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On the Green Function of a Complete Riemannian or Kähler Manifold with Asymptotically Negative Constant Curvature and Applications

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Introduction

In this paper we shall consider a complete noncompact Riemannian or kähler manifold whose curvature tensor is asymptotically close to that of the real or complex space form of negative curvature. Examples of such a manifold are supplied by strictly convex bounded domains with a certain metric in R^n and by strictly pseudoconvex bounded domains with the Bergman metric in C^n (see § 5 and Appendix B).

Our main concern is to find an asymptotic estimate of the Green function of such a manifold. The result is that it behaves just like the Green function of space forms (Theorems 2, 4).

As an application we give a differential geometric proof of Malliavin's estimate ([22]) of the Green function of a strictly pseudoconvex bounded domain relative to the Bergman metric (Corollary 1 in § 5). Namely, let D be such a domain with the smooth boundary ∂D and $G(p, q)$ be the Green function. Fix a point q. Then for some constants c_i, the inequalities

$$c_1 d_E(p, \partial D)^n \leqq G(p, q) \leqq c_2 d_E(p, \partial D)^n$$
$$|\nabla_p G(p, q)| \leqq c_3 d_E(p, \partial D)^n$$

are valid for all p away from q. Here $d_E(p, \partial D)$ is the euclidean distance to ∂D. Unfortunately our proof needs some assumption on the metric, which probably restricts the topological type of the domain.

Another application in the *real* case is to construct bounded harmonic functions (Corollary 2 in § 6). For that purpose we will give a geometric description of the Martin boundary and solve the Dirichlet problem for harmonic functions relative to this boundary (Theorem 8). In this case the curvature is assumed to be strictly negative and asymptotically negative

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constant. The author thinks that the construction of bounded harmonic functions in the kähler case will be proved in this approach.

Finally we mention some technical tools used in this paper. The estimate of the Green function is established relying on the estimate of the Laplacian of the distance function. And, as usual, the latter is reduced to the study of the Jacobi equation; the problem is to study asymptotic behavior of solutions of such a system with asymptotically constant coefficients.

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§ 1. Laplacian of the distance function

In this section we will state known results on the Laplacian of the distance function on a complete Riemannian manifold following mainly A. Kasue's paper [17].

Let M be a complete noncompact Riemannian manifold of dimension n. Let T_pM denote the tangent space of M at p and $\langle\ ,\ \rangle$ the Riemannian inner product. Set $\|X\|^2=\langle X, X\rangle$. For a C^2-function f on M its hessian $\nabla^2 f$ is defined by $\nabla^2 f(X, Y)=\nabla_X\nabla_Y f-(\nabla_X Y)f$. The trace of $\nabla^2 f$ is the Laplacian of f and denoted by Δf.

Let $R(X, Y)$ be the curvature operator. The sectional curvature of the plane spanned by independent tangent vectors X and Y is denoted by $K(X, Y)$. Namely, $\langle R(X, Y)Y, X\rangle=K(X, Y)\|X\wedge Y\|^2$. The Ricci curvature in the direction X is denoted by $\mathrm{Ric}(X)$. Other undefined terms and some properties used below are easily referred in the book [4] by Cheeger-Ebin.

Now let N be a hypersurface of M and γ be a geodesic in M starting at $q\in N$, $\gamma\colon [0, l]\to M$, $\gamma(0)=q$. The parameter t of γ is always assumed to be the length parameter. By V_γ we denote the set of all vector fields $Y(t)$ along γ with the properties $Y(0)\in T_qN$ and $\langle Y, \dot\gamma\rangle=0$. Let α denote the second fundamental form of N. Then the index form is defined by

$$I(X, Y)=\alpha(X(0), Y(0))+\int_0^l \{\langle \nabla_{\dot\gamma}X, \nabla_{\dot\gamma}Y\rangle-\langle R(X, \dot\gamma)\dot\gamma, Y\rangle\}dt$$

for X, Y in V_γ.

Let $\rho(p)=d(p, N)$ denote the distance function. Fix p and let γ be the geodesic realizing the distance $d(p, N)$. Assume the exponential mapping is a diffeomorphism near some convex open neighborhood in $T_pN^\perp$ of the preimage of γ under the exponential mapping. Then there is no focal point of N along γ and ρ is differentiable at p. We define functions $r(t)$ and $k(t)$ by the equations

$$r(t)=\mathrm{Ric}\,(\dot{\gamma})/(n-1). \tag{1.1}$$

$$k(t)=\max\{K(X,\dot{\gamma});\ X\in T_{\gamma}M,\ \langle X,\dot{\gamma}\rangle=0\}. \tag{1.2}$$

Define functions $f(t)$ and $g(t)$ as the solutions of the following equations respectively:

$$f''+rf=0 \ \text{ with } f(0)=1 \text{ and } f'(0)\geqq \mathrm{trace}\,\alpha_q/(n-1), \tag{1.3}$$

$$g''+kg=0 \text{ with } g(0)=1 \text{ and } g'(0)\leqq \min\{\text{eigenvalues of } \alpha_q\}. \tag{1.4}$$

Then we have the following lemmas.

Lemma 1. ([17]; cf. [12], [24]) $\Delta\rho\leqq(n-1)f'(\rho)/f(\rho)$.

Lemma 2. ([17]) *Assume* $g(t)>0$ *for all* t. *Then*

$$\nabla^2\rho(X,X)\geqq\{\|X\|^2-\langle\dot{\gamma},X\rangle^2\}g'(\rho)/g(\rho).$$

The proof will now be sketched. Let e_i, $2\leqq i\leqq n$, be parallel vector fields along γ so that $\{\dot{\gamma}, e_2, \cdots, e_n\}$ is an orthonormal basis. Let Y_i be N-Jacobi fields such that $Y_i(l)=e_i(l)$. Then it is known that $\Delta\rho=\sum_{i=2}^n I(Y_i,Y_i)$. First, note that f is positive. In fact, by definition

$$\sum I_s(fe_i,fe_i)=\alpha(fe_i(0),fe_i(0))+\int_0^s\{\|\nabla_{\dot{\gamma}}fe_i\|^2-K(fe_i,\dot{\gamma})|f|^2\}dt.$$

Here s is an arbitrary value in $(0,l]$ and the indices I_s are taken with respect to the geodesic $\gamma|_{[0,s]}$. By the conditions (1.1) and $f(0)=1$, we have

$$\sum I_s(fe_i,fe_i)\leqq \mathrm{trace}\,\alpha_q+(n-1)\int_0^s(f'^2-rf^2)dt.$$

In view of the equation (1.3), we have

$$\sum I_s(fe_i,fe_i)\leqq(n-1)f'(s)f(s).$$

Since the left hand side is positive by the condition the N has no focal point along γ, we see $f(s)\neq 0$. This makes possible to consider the vector field $(f(t)/f(l))e_i$. Then, by the index lemma ([4], p.24),

$$I(Y_i,Y_i)\leqq I(fe_i/f(l),fe_i/f(l)).$$

Hence we have $\sum I(Y_i,Y_i)\leqq(n-1)f'(l)/f(l)$. This shows Lemma 1.

The proof of Lemma 2 is done in the same way. We may assume $\langle X, \dot\gamma\rangle=0$. Let Y be the N-Jacobi field along γ such that $Y(l)=X$. Set $y(t)=\|Y(t)\|$. By the Schwarz inequality $|y'(t)|\leqq\|\nabla_{\dot\gamma}Y(t)\|$, and by definition

$$I(Y, Y)\geqq\alpha(Y(0), Y(0))+\int_0^l\{y'^2-K(Y, \dot\gamma)y^2\}dt. \tag{1.5}$$

Since $(g'y^2/g)'=y'^2-ky^2-(g'y/g-y')^2$, we can estimate the righthand side of (1.5) from below by $\alpha(Y(0), Y(0))+(g'y^2/g)|_0^l$, which is not smaller than $(g'(l)/g(l))\|X\|^2$ by (1.4). Since $\nabla^2\rho(X, X)=I(Y, Y)$, we have Lemma 2.

Remark 1. The Laplacian and Hessian of the distance function from a point o instead of from a hypersurface can be estimated analogously under the assumption that the exponential mapping at o is a diffeomorphism, i.e. o is a pole. Namely, fixing a geodesic $\gamma(t)$ from o, we define functions $k(t)$, $r(t)$ by (1.1) and (1.2). As for the functions $f(t)$, $g(t)$ we impose another initial conditions

$$f''+rf=0 \quad \text{with } f(0)=0, \quad f'(0)=1, \tag{1.3$'$}$$

$$g''+kg=0 \quad \text{with } g(0)=0, \quad g'(0)=1. \tag{1.4$'$}$$

Then, for the distance function $\rho(p)=d(o, p)$, one has the same statements as in Lemmas 1 and 2. More generally, as is done by Kasue, we can do calculations concerning the distance function from a closed submanifold of arbitary dimension. See [17] on these matters.

With Remark 1 in mind we set $\rho(p)=d(p, *)$ where $*=$a hypersurface N or one point o. Suppose the differentiability of ρ and $g(t)>0$ for all t. Then, making a direct use of these Lemmas, we have

Proposition 1. *For any non-increasing C^2-function ψ on* $[0, l]$, *the function $\psi(\rho(p))$ satisfies*

$$(\psi''+(n-1)\psi'f'/f)(\rho)\leqq\Delta\psi(\rho)\leqq(\psi''+(n-1)\psi'g'/g)(\rho).$$

Proof. This is seen because of $\Delta\psi(\rho)=\psi''+\psi'(\rho)\Delta\rho$ and the non-increasing property of ψ.

§ 2. Green function of a complete Riemannian manifold

Let M be a complete noncompact Riemannian manifold. Let $G(p, q)$ be the Green function of M if it exists. The aim of this section is to give estimates of $G(p, q)$ at infinity applying Proposition 1.

Let N be a closed hypersurface bounding a compact set B. We call the part $M-B$ the outward. We have made the assumption:

(A.1) The exponential mapping restricted to the set of outward normal vectors to N is a diffeomorphism.

In this section we need one more assumption:

(A.2) N is convex outward in the sense that the second fundamental form is positive definite with respect to the outward normals.

(A.2) is satisfied when N is a geodesic sphere and the sectional curvature is non-positive. (A.1) is satisfied when the sectional curvature is non-positive outside B under the assumption (A.2).

Let r and k be functions defined in (1.1) and (1.2) for a geodesic from $q \in N$. We put q to denote the reference point: r_q, k_q. Now define

$$r(t)=\min_{q\in N} r_q(t), \qquad k(t)=\max_{q\in N} k_q(t). \tag{2.1}$$

Next, define functions f and g as the solutions of the equations

$$f''+rf=0 \quad \text{with } f(0)=1, \quad f'(0)\geqq(\max_{q\in N} \text{trace } \alpha_q)/(n-1), \tag{2.2}$$

$$g''+kg=0 \quad \text{with } g(0)=1, \quad g'(0)\leqq(\min_{q\in N} \{\text{min eigenvalues of } \alpha_q\}. \tag{2.3}$$

We will in the following assume the conditions

$$f \text{ and } g \text{ are defined on } [0, \infty), \tag{2.4}$$

$$g(t) \text{ is positive for all } t, \tag{2.5}$$

and

$$\alpha_1=\int_0^\infty f^{1-n}dt \quad \text{and} \quad \alpha_2=\int_0^\infty g^{1-n}dt \quad \text{are finite.} \tag{2.6}$$

Now define a new function $\psi_1(t)$ by

$$\psi_1(t)=\frac{1}{\alpha_1}\int_t^\infty f(s)^{1-n}ds.$$

Then $\psi_1(0)=1$ and $\psi_1=0$ at infinity. Set $h_1(p)=\psi_1(\rho(p))$. Proposition 1 implies

$$\Delta h_1\geqq 0. \tag{2.7}$$

Similarly define $\psi_2(t)$ by

$$\psi_2(t)=\frac{1}{\alpha_2}\int_t^\infty g(s)^{1-n}ds,$$

and set $h_2(p)=\psi_2(\rho(p))$. Again by Proposition 1,

$$\Delta h_2 \leqq 0. \tag{2.8}$$

Next set $c_1=\inf\{G(o,p); p\in N\}$ and $c_2=\sup\{G(o,p); p\in N\}$. Both are positive and we have

$$c_1h_1 \leqq G(o,p) \leqq c_2h_2 \tag{2.9}$$

on the boundary N and at infinity. Hence we have by the maximum principle

Theorem 1. *Under the assumptions* (A.1), (A.2) *and the conditions* (2.4)–(2.6), *there exist constants* c_1 *and* c_2 *such that*

$$c_1h_1(p) \leqq G(o,p) \leqq c_2h_2(p) \qquad \textit{for all } p\in M-B.$$

Remark 2. (1) On any geodesic ball with center o and containing B we can obtain the likewise estimate of the Green function of this ball. Since the limit of this Green function when the radius diverges is the Green function of M with pole at o and the condition (2.6) assures the existence of this limit, it is not necessary to assume the existence of the Green function in advance. (2) Obviously constants c_1 and c_2 depend on the point o. But, as far as the point o remains in a compact set, these constants can be chosen dependent only on this set.

In the situation that M is a manifold with a pole, we have the estimates of the same kind using the estimates of the Laplacian of the distance function from one point. In order to state this estimate, we change some of definitions. Fix one point o. Let r and k be functions defined in (1.1) and (1.2) for a geodesic γ from o. Denote these by r_γ and k_γ. Set

$$r(t)=\min_\gamma r_\gamma(t), \qquad k(t)=\max_\gamma k_\gamma(t). \tag{2.1'}$$

Define functions f and g as the solutions of the equations

$$f''+rf=0 \quad \text{with } f(0)=0, \quad f'(0)=1, \tag{2.2'}$$

$$g''+kg=0 \quad \text{with } g(0)=0, \quad g'(0)=1. \tag{2.3'}$$

Instead of ψ_1, ψ_2 we define

$$\phi_1(t)=\frac{1}{\omega_{n-1}}\int_t^\infty f^{1-n}dt, \qquad \phi_2(t)=\frac{1}{\omega_{n-1}}\int_t^\infty g^{1-n}dt,$$

where ω_{n-1} is the volume of the euclidean $(n-1)$-dimensional unit sphere. Then we have

Theorem 1′ ([17]). *Suppose the point o is a pole and the functions $\phi_1(t)$, $\phi_2(t)$ are finite for $t>0$. Then, under the conditions* (2.4) *and* (2.5),

$$\phi_1(\rho(p)) \leqq G(o, p) \leqq \phi_2(\rho(p)).$$

For the proof one only needs to notice the singularity of $G(o, p)$: $G(o, p) \sim d(o, p)^{-n+2}/((n-2)\omega_{n-1})$ $(n \geqq 3)$ or $-1/2\pi \log d(o, p)$ $(n=2)$. Then the theorem follows from the maximum principle in view of Proposition 1.

§ 3. Complete manifold with asymptotically negative constant curvatures

Theorems 1 and 1′ show that some curvature conditions imply the existence of the Green function and restrict the order of decay. Let us recall that on a simply connected Riemannian manifold there always exists the Green function if its sectional curvature is non-positive $(n \geqq 3)$ or strictly negative $(n=2)$ ([1] or Theorem 1′). Moreover, when the sectional curvatures are bounded by negative constants from both sides, the Green functions are estimated in terms of the Green function of the unit ball with the constant curvature metric ([7]). This follows easily from Theorem 1.

In this section we will give the condition on the sectional curvature so that the functions h_1 and h_2 in Theorem 1 have the same order at infinity.

Let M be a complete noncompact Riemannian manifold and choose a point p. For a negative constant $-c^2$, we put

$$\chi(p; X, Y) = |K(X, Y) + c^2|$$

for $X, Y \in T_pM$ and put

$$\chi(p) = \max\{\chi(p; X, Y);\ X, Y \in T_pM\}. \tag{3.1}$$

Let $\gamma(t)$ be a geodesic in M tending to infinity, i.e. not remaining in any compact set and define

$$\chi_\gamma(t) = \max\{\chi(\gamma(t); X, \dot{\gamma});\ X \in T_{\gamma(t)}M\}. \tag{3.2}$$

Then

Definition 1. A complete noncompact Riemannian manifold is called of asymptotically negative constant curvature $-c^2$ if the sectional curvature is non-positive outside some compact set and

(C.1) $$\int^{\infty} \chi_{\gamma}(t)dt < \infty$$

for all geodesics γ tending to infinity.

The meaning of the condition (C.1) will be observed in the next lemma and the following arguments.

Let us consider the differential equation of the form

(3.3) $$y'' - (\lambda^2 + \chi(t))y = 0.$$

Here $\chi(t)$ is defined and continuous for $0 \leqq t < \infty$.

Lemma 3. *Assume* $\int_0^{\infty} |\chi| dt$ *is finite. Set* $\phi(t) = \int_t^{\infty} |\chi(s)| ds$. *Then,* (1) *Any solution of* (3.3) *can be written in the form*

(3.4) $$ae^{\lambda t}(1 + A(t)) + be^{-\lambda t}(1 + B(t))$$

where a, b are constants and $A(t)$, $B(t)$ are functions which tend to zero as $t \to \infty$. (2) *There exists a constant C such that*

(3.5) $$|A(t)|, |B(t)| \leqq C(e^{-\lambda t} + \phi(t/2)).$$

This lemma is a special case of Theorem 5.4.5 in Hille's book [14]; see Lemma 6. For the convenience of the reader we will reproduce the proof in Appendix A.

Now we shall apply this lemma to the estimate of the Green function of a complete noncompact Riemannian manifold with asymptotically negative constant curvature $-c^2$. Let M be such a manifold. We use notations in Section 2. As a hypersurface N we take a geodesic sphere $S = S(o, \rho)$ with center o and radius ρ for the sake of simplicity. Choose ρ sufficiently large so that the sectional curvature is non-positive outside the ball $B = B(o, \rho)$. Assume (A.2). Then the function $g(t)$ defined by (2.3) can be supposed to have the initial condition $g'(0) > 0$. Moreover we have

Proposition 2. *The functions $f(t)$ and $g(t)$ are positive and increasing for all t.*

This is seen by the next lemma.

Lemma 4. *Let $a(t)$ be defined and continuous for $t \in [0, \infty)$ and $h(t)$ be the solution of $h'' - ah = 0$ with $h(0) = 1$ and $h'(0) > 0$. Moreover suppose $a \geqq 0$. Then $h(t)$ is positive and increasing for all t.*

Proof. Set $h_1(t)=1+h'(0)t$. We have $h'h_1-hh_1'=\int_0^t -ahh_1\,dt$. If $h(t_0)=0$ for some $t_0>0$ for the first time, then $h'(t_0)h_1(t_0)\geqq 0$. Since $h'(t_0)<0$ and $h_1(t_0)>0$, this is a contradiction. Hence $h(t)>0$. Then $h''(t)=-ah$ is non-negative always and, hence, $h'(t)$ is positive. This implies that h is increasing.

On the other hand, by Lemma 3, f and g have the form (3.4). Choosing the radius ρ sufficiently large if necessary we may assume $|A(t)|<1$, where $A(t)$ is the function used in Lemma 3. Then the coefficient a in (3.5) must be positive by the increasing property of f and g. This means that both f and g increase like ae^{ct}. Hence the condition (2.6) in Section 2 is satisfied and Theorem 1 will imply in the present case

Theorem 2. *Let M be a complete noncompact Riemannian manifold of asymptotically negative constant curvature $-c^2$. Suppose every geodesic sphere $S(o, \rho)$ is strictly convex for a sufficiently large ρ,* (A.2). *Let K be a compact set in $B(o, \rho)$. Then there exist positive constants c_1 and c_2 such that*

$$c_1 e^{-c(n-1)d(q,p)} \leqq G(q, p) \leqq c_2 e^{-c(n-1)d(q,p)}$$

for p in $M-B$ and q in K.

Proof. Theorem 1 tells us that, for some constants c_1' and c_2',

$$c_1' \int_t^\infty f(s)^{1-n} ds \leqq G(o, \gamma(t)) \leqq c_t' \int_t^\infty g(s)^{1-n} ds.$$

By the asymptotic constancy of the curvature, we have seen

$$f(t)=a_1 e^{ct}(1+o(1))+o(1) \quad \text{and} \quad g(t)=a_2 e^{ct}(1+o(1))+o(1).$$

Then

$$\lim e^{c(n-1)t} G(o, \gamma(t)) \leqq \frac{c_2'}{c(n-1)} \lim e^{c(n-1)t} g(t)^{1-n} < \infty.$$

This implies the right side inequality (see Remark 2 (2)). The left side is proved similarly.

Remark 3. In the statement of the theorem it is not necessary to assume (A.2) for a geodesic sphere. It is sufficient to assume the existence of an arbitrary large hypersurface containing o and satisfying (A.2). This remark is valid also for Theorems 3 and 4 in Section 4.

Remark 4. Theorem 2 can be proved using Theorem 1′ under the assumption that the sectional curvatures are always non-positive instead of (A.2).

Remark 5 (Gradient estimate of the Green function). In [5] Cheng and Yau proved the next theorem: (a special case of Theorem 6 in p. 350) *Let M be a n-dimensional complete Riemannian manifold. Let f be a non-negative harmonic function defined on a geodesic ball $B(a)$ of redius a. Then we can find a constant c_n depending only on n such that*

$$|\nabla f(x)|\leqq c_n f(x)\left(\frac{a^2}{a^2-r^2}\right)(|K|+a^{-1})$$

where r is the distance from x to the center of $B(a)$ and K is the lower bound of the Ricci curvature on $B(a)$. From this theorem and Theorem 2 we have

$$|\nabla_p G(q,p)|\leqq c_3 e^{-(n-1)cd(p,q)}(1+d(p,q)^{-1})$$

for some $c_3>0$, if we assume further that the Ricci curvature is bounded from below.

Example. The real hyperbolic space form obviously satisfies the conditions in Theorem 2. It is classical that this space form has a realization as a unit ball with the Hilbert metric, so-called the Klein model. Generalizing this model, Loewner and Nirenberg defined on any strictly convex bounded domain Ω in R^n a canonical complete metric in terms of the unique negative convex solution u of the equation: $\det u_{ij}=(-u)^{-n-2}$ on Ω, $u=0$ on $\partial\Omega$. ([20], [6]). The metric is $-u^{-1}d^2u$. But the boundary regularity of u is not still well known even if the boundary $\partial\Omega$ is smooth. So we here consider another metric which seems somewhat artificial but looks like the above metric and, moreover, becomes equivalent to this if the boundary regularity of u^2 is established at the third order of differentiability. Namely we let $\Omega=\{\phi<0\}$ be such a domain with ϕ strictly convex and $d\phi\neq 0$ at the boundary. Set $v=\sqrt{-\phi}$ and define $ds^2=-v^{-1}d^2v$. When $\phi=|x|^2-1$ for example and Ω is the unit ball, ds^2 is the Hilbert metric. Calculations show that the curvature function K satisfies $|K+1|=O(\phi)$ near the boundary. Moreover we can see that every geodesic not remaining in any compact set tends to the boundary and touches transversally the boundary. These facts imply that the manifold Ω with ds^2 is of asymptotically negative constant curvature -1 and Theorem 2 holds for Ω. Calculations will be given in Appendix B.

§ 4. The kähler case

In this section M will be a complete kähler manifold of complex dimension n. J will denote the complex structure tensor. In order to define the asymptotic constancy of curvature in this case we will give some notations.

Let D be the unit ball with constant holomorphic sectional curvature $-c^2$. Choose a point $o \in D$ and denote by K_D the sectional curvature function at o. Let p be any point of M and fix arbitrarily a unitary isomorphism θ between T_pM and T_oD. Then we set

$$\chi(p; X, Y)=|K(X, Y)-K_D(\theta X, \theta Y)|$$

for $X, Y \in T_pM$ and

$$\chi(p)=\max\{\chi(p; X, Y); X, Y \in T_pM\}. \tag{4.1}$$

Let next $\gamma(t)$ be a divergent geodesic in M and define

$$\chi_\gamma(t)=\max\{\chi(\gamma(t); X, \dot{\gamma}); X \in T_{\gamma(t)}M\}. \tag{4.2}$$

Then we can state

Definition 2. A complete noncompact kähler manifold is called of asymptotically negative constant curvature $-c^2$, if the sectional curvature is nonpositive outside a compact set and, for any diverging geodesic γ, the function $\chi_\gamma(t)$ tends to zero and satisfies

$$\int^\infty \chi_\gamma(t)dt<\infty. \tag{C.1}$$

We are now in the same situation as in the preceding section except the following. The difference is seen in the limit values of r and k. Namely in the present case

$$\lim_{t\to\infty} r(t)=-((n+1)/(4n-2))c^2, \qquad \lim_{t\to\infty} k(t)=-c^2/4.$$

Hence we have the estimates of the Green function

$$c_1e^{-\sqrt{(n+1)(n-1/2)}cd(p,q)}\leqq G(p, q)\leqq c_2e^{-(n-1/2)cd(p,q)} \tag{4.3}$$

for some constants c_1 and c_2. But we can sharpen these estimates reasoning more carefully the treatments in Section 1 and Section 2.

Let M be as above. Recall the notations in Section 1. N is a real closed hypersurface in M. Set $\rho(p)=d(p, N)$. $\gamma(t)$ is a geodesic starting from one point q in N defined for $t \in [0, l]$, and $\dot{\gamma}(0)$ is normal to N. α_q is the second fundamental form of N at q. $X(t)$ is a vector field along γ. Define functions $\underline{s}_q(t)$ and $h_q(t)$ by

$$\underline{s}_q = \min\{K(\dot{\gamma}, X) \text{ with } \langle X, \dot{\gamma}\rangle = 0 \text{ and } \langle X, J\dot{\gamma}\rangle = 0\},$$
$$h_q = K(\dot{\gamma}, J\dot{\gamma}).$$

Set $\lambda_q = \max\{\text{eigenvalues of } \alpha_q\}$. Let $a_1(t)$ and $b_1(t)$ be any functions satisfying differential equations

$$a_1'' + \underline{s}_q(t) a_1 = 0 \quad \text{with} \quad a_1(0) = 1 \quad \text{and} \quad a_1'(0) \geqq \lambda_q,$$
$$b_1'' + h_q(t) b_1 = 0 \quad \text{with} \quad b_1(0) = 1 \quad \text{and} \quad b_1'(0) \geqq \lambda_q.$$

Now let e be a parallel vector field along γ such that $\|e\| = 1$, $\langle e, \dot{\gamma}\rangle = 0$ and $\langle e, J\dot{\gamma}\rangle = 0$, and Y be the N-Jacobi field satisfying $Y(l) = e(l)$. Then we have seen already in Section 1 that

$$I(Y, Y) \leqq I(a_1/a_1(l)e,\ a_1/a_1(l)e) \leqq a_1'(l)/a_1(l). \tag{4.4}$$

If we denote by Z the N-Jacobi field satisfying $Z(l) = J\dot{\gamma}(l)$, we can see

$$I(Z, Z) \leqq b_1'(l)/b_1(l). \tag{4.5}$$

The inequalities (4.4) and (4.5) imply

Lemma 5. $\Delta\rho(\gamma(l)) \leqq (2n-2) a_1'(l)/a_1(l) + b_1'(l)/b_1(l)$.

The proof is similar to that of Lemma 1. Next, following Section 2, we set

$$\underline{s}(t) = \min_q \underline{s}_q(t) \quad \text{and} \quad h(t) = \min_q h_q(t)$$

and define functions a and b by

$$a'' + \underline{s}a = 0 \quad \text{with} \quad a(0) = 1 \quad \text{and} \quad a'(0) = \max_q \lambda_q$$
$$b'' + hb = 0 \quad \text{with} \quad b(0) = 1 \quad \text{and} \quad b'(0) = \max_q \lambda_q.$$

Moreover we define

$$\psi(t) = \frac{1}{\alpha}\int_t^\infty a^{2(1-n)} b^{-1} dt,$$

where $\alpha = \int_0^\infty a^{2(1-n)} b^{-1} dt$ is supposed to be finite. Then we have the estimate

$$c'\psi(\rho(p)) \leqq G(o, p) \tag{4.6}$$

as in Theorem 1. It is necessary, of course, to assume (A.1) and (A.2).

Let us now proceed to the case of asymptotically negative constant curvature $-c^2$. In this case $\underline{s}$ and h have limits:

$$\lim_{t\to\infty} \underline{s}(t) = -c^2/4 \quad \text{and} \quad \lim_{t\to\infty} h(t) = -c^2.$$

Hence the functions a and b have asymptotic behavior such as $e^{ct/2}$ and e^{ct} respectively up to positive constants. These imply that

$$-\psi' \sim e^{-nct}.$$

Therefore, with (4.3) and (4.6), we have

Theorem 3. *Let M be a complete noncompact kähler manifold with asymptotically negative constant curvature $-c^2$. Assume the strict convexity of a geodesic sphere with sufficiently large radius. Let K be a compact set. Then there exist positive constants c_1 and c_2 such that*

$$c_1 e^{-ncd(p,q)} \leqq G(p, q) \leqq c_2 e^{-(n-1/2)cd(p,q)} \tag{4.7}$$

for p away from K and q in K.

In the rest of this section we shall give the condition to improve the upper estimate. The problem is to improve the lower estimate of $\Delta\rho$. For the sake of simplicity we take a geodesic sphere S as a hypersurface N. Suppose the radius is sufficiently large so that the curvature is nonpositive outside the ball B and suppose S is strictly convex, (A.2). $\gamma(t)$ is a geodesic from $q \in S$ defined for $t \in [0, \infty)$. Fix a positive number l. Let $Y^l(t)$ denote any one of S-Jacobi fields along $\gamma|_{[0,l]}$ such that $\langle Y^l(l), \dot\gamma(l)\rangle = \langle Y^l(l), J\dot\gamma(l)\rangle = 0$ and $Z^l(t)$ be the S-Jacobi field along $\gamma|_{[0,l]}$ such that $Z^l(l) = J\dot\gamma(l)$. Define functions $\bar{s}^l_q$ and k^l_q by

$$\bar{s}^l_q(t) = \max_{Y^l} \{K(\dot\gamma(t), Y^l(t))\} \quad \text{and} \quad k^l_q(t) = K(Z^l(t), \dot\gamma(t)),$$

and set

$$\bar{s}_q(t) = \max_l \bar{s}^l_q(t) \quad \text{and} \quad k_q(t) = \max_l k^l_q(t).$$

Also set $\mu_q = \min\{\text{eigenvalues of } \alpha_q\} > 0$. If we define functions c_1 and d_1 by equations

$$c_1'' + \bar{s}^l_q c_1 = 0 \quad \text{with} \quad c_1(0) = 1 \quad \text{and} \quad 0 < c_1'(0) \leqq \mu_q,$$
$$d_1'' + k^l_q d_1 = 0 \quad \text{with} \quad d_1(0) = 1 \quad \text{and} \quad 0 < d_1'(0) \leqq \mu_q,$$

then by the reasoning in Section 1 for Lemma 2, we have

$$I(Y^l, Y^l) \geqq c_1'(l)/c_1(l) \quad \text{and} \quad I(Z^l, Z^l) \geqq d_1'(l)/d_1(l).$$

Note that c_1 and d_1 are positive by the assumption on curvature. Moreover if we define functions c_2 and d_2 by equations

$$c_2'' + \bar{s}_q c_2 = 0 \quad \text{with} \quad c_2(0) = 1 \quad \text{and} \quad c_2'(0) = c_1'(0),$$
$$d_2'' + k_q d_2 = 0 \quad \text{with} \quad d_2(0) = 1 \quad \text{and} \quad d_2'(0) = d_1'(0),$$

then, by the Sturm-Liouville comparison theorem, we have

$$c_1'(l)/c_1(l) \geqq c_2'(l)/c_2(l) \quad \text{and} \quad d_1'(l)/d_1(l) \geqq d_2'(l)/d_2(l).$$

Hence we have shown

Lemma 5′. $\Delta\rho(\gamma(l)) \geqq (2n-2)c_2'(l)/c_2(l) + d_2'(l)/d_2(l)$.

Next, set

$$k(t) = \max_q k_q(t) \quad \text{and} \quad \bar{s}(t) = \max_q \bar{s}_q(t), \tag{4.8}$$

and define functions c and d by equations

$$\begin{aligned} &c'' + \bar{s}c = 0 \quad \text{with} \quad c(0) = 1 \quad \text{and} \quad c'(0) = \min_q \mu_q, \\ &d'' + kd = 0 \quad \text{with} \quad d(0) = 1 \quad \textit{and} \quad d'(0) = \min_q \mu_q. \end{aligned} \tag{4.9}$$

Here we have supposed $\min \mu_q > 0$. Then, defining a function ψ_1 by

$$\psi_1 = \frac{1}{\alpha_1} \int_t^\infty c^{2(1-n)} d^{-1} dt$$

where $\alpha_1 = \int_0^\infty c^{2(1-n)} d^{-1} dt$ is supposed to be finite, we have by Lemma 5′ before the estimate

$$G(o, p) \leqq c'' \psi_1(\rho(p)). \tag{4.10}$$

In order to estimate $\psi_1(t)$, we quote a lemma on a system of differential equations:

$$\underline{Y}'(t) = (A_0 + A_1(t))\underline{Y}(t) \tag{4.11}$$

where $\underline{Y}(t)$ is a n-vector, A_0 and A_1 are n by n matrices and A_0 is constant.

Lemma 6 ([14], Theorem 5.4.5 and [19], Theorem 2). *Suppose A_0 is diagonalizable with real eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n$ and that $\int^\infty \|A_1(t)\| dt < \infty$.*

Then (1) *the equation* (4.11) *has a solution of the form*

$$\underline{Y}(t)=\sum_{k=1}^{n}(E_k+R_k(t))e^{\lambda_k t} \tag{4.12}$$

where E_k *is the eigenvector of* A_0 *belonging to* λ_k *and* $\lim_{t\to\infty}\|R_k(t)\|=0$. (2) *Define* $\beta_k=(\lambda_k-\lambda_{l+1})$ *when* $\lambda_k=\cdots=\lambda_l>\lambda_{l+1}$ *and* $\beta_k=\infty$ *when* $\lambda_k=\cdots=\lambda_n$. *Then, for some constants* C_k,

$$\|R_k(t)\|\leqq C_k\left(\int_0^t \exp(-\beta_k(t-s))\|A_1(s)\|ds+\int_t^\infty \|A_1(s)\|ds\right). \tag{4.13}$$

Remark 6. The proof is done analogously to that of Lemma 3. It is not necessary to suppose that eigenvalues are real ([19]). The part (2) is contained in the proof of Theorem 2 in [19], p. 177.

We shall apply this lemma in the following situation. Choose parallel vector fields $e_3, \cdots, e_{2n}$ along γ so that $\{\dot{\gamma}, J\dot{\gamma}, e_3, \cdots, e_{2n}\}$ is an orthonormal frame. Let $Y=f_2J\dot{\gamma}+\sum_{i\geqq 3}f_ie_i$ be a vector field orthogonal to $\dot{\gamma}$. Define a vector function $\underline{Y}(t)=(f_2, f_2', \cdots, f_{2n}, f_{2n}')$. Then the Jacobi equation for Y is written as a system of differential equations of type (4.11). Here note that $\langle R(e_i, \dot{\gamma})\dot{\gamma}, J\dot{\gamma}\rangle=\langle R(e_i, \dot{\gamma})\dot{\gamma}, e_j\rangle(i\neq j)=0$ for the space of constant holomorphic sectional curvature. The eigenvalues of A_0 are c, $c/2, \cdots, c/2(2n-2\text{-times})$, $-c/2, \cdots, -c/2$ $(2n-2\text{-times})$, $-c$. The values c and $-c$ correspond to the direction $J\dot{\gamma}$ and others correspond to e_i. The absolute value of each component of $A_1(t)$ is bounded by a constant times of $\chi_\gamma(t)$, (4.1). Hence

$$\|A_1(t)\|\leqq c_1\chi_\gamma(t). \tag{4.14}$$

Lemma 6 then shows that there exist $4n-2$ linear independent solutions of the form $e^{ct}((1+r_2)J\dot{\gamma}+R_2)$, $e^{-ct}((1+s_2)J\dot{\gamma}+S_2)$ and $e^{ct/2}((1+r_i)e_i+R_i)$, $e^{-ct/2}((1+s_i)e_i+S_i)$, $3\leqq i\leqq 2n$; where r_i, s_i are functions and R_2, S_2, R_i, S_i are vectors orthogonal to $J\dot{\gamma}$, $J\dot{\gamma}$, e_i, e_i respectively. By Lemma 6, $|r_i|$, $|s_i|$, $\|R_i\|$, $\|S_i\|$ are all estimated like (4.13). Since the norms of S-Jacobi fields under question will diverge by the non-positivity of curvatures and the strict convexity of the geodesic sphere (cf. Lemma 4), we come to the situation that there exist $2n-1$ independent S-Jacobi fields

$$\begin{aligned} J_2&=e^{ct}((1+r_2)J\dot{\gamma}+R_2(t)),\\ J_i&=e^{ct/2}((1+r_i)e_i+R_i(t)), \qquad 3\leqq i\leqq 2n, \end{aligned} \tag{4.15}$$

where r_i and R_i are probably different from the above.

Let us fix l. Then

$$(e^{-cl}J_2, e^{-cl/2}J_3, \cdots, e^{-cl/2}J_{2n})=(J\dot{\gamma}(l), e_3(l), \cdots, e_{2n}(l))(1+E(l)),$$

where $E(l)$ is a $2n-1$ by $2n-1$ matrix whose components are linear combinations of $r_i(l)$ and the coefficients of $R_i(l)$. Choose a sufficiently large l so that $(1+E(l))$ is invertible, which is possible by Lemma 6. Then

$$\begin{aligned} J\dot{\gamma}(l) &= (1+\phi_2(l))e^{-cl}J_2(l)+\textstyle\sum_{j\geqq 3}\phi_j(l)e^{-cl/2}J_j(l), \\ e_i(l) &= (1+\phi_{ii}(l))e^{-cl/2}J_i(l)+\phi_{i2}(l)e^{-cl}J_2(l) \\ &\quad +\textstyle\sum_{j\geqq 3, j\neq i}\phi_{ij}(l)e^{-cl/2}J_j(l). \end{aligned}$$

Here and hereafter $\phi(t)$, $\phi_i(t)$ and $\phi_{ij}(t)$ denote terms whose absolute value is bounded from above by a linear combination of $|r_k(t)|$ and $\|R_k(t)\|$, $2\leqq k\leqq 2n$. Now set

$$\begin{aligned} Z(t) &= (1+\phi_2(l))e^{-cl}J_2(t)+\textstyle\sum \phi_i(l)e^{-cl/2}J_i(t), \\ Y_i(t) &= (1+\phi_{ii}(l))e^{-cl/2}J_i(t)+\phi_{i2}(l)e^{-cl}J_2(t) \\ &\quad +\textstyle\sum \phi_{ij}(l)e^{-cl/2}J_j(t). \end{aligned} \tag{4.16}$$

Z is the S-Jacobi field with $Z(l)=J\dot{\gamma}(l)$; namely, the Jacobi field Z^l which we defined before.

In order to estimate $K(Z, \dot{\gamma})$ and $K(Y_i, \dot{\gamma})$, we pose one more assumption on the curvature:

(C.2) $\quad \chi_\gamma(t)\leqq c_2e^{-(c+\varepsilon)t/2}$ for some positive constants ε and c_2.

This assumption then implies by (4.14)

$$\|A_1(t)\|\leqq c_1c_2e^{-(c+\varepsilon)t/2}. \tag{4.17}$$

Next note that β_i's in Lemma 6 are, in our case, $\beta_1=c/2$, $\beta_2=\cdots=\beta_{2n-1}=c$, $\beta_{2n}=\cdots=\beta_{4n-3}=c/2$ and $\beta_{4n-2}=\infty$. Hence, by (4.13),

$$|r_k|,\ \|R_k\|\leqq c'_ke^{-ct/2}. \tag{4.18}$$

Therefore (4.17) implies

$$\phi_i(t),\ \phi_{ij}(t)\leqq c_3e^{-ct/2}. \tag{4.19}$$

Combining (4.15) and (4.16), we have

$$\begin{aligned} Z(t)e^{c(l-t)} &= (1+\phi_2(l))((1+r_2)J\dot{\gamma}+R_2(t)) \\ &\quad +\textstyle\sum \phi_i(l)e^{c(l-t)/2}((1+r_i)e_i+R_i(t)). \end{aligned}$$

Set $R(t)=\sum \|R_k(t)\|$. Then, by calculations, we can see

$$\|Z\|^2e^{2c(l-t)}=(1+\phi_2(l))^2(1+r_2(t))^2+O(1+\phi(l)e^{cl/2})(R(t)+e^{-ct/2}),$$

and

$$\begin{aligned}&\langle R(Z, \dot{\gamma})\dot{\gamma}, Z\rangle e^{2c(l-t)}\\&\quad=\{(1+\phi_2(l))^2(1+r_2(t))^2+O(\phi(l)e^{cl/2})e^{-ct/2}\}K(\dot{\gamma}(t), J\dot{\gamma}(t))\\&\qquad+O(\phi(l)e^{cl/2})(R(t)+e^{-ct/2}),\end{aligned}$$

where $\phi(l)$ is a certain linear combination of $\phi_i(l)$ and $\phi_{ij}(l)$. Therefore by (4.18) and (4.19) we have

$$K(Z, \dot{\gamma})=K(\dot{\gamma}, J\dot{\gamma})+O(e^{-ct/2}). \tag{4.20}$$

As for Y_i the calculations will be the same. Let Y be one of Y_i, say Y_3. Then it has the form

$$\begin{aligned}Y(t)e^{c(l-t)/2}&=(1+\phi_{33}(l))(1+r_3)e_3+\sum O(1+\phi)(R_j+e_j)\\&\quad+O(\phi)e^{-c(l-t)/2}((1+r_2)J\dot{\gamma}+R_2).\end{aligned}$$

Then we can see

$$\| Y\|^2e^{c(l-t)}=(1+\phi_{33}(l))^2(1+r_3)^2+O(\phi(l))R(t)+O(\phi(l)^2),$$

and

$$\begin{aligned}&\langle R(Y, \dot{\gamma})\dot{\gamma}, Y\rangle e^{c(l-t)}\\&\quad=(1+\phi_{33}(l))^2(1+r_3)^2K(e_3, \dot{\gamma})+O(1+\phi(l))R(t)+O(\phi(l)).\end{aligned}$$

Since $l\geqq t$ in our consideration, $\phi(l)\leqq c'e^{-ct/2}$ for some c' by (4.19). Therefore we have

$$K(Y_i, \dot{\gamma})=K(e_i, \dot{\gamma})+O(e^{-ct/2}). \tag{4.21}$$

Now recall the definition of k and $\bar{s}$. See (4.8). Then the identities (4.20) and (4.21) imply

$$\lim_{t\to\infty} k(t)=-c^2 \quad\text{and}\quad \lim_{t\to\infty} \bar{s}(t)=-c^2/4,$$

and

$$|k(t)+c^2|,\ |\bar{s}(t)+c^2/4|\leqq c_4e^{-ct/2} \tag{4.22}$$

for some $c_4>0$. Hence the functions c and d defined in (4.9) have the asymptotic behavior such as $e^{ct/2}$ and e^{ct} respectively up to positive constants. Then this implies

$$-\psi_1'\sim e^{-nct},$$

and we have by (4.10) the upper estimate of the Green function. Namely

Theorem 4. *Let M be a complete noncompact kähler manifold satisfying* (C.2): $\chi_\gamma(t) \leqq c' e^{-(c+\varepsilon)t/2}$ *for some positive constants ε and c'. Assume the strict convexity of a geodesic sphere with sufficiently large radius and the non-positivity of curvature outside some compact set. Let K be a compact set. Then there exist positive constants c_1 and c_2 such that*

$$c_1 e^{-cnd(p,q)} \leqq G(p, q) \leqq c_2 e^{-cnd(p,q)} \tag{4.23}$$

for p away from K and q in K.

Remark 7. The condition (C.1) follows trivially from the condition (C.2).

Remark 8. Theorems 3 and 4 can be proved under the assumption that the sectional curvatures are always non-positive instead of the assumption (A.2). Modifications necessary are on the definition of functions $a, b, c, d, \cdots$. Let e be one of these. Then the initial conditions must be $e(0)=0$ and $e'(0)=1$. Remaining arguments are the same.

Remark 9. The gradient estimate of $G(p, q)$ is shown in the same way as in Remark 5 of Section 3.

§ 5. Example: A strictly pseudoconvex bounded domain in C^n with the Bergman metric

Let $D=\{\psi>0\}$ be a strictly pseudoconvex bounded smooth domain in C^n, where ψ is a C^∞-function defined in a neighborhood of D. $-\psi$ is strictly plurisubharmonic on $\bar{D}$, i.e. $(-\psi_{i\bar{j}})>0$, and $|d\psi|\neq 0$ on the boundary ∂D. Denote by $K(z, \bar{w})$ the Bergman kernel function of the domain D. Fefferman [9] has proved that $K(z, \bar{z})=\Phi(z)\psi^{-(n+1)}(z)\times(1+\Psi(z)\psi^{(n+1)}\log\psi(z))$ where, $\Phi, \Psi \in C^\infty(U)$ with $\Phi>0$, U being a neighborhood of the domain D. Let $\phi=\psi(\Phi(1+\Psi\psi^{(n+1)}\log\psi))^{-1/(n+1)}$. Then $K(z, \bar{z})=\phi(z)^{-(n+1)}$. Note that the rate of $\phi\to 0$ is the same as the rate of $\psi\to 0$. Put $h_{i\bar{j}}=-\phi_{i\bar{j}}/\phi+\phi_i\phi_{\bar{j}}/\phi^2$ and $g_{i\bar{j}}=(n+1)h_{i\bar{j}}$. The Bergman metric is given by $ds^2=\sum g_{i\bar{j}}dz^i d\bar{z}^j$. Fix a point q in ∂D and choose a holomorphic coordinate $(z_1, \cdots, z_n)$ near q such that $\partial/\partial y_1$ is outward normal to ∂D at q and $\partial/\partial x_1$, $\partial/\partial x_i$ $(i\geqq 2)$ and $\partial/\partial y_i (i\geqq 2)$ are tangent to ∂D at q. Here we have set $z_i=x_i+\sqrt{-1}y_i$. Hence

$$|\partial\phi/\partial y_1|>0 \qquad \text{near } q. \tag{5.1}$$

Let $\gamma(t)$ be a divergent geodesic with respect to the Bergman metric.

Then Fefferman proved $\gamma(t)$ tends to the unique boundary point as $t \to \infty$ and, moreover, the geodesic is transversal to the boundary at the limit point (Lemma 3 in [10], p.57). Letting this point be q, we have

(5.2) $\quad d\phi/dy_1 = d\phi(\gamma(t))/dt/dy_1(\gamma(t))/dt$ has a positive limit at q.

Now we will follow Klembeck's calculations for a while, [18]. The curvature tensor $S_{i\bar{j}k\bar{l}}$ of the metric $\sum h_{i\bar{j}} dz^i d\bar{z}^j$ is given by the formula

$$(5.3)\qquad -\frac{1}{2} S_{i\bar{j}k\bar{l}} = (h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}) - (\phi\phi_{i\bar{j}k\bar{l}} - \phi_{ik}\bar{\phi}_{jl})/\phi^2$$
$$- \sum h^{m\bar{n}} (\phi\phi_{ik\bar{m}} - \phi_{ik}\phi_{\bar{m}})(\phi\phi_{\bar{j}\bar{l}n} - \phi_{\bar{j}\bar{l}}\phi_n)/\phi^4$$

and Klembeck has proved

$$(5.4)\qquad \frac{1}{2} S_{i\bar{j}k\bar{l}} + (h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}) = O(1/\phi)$$

as the point tends to ∂D. He used (5.1) and the fact that the eigenvalues of $(h_{i\bar{j}})$ go to infinity at least as fast as $1/\phi$. By the equality $g_{i\bar{j}} = (n+1) h_{i\bar{j}}$ the curvature tensor $R_{i\bar{j}k\bar{l}}$ of the Bergman metric ds^2 satisfies

$$(5.5)\qquad R_{i\bar{j}k\bar{l}} + \frac{2}{n+1}(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) = O(1/\phi)$$

near the boundary. Hence the function χ defined in (4.1) satisfies

$$(5.6)\qquad \chi(p) = O(\phi(p)).$$

Here the curvature constant is $-c^2 = -4/(n+1)$. Especially for any geodesic γ we have

$$(5.7)\qquad \chi_\gamma(t) = O(\phi(\gamma(t))).$$

We will next examine the condition (C.2). Let again $\gamma(t)$ be a geodesic tending to the boundary point q. We fix a point $p_0 = \gamma(t_0)$ for a large t_0. Since $\partial/\partial y_1$ is normal to ∂D, we have

$$ds^2 = ((n+1)/4)\phi_{y_1}^2/\phi^2 dy_1^2 + O(1/\phi)|dz|^2$$

near q. The first term is not zero by (5.2). Therefore, for sufficiently large t_0 and t,

$$d(\gamma(t), p_0) = \int_{t_0}^{t} ds = -\frac{\sqrt{n+1}}{2}\log(\phi(\gamma(t))/\phi(p_0)) + O(1).$$

This means

$$\phi(\gamma(t)) \sim c e^{-2t/\sqrt{n+1}} \tag{5.8}$$

for all large t. Since the curvature constant is $c=2/\sqrt{n+1}$, we can see, combining (5.7) and (5.8), the condition (C.2) holds for $\varepsilon=c/2$. Therefore we have

Theorem 5. *A strictly pseudoconvex bounded domain is of asymptotically negative constant curvature* $-c^2=-4/(n+1)$ *with respect to the Bergman metric. Moreover it satisfies the condition* (C.2).

Theorem 6. *Let* $G(p, q)$ *be the Green function of a strictly pseudoconvex bounded domain with the Bergman metric. Assume every geodesic sphere with a sufficiently large radius is strictly convex. Then, for any compact set* K, *there exist constants* c_1 *and* c_2 *such that*

$$c_1 e^{-2n/\sqrt{n+1}\, d(p,q)} \leqq G(p, q) \leqq c_2 e^{-2n/\sqrt{n+1}\, d(p,q)} \tag{5.9}$$

for p *away from* K *and* q *in* K.

Since $\phi(p)$ and the euclidean distance $d_E(p, \partial D)$ to the boundary ∂D behave similarly to each other near ∂D, we obtain from Theorem 6 and Remark 9.

Corollary 1. *With notations and assumptions in Theorem* 6

$$\begin{gathered} c_1 d_E(p, \partial D)^n \leqq G(p, q) \leqq c_2 d_E(p, \partial D)^n, \\ |\nabla_p G(p, q)| \leqq c_3 d_E(p, \partial D)^n \end{gathered} \tag{5.10}$$

for some $c_i > 0$.

Remark 10. This corollary assumes the strict convexity of a geodesic sphere. This is satisfied when the sectional curvature of the Bergman metric is non-positive, which, however, is not always the case. Hence, this corollary is weaker than Malliavin's estimate in this sense. According to Remark 3, in order to avoid this assumption, it is enough to show the existence of the closed strictly convex hypersurface which is arbitrarily large. But the author does not know anything about this problem.

§ 6. An application in the Riemannian case: Martin boundary and bounded harmonic functions

The aim of this section is to construct nonconstant bounded harmonic functions on the manifold considered in Section 3. This is done by the geometric description of the Martin boundary.

In Section 3 we have proved the estimate $c_1 e^{-kd(p,q)} \leqq G(p, q) \leqq c_2 e^{-kd(p,q)}$ for the Green function $G(p, q)$, where $k=c(n-1)$. Here constants c_i generally depend on the initial point, say p, and this estimate is valid for $d(p, q) \geqq c_3 > 0$, c_3 being a constant also depending on p.

First we shall control this dependence under more strong conditions. Let M be a simply connected noncompact complete Riemannian manifold of non-positive curvature. In Section 3 we defined the function χ which measure the difference between the curvature and the given constant $-c^2$. Let p_0 be a point which we fix once and for all. Define a new function $\chi_0(t)$ by

$$\chi_0(t) = \max \{\chi(\gamma(t)) \text{ for all geodesics } \gamma \text{ from } p_0\}.$$

Then we will set the following condition:

(C.3) There exists a non-increasing function $\chi_1(t)$ such that $\chi_1(t) \geqq \chi_0(t)$ and $\int_0^\infty \chi_1(s) ds =: a$ is finite.

Remark 11. Since we are assuming the non-positivity of curvature, we can see that, if M satisfies (C.3), then M is of asymptotically negative constant curvature $-c^2$ (see (6.2)).

Theorem 7. *Let M be a simply-connected noncompact complete Riemannian manifold of non-positive curvature and satisfying the condition* (C.3) *for a constant $-c^2$. Assume $\chi_1 \leqq b^2$ for some constant b. Then there exist constants c_1 and c_2 depending only on a, b and c such that*

$$c_1 e^{-kd(p,q)} \leqq G(p, q) \leqq c_2 e^{-kd(p,q)} \tag{6.1}$$

for $d(p, q) \geqq 1$.

The proof relies on the next lemma, which we prove in Appendix A.

Lemma 7. *Let y be the solution of $y'' - (\lambda^2 + \chi(t))y = 0$ with the initial conditions $y(0)=0$ and $y'(0)=1$. Assume $0 \leqq \lambda^2 + \chi \leqq b^2$ and $a = \int_0^\infty |\chi| ds < \infty$. Then there exist constants c_1 and c_2 depending on λ and b such that*

$$c_1^a e^{\lambda t} \leqq y(t) \leqq c_2^a e^{\lambda t} \qquad \textit{for } t \geqq 1.$$

Proof of Theorem 7. We first see

$$\int_{-\infty}^{\infty} \chi(\gamma(t)) dt \leqq 2a \tag{6.2}$$

for any normal geodesic γ. In fact, when γ is through p_0, the assertion is the condition (C.3) itself. Assume γ is not through p_0 and the geodesic joining p_0 and $\gamma(0)$ is perpendicular to $\dot{\gamma}$ (translate parameter if necessary). Since curvature is non-positive, we have $d(p_0, \gamma(t)) \geqq |t|$ by the triangle inequality. Hence $\chi(\gamma(t)) \leqq \chi_1(d(p_0, \gamma(t)) \leqq \chi_1(|t|)$. This implies (6.2) by (C.3).

Now the proof is immediate. Recall that the estimate of the Green function is given by the estimate of functions f and g defined by (2.2)′ and (2.3)′ in Section 2. See Remark 4 in Section 3. So it is sufficient to get estimates of these functions which are dependent only on a, b and c. But this is accomplished in Lemma 7 in view of (6.2).

To state the next theorem let us first recall the visibility boundary of M. Let γ and δ be two geodesic rays. They are said to be asymptotic if $d(\gamma(t), \delta(t))$ is bounded. Then the *visibility boundary* is by definition the set of all asymptotic classes of geodesic rays ([9]). We denote it by $M(\infty)$. Since we are assuming that the curvature is non-positive, to every geodesic ray δ, there exists a unique geodeisc ray γ from a fixed point p_0 such that γ and δ are asymptotic. This means $M(\infty)$ can be identified with the set of all geodesic rays from p_0. We can give a topology on $M(\infty)$, taking as a subbase of the topology, the set of open cones of geodesic rays. With this topology $M \cup M(\infty)$ is compact and homeomorphic to a n-cell.

We will next recall the definition and some properties of the Martin boundary. Proofs and other properties can be found in the original paper of R. S. Martin [23] or in [16], [13]. Let M be for a while a noncompact complete Riemannian manifold admitting the Green function $G(p, q)$.

One chooses a reference point p_0 and sets

$$K(p, q) = G(p, q)/G(p_0, q) \qquad (=1 \text{ if } p = p_0 = q).$$

This is non-negative and harmonic on $M - \{q\}$ as the function of p. Consider a divergent sequence $\{q_n\}$ of points in M. In any bounded domain in M the functions $K(p, q_n)$ form a normal family by Harnack's principle. Hence a subsequence, say $K(p, q_{n'})$, is convergent to a harmonic function. Writing $\xi = \{q_{n'}\}$, we denote this limit by $K_\xi(p)$ and call this sequence ξ fundamental. Two fundamental sequences ξ and ξ' are called equivalent if $K_\xi = K_{\xi'}$. The set of all equivalence classes of fundamental sequences is called the *Martin boundary* of M and denoted by ∂M. The function K_ξ is called the Martin kernel function with pole ξ.

One can introduce a metric topology on $M \cup \partial M$ such that $K_\xi(p)$ is continuous with respect to (p, ξ). With this topology $M \cup \partial M$ is a compactification of M. A positive harmonic function h is called *minimal* if every non-negative harmonic function u with $u \leqq h$ is a constant multiple of h. The set $\partial_1 M = \{\xi \in \partial M;\ K_\xi \text{ is minimal}\}$ is called the minimal part of the boundary. K_ξ is minimal if and only if the reduced function of K_ξ relative to the set $\{\xi\}$ is equal to K_ξ itself. Then the Martin representation theorem says that every non-negative harmonic function h can be written as $h(p) = \int_{\partial M} K_\xi(p) d\mu(\xi)$ using some Borel measure μ on ∂M with its support in $\partial_1 M$. This measure is uniquely determined by h. We write by ν the measure corresponding to the function 1. Then one can solve the Dirichlet problem using ν as a reference measure on the boundary. Namely Brelot's theorem ([3], [12] Theorem 12.22) says: Every continuous function f on ∂M is resolutive; that is, the function $\int f(\xi) K_\xi(p) d\nu(\xi)$ is the Dirichlet solution for the boundary value f.

Now we can state

Theorem 8. *Let M be a simply-connected noncompact complete Riemannian manifold of strictly negative curvature and satisfying the condition* (C.3) *for a constant $-c^2$. Assume $\chi_1 \leqq b^2$ for some constant b. Then the visibility boundary $M(\infty)$ is homeomorphic to the Martin boundary ∂M and every boundary point is minimal.*

The proof is divided into several steps. Let $\gamma(t)$ be a geodesic ray from the fixed point p_0. When $\gamma(t) \neq p, p_0$, then $K(p, \gamma(t)) = G(p, \gamma(t))/G(p_0, \gamma(t))$ by definition. The inequality (6.1) implies

$$a^{-1} e^{-k(d(p,\gamma(t))-t)} \leqq K(p, \gamma(t)) \leqq a e^{-k(d(p,\gamma(t))-t)}, \tag{6.3}$$

where $k = c(n-1)$ and $a = c_2/c_1$. If $\xi = \{\gamma(t_n)\}$ is a fundamental sequence, then

$$a^{-1} e^{-k\psi_\gamma(p)} \leqq K_\xi(p) \leqq a e^{-k\psi_\gamma(p)}. \tag{6.4}$$

The function $\psi_\gamma(p)$ here is defined by

$$\psi_\gamma(p) = \lim_{t \to \infty} (d(p, \gamma(t)) - t)$$

called the Busemann function associated with a geodesic ray γ.

The estimate (6.3) enables us to consider a mapping $\Phi: \partial M \to M(\infty)$ as follows. Let $\xi = \{p_n\} \in \partial M$ be a fundamental sequence. γ_n denotes a unique geodesic ray joining p_0 and p_n. Take one of limits of $\{\gamma_n\}$, say γ. Then, by the continuity of $\psi_\gamma(p)$ with respect to p and γ ([8] Proposition 2.3) and by (6.3), we have (6.4) for γ. If $\{\gamma_n\}$ has another limit δ, (6.4) is valid also for δ. But this is possible only if $\gamma = \delta$ by the simple fact that

$\lim_{t\to\infty}\psi_\gamma(\gamma(t))=-\infty$ and $\lim_{t\to\infty}\psi_\gamma(\delta(t))=\infty$ for $\gamma\neq\delta$, which is the consequence of the strict negativity of curvature. Hence we have seen that a fundamental sequence $\xi\in\partial M$ determines a unique $\gamma\in M(\infty)$. Now define $\Phi(\xi)=\gamma$. Then we have

(a) The mapping Φ is surjective and continuous.

Proof. The surjectiveness is clear from the definition of ∂M. The continuity is an easy consequence of the continuity of $\psi_\gamma(p)$. Namely, let a sequence $\{\xi_n\}$ tend ξ and set $\gamma_n=\Phi(\xi_n)$ and $\gamma=\Phi(\xi)$. We have to show γ_n tends to γ. If δ is one of limits of $\{\gamma_n\}$, then $K_\xi\sim e^{-k\psi_\gamma}$ in the sense of (6.4). Hence $K_\xi\sim e^{-k\psi_\delta}$ shows $\gamma=\delta$.

Let us next see the injectiveness of Φ. Pick a compact set B in ∂M and let U be an open neighborhood of $\Phi(B)$ in $M\cup M(\infty)$. Then we have

(b) There exists a constant k such that $K_\xi(p)\leqq k$ for any $\xi\in B$ and $p\in U^c$.

In fact, by the estimate (6.4), it is enough to show $\psi_\gamma(p)\geqq k$ for some constant k when $p\in U^c$ and $\gamma\in\Phi(B)$. But this follows from the strict negativity of curvature; namely, for any positive constant ε and a constant k, there exists a constant t_0 such that, for any ray δ through p_0, the value $\psi_\gamma(\delta(t))$ is greater than k if the angle between $\dot\gamma(0)$ and $\dot\delta(0)$ is greater than ε and $t\geqq t_0$ (compare with the negative constant curvature case; [4]).

Let h be a non-negative harmonic function on M. It is written as

$$h(p)=\int_{\partial M}K_\xi(p)d\mu(\xi) \tag{6.5}$$

for some Borel measure μ on ∂M. Set $B=\operatorname{supp}\mu$. Then

(c) $$\lim_{p\to\gamma}h(p)=0 \qquad \text{for } \gamma\notin\Phi(B).$$

Proof. Choose a neighborhood U of $\Phi(B)$ such that $\gamma\notin U$. By the fact (b), $K_\xi(p)\leqq k$ for $\xi\in B$ and $p\in U^c$. The representation (6.5) implies $h(p)\leqq k\mu(\partial M)<\infty$ for $p\in U^c$. Hence we can take the limit of the integral when p tends to γ. But $\lim_{p\to\gamma}K_\xi(p)=0$ implies $\lim_{p\to\gamma}h(p)=0$.

Fix γ and asssme $\lim_{p\to\delta}h(p)=0$ for any $\delta\neq\gamma$. Then

(d) $$\operatorname{supp}\mu\subset\Phi^{-1}(\gamma).$$

Proof. Let B be a compact set in ∂M such that $B\cap\Phi^{-1}(\gamma)=\phi$. Define a new function h_1 by

$$h_1(p)=\int_{\partial M}K_\xi(p)\chi_B(\xi)d\mu(\xi),$$

where χ_B is the characteristic function of the set B. Obviously $h_1 \leqq h$. Hence $0 \leqq \lim_{p\to\delta} h_1 \leqq \lim_{p\to\delta} h = 0$ for $\delta \neq \gamma$. If $\gamma \notin \Phi(B)$, $\lim_{p\to\gamma} h_1 = 0$ by (b). Hence $h_1 = 0$ and especially $\mu(B) = h_1(p_0) = 0$, where p_0 is the reference point in the definition of the Martin boundary.

(e) The mapping Φ is injective and any boundary point is minimal.

Proof. The fact (d) implies $\Phi^{-1}(\gamma)$ contains at least one minimal point, since the measure μ can be chosen so that the supp μ is contained in the minimal part $\partial_1 M$. Let one of them be ξ and $\eta \in \Phi^{-1}(\gamma)$ be another point. The estimate (6.4) implies

$$a^{-2}K_\xi \leqq K_\eta \leqq a^2 K_\xi.$$

Hence $K_\eta = K_\xi$ and $\eta = \xi$. Namely, Φ is injective and every boundary point is minimal.

The facts (a) and (e) complete the proof of Theorem 8. Now we can confuse $\Phi(\xi)$ with ξ. Let f be a continuous function on $M(\infty) = \partial M$. Then $H_f(p) = \int f(\xi) K_\xi(p) d\nu(\xi)$ solves the Dirichlet problem for f. But, by virtue of the fact (c) above and a general theorem on the Martin boundary ([3]), Théorème 15), H_f is a solution in the strict sense. Namely,

Corollary 2. *With the above notations and under the assumptions in Theorem* 8, $\lim_{p\to\xi} H_f(p) = f(\xi)$.

Example. We continue the discussion on the example in Section 3. Let $\Omega = \{\phi < 0\}$ be a bounded strictly convex smooth domain in R^n with the metric defined there. We have seen that the curvature is asymptotically negative constant. Moreover it is not hard to see that the curvature assumptions in Theorem 8 are satisfied provided that Ω is a sufficiently small deformation of the unit ball in the sense of C^∞-topology. And, in this situation, the boundary $\Omega(\infty)$ is canonically identified with the geometric boundary $\partial\Omega$ (This is proved following arguments in p. 61–p. 64 of [10] with necessary modification and with use of results in Appendix B). Hence the Martin boundary of Ω with respect to the present metric is identical with $\partial\Omega$. The property $\lim H_f = f$ in Corollary 2 is proved more directly in this case. Choose $b \in \partial\Omega$ and fix an affine coordinate (x^i) with origin at b. Set $u = -(-\phi)^{1/3} - \varepsilon|x|^2$. Then one can see, by the straightforward calculation making use of the explicit form of the metric tensor and the Christoffel symbol given in Appendix B, that u is subharmonic near b for a sufficiently small ε. Since it is non-positive on $\overline{\Omega}$ and takes 0 only at b, u is a barrier function at b. This implies the above property ([13]).

Appendix

A. Proof of Lemmas 3 and 7.

Proof of Lemma 3. First choose a constant a such that $\phi(a)<2\lambda$, which is possible by the assumption $\phi(t)\to 0$ as $t\to\infty$. Define functions $y_1(t)$ and $y_2(t)$ for $t\in[a,\infty)$ by the integral equations

$$y_1(t)=\frac{1}{2\lambda}e^{-\lambda t}+\frac{1}{2\lambda}\int_t^\infty (e^{-\lambda(t-s)}-e^{\lambda(t-s)})\chi(s)y_1(s)\,ds,$$

$$y_2(t)=\frac{1}{2\lambda}e^{\lambda t}-\frac{1}{2\lambda}\left\{\int_a^t e^{-\lambda(t-s)}\chi(s)y_2(s)\,ds+\int_t^\infty e^{\lambda(t-s)}\chi(s)y_2(s)\,ds\right\}.$$

It is easily seen that these functions, if they exist, satisfy the equation (3.3). To see the existence, set $v_1(t)=e^{\lambda t}y(t)$ and $v_2(t)=e^{-\lambda t}y_2(t)$. Then v_i is given by

$$\text{(1)}\qquad \begin{aligned} v_1(t)-\frac{1}{2\lambda}&=\frac{1}{2\lambda}\int_t^\infty (1-e^{2\lambda(t-s)})\chi v_1 ds,\\ v_2(t)-\frac{1}{2\lambda}&=-\frac{1}{2\lambda}\left\{\int_a^t e^{-2\lambda(t-s)}\chi v_2 ds+\int_t^\infty \chi v_2 ds\right\}.\end{aligned}$$

Since $\phi(a)<2\lambda$, each of these integral equations has a unique bounded solution which can be seen by the usual iteration method. Denote its bound by L: $|v_i(t)|\leqq L$. Then, taking absolute values of both sides of (1), we have

$$\text{(2)}\qquad \begin{aligned} \left|v_1(t)-\frac{1}{2\lambda}\right|&\leqq\frac{L}{2\lambda}\phi(t),\\ \left|v_2(t)-\frac{1}{2\lambda}\right|&\leqq\frac{L}{2\lambda}\{e^{-\lambda t}\phi(a)+\phi(t/2)\}.\end{aligned}$$

Since these inequalities show that y_1 and y_2 are linearly independent, any solution of (3.3) has the form stated in (1) in the Lemma. The part (2) is seen from (2).

Proof of Lemma 7. For the proof of Lemma 7 we prepare another

Lemma. *Let y be the solution of $y''-(\lambda^2+\chi(t))y=0$ with initial conditions $y(t_0)=A\geqq 0$ and $y'(t_0)=B\geqq 0$, $(A+B\neq 0)$. Assume $\lambda^2+\chi\geqq 0$. Define t_1 by $\int_{t_0}^{t_1}|\chi(s)|ds=2\lambda k$ where k is a constant determined below. Set*

$t_1=\infty$ *if* $\int_{t_0}^{\infty}|\chi(s)|ds\leqq 2\lambda k$. *Then there exist constants* c_i, d_i $(i=1, 2)$ *depending on* λ *such that*

$$c_1 A\leqq y(t)e^{-\lambda(t-t_0)}\leqq d_1(A+B)$$
$$c_2 B\leqq y'(t)e^{-\lambda(t-t_0)}\leqq d_2(A+B)$$

for $t_0\leqq t\leqq t_1$.

Proof. We may assume $t_0=0$. Put $y_1(t)=A\cosh\lambda t+B/\lambda\sinh\lambda t$. The solution y is given by the equation

$$y(t)=y_1(t)+\frac{1}{\lambda}\int_0^t \sinh\lambda(t-s)\chi(s)y(s)ds. \tag{3}$$

Define

$$y_{i+1}(t)=y_1(t)+\frac{1}{\lambda}\int_0^t \sinh\lambda(t-s)\chi(s)y_i(s)ds$$

succesively. Then

$$e^{-\lambda t}|y_{i+1}-y_i|\leqq k^i \max\{e^{-\lambda s}y_1(s);\ 0\leqq s\leqq t\}.$$

If we assume $k<1$, this implies

$$|y-y_1|\leqq\sum_{i=1}^{\infty}|y_{i+1}-y_i|\leqq\frac{k}{1-k}\left(A+\frac{B}{2\lambda}\right)e^{\lambda t} \qquad \text{for } 0\leqq t\leqq t_1.$$

Hence

$$y(t)\leqq\left(A+\frac{B}{2\lambda}\right)e^{\lambda t}/(1-k). \tag{4}$$

Differentiating (3) and substituting (4) we obtain

$$\begin{aligned} y'(t)&\leqq y_1'(t)+\int_0^t \cosh\lambda(t-s)|\chi(s)|\left(A+\frac{B}{2\lambda}\right)e^{\lambda s}/(1-k)ds \\ &\leqq\left(\lambda A\frac{1+3k}{2-2k}+B\frac{1}{1-k}\right)e^{\lambda t}. \end{aligned} \tag{5}$$

To obtain lower estimates, define z (resp. w) to be the solution of the present equation with $z(0)=A$ and $z'(0)=0$ (resp. $w(0)=0$ and $w'(0)=B$). Then $z\leqq y$ and $w'\leqq y'$ because $\lambda^2+\chi\geqq 0$. z (resp. w) is given by the above y setting $B=0$ (resp. $A=0$). Hence we can use estimates for y and we obtain

(6) $$z(t) \geqq \frac{1-3k}{2-2k} A e^{\lambda t} \quad \text{and} \quad w(t) \leqq \frac{B}{2\lambda(1-k)} e^{\lambda t}.$$

Substituting the latter inequality into

$$w'(t) = B \cosh \lambda t + \int_0^t \cosh \lambda(t-s) \chi(s) w(s) ds$$

we have

(7) $$w'(t) \geqq \frac{1-3k}{2-2k} B e^{\lambda t}.$$

Choosing k smaller than $1/3$, we complete the proof.

We give the proof of Lemma 7 applying the above lemma repeatedly. Start with the case $t_0 = 1$. The values A, B are estimated by absolute values due to the assumption on χ. t_1 is defined as in Lemma. Set $s_1 = t_1$. Then $c_1 e^{\lambda s_1} \leqq y(s_1) \leqq d_1 e^{\lambda s_1}$, $c_2 e^{\lambda s_1} \leqq y'(s_1) \leqq d_2 e^{\lambda s_1}$ for some constants c_i and d_i. Next, putting $t_0 = s_1$ in Lemma and define t_1 which we now write s_2. Let y_1 be the solution with $y_1(s_1) = c_1 e^{\lambda s_1}$ and $y_1'(s_1) = c_2 e^{\lambda s_1}$, then by comparison and by Lemma, we have

$$y(s_2) \geqq y_1(s_2) \geqq c_1 e^{\lambda(s_2 - s_1)} y_1(s_1) = c_1^2 e^{\lambda s_2},$$
$$y'(s_2) \geqq y_1'(s_2) \geqq c_2 e^{\lambda(s_2 - s_1)} y_1'(s_1) = c_2^2 e^{\lambda s_2}.$$

Repeat this process. By the assumption of finiteness of $a = \int_0^\infty |\chi| ds$, this process will terminate at the $a/2\lambda k$ -th step. This finishes the proof of the lower estimate. The upper estimate is given similarly.

B. Curvature behavior of the metric $-(1/v)d^2v$.

Let $\Omega = \{\phi < 0\}$ be a smooth strictly convex bounded domain in R^n. The defining function ϕ is strictly convex in some neighborhood of Ω. We set

$$v = \sqrt{-\phi}$$

on Ω. In this Appendix we consider the metric

$$ds^2 = -\frac{1}{v} d^2 v = -\frac{1}{v} \sum v_{ij} dx^i dx^j$$

defined on Ω. We first calculate the curvature tensor of this metric and, second, we investigate the boundary bahavior of the geodesics.

Remark 1. The metric ds^2 depends on the choice of the defining function. But it has a "projective invariance" in the following sense. Let $A: R^n \to R^n$ be a projective transformation defined as $(Ax)^i = (\sum a^i_j x^j + a^i)/(\sum a_i x^i + a)$; $a^i_j, \cdots$ being constants. Set $k(x) = \sum a_i x^i + a$. For a given strictly convex domain $\Omega = \{\phi < 0\}$ we define on the domain $A^{-1}\Omega$ the function ψ by $\psi(x) = k^2(x)\phi(Ax)$. We denote by ds^2_ψ (resp. ds^2_ϕ) the metric defined by ψ (resp. ϕ) as above. Then the mapping A is an isometry from $(A^{-1}\Omega, ds^2_\psi)$ to (Ω, ds^2_ϕ).

We fix a coordinate system $(x^1, \cdots, x^n)$ of R^n.

The fundamental tensor of the metric ds^2 is $1/2(g_{ij})$ where

(1)
$$g_{ij} = -\phi_{ij}/\phi + \phi_i\phi_j/2\phi^2.$$

Proposition B-1. *The metric ds^2 is complete.*

Proof. Let (ϕ^{ij}) be the inverse matrix of (ϕ_{ij}). Set $|d\phi|^2 = \sum \phi^{ij}\phi_i\phi_j$ and $\phi^i = \sum \phi^{ij}\phi_j$. Then the inverse of (g_{ij}) is given by

(1)′
$$g^{ij} = -\phi\left(\phi^{ij} + \frac{\phi^i\phi^j}{2\phi - |d\phi|^2}\right).$$

Let $|dv|$ be the norm of grad v relative to the tensor g_{ij}. Then we have

$$|dv|^2 = \frac{\phi|d\phi|^2}{2\phi - |d\phi|^2} \leqq -\phi.$$

Hence $|dv|^2 \leqq v$. Let γ be a curve tending to $\partial\Omega$. Taking arc-length parameter t, we see

$$\text{the length of } \gamma = \int_\gamma dt \geqq \int_\gamma \frac{1}{v} dv = \infty.$$

Hence ds^2 is complete.

1. Calculations of the curvature tensor

For the sake of convenience we treat the metric $2ds^2$ for a while. The summation convention is used. The Christoffel coefficients and the curvature tensor are given by the formulas

(2)
$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m})$$

(3)
$$R_{ijkl} = \frac{1}{2}(g_{il,jk} + g_{jk,il} - g_{ik,jl} - g_{jl,ik}) + g_{mn}(\Gamma^m_{jk}\Gamma^n_{il} - \Gamma^m_{jl}\Gamma^n_{ik})$$

where $g_{mj,k}=\partial g_{mj}/\partial x^k$, $g_{il,jk}=\partial^2 g_{il}/\partial x^j \partial x^k$, $\cdots$. Taking derivatives of (1), we have

$$g_{ij,k}=-\phi_{ijk}/\phi+\phi_{ij}\phi_k/\phi^2+(\phi_{ik}\phi_j+\phi_i\phi_{jk})/2\phi^2-\phi_i\phi_j\phi_k/\phi^3, \tag{4}$$

$$\begin{aligned} g_{ij,kl}=&-\phi_{ijkl}/\phi+(\phi_{ijk}\phi_l+\phi_{ijl}\phi_k+\phi_{ij}\phi_{kl})/\phi^2\\ &+(\phi_{ikl}\phi_j+\phi_{ik}\phi_{jl}+\phi_{il}\phi_{jk}+\phi_i\phi_{jkl})/2\phi^2\\ &-(2\phi_{ij}\phi_k\phi_l+\phi_{ik}\phi_j\phi_l+\phi_{jk}\phi_i\phi_l+\phi_{il}\phi_j\phi_k+\phi_{jl}\phi_i\phi_k+\phi_{kl}\phi_i\phi_j)/\phi^3\\ &+3\phi_i\phi_j\phi_k\phi_l/\phi^4. \end{aligned} \tag{5}$$

We put $A_{mjk}=g_{mj,k}+g_{mk,j}-g_{jk,m}$. Then by the substitution of (4)

$$A_{mjk}=-\phi_{mjk}/\phi+(\phi_{mk}\phi_j+\phi_{mj}\phi_k)/\phi^2-\phi_m\phi_j\phi_k/\phi^3. \tag{6}$$

Since $\Gamma^i_{jk}=g^{im}A_{mjk}/2$, we know the second term of the right hand side of (3) is equal to $g^{mn}(A_{mjk}A_{nil}-A_{mjl}A_{nik})/4$. Now we can express the curvature tensor in terms of derivatives of ϕ using (5) and (6). Then, using the identity (1) we have

Proposition B-2. *The curvature tensor is given by the formula:*

$$R_{ijkl}=(g_{il}g_{jk}-g_{ik}g_{jl})/2+g^{mn}(\phi_{jkn}\phi_{ilm}-\phi_{jln}\phi_{ikm})/4\phi^2. \tag{7}$$

With this formula we will estimate R_{ijkl} near the boundary. By equations (1) and (1)′, g_{ij} is at least as fast as $1/\phi$ on one hand, and g^{ij} is at most $O(\phi)$ on the other hand. So the main term of R_{ijkl} is the first one:

$$R_{ijkl}=(g_{il}g_{jk}-g_{ik}g_{jl})/2+O(1/\phi).$$

Since $g_{il}g_{jk}-g_{ik}g_{jl}$ is the curvature tensor of the metric with constant sectional curvature -1 and since R_{ijkl} is the curvature tensor of $2ds^2$, we have

Proposition B-3. *The sectional curvature of the metric ds^2 is equal to $-1+O(\phi)$ near the boundary.*

2. Boundary behavior of geodesics and asymptotic constancy of curvatures

In this part we shall see that every divergent geodesic has a limit in the boundary. The reasoning for that is already exhibited by Fefferman: Lemma 3 in [10]. We follow it with little modification. Since we are dealing with the real case, the argument is easier than that in [10]. To avoid minus sign we consider a positive defining function ϕ of the domain, which is strictly concave. We use the notation $\langle\ ,\ \rangle$ to denote the inner product of the metric ds^2 and the notation $\langle\ ,\ \rangle_E$ to denote the euclidean inner product relative to a fixed coordinate system.

Proposition B-4. *Let $\gamma(t)$ be a divergent normal geodesic. Then there exist positive constants C and c such that*

$$C\phi(\gamma(t)) \geqq -d\phi(\gamma(t))/dt \geqq c\phi((t)) \tag{8}$$

for sufficiently large t.

Proof. Since $-\phi^{-1}d\phi/dt = \langle -\phi^{-1}\operatorname{grad}\phi, \dot{\gamma}\rangle_E$ and $\langle\dot{\gamma}, \dot{\gamma}\rangle = 1$ by the normality, the left hand side inequality is a simple consequence of the definition of g_{ij}. In the sequel we show the right hand side inequality. Assume that for a point $p_0 = \gamma(t_0)$ near $\partial\Omega$, we have

$$d\phi(\gamma(t))/dt\,|_{t_0} \geqq -c_1\phi(p_0)^{3/2}. \tag{9}$$

Then the estimate

$$-d\phi(\gamma(t))/dt \geqq c_2\phi(\gamma(t)) \tag{10}$$

is valid for $t_0 + a \leqq t \leqq t_0 + 10a$. Here a is an absolute constant.

The first step is to prove (10) for the Hilbert metric of the unit ball. Let $\phi = 1 - |x|^2$ and $B = \{\phi > 0\}$. Fix $p_0 \in B$. Since it is known that any geodesic is a segment of an affine line, the geodesic $\gamma(t)$ through p_0 is written as

$$\gamma(t) = p_0 + (p_\infty - p_0)\frac{e^t - 1}{e^t + 1}.$$

p_∞ is the limit point of $\gamma(t)$ on the boundary. Then

$$\dot{\gamma}(t) = (p_\infty - p_0)\frac{2e^t}{(e^t + 1)^2}.$$

Hence we have

$$\begin{aligned}-d\phi(\gamma(t))/dt &= 2\langle\gamma, \dot{\gamma}\rangle_E \\ &= \frac{4e^t}{(e^t + 1)^3}\{(e^t - 1)\langle p_\infty, p_\infty - p_0\rangle_E + 2\langle p_0, p_\infty - p_0\rangle_E\}\end{aligned}$$

and

$$\phi(\gamma(t)) = 1 - \langle\gamma, \gamma\rangle_E = \frac{4}{(e^t + 1)^2}\{e^t\langle p_\infty - p_0, p_\infty\rangle_E + \langle p_0, p_\infty - p_0\rangle_E\}.$$

Then, noting that (9) is equivalent to $\langle p_0, p_\infty - p_0\rangle_E \geqq -c_1\phi(p_0)^{3/2}$, we can easily verify that there is a constant a such that (10) is valid for $t \geqq t_0 + a$.

Next let Ω be a general domain. Fix a boundary point $q \in \partial\Omega$ and choose coordinates (x^i) with origin at q. Take an ellipsoid B which is tangent to Ω at the second order. Making a linear change of coordinates we may assume that B is the unit ball with the Hilbert metric (see Remark 1). Namely $B = \{\phi_B = 2x^1 - |x|^2 > 0\}$ and $\Omega = \{\phi_\Omega > 0\}$ for $\phi_\Omega = \phi_B + O(|x|^3)$. Put $p = (\delta, 0, \cdots, 0)$ for small δ. We compare ds^2 with ds_B^2 as is done in [10], p. 58–59. Choose new coordinates (y^i) at p by $y^1 = \delta^{-1}(x^1 - \delta)$, $y^i = \delta^{-1/2}x^i$, $i \geqq 2$. Then

$$ds_B^2 = \sum g_{ij} dy^i dy^j$$
$$ds^2 = \sum (g_{ij} + \delta h_{ij}) dy^i dy^j$$

near p. Set $N = \{x \in B;\, d_B(p, x) < 100a\}$. Then g_{ij} and h_{ij} are C^∞ on N and $\det g_{ij}$ is bounded from below by a positive constant depending on a. In the following the letter c is assumed to denote a positive number which, at each step, depends on a or on the defining functions.

Let $\gamma_\Omega(t)$ be a given normal geodesic with $\gamma_\Omega(0) = p$ and determine a normal geodesic $\gamma_B(t)$ relative to ds_B^2 by $\gamma_B(0) = p$ and $\dot\gamma_B(0) = \dot\gamma_\Omega(0)$. The perturbation result of ordinary differential equations show that

$$|\gamma_\Omega(t) - \gamma_B(t)|,\ |\dot\gamma_\Omega(t) - \dot\gamma_B(t)| < c\delta \qquad \text{for } 0 \leqq t \leqq 50a.$$

Coming back to the original coordinate (x^i), we have

$$\text{(11)} \qquad |\gamma_\Omega(t) - \gamma_B(t)|_E,\ |\dot\gamma_\Omega(t) - \dot\gamma_B(t)|_E < c\delta^{3/2} \qquad \text{for } 0 \leqq t \leqq 50a.$$

By the way of choice of B, $|\mathrm{grad}\,\phi_\Omega - \mathrm{grad}\,\phi_B|_E \leqq c|x|^2$, and we can see $|x|^2 \leqq c\delta$ on N. Therefore

$$\text{(12)} \qquad |\mathrm{grad}\,\phi_\Omega - \mathrm{grad}\,\phi_B|_E \leqq c\delta \qquad \text{on } N.$$

Since γ_B travels with unit speed, we have

$$\text{(13)} \qquad |\dot\gamma_B(t)|_E \leqq c\delta^{1/2} \qquad \text{on } N$$

for small δ. Then by (11)–(13), we have inequalities

$$\text{(14)} \qquad \begin{aligned} &|\langle \dot\gamma_\Omega(t), \mathrm{grad}\,\phi_\Omega(\gamma_\Omega(t))\rangle_E - \langle \dot\gamma_B(t), \mathrm{grad}\,\phi_B(\gamma_B(t))\rangle_E| \\ &\quad \leqq |\langle \dot\gamma_\Omega(t) - \dot\gamma_B(t), \mathrm{grad}\,\phi_\Omega(\gamma_\Omega(t))\rangle_E| \\ &\qquad + |\langle \dot\gamma_B(t), \mathrm{grad}\,\phi_\Omega(\gamma_\Omega(t)) - \mathrm{grad}\,\phi_B(\gamma_\Omega(t))\rangle_E| \\ &\qquad + |\langle \dot\gamma_B(t), \mathrm{grad}\,\phi_B(\gamma_\Omega(t)) - \mathrm{grad}\,\phi_B(\gamma_B(t))\rangle_E| \\ &\quad \leqq c\delta^{3/2}, \qquad 0 \leqq t \leqq 50a. \end{aligned}$$

Now assume (9) for Ω. Then the estimate (10) for the unit ball and (14) imply (10) for Ω. To finish the proof, find t_0 such that $\gamma(t_0)$ is near $\partial\Omega$ and $-d\phi/dt \geqq 0$ at t_0 and apply (10) repeatedly.

Proposition B-5. *Let $\gamma(t)$ be a divergent geodesic. Then there exists* $\lim_{t\to\infty} \gamma(t)$ *in* $\partial\Omega$.

Proof. We have set $\Gamma^i_{jk} = g^{im}A_{mjk}$ and A_{mjk} is defined in (6). By (1)

$$\Gamma^i_{jk} = -g^{im}\phi_{mjk}/2\phi - (\phi_j\delta_{ik} + \phi_k\delta_{ij})/2\phi. \tag{15}$$

Note that g^{im}/ϕ are bounded ((1)′). We define tangent vectors ν_i by $\nu_i = \partial\phi/\partial x^i$ and set $\dot\gamma = \sum Q^i\nu_i$ for a normal geodesic γ. Writing $\gamma(t) = (x^i(t))$ we have

$$\dot x^i = \phi Q^i, \qquad \ddot x^i + \Gamma^i_{jk}\dot x^j \dot x^k = 0. \tag{16}$$

Taking derivatives of Q^i we have

$$\begin{aligned}\dot Q^i &= -\dot x^i\dot\phi/\phi^2 + \ddot x^i/\phi = (-\phi_k\delta_{ij}/\phi^2 - \Gamma^i_{jk}/\phi)\dot x^j \dot x^k \\ &= (-(\phi_k\delta_{ij} + \phi_j\delta_{ik})/2\phi^2 - \Gamma^i_{jk}/\phi)\dot x^j\dot x^k.\end{aligned}$$

Then, by (15)

$$\dot Q^i = \frac{1}{2} g^{im}\phi_{mjk}Q^jQ^k. \tag{17}$$

We next set $N(t) = \sum |Q^i(\gamma(t))|^2$. Since $\phi g_{ij} \geqq -\phi_{ij}$ and ϕ is strictly concave, we have $1 = \langle\dot\gamma, \dot\gamma\rangle = \sum Q^iQ^j\phi^2 g_{ij} \geqq c\phi N$. This implies

$$N \leqq c\phi^{-1} \tag{18}$$

near the boundary.

Now we introduce a new time τ by

$$\tau = \int^t \phi(\gamma(t))dt.$$

Proposition B-4 shows $C \geqq -d\phi(\gamma)/d\tau \geqq c$. Hence for some finite value τ_∞,

$$\phi(\gamma(\tau)) \sim (\tau_\infty - \tau).$$

By (18),

$$N(\tau) \leqq c(\tau_\infty - \tau)^{-1}. \tag{19}$$

However

$$|dN/d\tau|=2|\sum Q^i dQ^i/d\tau|\leqq cN^{1/2}\max_i |dQ^i/d\tau|.$$

Then, from the equation (17), we have

$$|dN/d\tau|\leqq cN^{3/2}. \tag{20}$$

Substituting (19) in the right hand side and integrating this inequality we obtain

$$N(\tau)\leqq c(\tau_\infty-\tau)^{-1/2}.$$

Again substituting this into (20) we obtain

$$N(\tau)\leqq c(\tau_\infty-\tau)^{1/4}+c'.$$

Namely $N(\tau)$ and, hence, Q^i are bounded. On the other hand the ordinary differential equations (16) and (17) are written as

$$dx^i/d\tau=Q^i, \qquad dQ^i/d\tau=\frac{1}{2\phi}g^{im}\phi_{mjk}Q^jQ^k,$$

which shows the existence of lim Q^i and lim x^i.

Remark 2. Take another basis of tangent vectors μ_i such that $\mu_1=-\phi\,\mathrm{grad}\,\phi$. Defining P^i by $\dot{\gamma}=\sum P^i\mu_i$ we can show that P^1 tends to a positive constant by Proposition B-4 and, by this fact, that a divergent geodesic hits the boundary transversally.

Proposition B-6. *A strictly convex bounded domain with the metric ds^2 is of asymptotically negative constant curvature* -1.

Proof. We have by Proposition B-4

$$\phi(\gamma(t))\sim c'e^{-ct}$$

for some positive constants c and c'. Since Proposition B-3 implies $|K+1|\sim\phi(\gamma)$, we have the desired result.

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