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A Geometric Significance of Total Curvature on Complete Open Surfaces

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1. Let M be a 2-dimensional complete non-compact Riemannian manifold with non-negative Gaussian curvature K. Then the total curvature of M satisfies the inequality

$$\int_M K\,dv \leqq 2\pi,$$

where dv is the volume element of M induced from the Riemannian metric on M. This was proved by Cohn-Vossen in [2]. Obviously in contrast with compact case, the total curvature of M is not a topological invariant when M is non-compact and it depends on the Riemannian structures on M. Concerning this fact, in [5], [7], we showed that the total curvature of M is expressing a certain curvedness of M. We will state it in the following.

For a point $q \in M$, put $S_q(M) := \{v \in T_q(M);\ \text{norm of } v = 1\}$, where $T_q(M)$ is the tangent space of M at q. From the Euclidean metric on $T_q(M)$, $S_q(M)$ becomes a Riemannian submanifold of $T_q(M)$ isometirc to the standard unit circle. Thus we can consider the Riemannian measure on $S_q(M)$. Let $A(q) \subset S_q(M)$ be the set defined as

$$\{v \in S_q(M);\ \text{geodesic } \gamma\colon [0, \infty) \to M \text{ given by } \gamma(t) = \exp_q tv \text{ is a ray}\}.$$

Here $\exp_q\colon T_q(M) \to M$ is the exponential mapping of M and geodesic γ is called a ray when any subarc of γ is a shortest connection between its end points. Using these notations, the facts mentioned above are stated as follows.

Fact 1. Let M be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to a Euclidean plane. Then for any point $q \in M$,

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$$\text{measure } A(q) \geqq 2\pi - \int_M K\,dv.$$

Note that from classification by Cohn-Vossen, 2-dimensional complete non-compact Riemannian manifold with non-negative Gaussian curvature is diffeomorphic to a Euclidean plane or isometric to a flat cylinder or a flat Möbius band.

And in [6], we have tried to estimate the measure $A(q)$ from above;

Fact 2. Let M be a 2-dimensional complete non-compact Riemannian manifold with non-negative Gaussian curvature K. Then it holds

$$\inf_{q \in M} \text{measure } A(q) \leqq 3\pi - \int_M K\,dv.$$

Here note that for each value $u \in (0, 2\pi]$, we can easily construct a complete non-compact rotation surface in 3-dimensional Euclidean space with non-negative Gaussian curvature K satisfying $\int_M K\,dv = u$ and with point $q \in M$ satisfying measure $A(q) = 2\pi$. Thus it will be reasonable to consider on an estimate of $\inf_{q \in M}$ measure $A(q)$. And as is easily seen, the estimation in Fact 2 is very rough. So in this paper, we will give a more sharper estimation which is

Theorem. *Let M be a 2-dimensional complete non-compact Riemannian manifold with non-negative Gaussian curvature K. Then it holds*

$$\inf_{q \in M} \text{measure } A(q) \leqq 2\pi - \int_M K\,dv.$$

An upper bound $2\pi - \int_M K\,dv$ is optimal in this type of estimation, because together with Fact 1, we have

Corollary. *Let M be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to a Euclidean plane. Then it holds*

$$\int_M K\,dv = 2\pi - \inf_{q \in M} \text{measure } A(q).$$

Thus we get a geometrical significance of the total curvature of M. Another trials to give a geometrical significance of the total curvature are done by K. Shiohama, see [8], [9].

2. We will give the proof of the Theorem. For convenience of the proof, we will restate a main part of the proof of Fact 1, following in [6]. From classification by Cohn-Vossen, it sufficies to prove when M is diffeomorphic to a Euclidean plane. Then from [4], we have a family of compact domains $\{Q_{r_i}\}_{i=1,2,\dots}$ satisfying

(1) the boundary of Q_{r_i} is a geodesic quadrateral, $i=1, 2, \cdots$

(2) $Q_{r_i} \subset Q_{r_{i+1}}$ for $i=1, 2, \cdots$ and

(3) $\bigcup_{i=1}^{\infty} Q_{r_i}=M$.

For this family $\{Q_{r_i}\}$, we have

Lemma 1. *If M is not flat, then for each r_i, there exists $r_{i_j}>r_i$ such that every ray starting from any point of the complement of $Q_{r_{i_j}}$ does not meet Q_{r_i}.*

The proof of this lemma is done by using Toponogov's splitting theorem [6; p.4].

Now for any small positive $\varepsilon>0$, there exists a number i_0 such that

$$\int_{Q_{i_0}} K\,dv \geqq \int_M K\,dv-\varepsilon.$$

This follows from the property (2) for $\{Q_{r_i}\}$. For this $Q_{r_{i_0}}$, we apply Lemma 1. Then we get $Q_{r_{j_0}}$ which satisfies the following; for any point $q \in (Q_{r_{j_0}})^c$ (=the complement of $Q_{r_{j_0}}$), any ray starting from q does not meet $Q_{r_{i_0}}$. If $\# A(q)$ (=number of the elements of $A(q)$)$=1$, then there is nothing to prove. So we consider the case $\# A(q)\geqq 2$. So $S_q(M)-A(q)$ is disjoint union of connected open subsets $F_{\lambda, \lambda\in\Lambda}$ of $S_q(M)$ i.e. $\bigcup_{\lambda\in\Lambda}F_\lambda =S_q(M)-A(q)$, becauses $A(q)$ is a closed subset of $S_q(M)$. For each $\lambda \in \Lambda$, ∂F_λ consists of two vectors $v_1^\lambda, v_2^\lambda \in A(q)$. Let $\gamma_i^\lambda\colon [0, \infty)\to M$ be the ray defined by $\gamma_i^\lambda(t)=\exp_q tv_i^\lambda$, $i=1, 2$. Since $\gamma_1^\lambda, \gamma_2^\lambda$ are rays, γ_1^λ and γ_2^λ do not meet other than q. Let $\delta>0$ be the convexity radius of q. Then from above facts, we get domains $D_{\lambda, \lambda\in\Lambda}$ whose boundary is $\gamma_1^\lambda([0, \infty)) \cup \gamma_2^\lambda([0, \infty))$ and which satisfies $\exp_q\{tv; v \in F_\lambda, 0<t\leqq\delta\}\subset D_{\lambda, \lambda\in\Lambda}$ and $\bigcup_{\lambda\in\Lambda} \bar{D}_\lambda=M$.

Now, let $\{C_t\}_{t\geqq 0}$ be the family of compact totally convex subsets of M defined by

$$C_t=\bigcap_{c\in A}(M-B_{c_t})$$

where by definition, $B_{c_t}:=\bigcup_{s>0} B_s(c(t+s))$ ($B_r(x)$ is the open geodesic ball in M with radius r centered at x) and A is the set of all rays starting from q. In this paper, all geodesics have arc-length as their parameter. Since C_t is totally convex, C_t is a topological manifold and hence ∂C_t is homeomorphic to a circle for $t>0$, because dim $M=2$, see [1]. For this family of totally convex sets $\{C_t\}_{t\geqq 0}$, we have shown in [5] that for each $D_{\lambda,\lambda\in\Lambda}$, there exists a divergent sequence $\{t_i\}$ ($t_i\uparrow\infty$) and minimal geodesics $\gamma^+_{ti}, \gamma^-_{ti}\colon [0, s_i]\to\bar{D}_\lambda$, $i=1, 2, \cdots$ satisfying the following conditions;

(1) $\gamma^+_{ti}, \gamma^-_{ti}\colon (0, s_i]\to D_\lambda$, $i=1, 2, \cdots$

(2) $\gamma^+_{ti}(0)=\gamma^-_{ti}(0)=q$, $\gamma^+_{ti}(s_i)=\gamma^-_{ti}(s_i)\in\partial C_{t_i}$, $i=1, 2, \cdots$

(3) $\gamma^+_{ti}\to\gamma^\lambda_1$, $\gamma^-_{ti}\to\gamma^\lambda_2$ as $i\to\infty$.

For these $\gamma^+_{ti}, \gamma^-_{ti}$, it holds;

Lemma 2. $\lim_{t_i\to\infty}\measuredangle(-\dot\gamma^+_{ti}(s_i), -\dot\gamma^-_{ti}(s_i))=0$.

Proof. Step 1. From the definition of C_{t_i} and the fact that $\gamma^\lambda_1, \gamma^\lambda_2\in A$, we can easily see that $\gamma^\lambda_1(t_i), \gamma^\lambda_2(t_i)\in\partial C_{t_i}$ for each t_i and $\gamma^\lambda_1|[0, t_i]$, $\gamma^\lambda_2|[0, t_i]$ is a shortest connection between q and ∂C_{t_i}.

Step. 2. Fix a number t_i. We consider a function $\varphi_i\colon [0, s_i]\to R$ defined by $\varphi_i(s):=d(\gamma^+_{ti}(s), \partial C_{t_i})$ (d is the distance function on M). Since $\gamma^+_{ti}([0, s_i])\subset C_{t_i}$, from [1; Th. 1.10], φ_i is a concave function, that is, for any $a\geqq 0$, $b\geqq 0$, $a+b=1$ and $s<s'$,

$$\varphi_i(as+bs')\geqq a\varphi_i(s)+b\varphi_i(s').$$

And from Step 1, we see

$$d(\gamma^\lambda_1(s), \partial C_{t_i})=t_i-s.$$

So

$$\begin{aligned}\varphi_i(s)&\leqq d(\gamma^+_{ti}(s), \gamma^\lambda_1(s))+d(\gamma^\lambda_1(s), \partial C_{t_i})\\&=d(\gamma^+_{ti}(s), \gamma^\lambda_1(s))+t_i-s.\end{aligned}$$

Using comparison theorem by Toponogov, we have

$$d(\gamma^\lambda_1(s), \gamma^+_{ti}(s))\leqq\sqrt{2(1-\cos\theta)}\cdot s,$$

where we put $\theta:=\measuredangle(\dot\gamma^\lambda_1(0), \dot\gamma^+_{ti}(0))$. Thus

$$\begin{aligned}\varphi_i(s)&\leqq t_i-s+\sqrt{2(1-\cos\theta)}\cdot s\\&=t_i-(1-\sqrt{2(1-\cos\theta)})s.\end{aligned}$$

Put $m(\theta):=1-\sqrt{2(1-\cos\theta)}$. Then $m(\theta)<1$ and $m(\theta)\to 1$ as $\theta\to 0$. Hence we have

$$\varphi_i(s)\leqq t_i-m(\theta)s$$

for $s\in[0, t_i]$ and hence $s\in[0, s_i]$ because of the concavity of φ_i.

Step 3. By using the concavity of φ_i and the inequality $\varphi_i(s)\leqq t_i-m(\theta)s$, we can easily see that for any s, s', $s<s'\leqq s_i$,

$$\frac{\varphi_i(s')-\varphi_i(s)}{s'-s}\leqq -m(\theta).$$

So putting $s'=s_i$ in the above inequality, we have

$$\varphi_i(s)\geqq m(\theta)(s_i-s) \qquad \text{for any } s\in[0, s_i],$$

because $\varphi_i(s_i)=0$.

Step 4. For a small $\delta'>0$, let $c\colon[-\delta', \delta']\to M$ be a geodesic such that $c(0)=\gamma^+_{ti}(s_i)\in\partial C_{t_i}$ and

$$c([-\delta', \delta'])\subset(C_{t_i})^c\cup\partial C_{t_i}.$$

Such a c is obtained as follows. Since C_{t_i} is totally convex, tangent cone

$$C_{\gamma^+_{ti}(s_i)}=\left\{\begin{matrix} v\in T_{\gamma^+_{ti}(s_i)}(M);\ \exp tv/\|v\|\in \operatorname{int} C_{t_i} \text{ for some} \\ \text{positive } t<r(\gamma^+_{ti}(s_i))\end{matrix}\right\}\cup\{0\}$$

at $\gamma^+_{ti}(s_i)\in\partial C_{t_i}$ is a convex cone in $T_{\gamma^+_{ti}(s_i)}(M)$, where $r(\gamma^+_{ti}(s_i))$ is the convexity radius at $\gamma^+_{ti}(s_i)$, see [1; Prop. 1.8]. Let $v\in\partial C_{\gamma^+_{ti}(s_i)}-\{0\}$ and define $c(t)=\exp tv/\|v\|$. Then c is a desired one.

Now, choosing s sufficiently close to s_i and fixing it for a moment, we can assume that end point of the minimal geodesic $c_1\colon[0, d(\gamma^+_{ti}(s), c([-\delta', \delta']))]\to M$ from $\gamma^+_{ti}(s)$ to $c([-\delta', \delta'])$ is $c(s_0)$, $s_0\in(-\delta', \delta')$. We only consider the case $s_0\geqq 0$. If $s_0<0$, then putting $\tilde{c}(s):=c(-s)$, we can obtain same conclusion. Put $d(\gamma^+_{ti}(s), c([-\delta', \delta']))=d(\gamma^+_{ti}(s), c(s_0))=: s_1$. Then $\measuredangle(\dot{c}(s_0), \dot{c}_1(s_1))=\pi/2$. If $c_1=\gamma^+_{ti}|_{[s,s_i]}$, then there is nothing to prove as is seen in the following. So we consider the case $c_1\neq\gamma^+_{ti}|_{[s,s_i]}$. Put $s_i-s=: s_2$. Then because of the property of c, minimal geodesic c_1 from $\gamma^+_{ti}(s)$ to $c([-\delta', \delta'])$ meet ∂C_{t_i}. Thus

$$s_1\geqq\varphi_i(s)\geqq m(\theta)(s_i-s)=m(\theta)s_2,$$

i.e. $s_1\geqq m(\theta)s_2$. Let D be the compact domain surrounded by the geodesic triangle $(\gamma^+_{ti}|_{[s,s_i]}, c|_{[0,s_0]}, c_1)$. Put $\alpha:=\measuredangle(-\dot{\gamma}^+_{ti}(s_i), \dot{c}(0))$,

$$\beta:=\measuredangle(\dot{\gamma}^+_{ti}(s), \dot{c}_1(0)) \quad\text{and}\quad \gamma:=\measuredangle(-\dot{c}(s_0), -\dot{c}_1(s_1)) \quad (=\pi/2).$$

Now for any small $\varepsilon'>0$, we choose s again sufficiently close to s_i satisfying $\varepsilon'\geqq\int_D K\,dv$. Then applying Gauss-Bonnet Theorem to D, we have

$$\varepsilon'\geqq\int_D K\,dv=\alpha+\beta+\gamma-\pi=\alpha+\beta-\frac{\pi}{2}\geqq 0.$$

Thus

$$\frac{\pi}{2} \leqq \alpha + \beta \leqq \frac{\pi}{2} + \varepsilon'.$$

In particular

$$\alpha \leqq \frac{\pi}{2} + \varepsilon'.$$

From Toponogov's comparison theorem, if we construct a triangle in a Euclidean plane with sides having lengths s_2, s_0, s_1 corresponding to the geodesic triangle $(\gamma_{ti}^+|_{[s,s_i]}, c|_{[0,s_0]}, c_1)$ and if $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ are the corresponding angles to α, β, γ respectively, then

$$\alpha \geqq \tilde{\alpha}, \qquad \beta \geqq \tilde{\beta}, \qquad \gamma \geqq \tilde{\gamma}.$$

So

$$\tilde{\alpha} + \tilde{\beta} \leqq \frac{\pi}{2} + \varepsilon' \quad \text{and} \quad \tilde{\gamma} \leqq \frac{\pi}{2}.$$

Thus, using $\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} = \pi$ we have

$$\frac{\pi}{2} - \varepsilon' \leqq \tilde{\gamma} \leqq \frac{\pi}{2}.$$

On the other hand, from Sine formula

$$\frac{\sin \tilde{\gamma}}{s_2} = \frac{\sin \tilde{\alpha}}{s_1}.$$

So

$$\sin \tilde{\alpha} = \frac{s_1}{s_2} \sin \tilde{\gamma} \geqq m(\theta) \cdot \sin\left(\frac{\pi}{2} - \varepsilon'\right).$$

Thus we have

$$\sin^{-1}\left(m(\theta) \cdot \sin\left(\frac{\pi}{2} - \varepsilon'\right)\right) \leqq \alpha \leqq \frac{\pi}{2} + \varepsilon',$$

where $\sin^{-1}(\quad)$ is the principal value. Since ε' is arbitraly letting $\varepsilon' \to 0$, we have

$$\sin^{-1}(m(\theta)) \leqq \alpha \leqq \frac{\pi}{2}, \quad \text{i.e.}$$

$$\sin^{-1}(m(\theta)) \leqq \measuredangle(-\dot{\gamma}_{ti}^+(s_i), \dot{c}(0)) \leqq \frac{\pi}{2}.$$

So together with the case $s_0 < 0$, we have

$$\measuredangle(-\dot{\gamma}_{ti}^{+}(s_i),\ \dot{c}(0)) \geqq \sin^{-1}(m(\theta))$$

and
$$\measuredangle(-\dot{\gamma}_{ti}^{+}(s_i),\ -\dot{c}(0)) \geqq \sin^{-1}(m(\theta)).$$

Similary for γ_{ti}^{-}, we have

$$\measuredangle(-\dot{\gamma}_{ti}^{-}(s_i),\ \dot{c}(0)) \geqq \sin^{-1}(m(\theta))$$

and
$$\measuredangle(-\dot{\gamma}_{ti}^{-}(s_i),\ -\dot{c}(0)) \geqq \sin^{-1}(m(\theta)).$$

So
$$\measuredangle(-\dot{\gamma}_{ti}^{+}(s_i),\ -\dot{\gamma}_{ti}^{-}(s_i)) \leqq 2\left(\frac{\pi}{2} - \sin^{-1}(m(\theta))\right).$$

Now if $i \to \infty$, then $\theta \to 0$ and hence $m(\theta) \to 1$. Thus

$$2\left(\frac{\pi}{2} - \sin^{-1}(m(\theta))\right) \to 0.$$

q.e.d.

Now, let $\Delta(t_i)$ be the compact domain surrounded by γ_{ti}^{+} and γ_{ti}^{-} contained in $\bar{D}_\lambda$. Applying Gauss-Bonnet Theorem to $\Delta(t_i)$, we have

$$\int_{\Delta(t_i)} K\,dv = \measuredangle(\dot{\gamma}_{ti}^{+}(0),\ \dot{\gamma}_{ti}^{-}(0)) + \measuredangle(-\dot{\gamma}_{ti}^{+}(s_i),\ -\dot{\gamma}_{ti}^{-}(s_i)).$$

If $t_i \to \infty$, then $\Delta(t_i) \to \bar{D}_\lambda$. Thus from Lemma 2, we have

Lemma 3. $\int_{\bar{D}_\lambda} K\,dv = \measuredangle(\dot{\gamma}_1^\lambda(0),\ \dot{\gamma}_2^\lambda(0)) = \text{measure } F_\lambda.$

From the choice of the point $q \in (Q_{r_{j_0}})^c$, any ray starting from q does not meet $Q_{r_{i_0}}$. So we can find D_{λ_0} such that $Q_{r_{i_0}} \subset D_{\lambda_0}$. Thus we have

$$\begin{aligned}
\int_M K\,dv - \varepsilon &\leqq \int_{Q_{i_0}} K\,dv \\
&\leqq \int_{\bar{D}_{\lambda_0}} K\,dv \\
&\leqq \sum_\lambda \int_{\bar{D}_\lambda} K\,dv \\
&= \sum_\lambda \text{measure } F_\lambda \\
&= \text{measure } \bigcup_\lambda F_\lambda \\
&= \text{measure } (S_q(M) - A(q)) \\
&= 2\pi - \text{measure } A(q).
\end{aligned}$$

That is,

$$\text{measure } A(q) \leqq 2\pi - \int_M K\,dv + \varepsilon.$$

Hence
$$\inf_{q \in M} \text{measure } A(q) \leqq 2\pi - \int_M K\,dv. \qquad \text{q.e.d.}$$

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