_0, v_1 \cdots, v_n, v_1, \cdots, v_n)$. Then $t \in T$ and $t^{-1}k \in K_{\tilde{e}}$. Since $t^{-1}a=at^{-1}$ and $T\subset H$, if we put $Z=\mathrm{Ad}(t^{-1}h)X$, then $t^{-1}k\pi(x, X)=\pi(a, Z)$ and $\Phi(\pi(a, Z))=C$. Now we put $a=\exp(D(q))$ with $D(q) \in \bar{\mathfrak{a}}_+$. Then the last identity implies that $D(q) \in \mathfrak{a}_+$ and Z is of the form $Z(q, p)$. The uniqueness of $k \in K_{\tilde{e}}$ and $(q, p) \in D\times \boldsymbol{R}^n$ and the assertion (ii) are proved in the same manner as in Proposition 4.2.

Theorem 6.3. *Keeping the assumption* (6.3), *we denote by $M(C)$ the set of K_C-orbits in $\Phi^{-1}(C)$ and by π_C the canonical projection of $\Phi^{-1}(C)$ onto $M(C)$. Define a map φ of $D\times \boldsymbol{R}^n$ into $M(C)$ by*

$$\varphi(q, p)=\pi_C\circ\pi(\exp(D(q)), Z(q, p)). \tag{6.9}$$

Then φ is a bijection and hence $M(C)$ has a smooth manifold structure under which φ is a diffeomorphism and π_C is a submersion. Thus $M(C)$ is a reduced phase space with the symplectic structure ω_C.

(ii) *It holds that* $\varphi^*\omega_C=\sum_{i=1}^n dq_i\wedge dp_i$ *and hence* φ *is a symplectic diffeomorphism.*

(iii) *Define a G-invariant Hamiltonian F on M by*

$$F(\pi(x, X))=\frac{1}{2}\langle X, X\rangle$$

and denote the reduced Hamiltonian on $M(C)$ *by* F^C. *Then we have*

$$F^C\circ\varphi=H_{(B_n, \varepsilon_m)}\quad \text{if } c_2=0$$

and

$$F^C\circ\varphi=H_{(BC_n, \varepsilon_m)}\quad \text{if } c_2\neq 0.$$

Proof. (i) From Proposition 6.2, it is clear that φ is bijective, so that we can define a C^∞-structure on $M(C)$ under which φ is a diffeomorphism. If we define a smooth map $\tilde{\varphi}$ of $D\times \boldsymbol{R}^n$ into $G\times\mathfrak{q}$ by $\tilde{\varphi}(q, p)=(\exp(D(q)), Z(q, p))$, then it holds that $\varphi=\pi_C\circ\pi\circ\tilde{\varphi}$ and $i_C\circ\pi\circ\tilde{\varphi}=\pi\circ\tilde{\varphi}$. Hence π_C is a submersion and $M(C)$ is a reduced phase space.

(ii) We define $V(q)\in\mathfrak{h}$ by

$$\text{(6.10)}\qquad V(q)=\begin{bmatrix} 0 & -V_0(q)^*J_m & -V_0(q)^* \\ V_0(q) & V_1(q) & V_2(q) \\ -J_mV_0(q) & J_mV_2(q)J_m & J_mV_1(q)J_m \end{bmatrix}$$

where $V_0(q)={}^t(V_0(q)_1, \cdots, V_0(q)_n)$ with

$$\text{(6.11)}\qquad V_0(q)_j=\begin{cases} -\sqrt{-1}\,c_0\,\mathrm{ch}(q_j)\,\mathrm{sh}^{-2}(q_j) & (1\leqq j\leqq m), \\ -\sqrt{-1}\,c_0\,\mathrm{sh}(q_j)\,\mathrm{ch}^{-2}(q_j) & (m+1\leqq j\leqq n) \end{cases}$$

and $V_1(q)$ (resp. $V_2(q)$) is the same as in (5.13) (resp. (5.14)). Then the differential $d\tilde{\varphi}$ is again given by the same formula as (4.24). Hence the proof of (ii) is parallel to that of Theorem 4.3.

(iii) In the same manner as in Theorem 4.3, we have

$$F^C(\varphi(q, p))=2^{-1}\langle Z(q, p), Z(q, p)\rangle=4^{-1}\mathrm{tr}(Z(q, p)^2).$$

From (6.6) and the fact that the diagonal part of $Z(q)$ is equal to zero, it follows that $4^{-1}\mathrm{tr}(Z(q, p)^2)=4^{-1}\mathrm{tr}(D(p)^2)+4^{-1}\mathrm{tr}(Z(q)^2)$. Since $Z(q)$ is given by (6.6), we obtain

$$4^{-1}\mathrm{tr}(Z(q)^2)=\mathrm{tr}(Z_0(q)^*J_mZ_0(q))+2^{-1}(\mathrm{tr}(Z_1(q)^2)-\mathrm{tr}((J_mZ_2(q))^2)).$$

The first term is equal to

$$c_0^2\left(\sum_{j=1}^{m}\mathrm{sh}^{-2}(q_j)-\sum_{j=m+1}^{n}\mathrm{ch}^{-2}(q_j)\right).$$

The second term is computed as in Theorem 5.3 (iii). The result is

$$c_1^2(\textstyle\sum_{(1)}(\mathrm{sh}^{-2}(q_{jk})+\mathrm{sh}^{-2}(\hat{q}_{jk}))$$
$$-\textstyle\sum_{(2)}(\mathrm{ch}^{-2}(q_{jk})+\mathrm{ch}^{-2}(\hat{q}_{jk})))+c_2^2/2\displaystyle\sum_{j=1}^{n}\mathrm{sh}^{-2}(2q_j).$$

From these we can deduce (iii).

We define G-invariant smooth functions $F_1, \cdots, F_n$ on M by

$$F_k(\pi(x, X))=\bar{F}_k(X)=(2k)^{-1}\mathrm{tr}\,(X^{2k}). \tag{6.12}$$

Then they have the same properties described in Section 5. Moreover if we define $I_k(q, p)=F_k^C(\varphi(q, p))$ $((q, p)\in D\times \boldsymbol{R}^n,\ 1\leqq k\leqq n)$, then

$$I_k(q, p)=(2k)^{-1}\mathrm{tr}\,(Z(q, p)^{2k})\quad (1\leqq k\leqq n). \tag{6.13}$$

They are clearly rational functions of $p_1, \cdots, p_n, \exp(q_1), \cdots, \exp(q_n)$. Hence the following corollary is valid.

Corollary 6.4. *The Hamiltonian systems attached to the root systems with signature* (B_n, ε_m) *and* (BC_n, ε_m) *are completely integrable under the assumption* (6.3). *The above rational functions* $I_1, \cdots, I_n$ *are mutually involutive integrals of motion, which are generically functionally independent.*

For $q\in D$ we set

$$B(q)=\begin{bmatrix} B_{00}(q) & -B_0(q)^* & -B_0(q)^* \\ B_0(q) & B_1(q) & B_2(q) \\ B_0(q) & B_2(q) & B_1(q) \end{bmatrix} \tag{6.14}$$

where

$$B_{00}(q)=2\sqrt{-1}\,c_1\left(\sum_{j=1}^{m}\mathrm{sh}^{-2}(q_j)-\sum_{j=m+1}^{n}\mathrm{ch}^{-2}(q_j)\right) \tag{6.15}$$

and $B_0(q)={}^t(B_0(q)_1, \cdots, B_0(q)_n)$ such that

$$B_0(q)_j=\begin{cases} -\sqrt{-1}\,c_0\,\mathrm{sh}^{-2}(q_j) & (1\leqq j\leqq m), \\ \sqrt{-1}\,c_0\,\mathrm{ch}^{-2}(q_j) & (m+1\leqq j\leqq n) \end{cases} \tag{6.16}$$

and $B_1(q)_{jk}$ $(1\leqq j\neq k\leqq n)$ are given by (5.20) and $B_2(q)_{jk}(1\leqq j, k\leqq n)$ are given by (5.21) and finally

$$B_1(q)_{jj} = -2\sqrt{-1}\,c_0\Big(\sum_{j=1}^{m} \mathrm{sh}^{-2}(q_j) - \sum_{j=m+1}^{n} \mathrm{ch}^{-2}(q_j)\Big) - \sum_{i\neq j} B_1(q)_{ij} - \sum_{i} B_2(q)_{ij}.$$

for $1 \leqq j \leqq n$. Then we can check $B(q) \in \mathfrak{k}_{\varepsilon}$ Furthermore we define $U(q) \in \mathfrak{h}$ $(q \in D)$ by

(6.17) $$U(q) = \frac{1}{2}(\mathrm{Ad}(\exp(-D(q)))B(q) + \sigma(\mathrm{Ad}(\exp(-D(q)))B(q))).$$

Then it holds that $U(q)_{jj} = B(q)_{jj}$ $(0 \leqq j \leqq 2n)$ and $U(q)_{jk} = V(q)_{jk}$ $(0 \leqq j \neq k \leqq 2n)$ where $V(q)$ is given by (6.10). Moreover we have

(6.18) $$Z(q) = \frac{1}{2}(\mathrm{Ad}(\exp(-D(q)))B(q) - \sigma(\mathrm{Ad}(\exp(-D(q)))B(q)))$$

and hence

(6.19) $$\mathrm{Ad}(\exp(-D(q)))B(q) = U(q) + Z(q).$$

Corollary 6.5. *Under the assumption* (6.3), *let* $(q(t), p(t)) = \phi_t(q, p)$ *be the trajectory of the Hamiltonian flow starting from* $(q, p) \in D \times \boldsymbol{R}^n$. *Then we have*

(6.20) $$\exp(D(q)) \exp(2tZ(q, p)) \exp(D(q))J = k(t) \exp(2D(q(t)))Jk(t)^{-1}$$

where $k(t)$ *is a curve in* K_{ε} *given by*

(6.21) $$\frac{d}{dt}k(t) = k(t)B(q(t)),\ k(0) = 1_{2n+1}.$$

Moreover $Z(q(t), p(t))$ *satisfies the following Lax's isospectral deformation equation;*

(6.22) $$\frac{d}{dt}Z(q(t), p(t)) + [U(q(t)), Z(q(t), p(t))] = 0.$$

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