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# Configurations and Invariant Theory of Gauß-Manin Systems

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Let $f_0, f_1, \cdots, f_m$ be polynomials. The integral

$$\int \exp[f_0(x_1, \cdots, x_n)] \cdot f_1(x)^{\lambda_1} \cdots f_m(x)^{\lambda_m} dx_1 \cdots dx_n \tag{0.1}$$

satisfies Gauß-Manin system or holonomic system as a function of coefficients of $f_0, f_1, \cdots, f_m$. $GL_n(\boldsymbol{C})$ naturally acts on the space of coefficients, so that the above integral is written in invariant expression. This is a similar situation to *D. Mumford's geometric invariant theory* [1]. (See also [2] *in relation to Cayley forms*).

Let $T$, $X$ be non-singular algebraic spaces of dimension $n$ and $l$ respectively. Let $W$ be an analytic subset of codimension 1 such that the complement $V = T \times X - W$ is affine. We denote by $\rho$ the natural projection:

$$\rho: V = T \times X - W \longmapsto T. \tag{0.2}$$

Then *by the isotopy theorem due to R. Thom* ([3], See [4] for further developments.) there exists a natural stratification of the morphism $(V, T, \rho)$ satisfying the following property:

There exists an analytic subset $T_0$ of codimension 1 in $T$ such that for arbitrary $t \in T - T_0$, the morphism

$$\rho: f^{-1}(T - T_0) \longmapsto T - T_0$$

is a topological fibre bundle whose fibre $V_t = \rho^{-1}(t)$ is non-singular:

We shall denote by $\Omega^p(V, F)$ the space of rational $p$-forms in a compactification of $V$ and holomorphic in $V$ with values in a sheaf $F$.

## § 1.

Let a $\mathfrak{gl}(m, \boldsymbol{C})$-valued rational 1-form in $T \times X$ which is holomorphic in $V$,

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$$\omega(t, dt; x, dx) \in \Omega^1(T\times X - W)\otimes \mathfrak{gl}(n, C),$$

$(t, x) \in T\times X$, is given such that $\omega$ satisfies the *integrability condition*

$$(1.1) \qquad d\omega + \omega\wedge\omega = 0 \quad \text{in } V.$$

$\omega_t = \omega|_{\rho^{-1}(t)}$ defines *a Gauß-Manin connection on* $\rho^{-1}(t) = X - W_t$ which is denoted by $V_t$:

$$(1.2) \qquad d\omega_t + \omega_t\wedge\omega_t = 0 \quad \text{in } V_t.$$

*Consider the linear differential system on* $V_t$ *of order* $m$:

$$(\varepsilon) \qquad dy = y\omega_t \qquad (y \in C^m)$$

and *the twisted rational de Rham cohomology* $H^*(V_t, \nabla_{\omega_t})$ in $V_t$ with coefficients in solution sheaves $\mathcal{S}$ of $(\varepsilon)$:

$$(1.3) \qquad 0 \longmapsto \Omega^0(V_t)\otimes C^m \xrightarrow{\nabla_{\omega_t}} \Omega^1(V_t)\otimes C^m \longmapsto \cdots.$$

Let $\mathcal{S}^*$ be the dual sheaf of $\mathcal{S}$ defined on $V_t$ (a local system in $V_t$ if $(\varepsilon)$ is regular singular and a relative local system in $X$ modulo $W_t$ if $(\varepsilon)$ is irregular singular of simple type).

A systematic study of these kinds of cohomologies has been done by several authors ([5]~[10]). By *the comparison theorem* we have the isomorphism ([5])

$$(1.4) \qquad H^*(V_t, \nabla_{\omega_t}) \simeq H^*(V_t, \mathcal{S}).$$

From micro local point of view many important results have been established ([9], [10]). But from our point of view that they give "*regularization*" or "*finite part*" of integrals in the sense of J. Hadamard-J. Leray ([11]~[12]), these will be made clear in a concrete way in the foregoing.

We shall restrict ourselves to cases *where the cohomologies* $H^*(V_t, \mathcal{S})$ *are finite dimensional.*

Let $Y_t = {}^t(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ be fundamental solutions of $(\varepsilon)$ such that

$$(1.5) \qquad Y_t^{-1}\cdot dY_t = \omega_t.$$

Then the cohomology $H^p(V_t, \mathcal{S})$ and $H_p(V_t, \mathcal{S}^*)$ are dual to each other. The pairing of $\xi = \Sigma\xi_j\otimes\Delta_j \in H_p(V_t, \mathcal{S}^*)$ and $Y\cdot\varphi \in H^p(V_t, \mathcal{S})$, $\varphi \in \Omega^p(V_t)$ is given as follows:

$$(1.6) \qquad \langle\xi, Y\varphi\rangle = \sum_j \int_{\Delta_j} \xi_j\cdot Y\varphi$$

where $\Delta_j$ denotes $p$-chains in $V_t$ and $\xi_j \in \mathscr{S}^*_{\Delta_j}$ (stalk over $\Delta_j$) which is isomorphic to $\boldsymbol{C}^m$.

Let $G$ be a connected algebraic group over $\boldsymbol{C}$ acting in $T$, $X$ and $W$ such that the following holds:

($\mathscr{H}$) *There exists* $\Phi \in \Omega^n(V_t)\otimes\mathfrak{gl}(m, \boldsymbol{C})$ *such that* $Y\cdot\Phi$ *has the invariant property*:

$$Y\cdot\Phi(g^{-1}t, g^{-1}x; d(g^{-1}x)) = Y\cdot\Phi(t, x; dx)\cdot\lambda(g), \qquad g \in G \tag{1.7}$$

*where* $\lambda(g)$ *denotes a representation of* $G$ *into* $GL_m(\boldsymbol{C})$. We put

$$\tilde{\Phi}(t) = \langle \xi, Y\Phi \rangle. \tag{1.8}$$

Then

$$\begin{aligned} \tilde{\Phi}(g^{-1}t) &= \sum \int_{\Delta_j} \xi_j \cdot Y\Phi(g^{-1}t, x; dx) \\ &= \sum \int_{g\Delta_j} \xi_j Y\Phi(g^{-1}t, g^{-1}x; d(g^{-1}x)) \\ &= \sum \int_{g\Delta_j} \xi_j Y\Phi(t, x; dx)\cdot\lambda(g) \\ &= \sum_{\Delta_j} \xi_j Y\Phi(t, x; dx)\cdot\lambda(g) \end{aligned} \tag{1.9}$$

because $\sum_j \xi_j\otimes g\Delta_j$ is homologous to $\sum_j \xi_j\otimes\Delta_j$ in $H_n(V_t, \mathscr{S}^*)$, seeing that $G$ is connected.

We shall only consider the forms $Y\Phi$ satisfying ($\mathscr{H}$).

We assume now that $\tilde{\Phi}(t)$ satisfies the Gauß-Manin system in $T-T_0$, namely that there exists a system of matrix functions of integrals $\tilde{\Phi}_1(t), \cdots, \tilde{\Phi}_\mu(t)$ such that

$$d_t\tilde{\Phi}_i(t) = \sum_{j=1}^{\mu} \tilde{\Phi}_j\Theta_{ji}(t, dt) \tag{1.10}$$

where $\Theta = (\Theta_{ij}(t, dt))_{1\le i, j\le\mu}$ satisfies

$$d\Theta + \Theta\wedge\Theta = 0. \tag{1.11}$$

Then we have the following proposition.

**Proposition 1.** *$\Theta(t, dt)$ is invariant with respect to the action of $G$ in $T$*:

$$\Theta(g^{-1}t, d(g^{-1}t)) = \Theta(t, dt). \tag{1.12}$$

This invariant expression gives us a powerful tool for explicit expressions of Gauß-Manin systems for certain integrals of (0.1).

We shall give it from now on. *We shall only consider the case where* $m=1$. So we may assume $\lambda(g)\in C^*$.

## § 2. (Example 1) Configurations of hyperplanes in projective spaces

Let $X=CP^n$ and $f_1,\cdots,f_m$ be a sequence of linear functions

(2.1) $$f_j=\sum_{\nu=1}^{n}a_{j\nu}x_\nu+a_{j0}.$$

We consider the integral

(2.2) $$F(t)=\int f_1^{\lambda_1}\cdots f_m^{\lambda_m}dx_1\wedge\cdots\wedge dx_n,\ \text{for}\ \lambda_j\in C.$$

The space $T$ is isomorphic to $C^{(n+1)m}$ consisting of points of coefficients $t=((a_{j\nu}))_{1\le j\le m,0\le\nu\le n}$. The *integral* $F(t)$ *admits of the action of* $GL_n(C)$ *in* $T$ *and* $X$ *respectively such that* $f_j(x)$ *is invariant.*

$W$ is defined as follows:

(2.3) $$W=\{(t,x)\in T\times X\,|\,f_j(x)=0,\ 0\le j\le m\}$$

where $f_0$ denotes the hyperplane at infinity in $CP^n$. The space $T_0$ consists of points $t\in T$ such that

(2.4) $$\left[i_1\cdots i_n\right]=\left[\begin{vmatrix} a_{i_1 1}\cdots a_{i_1 n}\\ \vdots \\ a_{i_n 1}\cdots a_{i_n n}\end{vmatrix}\right]\neq 0$$

(2.5) $$\left[i_0 i_1\cdots i_n\right]=\left[\begin{vmatrix} a_{i_0 0} & a_{i_0 1}\cdots a_{i_0 n}\\ \vdots & \\ a_{i_n 0} & a_{i_n 1}\cdots a_{i_n n}\end{vmatrix}\right]\neq 0$$

for $0<i_1<\cdots<i_n\le m$ or $0<i_0<i_1<\cdots<i_n\le m$.

A basis of integrands are given by the forms

(2.6) $$\varphi(i_1\cdots i_n)=d\log f_{i_1}\wedge\cdots\wedge d\log f_{i_n}$$

with the fundamental relations

(2.7) $$\sum_{j=1}^{m}\lambda_j\varphi(ji_1\cdots i_{n-1})\sim 0$$

in $H^n(V_t,\nabla_{\omega_t})$.

Then *the Gauß-Manin connection is simply given in invariant expression by*

$$d\tilde{\varphi}(I)=\sum_{\kappa=0}^{n}(-1)^{\kappa}d\log\left(\frac{[i_0, I]}{\partial_{\kappa}[i_0, I]}\right)\cdot\tilde{\varphi}(\partial_{\kappa}(i_0, I)) \tag{2.8}$$

*for* $I=(i_1, i_2, \cdots, i_n)$, *where* $\partial_{\kappa}(i_1, \cdots, i_m)$ *denotes the* $\kappa$*-th deleted sequence* $(i_1, \cdots, i_{\kappa-1}, i_{\kappa+1}, \cdots, i_m)$. This is just *a generalization of classical Pochhammer type* ([14]).

## § 3. (Example 2) Configurations of a quadric and hyperplanes in $C^n$

Let $X=C^n$,

$$f_0=-\frac{1}{2}\sum_{j=1}^{n}x_j^2 \tag{3.1}$$

and $f_1, f_2, \cdots, f_m$ be linear functions:

$$f_j=\sum_{\nu=0}^{n}u_{j\nu}x_{\nu}+u_{j0}. \tag{3.2}$$

*Let G be* $SO_n(C)$ *which leaves invariant* $f_0$. Let $A$ be a symmetric matrix consisting of *basic algebraic invariants* $a_{ij}$, where we denote

$$\begin{cases} a_{ij}=(f_i, f_j)=\sum_{\nu=1}^{n}u_{i\nu}u_{j\nu} \\ a_{0i}=a_{i0}=u_{i0} \end{cases}. \tag{3.3}$$

We normalize $u_{i\nu}$ such that

$$a_{ii}=\sum_{\nu=1}^{n}u_{i\nu}^2=1. \tag{3.4}$$

We abbreviate by $A\begin{pmatrix} I \\ J \end{pmatrix}$ and $A(I)$ the determinants

$$\begin{vmatrix} a_{i_1j_1} & \cdots & a_{i_1j_p} \\ \vdots & & \vdots \\ a_{i_pj_1} & \cdots & a_{i_pj_p} \end{vmatrix} \tag{3.5}$$

and $A\begin{pmatrix} I \\ I \end{pmatrix}$ respectively. We denotes by $T$ *the determinantal variety defined by*

$$T=\{A \mid A(I)=0 \quad \text{for } |I|>m\} \tag{3.6}$$

and

(3.7) $$T_0=\{A \in T \mid A(I)=0 \text{ for some } I, |I|<m\}.$$

(See [15].)

Then the integral

(3.8) $$F(t)=\int \exp [f_0(x)] f_1^{\lambda_1}(x)\cdots f_m^{\lambda_m}(x) dx_1\wedge\cdots\wedge dx_n$$

satisfies the Gauß-Manin connection on $T-T_0$, which is expressed by means of basic invariants $a_{ij}$:

For $(i_1<\cdots<i_p)\subset\{1, 2, \cdots, m\}$, $0\leq p\leq n$ we write

(3.9) $$\varphi(I)=\frac{d\tau}{f_{i_1}\cdots f_{i_p}}$$

for $d\tau=dx_1\wedge\cdots\wedge dx_n$. Then $\varphi(I)$ give a basis of $\sum_{\nu=0}^m \binom{m}{\nu}$ linearly independent forms in $\boldsymbol{H}^n(V_t, \nabla_{\omega_t})$. We firstly give a basic formula:

(3.10) $$d\tilde{\varphi}(\phi)=\sum_{j=1}^m da_{j0}\lambda_j\tilde{\varphi}(j)+\frac{1}{2}\sum_{1\leq j\neq k\leq m} da_{jk}\lambda_j\lambda_k\tilde{\varphi}(jk).$$

More generally by using (3.10) we have for $I=(i_1, \cdots, i_p)$,

(3.11) $$\begin{aligned} A(I)d\tilde{\varphi}(I)=&\frac{1}{2}\sum_{\substack{j\neq k\\(j,k)\in I^c}}\theta\binom{I}{I, j, k}\lambda_j\lambda_k\tilde{\varphi}(I, j, k)+\theta(I)\tilde{\varphi}(I)\\ &+\frac{1}{2}\sum_{\substack{\mu,\nu\leq p\\ \mu\neq\nu}}\theta\binom{I}{\partial_\mu\partial_\nu I}\tilde{\varphi}(\partial_\mu\partial_\nu I)+\sum_{\substack{k\notin I\\1\leq\nu\leq p}}\lambda_k\theta\binom{I}{k, \partial_\nu I}\tilde{\varphi}(k, \partial_\nu I)\\ &+\sum_{1\leq\nu\leq p}\theta\binom{I}{\partial_\nu I}\tilde{\varphi}(\partial_\nu I)+\sum_{k\notin I}\lambda_k\theta\binom{I}{k, I}\tilde{\varphi}(k, I).\end{aligned}$$

$\theta\binom{I}{I, j, k}$, $\theta(I)$ and $\theta\binom{I}{\partial_\mu\partial_\nu I}$ denote rational 1-forms defined in $T$ which *depend only on* $f_{i_1}, \cdots, f_{i_p}$ *and don't depend on n*. See [16] for the precise expressions.

As for domains of integration, *we assume that* $f_j$ *are all real and in general position. Then a basis of* $H_n(V, \mathcal{S}^*)$ *can be chosen to be just connected components of* $\boldsymbol{R}^n-\bigcup_{j=1}^m (f_j=0)$ *provided* $\lambda_1, \cdots, \lambda_m$ *all greater than* $-1$.

## § 4. (Example 3) A degenerate case of Section 3

Let $f_0$ be the quadratic form $\sum_{i=1}^n x_i y_i$. Consider the integral

$$(4.1)\qquad F(t)=\int \exp(-f_0(x, y))f_1^{\lambda_1}(x)\cdots f_s^{\lambda_s}(x)f_{s+1}^{\lambda_{s+1}}(y)\cdots f_{s+t}^{\lambda_{s+t}}(y)dx_1\wedge\cdots\wedge dx_n\wedge dy_1\wedge\cdots\wedge dy_n$$

for $f_j(x)=\sum_{\nu=1}^n u_{j\nu}x_\nu$ and $f_{s+k}(y)=\sum_{\nu=1}^n u_{s+k,\nu}y_\nu$.

Then $X=C^{2n}$ and $T$ becomes the affine space $C^{st}$ consisting of matrices of $(s+t)$ order:

$$(4.2)\qquad \mathscr{A}=\begin{pmatrix} 0 & A \\ {}^tA & 0\end{pmatrix}=\left(\begin{pmatrix} 0 & ((a_{ij'})) \\ ((a_{i'j})) & 0\end{pmatrix}\right)$$

where $a_{j'i}=a_{ij'}=\sum_{\nu=1}^n u_{i\nu}u_{j'\nu}$.

*$H^{2n}(V_t, \nabla_{\omega t})$ is spanned by linearly independent forms $\varphi(I, J')=A\binom{I}{J'}\varphi_*(I, J')$, where $\varphi_*(I, J')$ denotes*

$$(4.3)\qquad \varphi_*(I, J')=\frac{dx_1\wedge\cdots\wedge dx_n\wedge dy_1\wedge\cdots\wedge dy_n}{f_{i_1}(x)\cdots f_{i_p}(x)f_{j'_1}(y)\cdots f_{j'_p}(y)}$$

*for $I=(i_1<\cdots<i_p)\subset\{1, 2, \cdots, s\}$, $J'=(j'_1<\cdots<j'_p)\subset\{s+1, \cdots, s+t\}$. This number is equal to $\sum_{p=0}^{n-1}\binom{s-1}{p}\binom{t-1}{p}$. The Gauß-Manin system for $\tilde\varphi(I, J')$ is described as follows:*

$$\begin{aligned} d\tilde\varphi(I, J') = & \sum_{i\notin I, j'\notin J'} \lambda_i\lambda_{j'} d\log A\binom{i, I}{j', J'}\tilde\varphi(i, I; j', J') \\ & +\sum_{\mu,\nu}(-1)^{\mu+\nu-1} d\log A\binom{\partial_\mu I}{\partial_\nu J'}\tilde\varphi(\partial_\mu I; \partial_\nu J') \\ & +\sum_{k\notin I}\sum_{\nu=1}^{|I|}(-1)^{\nu-1}\lambda_k d\log A\binom{k, \partial_\nu I}{J'}\tilde\varphi(k, \partial_\nu I; J') \\ & +\sum_{k'\notin J'}\sum_{\nu=1}^{|J'|}(-1)^{\nu-1}\lambda_{k'} d\log A\binom{I}{k', \partial_\nu J'}\tilde\varphi(I; k', \partial_\nu J') \\ & +\sum_{\nu=1}^{|I|}\lambda_{i_\nu} d\log A\binom{I}{J'}\tilde\varphi(I; J')] \\ & +\sum_{\nu=1}^{|J'|}\lambda_{j'_\nu} d\log A\binom{I}{J'}\tilde\varphi(I; J') \end{aligned}$$

*for* $|I|=|J'|\leq n-1$. (See [16] p. 280.)

## § 5. Applications

*Hypergeometric functions of Mellin-Sato type* are defined by Mellin transforms of products of $\Gamma$-functions

$$\Gamma\Big(\sum_{j=0}^{m} \alpha_j s_j + \alpha_0\Big) \tag{5.1}$$

where $\alpha_j \in \boldsymbol{Q}$ ([17]). They are described by means of convolution integrals of elementary functions ([18]). We shall show that some of these are expressed by means of integrals discussed in the preceding sections.

1) (Pochhammer [19]) *Goursat hyper-geometric functions* are defined by the following series.

$$F\begin{pmatrix}\alpha_1, \cdots, \alpha_m \\ \beta_1, \cdots, \beta_m\end{pmatrix}\!\!\Big|\, z\Big) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\cdots\Gamma(\alpha_m+n)}{\Gamma(\beta_1+n)\cdots\Gamma(\beta_m+n)} z^n. \tag{5.2}$$

By substitution of the well-known formula

$$\frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)} = \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 x^{\alpha+n-1}(1-x)^{\beta-\alpha-1} dx \tag{5.3}$$

we have

$$\begin{aligned} F\begin{pmatrix}\alpha_1, \cdots, \alpha_m \\ \beta_1, \cdots, \beta_m\end{pmatrix}\!\!\Big|\, z\Big) &= \sum_{n=0}^{\infty} \frac{1}{\prod_{j=1}^{m} \Gamma(\beta_j-\alpha_j)} \int_0^1 x_1^{\alpha_1+n-1}(1-x)^{\beta_1-\alpha_1-1} \\ &\quad \cdots x_m^{\alpha_m+n-1}(1-x_m)^{\beta_m-\alpha_m-1} dx_1 \wedge \cdots \wedge dx_m z^n \\ &= \frac{1}{\prod_{j=1}^{m} \Gamma(\beta_j-\alpha_j)} \int_0^1 \cdots \int_0^1 \prod_{j=1}^{m} (1-x_j)^{\beta_j-\alpha_j-1} \\ &\quad \cdot x_j^{\alpha_j-1} \cdot (1-z x_1 \cdots x_m)^{-1} dx_1 \cdots dx_m. \end{aligned} \tag{5.4}$$

By change of variables of integration,

$$\begin{cases} y_1 = x_1 \cdots x_m \\ y_2 = x_2 \cdots x_m \\ \vdots \\ y_m = x_m \end{cases} \tag{5.5}$$

we have

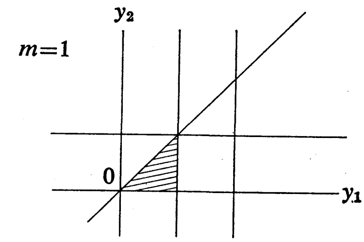
$$\begin{aligned} &F\begin{pmatrix}\alpha_1 \cdots \alpha_m \\ \beta_1 \cdots \beta_m\end{pmatrix}\!\!\Big|\, z\Big) \\ &= \frac{1}{\prod_{j=1}^{m} \Gamma(\beta_j-\alpha_j)} \int \cdots \int_{0<y_m<\cdots<y_1<1} (1-y_1)^{\beta_1-\alpha_1-1} \cdot (y_1-y_2)^{\beta_2-\alpha_2-} \end{aligned} \tag{5.6}$$

$$\cdots(y_{m-2}-y_{m-1})^{\beta_{m-1}-\alpha_{m-1}-1}(y_{m-1}-y_m)^{\beta_m-\alpha_m-1}\cdot y_1^{\alpha_1-\beta_2}$$
$$\cdots y_{m-1}^{\alpha_{m-1}-\beta_m}y_m^{\alpha_m-1}\frac{dy_1\cdots dy_{m-1}dy_m}{1-zy_m}.$$

This is a degenerate case of (2.2), where we put $X=\boldsymbol{C}^m$, $T=\boldsymbol{C}$ an

$$W=(x_1=0)\cup\cdots\cup(x_m=0)\cup(1-x_1=0)\cup(x_1-x_2=0)\cup\cdots \cup(x_{m-1}-x_m=0)\cup(1-zx_m=0) \tag{5.7}$$

for $z\in\boldsymbol{C}$ and $(x_1,\cdots,x_m)\in\boldsymbol{C}^m$.




*A basis of linearly independent integrands in* $\boldsymbol{H}^m(V_t, V_{\omega_t})$ can be chosen among logarithmic forms, for example, as follows

$$(5.7)\quad \begin{cases} d\log x_1\wedge d\log x_2\wedge\cdots\wedge d\log x_m, \\ d\log(1-x_1)\wedge d\log x_2\wedge\cdots\wedge d\log x_m, \\ d\log(1-x_1)\wedge d\log(x_1-x_2)\wedge d\log x_3\wedge\cdots\wedge d\log x_m, \\ \cdots\cdots\cdots \\ d\log(1-x_1)\wedge d\log(x_1-x_2)\wedge\cdots\wedge d\log(x_{m-1}-x_m) \end{cases}$$

so that $F\left(\begin{matrix}\alpha_1\cdots\alpha_m\\ \beta_1\cdots\beta_m\end{matrix}\middle| z\right)$ satisfies *the Gauβ-Manin connection of order* $(m+1)$.

When $\beta_m=1$, then the integral is reduced to the $(m-1)$-dimensional integral

$$F\left(\begin{matrix}\alpha_1 & \cdots\cdots & \alpha_m\\ \beta_1\cdots\beta_{m-1} & & 1\end{matrix}\middle| z\right)$$
$$=\frac{1}{\prod_{j=1}^m\Gamma(\beta_j-\alpha_j)}\int\cdots\int_{0<y_m<\cdots<y_1<1}(1-y_1)^{\beta_1-\alpha_1-1}(y_1-y_2)^{\beta_2-\alpha_2-1} \tag{5.8}$$
$$\cdots(y_{m-2}-y_{m-1})^{\beta_{m-1}-\alpha_{m-1}-1}\cdot y_1^{\alpha_1-\beta_2}\cdots y_{m-2}^{\alpha_{m-2}-\beta_{m-1}}$$
$$\cdot y_{m-1}^{\alpha_{m-1}-1}\cdot(zy_{m-1}-1)^{-\alpha_m}dy_1\wedge dy_2\wedge\cdots\wedge dy_{m-1}$$

spanned by

$$(5.9)\quad \begin{cases} d\log y_1\wedge\cdots\wedge d\log y_{m-1}, \\ d\log(1-y_1)\wedge d\log y_2\wedge\cdots\wedge d\log y_{m-1}, \\ \cdots\cdots\cdots \\ d\log(1-y_1)\wedge d\log(y_1-y_2)\wedge\cdots\wedge d\log(y_{m-2}-y_{m-1}) \end{cases}$$

so that $F$ satisfies *the Gauß-Manin connection of order m.* Exact expressions are both rather complicated although these are elementarily computable. See [20] and [21] for other kinds of hypergeometric functions.

2) *Lauricella hypergeometric functions* are defined to be the following series ([17])

$$(5.10)\quad \sum_{\nu_1\geq 0,\cdots,\nu_n\geq 0} \frac{\Gamma(\lambda_0+\nu_1+\cdots+\nu_m)\prod_{j=1}^m\{\Gamma(\lambda_j+\nu_j)\Gamma(\lambda'_j)\}}{\prod_{j=1}^m \Gamma(\lambda_j+\lambda'_j+\nu_j)\cdot\Gamma(\lambda_0)} \cdot x_1^{\nu_1}x_2^{\nu_2}\cdots x_n^{\nu_n}$$

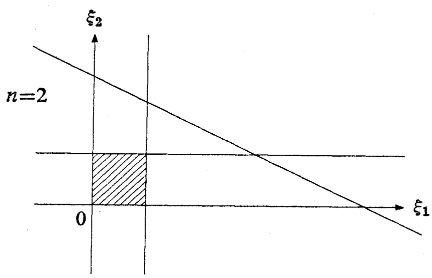
which can be expressed by the integrals

$$(5.11)\quad \int_0^1\cdots\int_0^1\left(1-\sum_{j=1}^n x_j\xi_j\right)^{\lambda_0}\prod_{j=1}^n \xi_j^{\lambda_j-1}(1-\xi_j)^{\lambda'_j-1}d\xi_1\wedge\cdots\wedge d\xi_n.$$

for $X=\boldsymbol{C}^n$ and $T=\boldsymbol{C}^n$,

where $W$ is defined by

$$\left(1-\sum_{j=1}^n x_j\xi_j=0\right)\cup\bigcup_{n\geq j\geq 1}(\xi_j=0)\cup\bigcup_{n\geq j\geq 1}(1-\xi_j=0)$$




A basis of the cohmology $H^n(V_t, \mathcal{V}_{\omega_t})$ can be chosen by means of spin variables as follows:

$$(5.12)\quad \varphi(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=\frac{d\xi_1}{\xi_1-\varepsilon_1}\wedge\frac{d\xi_2}{\xi_2-\varepsilon_2}\wedge\cdots\wedge\frac{d\xi_n}{\xi_n-\varepsilon_n}$$

where $\varepsilon_\nu$ denotes 0 or 1. Therefore $H^n(V_t, \mathcal{S})$ *is identified with the n-th* tensor product $\underbrace{\boldsymbol{C}^2\otimes\cdots\otimes\boldsymbol{C}^2}_{n}$. Then the Gauß-Manin connection in $T-T_0$ is described as follows:

(5.13) $$d\tilde{\varphi}(\varepsilon_1, \cdots, \varepsilon_n)=\sum_{\varepsilon'\in Z_2^n} \tilde{\varphi}(\varepsilon_1', \cdots, \varepsilon_n')\theta\begin{pmatrix}\varepsilon_1'\cdots\varepsilon_n'\\ \varepsilon_1\cdots\varepsilon_n\end{pmatrix}$$

*where* $\Theta=\left(\left(\theta\begin{pmatrix}\varepsilon_1'\cdots\varepsilon_n'\\ \varepsilon_1\cdots\varepsilon_n\end{pmatrix}\right)\right)$ *denotes the* $2^n$*-order matrix valued logarithmic* 1*-form*

(5.14) $$\Theta=\sum_{\nu=1}^{n} A_\nu\cdot d\log x_\nu+\sum A_0\cdot\hat{\Theta}$$

*through the formulae*

(5.15) $$\begin{aligned}A_0&=\lambda_0 \boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2+\sum_{\sigma=1}^{n} \underbrace{\boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2}_{\sigma-1}\otimes\begin{pmatrix}\lambda_\sigma-1, & \lambda_\sigma-1\\ \lambda_\sigma'-1, & \lambda_\sigma'-1\end{pmatrix}\otimes\underbrace{\boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2}_{n-\sigma}\\ &=\lambda_0 \boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2-\sum_{\sigma=1}^{n} A_\sigma\end{aligned}$$

(5.16) $$A_\sigma=-\underbrace{\boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2}_{\sigma-1}\otimes\begin{pmatrix}\lambda_\sigma-1, & \lambda_\sigma'-1\\ \lambda_\sigma-1, & \lambda_\sigma'-1\end{pmatrix}\otimes\underbrace{\boldsymbol{l}_2\otimes\cdots\otimes \boldsymbol{l}_2}_{n-\sigma}$$

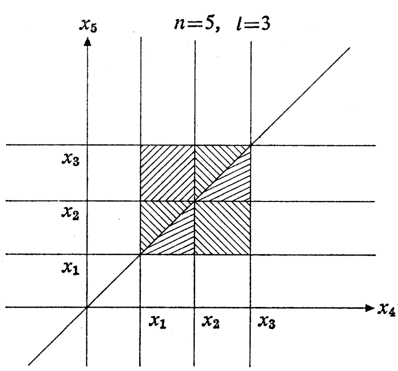
*and the diagonal* 1*-form* $\hat{\Theta}=((\hat{\theta}(\varepsilon_1, \cdots, \varepsilon_n)), \varepsilon_\nu=0$ *or* 1 *with*

$$\hat{\theta}(\varepsilon_1, \cdots, \varepsilon_n)=d\log(1-x_1\varepsilon_1-\cdots-x_n\varepsilon_n).$$

Here $\boldsymbol{l}_2$ dnotes the identity matrix of 2nd order.

3) *Correlation functions for random matrices.*
Consider the integral

(5.17) $$F(x_1, \cdots, x_l\,|\,\beta)=\int\prod_{1\le i<j\le n}(x_i-x_j)^\beta dx_{l+1}\wedge\cdots\wedge dx_n, \quad l\ge 2$$



where $X=\boldsymbol{C}^{n-l}$ and $T=\boldsymbol{C}^{l}$. $W$ is defined by the equations

$$x_i-x_j=0 \quad \text{for } 1\leq i\leq n, \quad l+1\leq j\leq n. \tag{5.18}$$

We denote by

$$\varphi(i_1, i_2, \cdots, i_{n-l})=d\log(x_{i_1}-x_{l+1})\wedge\cdots\wedge d\log(x_{i_{n-l}}-x_n) \tag{5.19}$$

for $i_1\leq l$, $i_2\leq l+1, \cdots, i_{n-l}\leq n-1$. These are not linearly independent in $\boldsymbol{H}^{n-l}(V_t, \boldsymbol{V}_{\omega_t})$. Actually *we can choose a basis as follows*:

$$\varphi(i_1, \cdots, i_{n-l})=d\log(x_{i_1}-x_{l+1})\wedge\cdots\wedge d\log(x_{i_{n-l}}-x_n) \tag{5.20}$$

*for* $i_1<l, \cdots, i_{n-l}<n-1$, *so that* $H^{n-l}$ *has dimension* $(l-1)\cdots(n-2)$.

**Lemma.** *Let* $Y(x)$ *a matrix valued function of* $x$ *satisfying*

$$Y^{-1}\frac{dY}{dx}=\sum_{i=1}^{N}\frac{U_i}{x-\alpha_i}, \tag{5.21}$$

*where* $U_i$ *denotes constant matrices of order* $n$ *satisfying the equations of Schlesinger-Lappo-Danilevski* (See [22].)

$$\sum_{j\neq i}[U_i, U_j]d\log(\alpha_i-\alpha_j)=0 \tag{5.22}$$

*for each* $i$. *Let* $e_\mu$ $(1\leq\mu\leq n)$ *be the* $\mu$*-th unit column* $n$*-vector. We put*

$$\tilde{y}_{i,\mu}=\int Ye_\mu d\log(x-\alpha_i). \tag{5.23}$$

*Then the line vector* $\tilde{y}_i=(\tilde{y}_{i,1}, \cdots, \tilde{y}_{i,n})$ *satisfies a linear differential equation of Pochhammer type*:

$$d\tilde{y}_{i,\mu}=\sum_{j\neq i}d\log(\alpha_j-\alpha_i)(\tilde{y}_{i,\mu}-\tilde{y}_{j,\mu})U_j \tag{5.24}$$

(See [23]).

By repeated application of this Lemma, *we can prove the following*

$$\begin{aligned} d\tilde{\varphi}(i_1, i_2, \cdots, i_{n-l}) = & \sum_{1\leq\sigma,\tau\leq p,\, j_1\leq l\cdots j_{n-l}\leq n-1}\tilde{\varphi}(j_1, j_2, \cdots, j_{n-l}). \\ & \cdot U_{\sigma\tau}^{(l)}\begin{pmatrix} j_1\cdots j_{n-l} \\ i_1\cdots i_{n-l}\end{pmatrix} d\log(\alpha_\sigma-\alpha_\tau)\end{aligned} \tag{5.25}$$

*for* $i_1\leq l, \cdots, i_{n-l}\leq n-1$. $U_{\sigma\tau}^{(l)}$ are determined recursively by

$$(5.26) \quad \begin{aligned} U_{\sigma\tau}^{(p)} &= (e_{\sigma,\sigma}^{(p)} - e_{\tau,\sigma}^{(p)}) \otimes (U_{\tau,p+1}^{(p+1)} + \beta \cdot \boldsymbol{l}_{N_{p+1}}) \\ &\quad + (-e_{\sigma,\tau}^{(p)} + e_{\tau,\tau}^{(p)}) \otimes (U_{\sigma,p+1}^{(p+1)} + \beta \cdot \boldsymbol{l}_{N_{p+1}}) + \boldsymbol{l}_p \otimes U_{\sigma\tau}^{(p+1)} \end{aligned}$$

for $1 \leq \sigma$, $\tau \leq p$, $l \leq p \leq n$, $N_p = p(p+1)\cdots(n-1)$, where we put $U_{\sigma,\tau}^{(n)} = \beta$ and $e_{\sigma\tau}^{(p)}$ denotes the unit matrix of order $p$ of $(\sigma, \tau)$ non-zero component. The symmetric group $\mathfrak{S}_{n-l}$ acts faithfully on $\boldsymbol{H}^{n-l}(V_t, \mathscr{S})$. (5.25) is not invariant by this action. *It seems interesting to compute the Gauß-Manin system for its invariant part* $[H^{n-l}(V_t, \mathscr{S})]^{\mathfrak{S}_{n-l}}$, for it is related to the correlation functions for random matrices ([24]~[25])

$$(5.27) \quad \int \prod_{\substack{x_1 \leq x_j \leq x_2 \\ l+1 \leq j \leq n}} |x_i - x_j|^\beta dx_{l+1} \cdots dx_n, \qquad l \geq 2.$$

*When* $l=2$, $\dim H^n(V_t, \mathscr{S})$ *is equal to* $(n-2)!$, *so that* $\dim [H^n(V_t, \mathscr{S})]^{\mathfrak{S}_n}$ is just equal to 1. Therefore from a result in [11], (5.27) is reduced to a finite product of $\Gamma$-factors. The exact expression has been known since [26]. It seems to be interesting to compute linear difference equations or Gauß-Manin connections of (5.27) of the variable $\beta$ or $(x_1, \cdots, x_l)$ in invariant expression with respect to the action of $\mathfrak{S}_{n-l}$.

4) *Correlation functions for random matrices* ([24]).

$$(5.28) \quad \int \exp\left[-\frac{1}{2}\sum_{j=1}^{n} x_j^2\right] \prod_{1 \leq i < j \leq n} (x_i - x_j)^\beta dx_{l+1} \wedge \cdots \wedge dx_n$$

as a function of $x_1, \cdots, x_l$ for $0 \leq l < n$. This is a degenerate case of the integral (3.8).

$X$, $T$ and $W$ are defined as in the preceding case. *A basis of* $\boldsymbol{H}^{n-l}(V_t, \boldsymbol{V}_{\omega_t})$ *can be chosen as follows: for* $p \geqq 0$,

$$(5.29) \quad \varphi((i_1 j_1) \cdots (i_p j_p)) = \frac{dx_{l+1} \wedge \cdots \wedge dx_n}{(x_{i_1} - x_{j_1}) \cdots (x_{i_p} - x_{j_p})}$$

*where* $i_1 < j_1, \cdots, i_p < j_p$ *and* $l+1 \leq j_1 < \cdots < j_p \leq n$. *Its dimension is equal to* $(l+1)\cdots n$. In particular, *when* $l=0$, *it is equal to* $n!$. *The invariant part* $\boldsymbol{H}^n(V_t, \boldsymbol{V}_{\omega_t})^{\mathfrak{S}_n}$ *is just* 1-*dimensional*. It is known that the corresponding integral is equal to a product of $\Gamma$-factors ([25]).

5) *Correlation functions for* 2-*dimensional vortex system in statistical mechanics* ([27], [28]).

$$\int \exp\left[-\frac{1}{2}\sum z_j \bar{z}_j\right] \sum_{1 \leq i < j \leq n} |z_i - z_j|^\beta d\bar{z}_{l+1} dz_{l+1} \cdots d\bar{z}_n dz_n.$$

Here we have $X = \boldsymbol{C}^{2(n-l)}$, $T = \boldsymbol{C}^{2l}$ and

$$W=\bigcup_{i<j}(z_i-z_j=0)\cup\bigcup_{i<j}(\bar{z}_i-\bar{z}_j=0).$$

*The cohomology* $\boldsymbol{H}^n(X-W_t, \nabla_{\omega_t})$ *has a basis of the forms*

$$\varphi((i_1j_1)\cdots(i_pj_p)\,|\,(i_1'j_1')\cdots(i_p'j_p'))$$
$$=\frac{dz_{l+1}\wedge d\bar{z}_{l+1}\wedge\cdots\wedge dz_n\wedge d\bar{z}_n}{(z_{i_1}-z_{j_1})\cdots(z_{i_p}-z_{j_p})(\bar{z}_{i_1'}-\bar{z}_{j_1'})\cdots(\bar{z}_{i_p'}-\bar{z}_{j_p'})},\quad 0\leq p\leq n$$

*for* $i_1<j_1,\ \cdots,\ i_p<j_p,\ i_1'<j_1',\ \cdots,\ i_p'<j_p',\ l+1\leq j_1<\cdots<j_p\leq n,\ l+1\leq j_1'<\cdots<j_p'\leq n$, *so that its dimension is equal to* $\{(l+1)\cdots n\}^2$. *The semi-direct product of the group* $\mathfrak{S}_{n-l}$ *and* $\boldsymbol{Z}_2^{n-l}$ *faithfully acts on* $\boldsymbol{H}^{n-l}(V_t,\nabla_{\omega_t})$. In particular when $l=0$, we have

$$\dim \boldsymbol{H}^n(V_t, \nabla_{\omega_t})=(n!)^2>n!\,2^n.$$

Namely $\dim \boldsymbol{H}^n(V_t, \nabla_{\omega_t})^{\mathfrak{S}_{n-l}\cdot \boldsymbol{Z}_2^{n-l}}>1$. This fact strongly suggests that the partition function of the 2-dimensional vortex system

$$\int_{\boldsymbol{C}^n}\exp\left(-\sum_{j=1}^n z_j\bar{z}_j\right)\prod_{i<j}|z_i-z_j|^\beta d\bar{z}_1\wedge dz_1\wedge\cdots\wedge d\bar{z}_n\wedge dz_n$$

can not be described by any product of $\Gamma$-factors as a function of $\beta$, although *this satisfies linear difference equations over rational functions of* $\beta$ (see [11]). From the view point of statistical mechanics it seems very interesting problem to compute the Gauß-Manin system of infinite order for correlation functions when $n$ tends to the infinity.

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