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Maximal Functions on Non-Compact, Rank One Symmetric Spaces

Radial Maximal Functions and Atoms

Takeshi Kawazoe

§ 1. Introduction

As in the case of a Euclidean space, the theories of maximal functions, Hardy spaces, atoms, etc. are rapidly expanding in the case of homogeneous groups, that is, connected and simply connected nilpotent Lie groups whose Lie algebras are endowed with a family of dilations (see G. B. Folland and E. M. Stein [5]) and more generally, spaces of homogeneous type (see R. R. Coifman and G. Weiss [3, § 2]). In this paper we shall attempt to develop these theories on a rank one, irreducible Riemannian symmetric space of non-compact type (see S. Helgason [8, Chs. V and VI]). Of course such a space is not of homogeneous type.

Let G be a connected, real rank one semisimple Lie group with finite center and $G=KAN$ an Iwasawa decomposition for G. Put $X=G/K$. Then X has a G-invariant measure $d\mu$ (resp. a metric) induced by the G-invariant measure dg on G (resp. the Killing form of the Lie algebra of G). For each locally integrable function f on X, the Hardy-Littlewood maximal function $M_{HL}f$ is defined by

$$M_{HL}f(x)=\sup_{0<\varepsilon<\infty}|B(x,\varepsilon)|^{-1}\int_{B(x,\varepsilon)}|f(g)|d\mu(g)\quad(x\in X),\tag{1.1}$$

where $B(x,\varepsilon)$ is the open ball on X around x with radius ε and $|B|$ is the volume of the ball. Then J. L. Clerc and E. M. Stein [1] and Jan-Olov Strömberg [12] showed that the operator M_{HL} is of type (L^p, L^p) for $p>1$ and of weak type (L^1, L^1) respectively. Now we shall define a radial maximal function as an extension of $M_{HL}f$ as follows. Since A is one dimensional, we can parametrize elements of A as a_t $(t\in \boldsymbol{R})$ and express the Cartan decomposition of x in G as $x=k_1a_{t(x)}k_2$ $(k_1, k_2\in K$ and $t(x)\geqq 0)$. Put $\Delta(t)=(\operatorname{sh} t)^{m_1}(\operatorname{sh} 2t)^{m_2}$, where m_1 and m_2 are the multiplicities of the roots α and 2α of (G, A) respectively. Then for f in $C_c^\infty(G)$,

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$$(1.2)\qquad \int_G f(g)dg = c\int_K\int_0^\infty\int_K f(k_1a_tk_2)\Delta(t)dk_1dtdk_2,$$

where dk_i $(i=1, 2)$ (resp. dt) is a Haar measure on K (resp. a Euclidean measure on $\boldsymbol{R}$) (see [8, Ch. X, § 1.4]). Let ϕ be a K-biinvariant function on G and put

$$(1.3)\qquad \phi_\varepsilon(x) = \frac{1}{\varepsilon}\Delta\left(\frac{t(x)}{\varepsilon}\right)\Delta(t(x))^{-1}\phi(a_{t(x)/\varepsilon}) \qquad (\varepsilon>0).$$

Then for each right K-invariant function f on G which is identified with a function on X, the radial maximal function $M_\phi f$ is defined by

$$(1.4)\qquad M_\phi f(g) = \sup_{0<\varepsilon<\infty} |f*\phi_\varepsilon(g)| \qquad (g\in G),$$

where $*$ is the convolution on G. Obviously if we replace ϕ_ε by $|B(e,\varepsilon)|^{-1}\chi_{B(e,\varepsilon)}$, where χ_B is the characteristic function of B on G, M_ϕ coincides with M_{HL}. In Section 3 we shall show that the operator M_ϕ is of type (L^p, L^p) for $p>1$ and of weak type (L^1, L^1) under the assumption that ϕ satisfies the estimate: $|\phi(x)|\leqq ce^{-(m_1+2m_2t(x))/\delta}(x\in G)$ for $0<\delta<1$.

Next let $G\neq SL(2, R)$. We fix a p in $0<p\leqq 1$ and suppose that ϕ satisfies the above estimate and moreover a condition of regularity. Here we define a radial maximal operator M'_ϕ by taking the supremum of (1.4) over the finite interval: $0<\varepsilon<(1-1/\delta)(1-1/p)^{-1}$ instead of $0<\varepsilon<\infty$. Then in Section 4 we shall obtain a family of compactly supported functions f on G with finite L^q $(1<q\leqq\infty)$ norm such that $\|M'_\phi f\|_p<C$, where C does not depend on f and ϕ, and introduce an analogous concept of atoms on homogeneous groups (see [5, Chs. 2 and 3]). To obtain this we shall use some results about Jacobi functions on $\boldsymbol{R}$ (see M. Flensted-Jensen and T. H. Koornwinder [4]) and in particular, an explicit expression of a kernel for the integral of the convolution $*$ on G obtained by G. Gasper [6] and T. H. Koornwinder [9]. In Section 5 we restrict our attention to K-biinvariant functions on G. Then we shall give a sufficient condition by which a K-biinvariant function can be written as a sum of K-biinvariant atoms.

I would like to express my appreciation to H. Miyazaki for many fruitful conversations.

§ 2. Notations

Let G be a connected, real rank one semisimple Lie group with finite center and $G=KAN$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$) an Iwasawa decomposition for G (resp. the Lie algebra $\mathfrak{g}$ of G). We put $X=G/K$ and identify functions on X with right K-invariant ones on G. Let α be a reduced simple root of $(\mathfrak{g}, \mathfrak{a})$ and H_0 an element of A such that $\alpha(H_0)=1$. We parametrize elements of A by $a_t=\exp(tH_0)$ $(t \in \boldsymbol{R})$ and identify λ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ with $\lambda(H_0)$ in $\boldsymbol{R}$. Let m_1 and m_2 denote the multiplicities of α and 2α respectively. Then we can express the Cartan decomposition of x in G and the Jacobian $\Delta(x)$ of the integral formula (1.2) as in Section 1. If f is a K-biinvariant function on G, we abbreviate $f(x)=f(a_{t(x)})$ $(x=k_1a_{t(x)}k_2, k_1, k_2 \in K$ and $t(x)\geqq 0)$ as $f(t(x))$. Let $B(x, r)$ denote the open ball on G around x with radius r, that is, $B(x, r)=\{g \in G;\ t(x^{-1}g)<r\}$. If x is the unit element e of G, we put $B(r)=B(e, r)$. For any subset S of G we denote the volume of S by $|S|$. Then $|B(x, r)|$ is given by $|B(r)|=c\int_0^r \Delta(t)\,dt$ (see (1.2)) and the following properties are easily obtained.

Lemma 2.1.
(1) *For each* $a\geqq 1$ *and* $r_0>0$ *there exists a constant* A^-_{a,r_0} *such that* $|B(ar)|<A^-_{a,r_0}|B(r)|$ $(r\leqq r_0)$.
(2) *For each* $a\geqq 0$ *and* $r_0>0$ *there exists a constant* A^+_{a,r_0} *such that* $|B(r+a)|<A^+_{a,r_0}|B(r)|$ $(r\geqq r_0)$.
(3) $|B(r)|=\begin{cases} O(r^{m_1+m_2+1}) & (r\to 0) \\ O(e^{(m_1+2m_2)r}) & (r\to\infty). \end{cases}$

Let $L^p(G)$ $(0<p\leqq\infty)$ (resp. $L_c(G)$) denote the space of all functions f on G such that $\|f\|_p^p=\int_G |f(g)|^p dg<\infty$ (resp. with compact support on G) and put $L_c^p(G)=L^p(G)\cap L_c(G)$. When we restrict our attension to right (resp. bi) K-invariant functions on G, we use the symbol G/K (resp. $G//K$) instead of G.

We use the letter c to denote a constant which need not be the same in different occurences.

§ 3. Radial maximal functions

Let $A_{\delta,\lambda}=A_{\delta,\lambda}(G//K)$ $(0<\delta<1$ and $\lambda \in \boldsymbol{R})$ denote the space of all K-biinvariant measurable functions ϕ on G which satisfy the estimate $|\phi(x)| \leqq e^{-2\rho t(x)/\delta}(1+t(x))^{-\lambda}$ $(x \in G)$, where $\rho=(m_1+2m_2)/2$. Then for each $\phi \in A_{\delta,\lambda}$, the *radial* maximal function $M_\phi f$ for $f \in L^p(G/K)$ $(1\leqq p\leqq\infty)$ is defined by

$$M_\phi f(x)=\sup_{0<\varepsilon<\infty} |f*\phi_\varepsilon(x)| \qquad (x \in G), \tag{3.1}$$

where $\phi_\varepsilon(x)=\varepsilon^{-1}\Delta(t(x))^{-1}\Delta(\varepsilon^{-1}t(x))\phi(\varepsilon^{-1}t(x))$ $(x \in G)$ and $*$ is the convolution on G.

Lemma 3.1. *There exists a constant $C^1_{\delta,\lambda}$ such that $\|\phi_\varepsilon\|_1=\|\phi\|_1\leqq C^1_{\delta,\lambda}$ for all $\phi\in A_{\delta,\lambda}$.*

Proof.
$$\begin{aligned}\|\phi_\varepsilon\|_1&=\int_0^\infty|\phi_\varepsilon(t)|\,\Delta(t)dt\\&=\int_0^\infty|\phi(t)|\,\Delta(t)dt\qquad(=\|\phi\|_1)\\&\leqq c\int_0^\infty(1+t)^{-\lambda}e^{2(1-1/\delta)\rho t}dt=C^1_{\delta,\lambda}<\infty.\end{aligned}$$
Q.E.D.

Lemma 3.2.
(1) *For all $\phi\in A_{\delta,\lambda}$ and $\varepsilon>0$, $\lambda\geqq 0$*
$$|\phi_\varepsilon(t)|\leqq c\begin{cases}\varepsilon^{-(m_1+m_2+1)}t^{m_1+m_2}(1+t/\varepsilon)^{-\lambda}\Delta(t)^{-1}e^{2(1-1/\delta)\rho t/\varepsilon} & (0<t<\infty)\\ \varepsilon^{-1}(1+t)^{-\lambda}\Delta(t)^{-1}e^{2(1-1/\delta)\rho t/\varepsilon} & (t\geqq c)\,.\end{cases}$$
(2) *For each $\varepsilon_0>0$ there exists a K-biinvariant function Φ_{ε_0} on G such that* (i) $|\Phi_{\varepsilon_0}(t)|\leqq c(1+t)^{-\lambda}e^{-2\rho t}$ $(t\in \boldsymbol{R}^+)$ *and thus, Φ_{ε_0} belongs to $L^q(G//K)$ for all $1<q\leqq\infty$.* (ii) *For all $\phi\in A_{\delta,\lambda}$ and $\varepsilon\geqq\varepsilon_0$, $|\phi_\varepsilon(t)|\leqq\Phi_{\varepsilon_0}(t)$ $(t\in\boldsymbol{R}^+)$.*

Proof. Since $\operatorname{sh}t\leqq e^t$, te^t $(t\geqq 0)$, (1) is obvious from the definitions of ϕ_ε and Δ. We put $\Phi_{\varepsilon_0}(t)=c\varepsilon_0^{-(m_1+m_2+1)}t^{m_1+m_2}(1+t/\varepsilon_0)^{-\lambda}\Delta(t)^{-1}$ for $0\leqq t<1$ and $c\varepsilon_0^{-1}(1+t/\varepsilon_0)^{-\lambda}\Delta(t)^{-1}$ for $t\geqq 1$, and extend it naturally to a K-biinvariant function on G. Then Φ_{ε_0} satisfies all the conditions of (2). Q.E.D.

Theorem 3.3. *Let us suppose that $0<\delta<1$ and $\lambda\geqq 0$. Then there exist constants $C_{p,\lambda}$ $(1\leqq p\leqq\infty)$ such that for all $\phi\in A_{\delta,\lambda}$,*
(1) $|\{x\in G;\ M_\phi f(x)>\alpha\}|\leqq C_{1,\lambda}\|f\|_1/\alpha$ *for all $f\in L^1(G/K)$ and $\alpha>0$.*
(2) $\|M_\phi f\|_p\leqq C_{p,\lambda}\|f\|_p$ *for all $f\in L^p(G/K)$ $(1<p\leqq\infty)$.*

Proof. First we choose an ε_0 in the interval $(0, 1/\delta-1)$ and divide the supremum of (3.1) as follows.
$$\begin{aligned}M_\phi f(x)&\leqq\sup_{0<\varepsilon<\varepsilon_0}|f*\phi_\varepsilon(x)|+\sup_{\varepsilon_0\leqq\varepsilon<\infty}|f*\phi_\varepsilon(x)|\\&=M_1f(x)+M_2f(x).\end{aligned}$$
We shall show the assertion for each M_if $(i=1, 2)$. M_1: We divide the integral $f*\phi_\varepsilon(x)=\int_G f(g)\phi_\varepsilon(g^{-1}x)dg$ into a sum of integrals over $B(x,\varepsilon)\cup\bigcup_{k=0}^\infty B(x, 2^k\varepsilon, 2^{k+1}\varepsilon)$, where $B(x, r, r')=B(x, r)_c\cap B(x, r')$. Then it follows from Lemma 3.2 (1) that
$$|f*\phi_\varepsilon(x)|\leqq\int_{B(x,\varepsilon)}|f(g)\phi_\varepsilon(g^{-1}x)|\,dg+\sum_{k=0}^\infty\int_{B(x,2^k\varepsilon,2^{k+1}\varepsilon)}|f(g)\phi_\varepsilon(g^{-1}x)|\,dg$$

$$\leqq c\varepsilon^{-(m_1+m_2+1)}[|B(x,\varepsilon)|(|B(x,\varepsilon)|^{-1}\int_{B(x,\varepsilon)}|f(g)|dg)$$

$$+\sum_{k=0}^{\infty}(2^k\varepsilon)^{m_1+m_2}\Delta(2^k\varepsilon)^{-1}(1+2^k)^{-\lambda}e^{2(1-1/\delta)2^k\rho}|B(x,2^{k+1}\varepsilon)|$$

$$\times(|B(x,2^{k+1}\varepsilon)|^{-1}\int_{B(x,2^{k+1}\varepsilon)}|f(g)|dg)].$$

Here we note that $|B(x,2^k\varepsilon)|=\int_0^{2^k\varepsilon}\Delta(t)dt\leqq 2^k\varepsilon\Delta(2^k\varepsilon)\leqq c(2^k\varepsilon)^{m_1+m_2+1}e^{2\rho 2^k\varepsilon}$ $(k\geqq 0)$ and $\Delta(2^{k+1}\varepsilon)\leqq c\Delta(2^k\varepsilon)e^{2\rho 2^k\varepsilon}$. Hence using the fact that $0<\varepsilon<\varepsilon_0<(1/\delta-1)$, we see that the infinite sum is convergent and moreover

$$|f*\phi_\varepsilon(x)|\leqq c(1+\sum_{k=0}^{\infty}2^{k(m_1+m_2+1)}(1+2^k)^{-\lambda}e^{-2((1/\delta-1)-\varepsilon_0)\rho 2^k})$$

$$\times\sup_{k\geqq 0}|B(x,2^k\varepsilon)|^{-1}\int_{B(x,2^k\varepsilon)}|f(g)|dg$$

$$\leqq cM_{HL}f(x).$$

In particular we have $M_1f(x)\leqq cM_{HL}f(x)$ $(x\in G)$. Therefore the desired results are obvious from the results obtained by [1] and [12]. M_2: Let Φ_{ε_0} be the function obtained in Lemma 3.2 (2). Then it is easy to see that $|f*\phi_\varepsilon(x)|\leqq|f|*\Phi_{\varepsilon_0}(x)$ for $\varepsilon\geqq\varepsilon_0$. Since Φ_{ε_0} belongs to $L^q(G//K)$ for all $1<q\leqq\infty$, we see that the operator M_2 is of type (L^p, L^p) $(1<p\leqq\infty)$ (see [1, Lemma 2]). Moreover using the estimate of Lemma 3.2 (2), (i) and the same argument in [12], we see that M_2 is of weak type (L^1, L^1).

This concludes the proof of the theorem. Q.E.D.

Corollary 3.4. *Let δ and λ be as in Theorem 3.3. We put*

$$M_{\delta,\lambda}f(x)=\sup_{\phi\in A_{\delta,\lambda}}M_\phi f(x)\qquad(x\in G).\tag{3.2}$$

Then the operator $M_{\delta,\lambda}$ satisfies the same results in Theorem 3.3.

Let $\mathscr{C}(G//K)$ denote the K-biinvariant Schwartz space on G and $\mathscr{C}'(G//K)$ its dual space (see [7, § 15]). Then the following assertions are obtained by the similar arguments in Proposition 1.49 and Theorem 2.7 in [5].

Proposition 3.5. *Let us suppose that $\phi\in A_{\delta,\lambda}$ and $\int_G\phi(g)dg=a$. Then*

(1) *For each $f\in\mathscr{C}(G//K)$, $f*\phi_\varepsilon\to af$ $(\varepsilon\to 0)$ in $\mathscr{C}(G//K)$.*
(2) *For each $f\in\mathscr{C}'(G//K)$, $f*\phi_\varepsilon\to af$ $(\varepsilon\to 0)$ in $\mathscr{C}'(G//K)$.*

Proposition 3.6. *Let us suppose that $\phi\in A_{\delta,\lambda}$ and $\int_G\phi(g)dg=1$. Then* *if $M_\phi f$ belongs to $L^q(G//K)$ for $f \in \mathscr{C}'(G//K)$ and $1 \leqq q \leqq \infty$, f also belongs to $L^q(G//K)$.*

§ 4. Atoms on G/K

Let us suppose that $0<p\leqq 1$. In this section we shall construct atoms on G/K which are similar to ones on homogeneous groups (see [2], [3, §2], [5, Ch. 2] and [10]). In Section 5 we shall obtain a sufficient condition by which a K-biinvariant function can be decomposed into a sum of K-biinvariant atoms.

4.1. We choose α and β such that $m_1=2(\alpha-\beta)$ and $m_2=2\beta+1$ (see [4, p. 265]). Then the restrictions of the zonal spherical functions on G to $CL(A^+)=\{a_t;\ t\geqq 0\}$ are given by the Jacobi functions $\phi_\lambda^{(\alpha,\beta)}(t)$ ($t \in \boldsymbol{R}^+$) (see [4, §3]). Furthermore for $f, g \in C_c^\infty(G//K)$ the convolution $f*g \in C_c^\infty(G//K)$ is rewritten as follows (see [6], [9] and [13]):

$$f*g(z)=\int_0^\infty\int_0^\infty f(x)g(y)K(x, y, z)\Delta(x)\Delta(y)dxdy \qquad (z \in \boldsymbol{R}^+), \tag{4.1}$$

where for $(x, y, z) \in (\boldsymbol{R}^+)^3$, $\Delta(x)=2^{2\rho}(\operatorname{sh} x)^{2\alpha+1}(\operatorname{ch} x)^{2\beta+1}$,

$$K(x, y, z)=\begin{cases} \dfrac{2^{1/2-\alpha}\Gamma(\alpha+1)(\operatorname{ch}(x)\operatorname{ch}(y)\operatorname{ch}(z))^{\alpha-\beta-1}(1-B^2)^{\alpha-1/2}}{\Gamma(\alpha+1/2)(\operatorname{sh}(x)\operatorname{sh}(y)\operatorname{sh}(z))^{2\alpha}} & \\ \quad \times {}_2F_1(\alpha+\beta,\ \alpha-\beta;\ \alpha+1/2;\ (1-B)/2) & |x-y|<z<x+y \\ 0 & \text{otherwise,} \end{cases}$$

and $B=(\operatorname{ch}(x)^2+\operatorname{ch}(y)^2+\operatorname{ch}(z)^2-1)/2\operatorname{ch}(x)\operatorname{ch}(y)\operatorname{ch}(z)$. Then $K(x, y, z)$ is a C^∞-function on $z\neq|x\pm y|$ and has an expansion $K(x, y, z)=\sum_{n=0}^\infty h_n P_n(x)P_n(y)P_n(z)/(P_n(1))^3$, where $P_n=P_n^{(\alpha,\beta)}$ is the Jacobi polynomial (see [6]). If α is integer, that is, $G=SO_0(2n, 1)$ ($n\geqq 2$), $SU(n, 1)$ ($n\geqq 1$), $Sp(n, 1)$ ($n\geqq 2$) and $F_{4(-20)}$, the singularity of $K(x, y, z)$ on $z=|x\pm y|$ arises from the term: $(\operatorname{sh}(x)\operatorname{sh}(y)\operatorname{sh}(z))^{-2\alpha}(1-B^2)^{\alpha-1/2}$. Therefore it is easy to see that

$$\begin{aligned} K_0(x, y, z)&=(y-z-x)^{-\alpha+1/2}(y-z+x)^{-\alpha+1/2}(y+z-x)^{-\alpha+1/2} \\ &\quad\times(y+z+x)^{-\alpha+1/2}K(x, y, z)\Delta(x)\Delta(y)\Delta(z) \end{aligned}$$

is a C^∞-function on $(\boldsymbol{R}^+)^3$ and $K_0(x, y, z)=O(xyz)$ ($x, y, z\to 0$). Here we shall describe some rough estimates which will be used in the proof of the main theorem. We put

$$H_a(x, y, z)=(y-z-x)^{-\alpha+1/2}(y-z+x)^{-\alpha+1/2}K(x, y, z)\Delta(x)\Delta(y)\Delta(z).$$

Clearly $H_\alpha(x, y, z)$ is a C^∞-function on $D(z_0)=\{(x, y, z)\in(\boldsymbol{R}^+)^3;\ |x-y|<z<x+y$ and $2x<z<z_0\}$ and $H_\alpha(x, y, z)=O(xyz(y+z-x)^{\alpha-1/2}(y+z+x)^{\alpha-1/2})$. Since $y\leqq y+z\pm x$ and $z/2\leqq y+z\pm x$ on $D(z_0)$, it follows that on $D(z_0)$

$$\left(\frac{d}{dx}\right)^l\left(\frac{d}{dy}\right)^{\mathrm{m}} H_\alpha(x, y, z)$$
$$=O(x\sum_{\substack{l_1+l_2=l,\\ m_1+\mathrm{m}_2=\mathrm{m}}}(y+z-x)^{\alpha+1/2-l_1-m_1}(y+z+x)^{\alpha+1/2-l_2-m_2}).$$

Next we put $D'(z_0)=\{(x, y, z)\in(\boldsymbol{R}^+)^3;\ |x-y|<z<x+y,\ x<z_0,\ x+z_0<z\}$. Then since $y+z\pm x>z_0$ on $D'(z_0)$, it is easy to see that if $\alpha\geqq 1/2$, that is, $G\neq SL(2, \boldsymbol{R})$, $(d/dx)^l(d/dy)^m H_\alpha(x, y, z)=O(e^{2\rho z})$ on $D'(z_0)$. Furthermore in this case, on $(\boldsymbol{R}^+)^3$

$$H_{1/2}(x, y, z)=O(\operatorname{sh}(x)\operatorname{sh}(y)\operatorname{sh}(z)e^{(\alpha+\beta)(x+y+z)}).$$

If $G=SO_0(2n+1)$ $(n\geqq 1)$, $K(x, y, z)\Delta(x)\Delta(y)\Delta(z)$ is a C^∞-function on $(\boldsymbol{R}^+)^3$ and $O(\operatorname{sh}(x)\operatorname{sh}(y)\operatorname{sh}(z)e^{(\alpha+\beta)(x+y+z)})$.

4.2. We assume that $0<\delta<1$, $\lambda\geqq 0$ and $a\in N$. Let $A^a_{\delta,\lambda}=A^a_{\delta,\lambda}(G//K)$ denote the space of all functions $\phi\in A_{\delta,\lambda}$ such that $((d/dt)^l\phi(t))\in A_{\delta,\lambda}$ for all $0\leqq l\leqq a+1$. We put $\varepsilon_p=(1-1/\delta)(1-1/p)^{-1}$, where $\varepsilon_1=\infty$. Then for each $\phi\in A^a_{\delta,\lambda}$ we define the radial maximal function $M'_\phi f$ for $f\in L^q(G//K)$ $(1\leqq q\leqq\infty)$ as follows.

$$M'_\phi f(x)=\sup_{0<\varepsilon<\varepsilon_p}|f*\phi_\varepsilon(x)| \qquad (x\in G). \tag{4.2}$$

If $p=1$, M'_ϕ coincides with M_ϕ in Section 3.

4.3. Let (p, q, a) be an ordered triplet such that $0<p\leqq 1$, $2(\alpha+1)/3<q\leqq\infty$, $a\in N$ and $a\geqq[2(\alpha+1)(1/p-1)]$, where $[\cdot]$ is the Gauss symbol. We put $r_p=p/2\rho(1-p)$, where $r_1=\infty$. Then we shall say that a function f on G/K is a *(p, q, a)-atom* if f belongs to $L^q_c(G/K)$ and satisfies

(1) there exists a ball $B(x, r)$ whose closure contains $\operatorname{supp}(f)$ and $\|f\|_q\leqq|B(x, r)|^{1/q-1/p}$,

(2) if $r<r_p$, $\int_0^\infty f_{x,K}(t)t^{2\alpha+1+n}dt=0$ for all $0\leqq n\leqq a$, where $f_{x,K}(g)=\int_K f(xkg)dk\in L^q_c(G//K)$.

If B is any ball satisfying the condition (1), we shall say that f is associated to B.

Theorem 4.1. *Let $G\neq SL(2, \boldsymbol{R})$ and (p, q, a) be as above. Suppose that $0<\delta<1$ and $\lambda>1/p$ if $0<p<1$. Then there exists a constant $C=C(p, q, r, \delta, \lambda)$ such that $\|M'_{\phi}f_{x,K}\|_p\leqq C$ for all $\phi\in A^{\alpha}_{\delta,\lambda}$ and (p, q, a)-atoms f on G/K.*

Proof. Let f be a (p, q, a)-atom associated to $B(x, r)$. Since $f_{x,K}=(f_x)_{e,K}$, where $f_x(g)=f(xg)$ $(g\in G)$, and f_x is a (p, q, a)-atom associated to $B(r)$, we may assume without loss of generality that $x=e$. We put $f_K=f_{e,K}$. Since $G\neq SL(2, \boldsymbol{R})$, $\alpha\geqq 1/2$. In the following we shall give a proof for the case that α is integer, because a quite similar argument is applicable for the other case. Thus let us suppose that $\alpha\in\boldsymbol{N}$, and show that

$$\int_G M'_{\phi}f_K(g)^p dg<C.$$

First we shall consider the case of $0<p<1$. We shall divide into two cases according to the radius r.

[*Case* I: $r<r_p$] We write the integral over G as a sum of three integrals over (i) $B(2r)$, (ii) $B(r_p+r)\cap B(2r)_c$ and (iii) $B(r_p+r)_c$.

(i) We note that $M'_{\phi}f_K(g)\leqq M_{\phi}f_K(g)$ $(g\in G)$, $\|f_K\|_q=\|f\|_q$ and $q>1$. Then by using Hölder's inequality, Theorem 3.3, Lemma 2.1 (1) and the condition 4.3 (1) of atoms in that order, we have

$$\begin{aligned}\int_{B(2r)} M'_{\phi}f_K(g)^p dg &\leqq\left(\int_{B(2r)} M_{\phi}f_K(g)^p dg\right)^{p/q}|B(2r)|^{1-p/q}\\ &\leqq\|M_{\phi}f_K\|_q^p(|B(2r)|/|B(r)|)^{1-p/q}|B(r)|^{1-p/q}\\ &\leqq C_{q,\lambda}{}^p A_{2,r_p}^{-}{}^{1-p/q}\|f_K\|_q|B(r)|^{1-p/q}\\ &\leqq C_{q,\lambda}{}^p A_{2,r_p}^{-}{}^{1-p/q}<\infty.\end{aligned}$$

(ii) Since

$$\int_{B(r_p+r)\cap B(2r)_c} M'_{\phi}f_K(g)^p dg=\int_{2r}^{r_p+r}(\sup\nolimits_{0<\varepsilon<\varepsilon_p}|f*\phi_{\varepsilon}(z)|)^p\Delta(z)dz,$$

we shall obtain an estimate of $\sup_{0<\varepsilon<\varepsilon_p}|f*\phi_{\varepsilon}(z)|$. It follows from 4.1 that for $z\geqq 2r$

$$\begin{aligned}f_K*\phi_{\varepsilon}(z)&=\int_0^r f_K(x)\int_{z-x}^{z+x}\phi_{\varepsilon}(y)K(x, y, z)\Delta(x)\Delta(y)dxdy\\ &=\int_0^r f_K(x)\int_{z-x}^{z+x}(y-z-x)^{\alpha-1/2}(y-z+x)^{\alpha-1/2}\phi_{\varepsilon}(y)\\ &\qquad\times H_{\alpha}(x, y, z)dxdy\Delta(z)^{-1},\end{aligned}\tag{4.3}$$

because $\operatorname{supp}(f_K)\subset B(r)$ and $z-x>r>0$. Here we shall prove two lemmas.

Lemma 4.2. *There exists a constant C such that for all $\phi \in A^{a}_{\delta,\lambda}$ and* $0 \leqq l \leqq a+1$,

$$\sup_{0<\varepsilon<\varepsilon_p} \left| \left(\frac{d}{dt}\right)^l \phi_\varepsilon(t) \right| \leqq C \begin{cases} (1) & t^{-2(\alpha+1)-l} e^{-2\rho t} \quad (0<t<r_p) \\ (2) & (1+t)^{-\lambda} e^{-2\rho t/p} \quad (t \geqq r_p). \end{cases}$$

Proof. If $l=0$, we use the estimate in Lemma 3.2 (1). Obviously it, as a function of $0<\varepsilon<\infty$, takes the maximum value $Ct^{-2(\alpha+1)}e^{-2\rho t}$ at $\varepsilon_{\max}=(1/\delta-1)\rho t/(\alpha+1)$ and $C(1+t)^{-\lambda}e^{-2\rho t}$ at $\varepsilon'_{\max}=2(1/\delta-1)\rho t$ respectively. We recall that we take the supremum of (4.2) in the finite interval $0<\varepsilon<\varepsilon_p$. Therefore, if $\varepsilon'_{\max} \geqq \varepsilon_p$, that is, $t \geqq r_p$, the maximal value must be $C(1+t)^{-\lambda}e^{-2\rho t/p}$ at $\varepsilon=\varepsilon_p$. Thus we obtain the case of $l=0$. Next we note that $(d/dt)\,\mathrm{sh}\,(t/\varepsilon) \leqq (1/t)\,\mathrm{sh}\,(t/\varepsilon)$ for $t \leqq r_p$ and $\leqq C(1/\varepsilon)\,\mathrm{sh}\,(t/\varepsilon)$ for $t \geqq r_p$. Hence the assertion for $0<l \leqq a+1$ is obvious from the definition of $A^{a}_{\delta,\lambda}$ and the first case. Q.E.D.

Lemma 4.3. *Let us suppose that $l-2(\alpha+1)/q+1>0$. Then there exists a constant C such that for $0<r<r_p$*

$$\int_0^r |f_K(x)|\, x^l dx \leqq C r^{l+1-2(\alpha+1)/p}.$$

Proof. Since $r<r_p$ and $\Delta(x)=O(x^{2\alpha+1})$ $(0<x<r_p)$, we see that

$$\begin{aligned} \int_0^r |f_K(x)|\, x^l dx &= \int_0^r (|f_K(x)|\, \Delta(x)^{1/q})(x^l \Delta(x)^{-1/q}) dx \\ &\leqq c\, \|f_K\|_q \left(\int_0^r x^{(l-(2\alpha+1)/q)\,q'} dx \right)^{1/q'} \\ &\qquad \text{by Hölder's inequality } \left(\frac{1}{q}+\frac{1}{q'}=1\right) \\ &\leqq c\, |B(r)|^{1/q-1/p} r^{l-(2\alpha+1)/q+1/q'} \qquad \text{by 4.3 (1)} \\ &\leqq C r^{l+1-2(\alpha+1)/p} \qquad \text{by Lemma 2.1 (3).} \end{aligned}$$

Q.E.D.

Now we return to (4.3) and consider the Taylor expansion of $\phi_\varepsilon(y) \times H_\alpha(x, y, z)$, as a function of (x, y), around $(0, z)$. Then since $H_\alpha(x, y, z)$ has a zero point at $x=0$ corresponding to $\mathrm{sh}\,(x)$, there exists a polynomial $P(x, y; z)$, whose degree with respect to x and y is less than or equal to a, such that

$$|\phi_\varepsilon(y) H_\alpha(x, y; z) - x P(x, y; z)| \leqq c \sum_{l+m=a+1} x^{l+1}(y-z)^m \times \left(\frac{d}{dx}\right)^l \left(\frac{d}{dy}\right)^m (\phi_\varepsilon(y) H_\alpha(x, y, z))\Big|_{\substack{x=x_0(z)\\ y=y_0(z)}}$$

where $(x_0(z), y_0(x))$ is a point on the segment of the line joining $(0, z)$ and (x, y) (see Fig. 1).

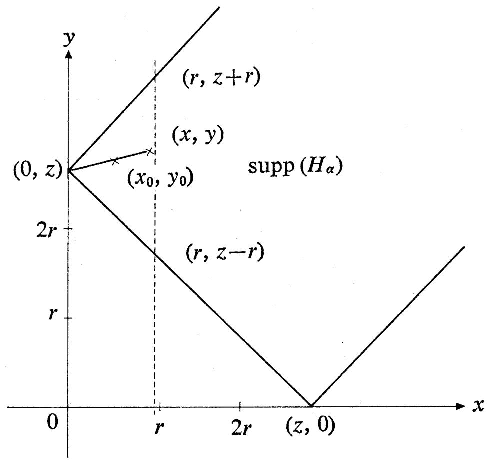

Fig. 1.

Since $\int_{z-x}^{z+x}(y-z-x)^{\alpha-1/2}(y-z+x)^{\alpha-1/2}(y-z)^n dy=cx^{2\alpha+n}$ if n is even, $=0$ if n is odd, it is easy to see from the condition 4.3 (2) of atoms that $\int_0^r f_K(x)\int_{z-x}^{z+x}(y-z-x)^{\alpha-1/2}(y-z+x)^{\alpha-1/2}xP(x, y; z)\,dxdy=0$. Therefore we can replace $\phi_\varepsilon(y)H_\alpha(x, y, z)$ of (4.3) by $\phi_\varepsilon(y)H_\alpha(x, y, z)-xP(x, y; z)$ and, applying the above estimate, we can obtain that

$$|f_K*\phi_\varepsilon(x)|\leqq c\sum_{l+m=\alpha+1}\int_0^r |f_K(x)|x^{l+1}$$
$$\times\int_{z-x}^{z+x}|(y-z-x)^{\alpha-1/2}(y-z+x)^{\alpha-1/2}(y-z)^m|\,dxdy$$
$$\times\left|\left(\frac{d}{dx}\right)^l\left(\frac{d}{dy}\right)^m(\phi_\varepsilon(y)H_\alpha(x, y, z))_{\substack{x=x_0(z)\\ y=y_0(z)}}\right|\Delta(z)^{-1}.$$

Here we note that $0<x_0(z)\leqq x<r<r_p$, $z/2<z-r\leqq y_0(z)\leqq z+r<3z/2<3r_p$, $y+z-x>y+z-r>z/2$ and $y+z+x>z$ for $y>0$ and $2r<z<r_p+r$. Hence, using the estimate of $H_\alpha(x, y, z)$ on $D(2r_p)$ in 4.1 and the one of ϕ_ε in Lemma 4.2 (1), we obtain that

$$\sup_{0<\varepsilon<\varepsilon_p}|f_K*\phi_\varepsilon(z)|\leqq cz^{-2(\alpha+1)+(2\alpha+1)-(\alpha+1)}\sum_{l+m=\alpha+1}\int_0^r|f_K(x)|x^{l+1}$$

$$\times\int_{z-x}^{z+x}|(y-z-x)^{\alpha-1/2}(y-z+x)^{\alpha-1/2}(y-z)^m|\,dxdy\Delta(z)^{-1} \quad (4.4)$$

$$\leqq cz^{-2(\alpha+1)+(2\alpha+1)-(\alpha+1)}\int_0^r |f_K(x)|x^{a+2+2\alpha}dx\Delta(z)^{-1}$$

$$\leqq cr^{a+1+2(\alpha+1)(1-1/p)}z^{-2(\alpha+1)+(2\alpha+1)-(\alpha+1)}\Delta(z)^{-1}$$

by Lemma 4.3

for $2r<z<r_p+r$. Therefore, since $a+1+2(\alpha+1)(1-1/p)\geqq 0$ by the assumption, we have

$$\int_{B(r_p+r)\cap B(2r)_c} M'_\phi f_K(g)^p dg$$

$$\leqq cr^{p(a+1+2(\alpha+1)(1-1/p))}\int_{2r}^{r_p+r} z^{-2p(\alpha+1)+p(2\alpha+1)-p(\alpha+1)}\Delta(z)^{1-p}dz$$

$$\leqq cr^{p(a+1)+2p(\alpha+1)-2(\alpha+1)}\int_{2r}^{r_p+r} z^{-2p(\alpha+1)+p(2\alpha+1)-p(\alpha+1)+(2\alpha+1)(1-p)}dz$$

$$\leqq c<\infty.$$

(iii) We note that $z-x>z-r>r_p$ and $y>z-x>z-r_p>0$. Therefore, using the estimate of $H_a(x, y, z)$ on $D'(r_p)$ in 4.1 and the similar argument in (ii) (we use Lemma 4.2 (2) instead of (1)), we obtain that

$$\sup_{0<\varepsilon<\varepsilon_p} |f_K*\phi_\varepsilon(z)|\leqq c(1+z-r)^{-\lambda}e^{-2\rho z/p}e^{2\rho z}\Delta(z)^{-1} \tag{4.5}$$

for $z\geqq r_p+r$. Then since $\lambda p>1$, we conclude that

$$\int_{B(r_p+r)_c} M'_\phi f_K(g)^p dg\leqq c\int_{r_p+r}^{\infty} (1+z-r)^{-\lambda p}e^{2(p-1)\rho z}\Delta(z)^{1-p}dz$$

$$\leqq c\int_{r_p}^{\infty}(1+z)^{-\lambda p}dz<\infty.$$

[*Case* II: $r\geqq r_p$] We write the integral over G as a sum of two integrals over (i) $B(r_p+r)$ and (ii) $B(r_p+r)_c$.

(i) Since $|B(r_p+r)|/|B(r)|\leqq A^+_{r_p,r_p}$ for $r\geqq r_p$ (see Lemma 2.1 (2)), we see that $\int_{B(r_p+r)} M'_\phi f_K(g)^p dg\leqq C_{q,\lambda}{}^p A^+_{r_p,r_p}{}^{1-p/q}$ as in I. (i).

(ii) We note that $z-x>z-r>r_p$ and $f_K*\phi_\varepsilon(z)=\int_0^r f_K(x)\int_{z-x}^{z+x}\phi_\varepsilon(y)\times H_{1/2}(x, y, z)dxdy\Delta(z)^{-1}$ for $z\geqq r_p+r$. Then applying the estimate of $H_{1/2}(x, y, z)$ on $(\boldsymbol{R}^+)^3$ in 4.1 and the one of Lemma 4.2 (2) for $l=0$ (we don't use the Taylor expansion in this case), we obtain that

$$\sup_{0<\varepsilon<\varepsilon_p} |f_K*\phi_\varepsilon(z)|\leqq\int_0^r |f_K(x)|\,\mathrm{sh}\,(x)e^{(\alpha+\beta)x}\int_{z-x}^{z+x}(1+y)^{-\lambda}e^{(1-2/p)\rho y}dxdy$$

$$\times e^{\rho z}\Delta(z)^{-1}$$

$$\leqq c(1+z-r)^{-\lambda}e^{2(1-1/p)\rho z}\Delta(z)^{-1}\int_0^r |f_K(x)|\,\mathrm{sh}\,(x)$$
$$\times e^{(\alpha+\beta)x}\,\mathrm{sh}\,((2/p-1)\rho x)dx.$$

Now we write the last integral as a sum of two integrals $\int_0^{r_p}$ and $\int_{r_p}^r$. Then since $r\geqq r_p$, $q>2(\alpha+1)/3$ and $\alpha\geqq 1/2$, as in the proof of Lemma 4.3,

$$\begin{aligned}\int_0^{r_p} &|f_K(x)|\,\mathrm{sh}\,(x)e^{(\alpha+\beta)x}\,\mathrm{sh}\,((2/p-1)\rho x)dx\\ &\leqq c\int_0^{r_p}|f_K(x)|x^2dx\\ &\leqq cr_p^{3-2(\alpha+1)/p}\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{r_p}^r &|f_K(x)|\,\mathrm{sh}\,(x)e^{(\alpha+\beta)x}\,\mathrm{sh}\,((2/p-1)\rho x)dx\\ &\leqq c\int_{r_p}^r (|f_K(x)|\Delta(x)^{1/q})(e^{2\rho/p}\Delta(x)^{-1/q})dx\\ &\leqq c\,\|f_K\|_q\Big(\int_{r_p}^r e^{2q'(1/p-1/q)\rho x}dx\Big)^{1/q'} \qquad (1/q+1/q'=1)\\ &\leqq c|B(r)|^{1/q-1/p}e^{2(1/p-1/q)\rho r}\leqq c \qquad \text{by Lemma 2.1 (3).}\end{aligned}$$

Hence we see that $\sup_{0<\varepsilon<\varepsilon_p}|f_K*\phi_\varepsilon(z)|\ (z\geqq r_p+r)$ satisfies the same estimate (4.5) and in particular, $\int_{B(r_p+r)_c} M'_\phi f(g)^p\,dg\leqq c\int_{r_p}^\infty (1+z)^{-\lambda p}dz<\infty$.

Therefore we can obtain the result for $0<p<1$. If $p=1$, we take $r_0>0$ arbitrary and consider the cases [I: $r<r_0$], $G=B(2r)\cup B(2r)_c$ and [II: $r\geqq r_0$], $G=B(r_p+r)\cup B(r_p+r)_c$. Then, applying the similar arguments as above, we can obtain the desired result for $p=1$.

This completes the proof of the theorem. Q.E.D.

Remark. If $G=SL(2, \boldsymbol{R})$, $H_{1/2}(x, y, z)$ does not satisfy the estimate in 4.1, because it has unboundedness corresponding to $(1-B^2)^{-1/2}$. Hence the argument in II. (ii) is not applicable.

Corollary 4.4. *We put*

$$M'_{\delta,\lambda}f(x)=\sup_{\phi\in A^a_{\delta,\lambda}} M'_\phi f(x) \qquad (x\in G). \tag{4.6}$$

Then under the same assumption in Theorem 4.1, $\|M'_{\delta,\lambda}f_{x,K}\|_p\leqq C$ *for all* (p, q, a)*-atoms* f *on* G/K.

Corollary 4.5. *Let $G \neq SL(2, \boldsymbol{R})$ and $(p, q, a, \delta, \lambda)$ be as in Theorem 4.1. Then there exists a constant $C = C(p, q, a, \delta, \lambda)$ such that $\|M'_{\delta,\lambda}f\|_p \leqq C$ for all K-biinvariant (p, q, a)-atoms f associated to $B(r)$ and all (p, q, a)-atoms f associated to $B(x, r)$ $(r \geqq r_p)$.*

Proof. If f is a K-biinvariant (p, q, a)-atom associated to $B(r)$, $f_{x,K} = f_{e,K} = f$. Therefore the assertion is obvious from Theorem 4.1. Now let f be a (p, q, a)-atom associated to $B(x, r)$ $(r \geqq r_p)$. As before we may assume that $x = e$. Then by the same argument in II. (i) of the proof of Theorem 4.1 we see that $\int_{B(r_p+r)} M'_{\delta,\lambda}f(g)^p dg \leqq C_{q,\lambda}{}^p A^+_{r_p,r_p}{}^{1-p/q}$. On the other hand, since $B(r_p+r)_c$ is a K-biinvariant set of G,

$$\int_{B(r_p+r)_c} M'_{\delta,\lambda}f(g)^p dg = \int_{B(r_p+r)_c} \left(\int_K M'_{\delta,\lambda}f(kg)^p dk\right) dg$$
$$\leqq \int_{B(r_p+r)_c} \left(\int_K M'_{\delta,\lambda}f(kg)\, dk\right)^p dg$$
$$\leqq \int_{B(r_p+r)_c} (|f|_K * (\sup_{\phi \in A^a_{\delta,\lambda}} \sup_{0<\varepsilon<\varepsilon_p} |\phi_\varepsilon|)(g))^p dg.$$

Here we note that $\mathrm{supp}\,(|f|_K) \subset B(r)$ $(r \geqq r_p)$ and $|f|_K$ is a (p, q, a)-atom associated to $B(r)$ (this is not true if $r < r_p$). Therefore, as in II. (ii) of the proof of Theorem 4.1, we see that $\int_{B(r_p+r)_c} M'_{\delta,\lambda}f(g)^p dg \leqq c \int_{r_p}^{\infty} (1+z)^{-\lambda p} dz < \infty$. Q.E.D.

§ 5. H^p and $H^p_{q,a}$ spaces on $G//K$

As before we assume that $\alpha \geqq 1/2$ $(G \neq SL(2, \boldsymbol{R}))$, $0 < p \leqq 1$, $2(\alpha+1)/3 < q \leqq \infty$, $a \in N$, $a \geqq [2(\alpha+1)(1/p-1)]$, $0 < \delta < 1$ and $\lambda > 1/p$ if $0 < p < 1$.

Now let $L^p_+(G//K)$ denote the subspace of $L^p(G//K)$ consisting of all $f \in L^p(G//K)$ such that each $|f|$ is dominated by a non-increasing function $f^+ \in L^p(G//K)$. We shall say that f has a non-increasing dominator f^+ in $L^p(G//K)$. Here we shall define two spaces $H^p(G//K)$ and $H^p_{q,a}(G//K)$ as follows;

$$H^p(G//K) = \{f \in \mathscr{C}'(G//K);\ M'_{\delta,\lambda}f \in L^p_+(G//K)\},$$
$$H^p_{q,a}(G//K) = \Big\{f = \sum_{i=1}^{\infty} \lambda_i f_i;\ \lambda_i \geqq 0,\ \sum \lambda_i{}^p < \infty \text{ and each } f_i \text{ is}$$
$$\text{a } K\text{-biinvariant } (p, q, a)\text{-atom associated to } B(r_i)\Big\}.$$

If $f \in H^p_{q,a}(G//K)$, the representation $f = \sum_{i=1}^{\infty} \lambda_i f_i$ is not unique. For any such representation we shall say that f has a K-biinvariant (p, q, a)-atomic decomposition. Let ρ^p and $\rho^p_{q,a}$ denote the quasi-norms of $H^p(G//K)$ and $H^p_{q,a}(G//K)$ defined by $\rho^p(f)=\|M'_{\delta,\lambda}f\|^p_p$ $(f \in H^p(G//K))$ and $\rho^p_{q,a}(f)=\inf\{\sum \lambda_i{}^p\}(f \in H^p_{q,a}(G//K))$ respectively, where we take the infimum over all K-biinvariant (p, q, a)-atomic decomposition $f=\sum_{i=1}^{\infty}\lambda_i f_i$. Then it is easy to see that $H^p_{\infty,a}(G//K)\subset H^p_{q_1,a}(G//K)\subset H^p_{q_2,a}(G//K)$ for $q_1>q_2$ and these inclusion maps are continuous.

Proposition 5.1. *$H^p_{\infty,a}(G//K)\subset H^p(G//K)$ and the inclution map is continuous.*

Proof. Let $f=\sum_{i=1}^{\infty}\lambda_i f_i$ be in $H^p_{\infty,a}(G//K)$. Since each f_i is a (p, ∞, a)-atom, $\|M'_{\delta,\lambda}f_i\|_\infty \leqq C^1_{\delta,\lambda}\|f_i\|_\infty \leqq C^1_{\delta,\lambda}|B(r_i)|^{-1/p}$. Therefore it follows from the proof of Theorem 4.1 that if $r<r_p$,

$$M'_{\delta,\lambda}f_i(z)\leqq F_i(z)=\begin{cases}|B(r_i)|^{-1/p} & (0<z\leqq 2r_i)\\ \mathrm{Min}\,(|B(r_i)|^{-1/p},\ cr_i^{\alpha+1+2(\alpha+1)(1-1/p)} & \\ \qquad \times z^{-2(\alpha+1)+(2\alpha+1)-(\alpha+1)}\Delta(z)^{-1}) & (2r_i<z\leqq r_p+r_i)\\ \mathrm{Min}\,(|B(r_i)|^{-1/p},\ c'(1+z-r_i)^{-\lambda}e^{-2\rho z/p}) & (z>r_p+r_i)\end{cases}$$

and if $r\geqq r_p$,

$$M'_{\delta,\lambda}f_i(z)\leqq F_i(z)=\begin{cases}|B(r_i)|^{-1/p} & (0<z\leqq r_p+r_i)\\ \mathrm{Min}\,(|B(r_i)|^{-1/p},\ c(1+z-r_i)^{-\lambda}e^{-2\rho z/p}) & (z>r_p+r_i).\end{cases}$$

We shall denote each K-biinvariant extension of F_i to G by the same letter. Then we may assume that each F_i is non-increasing and $\|F_i\|_p\leqq C$ for all i (see § 4). Here we put $(M'_{\delta,\lambda}f)^+=\sum_{i=1}^{\infty}\lambda_i F_i$. Obviously $(M'_{\delta,\lambda}f)^+$ is also non-increasing. Moreover, by the subadditivity of the maximal operator, we see that $\rho^p(f)=\|M'_{\delta,\lambda}f\|^p_p\leqq\sum\lambda_i{}^p\|M'_{\delta,\lambda}f_i\|^p_p\leqq\sum\lambda_i{}^p\|F_i\|^p_p\leqq C^p\sum\lambda_i{}^p<\infty$. Therefore f belongs to $H^p(G//K)$ and $\rho^p(f)\leqq c\rho^p_{\infty,a}(f)$.

Q.E.D.

Now let $L^1_0=L^1_0(G//K)$ denote the subspace of $L^1(G//K)$ consisting of all $f \in L^1(G//K)$ such that $f(t)=O(t^{-\gamma})$ $(t\to 0)$ for $0\leqq\gamma<1$.

Proposition 5.2. *Let us suppose that* $0<p<1$. *Then*

$$L^p_+(G//K)\cap L^1_0=H^p_{\infty,a}(G//K)\cap L^1_0=H^p(G//K)\cap L^1_0.$$

Proof. First we shall prove that $L^p_+(G//K)\cap L^1_0\subset H^p_{\infty,a}(G//K)$. Let f be in $L^p_+(G//K)\cap L^1_0$. Then we may assume without loss of generality that $f^+(t)=Lt^{-\gamma}$ on $0<t<r_p$ $(L>0$ and $0\leqq\gamma<1)$. We put $\Omega_k=\{g \in G; f^+(g)>2^k\}$ $(k \in Z)$. Then since f^+ is non-increasing, there exists $r_k>0$ $(k \in Z)$

such that $\Omega_k=B(r_k)$ and $r_k\geqq r_{k'}$ $(k<k')$. Here we denote the largest integer k such that $r_k>r_p$ by k_0 and the characteristic functions of Ω_k, Ω_k^c and $\Omega_k\cap\Omega_{k'}^c$ by χ_k, χ_k^c and $\chi_{k,k'}$ respectively. Now we shall decompose $f\chi_{k_0}$ and $f\chi_{k_0}^c$ into a sum of K-biinvariant (p,∞,a)-atoms. $f\chi_{k_0}^c$: We put

$$f\chi_{k_0}^c=\sum_{k\leqq k_0-1}f\chi_{k,k+1}=\sum_{k\leqq k_0-1}\lambda_k f_k,$$

where $\lambda_k=2^{k+1}|B(r_k)|^{1/p}$ and $f_k=\lambda_k^{-1}f\chi_{k,k+1}$. Then $\operatorname{supp}(f_k)\subset\Omega_k\cap\Omega_{k+1}^c$ $\subset B(r_k)$ $(r_k>r_p)$ for $k\leqq k_0-1$. Moreover since $f^+(t)\leqq 2^{k+1}$ on $\Omega_k\cap\Omega_{k+1}^c$,

$$\begin{aligned}\|f_k\|_\infty&=\lambda_k^{-1}\|f_{k,k+1}\|_\infty\\&\leqq 2^{-(k+1)}|B(r_k)|^{-1/p}\|f^+\chi_{k,k+1}^c\|_\infty\leqq|B(r_k)|^{-1/p}.\end{aligned}$$

Therefore each f_k is a K-biinvariant (p,∞,a)-atom associated to $B(r_k)$. $f\chi_{k_0}$: First we shall prove the following lemma.

Lemma 5.3. *For each k there exist functions h_i^k $(0\leqq i\leqq a'=a+2\alpha+1)$ in $L_c^\infty(G//K)$ such that*

(1) $\operatorname{supp}(h_i^k)\subset B(r_k)$ $\quad(0\leqq i\leqq a')$,

(2) $\displaystyle\int_0^{r_k}h_i^k(t)t^j dt=\delta_{ij}$ $\quad(0\leqq i,j\leqq a')$,

(3) $\|h_i^k\|_\infty\leqq Cr_k^{-(i+1)}$ $\quad(0\leqq i\leqq a')$,

where C does not depend on k.

Proof. Let $P_{a'}$ denote the space of all polynomials whose degrees are less than or equal to a'. Then we can find a solution $\{P_i^k\in P_{a'};0\leqq i\leqq a'\}$ of (2). Here we denote each K-biinvariant extension of P_i^k to G by the same letter and put $h_i^k=P_i^k\chi_k$. Then it is easy to see that $\{h_i^k;0\leqq i\leqq a'\}$ satisfies all the conditions of the lemma. Q.E.D.

Now we put

$$\begin{aligned}f\chi_{k_0}&=f\chi_{k,k_0}+f\chi_k\\&=\Big(f\chi_{k,k_0}+\sum_{i=0}^{a'}f_i^k h_i^k\Big)+\Big(f\chi_k-\sum_{i=0}^{a'}f_i^k h_i^k\Big)=g_k+g_k',\end{aligned}$$

where $f_i^k=\displaystyle\int_0^{r_k}f(t)t^i dt$ $(0\leqq i\leqq a')$. Then since $\operatorname{supp}(f\chi_{k_0}-g_k)\subset\Omega_k$, $g_k\to f\chi_{k_0}$ $(k\to\infty)$ and in particular,

$$f\chi_{k_0}=g_{k_0}+\sum_{k\geqq k_0}(g_{k+1}-g_k)=g_{k_0}+\sum_{k\geqq k_0}(g_k'-g_{k+1}').$$

Here we note that $|f(t)|\leqq f^+(t)=Lt^{-\gamma}$ on $0<t<r_p$, $|f_i^k|\leqq L\displaystyle\int_0^{r_k}t^{i-\gamma}dt$

$\leqq (L/(1-\gamma))r_k^{i-\gamma+1}$ and $r_k=(L^{-1}2^k)^{-1/\gamma}$ for $k\geqq k_0$. Therefore it follows from Lemma 5.3 (3) that

$$\begin{aligned}\|g_k'-g_{k+1}'\|_\infty &\leqq \|f\chi_{k,k+1}\|_\infty+\sum_{i=0}^{a'}(|f_i^k|\,\|h_i^k\|_\infty+|f_i^{k+1}|\,\|h_i^{k+1}\|_\infty)\\ &\leqq 2^{k+1}+2((a'+1)/(1-\gamma))2^{k+1}\\ &\leqq C_0 2^{k+1},\end{aligned}$$

where C_0 does not depend on k, and by the same way $\|g_{k_0}\|_\infty\leqq C_0 2^{k+1}$. Here we put $\lambda_k=C_0 2^{k+1}|B(r_k)|^{1/p}$, $g_0=\lambda_{k_0}^{-1}g_{k_0}$ and $f_k=\lambda_k^{-1}(g_k'-g_{k+1}')$ $(k\geqq k_0)$, that is, $f\chi_{k_0}=\lambda_{k_0}g_0+\sum_{k\geqq k_0}\lambda_k f_k$. Then using the same argument in the first case we see that f_k and g_0 satisfy the condition 4.3 (1) of (p,∞,a)-atoms. Moreover since $\int_0^{r_k}g_{k+1}'(t)t^j dt=\int_0^{r_{k+1}}g_{k+1}'(t)t^j dt$ and

$$\begin{aligned}\int_0^{r_k}g_k'(t)t^j dt&=\int_0^{r_k}f\chi_k(t)t^j\,dt-\sum_{i=0}^{a'}f_i^k\int_0^{r_k}h_i^k(t)t^j dt\\ &=f_j^k-\sum_{i=0}^{a'}f_i^k\delta_{ij}=0\qquad(0\leqq j\leqq a'),\end{aligned}$$

each f_k $(k\geqq k_0)$ satisfies the condition 4.3 (2) of atoms. Hence we see that each f_k $(k\geqq k_0)$ (resp. g_0) is a K-biinvariant (p,∞,a)-atom associated to $B(r_k)$ (resp. $B(r_{k_0})$; in this case, since $r_{k_0}>r_p$, g_0 need not satisfy the condition 4.3 (2)).

Therefore we find a decomposition of f such that

$$f=\lambda_{k_0}g_0+\sum_{k=-\infty}^{\infty}\lambda_k f_k.$$

On the other hand,

$$\begin{aligned}\rho_{\infty,a}^p(f)&\leqq \lambda_{k_0}{}^p+\sum_{k=-\infty}^{\infty}\lambda_k{}^p\\ &\leqq C\sum_{k=-\infty}^{\infty}2^{(k+1)p}|B(r_k)|\\ &\leqq C2^{p+1}p^{-1}\int_{-\infty}^{+\infty}p\xi^{p-1}|\{g\in G;f^+(g)>\xi\}|\,d\xi\\ &\leqq C2^{p+1}p^{-1}\|f^+\|_p^p<\infty.\end{aligned}$$

Therefore the above decomposition is a K-biinvariant (p,∞,a)-atomic one and thus, f belongs to $H_{\infty,a}^p(G//K)$.

Hence combining Proposition 5.1, we see that $L_+^p(G//K)\cap L_0^1\subset$

$H^p_{\infty,a}(G//K)\cap L^1_0 \subset H^p(G//K)\cap L^1_0$. Now let us suppose that f is in $H^p(G//K)\cap L^1_0$. Then since $f\in L^1(G//K)$, we obtain from the same argument in [5, Proposition 1.20 and Theorem 2.6] that $|f|\leqq M'_{\delta,\lambda}f$. In particular $|f|\leqq(M'_{\delta,\lambda}f)^+\in L^p_+(G//K)$ and thus, $f\in L^p_+(G//K)$.

This completes the proof of the proposition. Q.E.D.

Remark. In the above proof we assume that $f^+(t)=Lt^{-r}$ $(0<t<r_p)$. Therefore it is not apparent that there exists a constant C such that for any dominator f^+ of f in $L^p_+(G//K)\cap L^1_0$, $\rho^p_{\infty,a}(f)\leqq C\|f^+\|^p_p$. Hence we can not apply the limiting method as in homogeneous groups (see [2], [10] and [5, Ch. 3]). If $f\in L^p_+(G//K)$ $(0<p<1)$ has a non-increasing dominator f^+ such that $\int_0^r |f(t)|\,dt\leqq rf^+(r)$ for $r\to 0$, we can also obtain a K-biinvariant (p,∞,a)-atomic decomposition of f.

Remark. The condition (2) of atoms in 4.3 can be replaced as follows; if $r<r_p$, $\int_0^\infty f_{x,K}(t)t^n\Delta(t)dt=0$ for all $0\leqq n\leqq a$. Moreover if we replace the inequality of the condition (1) by $\|f\|_q\leqq|B(x,r)|^{1/q-1/p}$ if $r<r_p$ and $r^{-1}|B(x,r)|^{1/q-1/p}$ if $r\geqq r_p$, all the results in Section 4 are valid for $G=SL(2,\boldsymbol{R})$.

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Department of Mathematics
Faculty of Science and Technology
Keio University
Hiyoshi, Yokohama 223
Japan