Advanced Studies in Pure Mathematics 4, 1984
Group Representations and Systems of Differential Equations
pp. 263–266

Infra-red Finiteness in Quantum Electro-Dynamics

Takahiro Kawai and Henry P. Stapp

In this talk we want to report some mathematical aspects of a recent solution of the infra-red catastrophe in quantum electro-dynamics. See Stapp [1], [2] for the physical implications and general ideas of this solution. Some mathematical claims made in [1] were verified in Kawai-Stapp [3], upon which we report here. We hope this report will introduce more mathematicians to the infra-red divergence problem, which is becoming increasingly important and intriguing, particularly in connection with the development of *QCD*.

A principal result in [1] is that the coordinate space Feynman function $F^D(x)$ can be separated into two factors the first of which is a unitary operator in photon space representing the classical electro-magnetic contribution to the amplitude, and the second of which is a residual factor representing the quantum fluctuation about the classical contribution.

The main objectives of [3] were to verify:

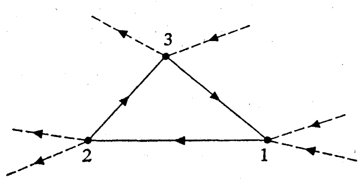
(i) the residual factor is free of infra-red divergences,

and

(ii) the dominant part of the singularity of the residual factor on the positive-α Landau surface $L^+(D)$ has the same analytic form as it would have if the photons were massive.

To explain these properties in more detail, let us first explain the recipe of [1] about how to separate the coordinate space Feynman function into its classical and residual quantum parts. To be specific, let us discuss the case where D is the following triangle graph.

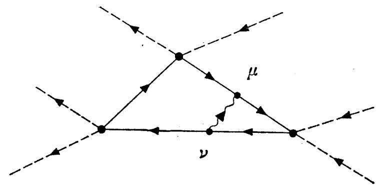
D:



Let D' denote the following graph corresponding to an electro-magnetic correction:

Received July 19, 1983.

D':



Here the wiggly line corresponds to a photon, and its end points are called photon vertices. These graphs may be regarded as space-time diagrams describing the trajectories of classical particles. (Coleman-Norton picture) To emphasize this fact, we denote by x_j ($j=1, 2, 3$) the point where the j-th vertex of D is located. The recipe of [1] tells us to decompose the coupling at a photon vertex into the classical part $C_\mu(k, z_l)$ and the quantum part $Q_\mu(k, z_l)$ according to the following rule (1). Here, and in what follows, e denotes a constant (electric charge), γ_μ ($\mu=0, 1, 2, 3$) denotes the gamma matrix and k denotes the energy-momentum vector associated with the photon line. The decomposition rule is

(1.a) $$-ie\gamma_\mu = C_\mu(k, z_l) + Q_\mu(k, z_l),$$

where

(1.b) $$z_l = x_{j^+(l)} - x_{j^-(l)}$$

and

(1.c) $$Q_\mu(k, z) = \left(\delta_\mu^\sigma - \frac{z_\mu k^\sigma}{z\cdot k + i0}\right)(-ie\gamma_\sigma).$$

Here l denotes the line on which the photon vertex in question lies and $j^+(l)$ (resp., $j^-(l)$) denotes the vertex at which the line l terminates (resp., begins).

This decomposition breaks $F^D(x)$ into four terms, the C-C, Q-C, C-Q and Q-Q contributions. The task required in [3] is to verify the infra-red finiteness etc. of the Q-C, C-Q and Q-Q contributions. The infra-red finiteness of the classical contribution C-C was proved in [1].

Since our main concern in [3] is to study the singularity structure in energy-momentum space, we calculate the Fourier transform of, say, the Q-Q term, and find that it is given by the following formula (2):

(2) $$F_{QQ}(q) = \int \frac{d^4k}{(2\pi)^4}\frac{g^{\mu\nu}}{k^2+i0}\int\frac{d^4p_1}{(2\pi)^4}$$
$$\mathrm{Tr}\left[F_{Q\mu}(p_1, k)V_1 F'_{Q\nu}(p_1+q_1, k)V_2 \frac{(p_1+q_1+q_2+m_3)}{(p_1+q_1+q_2)^2-m_3^2+i0}V_3\right]$$
$$\times(2\pi)^4\delta^4(q_1+q_2+q_3),$$

where $g^{\mu\nu}$ denotes the Minkowski metric tensor, $\not p = p^\mu \gamma_\mu = g^{\mu\nu}\gamma_\mu p_\nu$, V_j $(j=1, 2, 3)$ is some Dirac matrix, $F_{Q\mu}(p_1, k)$ is, by definition, given by

$$G_\mu(p_1, k) - \int_0^1 d\lambda G_\mu(p_1+\lambda k, 0) \tag{3}$$

with

$$G_\mu(p_1, k) = \frac{(\not p_1 + m_1)(-ie\gamma_\mu)(\not p_1 + \not k + m_1)}{(p_1^2 - m_1^2 + i0)((p_1+k)^2 - m_1^2 + i0)},$$

and $F'_{Q\nu}$ is the similar function for the other photon vertex.

One important feature of (3) is that the function $F_{Q\mu}$ is expressed as an integral over a compact interval, i.e., [0, 1]. This leads to the infra-red finiteness of $F_{Q\mu}$ (and hence to that of F_{QQ} etc.). (§ 3 of [3]) This compactness is a consequence of the fact that the definition of $Q_\mu(k, z)$ introduces a factor k into the numerator in the integrand of $F_{Q\mu}$. This factor enables us to apply the Ward-Takahashi identity in order to reduce the integral

$$\int_0^\infty \left[\frac{\partial}{\partial p^\mu}\left(\frac{(\not p + m_1)\not k(\not p + \not k + m_1)}{(p^2 - m_1^2 + i0)((p+k)^2 - m_1^2 + i0)}\right)\right]\Bigg|_{p=p_1+\lambda k} d\lambda$$

to the form

$$\int_0^\infty \left[\frac{\partial}{\partial p^\mu}\left(\frac{\not p + m_1}{p^2 - m_1^2 + i0} - \frac{\not p + \not k + m_1}{(p+k)^2 - m_1^2 + i0}\right)\right]\Bigg|_{p=p_1+\lambda k} d\lambda. \tag{4}$$

Although (4) is an integral over an infinite interval, it is equal to

$$\int_0^1 \left[\frac{\partial}{\partial p^\mu}\left(\frac{\not p + m_1}{p^2 - m_1^2 + i0}\right)\right]\Bigg|_{p=p_1+\lambda k} d\lambda,$$

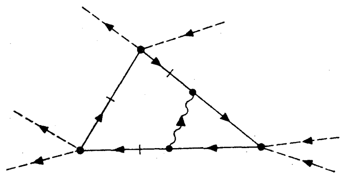
an integral over a compact region. See Section 2 of [3] for the details of the calculation.

By performing the integration explicitly, we find that $F_{Q\mu}(p_1, k)$ given by (3) contains a term Φ_μ which has a pole both on $\{p_1^2 = m_1^2\}$ and $\{(p_1+k)^2 = m_1^2\}$ and the singularity of $F_{Q\mu} - \Phi_\mu$ is weaker than that of Φ_μ. (§ 7 of [3]) The calculation shows that the dominant part Φ_μ has the form

$$\frac{(\not p_1 + m_1)(\gamma_\mu \not k - p_{1\mu} k^2 (p_1 \cdot k + i0)^{-1})}{(p_1^2 - m_1^2 + i0)((p_1+k)^2 - m_1^2 + i0)}. \tag{5}$$

This confirms a formula given in [1] ((7.32)), which was obtained by studying the asymptotic behavior of the function in coordinate space. An important feature of (5) is that, if we use Φ_μ as the dominant part of the propagator, then the dominant part of the associated discontinuity function corresponds to the diagram

D_0':



,

where the slashed line indicates that the corresponding energy-momentum vector should be kept on mass-shell. The residue of Φ_μ along $\{p_1^2=m_1^2\}$ is

$$\frac{(\not{p}_1+m_1)(\gamma_\mu k-p_{1\mu}k^2(p_1\cdot k+i0)^{-1})}{2p_1\cdot k+k^2+i0},$$

and, along $\{k^2=0\}$, it is homogeneous of degree 0 with respect to k, rather than of degree -1, as it is in the case for the full Feynman function. Therefore, even after multiplication by the factor $1/(k^2+i0)$, we obtain a convergent integral. The usual discontinuity function associated with D_0' is, in contrast, infra-red divergent. Since the singularity of $F_{Q\mu}-\Phi_\mu$ is weaker than that of Φ_μ, the dominant part of the discontinuity of $F_{Q\mu}$ along $L^+(D)$ is given by that of Φ_μ, which we have just described. Thus both tasks (i) and (ii) are achieved for F_{QQ}^D when D is the triangle graph. The analysis for the general case can be performed in a similar manner. (§ 9 of [3])

In conclusion, we want to emphasize that the property (ii) is very important in physical applications. Less careful calculations had apparently shown that the "stable particle poles" are not poles in the presence of soft photons. Such an effect would disrupt the normal connection between theory and experiment, and lead to a breakdown of the important reduction formulas of field theory.

References

[1] H. P. Stapp, Exact solution of the infrared problem, Phys. Rev. D., **28** (1983), 1386–1418.

[2] ——, Solution of the infrared problem, Phys. Rev. Lett., **50** (1983), 467–470.

[3] T. Kawai and H. P. Stapp, Infrared finiteness and analytic structure, to appear.

T. Kawai:
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606, Japan

H. P. Stapp:
Lawrence Berkeley Laboratory
University of California
Berkeley, CA 94720, USA