Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 105–124

Complete Integrability of the Geodesic Flows on Symmetric Spaces

Kiyotaka Ii and Shin-ichi Watanabe

Introduction

The geodesic flow of a Riemannian manifold M of dimension n is said to be *completely integrable* if there exist real-valued C^∞-functions $f_1, \cdots, f_n$ on the cotangent bundle T^*M which satisfy

(i) f_1 is the function assigning to each cotangent vector the square of its length,

(ii) $\{f_i, f_j\}=0$ for all $1 \leqq i, j \leqq n$, where $\{\ ,\ \}$ is the Poisson bracket, and

(iii) the set of critical points of the map $(f_1, \cdots, f_n)\colon T^*M \to \boldsymbol{R}^n$ has Liouville measure 0 in T^*M (cf. Def. 5.2.20 in [1]).

The classical examples of compact Riemannian manifolds with completely integrable geodesic flow are (a) compact surfaces of revolution, (b) $SO(3)$ with left invariant metric, (c) n-dimensional ellipsoids with different principal axes and (d) flat tori (cf. [5]). It is also known that the geodesic flow of a Zoll surface, which is not necessarily a surface of revolution, is completely integrable. In [6], Weinstein showed that the geodesic flow of the n-dimensional sphere S^n of constant curvature is completely integrable. Recently, Thimm showed that the geodesic flows of the following homogeneous spaces are completely integrable (cf. [4], [5]): (a) $G_{p,q}(\boldsymbol{R})$, (b) $G_{p,q}(\boldsymbol{C})$ (i.e., real and complex Grassmannians), (c) $SU(n)/SO(n)$, (d) a distance sphere in $P^n(\boldsymbol{C})$, (e) $SO(n)/SO(n-2)$. The method, exposed by Thimm, which allows the construction of families of first integrals in involution, is available for other homogeneous spaces.

In the present paper, we show that the geodesic flows of the following symmetric spaces are completely integrable: (a) $SU(n)$, (b) $SO(n)$, (c) $SU(2n)/Sp(n)$, (d) $SO(2n)/U(n)$. The procedure of the proof is as follows: Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. The set $C^\infty(\mathfrak{g}^*)$ of all C^∞-functions on $\mathfrak{g}^*$ has a naturally defined Poisson structure (§ 1). In the case that $\mathfrak{g}$ is the Lie algebra of $SU(n)$ or of $SO(n)$, we construct concretely a commutative Poisson subalgebra $\Lambda(\mathfrak{g})$ of $C^\infty(\mathfrak{g}^*)$ and a system $\{F_j^{(i)}\}$ of its

Received December 8, 1982.

generators (§ 2). In Section 3, we introduce a convex polyhedral domain D in a Euclidean space $\boldsymbol{R}^d$, $d=(\dim \mathfrak{g}+\operatorname{rank} \mathfrak{g})/2$, and a map $A: \mathfrak{g}^* \to \boldsymbol{D}$, so that the generators $\{F_j^{(i)}\}$ are obtained by the pull-back, by A, of the symmetric polynomial functions on D. In Section 4, for the symmetric spaces M mentioned above, we show that the image $A \circ \mu(T^*M)$ of T^*M under the composite map of the moment map $\mu: T^*M \to \mathfrak{g}^*$ with A, is itself a convex polyhedral domain of dimension equal to that of M. Next, we determine a set of symmetric polynomial functions on D, of which restrictions to $A \circ \mu(T^*M)$ are functionally independent. By the pull-back of these functions, we obtain a complete system of first integrals in involution for the geodesic flow of M. In our method, the polyhedral domain D plays a crucial role. By the introduction of this domain, the choice of a complete system and its independence become almost obvious. Here, we note that, as Thimm says in [5], the independence of the functions in question, i.e., (iii) in the definition of the complete integrability, was a difficult part to prove. In Appendix, it is stated that each point of a lattice in the convex polyhedral domain $A \circ \mu(T^*M)$ is seen to correspond to a one-dimensional subspace of $C^\infty(M)$, which is obtained as a simultaneous eigen-space of a system of differential operators on M.

The authors wish to thank Professor T. Sakai and Professor T. Sunada, who kindly informed us of the work of Thimm, [4]. We also have been informed, when our paper was in typing, by Professor T. Sunada of the following paper: A. S. Mishchenko, Integration of geodesic flows on symmetric spaces, Math. Notes, 31 (1982), 132–134.

§ 1. Poisson algebra and moment map

Let $\mathscr{F}$ be a commutative, associative algebra with identity over a field K. A Poisson bracket on $\mathscr{F}$ is a binary operation on $\mathscr{F}$: $(f_1, f_2) \mapsto \{f_1, f_2\}$ satisfying the following relations:

(i) $\{f, f\}=0$,
(ii) $\{f_1, f_2 f_3\}=\{f_1, f_2\} f_3+f_2\{f_1, f_3\}$,
(iii) $\{f_1, \{f_2, f_3\}\}+\{f_2, \{f_3, f_1\}\}+\{f_3, \{f_1, f_2\}\}=0$ and
(iv) $\{k, f\}=0$ for all $k \in K, f, f_1, f_2, f_3 \in \mathscr{F}$.

$\mathscr{F}$ is called a Poisson algebra over K if a Poisson bracket is defined on $\mathscr{F}$ (see [2]). Let $\mathfrak{g}^*$ be the dual space of a Lie algebra $\mathfrak{g}$ over the field $\boldsymbol{R}$ of real numbers. Then the algebra $C^\infty(\mathfrak{g}^*)$ of C^∞-functions on $\mathfrak{g}^*$ has a Poisson structure defined as follows (cf. [2]): Let define a bilinear map $\sigma_0: \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ by $(X, \sigma_0(Y, Z^*))=([X, Y], Z^*)$ for all $X \in \mathfrak{g}$, and a map $\sigma: \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^*$ by $\sigma(X, Z^*)=(\sigma_0(X, Z^*), Z^*)$. Identifying $\mathfrak{g} \times \mathfrak{g}^*$ (resp. $\mathfrak{g}^* \times \mathfrak{g}^*$) with the cotangent (resp. tangent) bundle to $\mathfrak{g}^*$, we may consider σ to be a bundle map of these bundles. For any $f \in C^\infty(\mathfrak{g}^*)$, $\sigma(df)$ is a tangent vector field to $\mathfrak{g}^*$. If we define a bracket product on $C^\infty(\mathfrak{g}^*)$ by $\{f_1, f_2\} = (\sigma(df_1))(f_2)$, then we have

Proposition 1.1 (cf. [2]). *$C^\infty(\mathfrak{g}^*)$ is a Poisson algebra under the bracket operation defined above. The ring $\mathbf{C}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}^*$ is a Poisson subalgebra of $C^\infty(\mathfrak{g}^*)$.*

Note that $\mathfrak{g}$ is naturally identified with a subspace of $\mathbf{C}[\mathfrak{g}]$ or of $C^\infty(\mathfrak{g}^*)$.

Proposition 1.2 (cf. [2]). *Let j_0 be a homomorphism of $\mathfrak{g}$ to a Poisson algebra Λ. Then j_0 is uniquely extended to a homomorphism j: $\mathbf{C}[\mathfrak{g}] \to \Lambda$ of Poisson algebras.*

Proposition 1.3. *Let j_0: $\mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of Lie algebras. Then j_0 is uniquely extended to homomorphisms j: $\mathbf{C}[\mathfrak{g}] \to \mathbf{C}[\mathfrak{g}']$ and j: $C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}'^*)$ of Poisson algebras.*

Let (P, Ω) be a symplectic manifold with a symplectic action Φ of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G. For any $X \in \mathfrak{g}$, a tangent vector field $\rho_\Phi(X)$ to P is defined by

$$(1.1) \qquad \rho_\Phi(X)(f)|_p = [(d/dt) f \circ \Phi(\exp(-tX), p)]_{t=0},$$

for any $f \in C^\infty(P)$ and $p \in P$. It is well-known that $C^\infty(P)$ has a natural Poisson structure. A pair (Φ, τ) is called a Hamiltonian action of G on P if τ: $\mathfrak{g} \to C^\infty(P)$ is a homomorphism of Lie algebras satisfying the following condition:

$$(1.2) \qquad \rho_\Phi(X)(f) = \{\tau(X), f\}$$

for all $f \in C^\infty(P)$. For each Hamiltonian action (Φ, τ) of G on P, the moment map μ: $P \to \mathfrak{g}^*$ is defined by $(X, \mu(p)) = (\tau(X))(p)$ for all $X \in \mathfrak{g}$.

Proposition 1.4 (cf. [2]). *The map μ^*: $C^\infty(\mathfrak{g}^*) \to C^\infty(P)$ induced by μ is a homomorphism of Poisson algebras.*

There does not, in general, exist a Hamiltonian action (Φ, τ) for any symplectic action Φ. However, in the following case, there exists a Hamiltonian action (Φ, τ) determined naturally by Φ. Let M be a C^∞-manifold and φ be a C^∞-action of a Lie group G on M. Then φ induces a symplectic action Φ of G on the cotangent bundle T^*M. For any $X \in \mathfrak{g}$, a tangent vector field $\rho_\varphi(X)$ to M is defined in the same manner as (1.1). The cotangent space at $m \in M$ is denoted by $T^*_m M$. If we define maps

$\tau: \mathfrak{g} \to C^\infty(T^*M)$ and $\mu: T^*M \to \mathfrak{g}^*$ by

$$(1.3) \qquad (X, \mu(\alpha)) = (\tau(X))(\alpha) = (\rho_\varphi(X)_m, \alpha)$$

for all $X \in \mathfrak{g}$, $\alpha \in T_m^*M$, $m \in M$, then we have

Proposition 1.5 (cf. [4]). *(Φ, τ) is a Hamiltonian action of G on T^*M and μ is the corresponding moment map.*

If M has a G-invariant Riemannian metric and $\mathfrak{g}$ has an $(\mathrm{Ad}\, G)$-invariant non-degenerate symmetric bilinear form, then T^*M and $\mathfrak{g}^*$ are identified naturally with TM and $\mathfrak{g}$, respectively. Thus, μ induces a map of TM to $\mathfrak{g}$. We shall identify this map with μ and call it the moment map.

§ 2. The center and a commutative subalgebra of $C[\mathfrak{g}]$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) is a G-module under the adjoint (resp. coadjoint) action of G. By Proposition 1.3, we have

Proposition 2.1. *The coadjoint action of G induces naturally automorphisms of Poisson algebras $C^\infty(\mathfrak{g}^*)$ and $C[\mathfrak{g}]$.*

Let Z^∞ and Z denote the centers of $C^\infty(\mathfrak{g}^*)$ and $C[\mathfrak{g}]$, respectively, i.e.,

$$Z^\infty = \{f \in C^\infty(\mathfrak{g}^*) \mid \{f, f'\} = 0 \text{ for all } f' \in C^\infty(\mathfrak{g}^*)\},$$
$$Z = \{f \in C[\mathfrak{g}] \mid \{f, f'\} = 0 \text{ for all } f' \in C[\mathfrak{g}]\}.$$

Then we have $Z = Z^\infty \cap C[\mathfrak{g}]$. The following is easily obtained from the definition of the bracket product on $C^\infty(\mathfrak{g}^*)$.

Proposition 2.2. *For any $f \in C^\infty(\mathfrak{g}^*)$, the following conditions are equivarent to each other:*

(i) $f \in Z^\infty$,

(ii) $\{X, f\} = 0$ *for all* $X \in \mathfrak{g}$,

(iii) *f is invariant under the coadjoint action of G.*

If $\mathfrak{g}$ has a non-degenerate symmetric bilinear form B invariant under the adjoint action of G, then a linear map $\iota: \mathfrak{g} \to \mathfrak{g}^*$ is defined by $(X, \iota(Y)) = B(X, Y)$ for all $X \in \mathfrak{g}$. We shall identify $C^\infty(\mathfrak{g}^*)$ (resp. $C[\mathfrak{g}]$) with $C^\infty(\mathfrak{g})$ (resp. $C[\mathfrak{g}^*]$) under the isomorphism ι^* induced from ι. In the following, for the cases of $\mathfrak{u}(i)$, $\mathfrak{su}(i)$ and $\mathfrak{so}(i)$, we give standard generators of the centers as Ad-invariant functions on these Lie algebras.

(1) For the Lie algebra $\mathfrak{g}_i=\mathfrak{u}(i)$ of the unitary group $U(i)$, let $F_j^{(i)} \in C[\mathfrak{g}_i^*]$, $1 \leqq j \leqq i$, be polynomial functions on $\mathfrak{g}_i$ defined by

$$\det(E_i-(-1)^{1/2}X\lambda)=1+\sum_{j=1}^{i} F_j^{(i)}(X)\lambda^j.$$

Let Z_i denote the center of $C[\mathfrak{g}_i]$. Then we have (under the identification of $C[\mathfrak{g}_i]$ with $C[\mathfrak{g}_i^*]$) $Z_i=C[F_1^{(i)}, \cdots, F_i^{(i)}]$; a ring of polynomials of i-variables.

(2) For the Lie algebra $\mathfrak{g}_i=\mathfrak{su}(i)$ of the special unitary group $SU(i)$, we have $F_1^{(i)}=0$ and $Z_i=C[F_2^{(i)}, \cdots, F_i^{(i)}]$; a ring of polynomials of $(i-1)$-variables.

(3) For the Lie algebra $\mathfrak{g}_i=\mathfrak{so}(i)$ of the special orthogonal group $SO(i)$, let $F_{2j}^{(i)} \in C[\mathfrak{g}_i^*]$, $1 \leqq j \leqq d$, $d=[i/2]$, be polynomial functions on $\mathfrak{g}_i$ defined by

$$\det(E_i+X\lambda)=1+\sum_{j=1}^{d-1} F_{2j}^{(i)}(X)\lambda^{2j}+\hat{F}_{2d}^{(i)}(X)\lambda^{2d},$$

and $F_{2d}^{(i)}=\hat{F}_{2d}^{(i)}$ for odd i and $(F_{2d}^{(i)})^2=\hat{F}_{2d}^{(i)}$ for even i. Here, note that, for even i, there exists actually a polynomial function $F_{2d}^{(i)}$ on $\mathfrak{g}_i$ satisfying $(F_{2d}^{(i)})^2=\hat{F}_{2d}^{(i)}$. Then we have $Z_i=C[F_2^{(i)}, \cdots, F_{2d}^{(i)}]$; a ring of polynomials of d-variables.

Let $\mathfrak{g}_i$ be either $\mathfrak{u}(i)$ or $\mathfrak{so}(i)$. There exists a series of natural inclusions: $\mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_2 \supset \mathfrak{g}_1$, where $\mathfrak{so}(1)=\{0\}$. Let $\pi_i: \mathfrak{g}_n^* \to \mathfrak{g}_i^*$ be the projection corresponding to the inclusion $\mathfrak{g}_i \subset \mathfrak{g}_n$. Let $\pi_i^*: C[\mathfrak{g}_i] \to C[\mathfrak{g}_n]$ be the homomorphism of rings induced by π_i. Then, by Proposition 1.3, it is a homomorphism of Poisson algebras. We identify, under π_i^*, $C^\infty(\mathfrak{g}_i^*)$ (resp. $C[\mathfrak{g}_i]$) with the Poisson subalgebra $\pi_i^*(C^\infty(\mathfrak{g}_i^*))$ (resp. $\pi_i^*(C[\mathfrak{g}_i])$) of $C^\infty(\mathfrak{g}_n^*)$ (resp. $C[\mathfrak{g}_n]$). Let denote the center of $C[\mathfrak{g}_i]$ by Z_i, and the composed algebra of $Z_1, \cdots, Z_n$ by $\Lambda(\mathfrak{g}_n)$. Then it is easily seen that $\Lambda(\mathfrak{g}_n)$ is a commutative subalgebra of the Poisson algebra $C[\mathfrak{g}_n]$. Let $\{F_j^{(i)}\}$ be the generators of Z_i defined above. Then we have

$$\Lambda(\mathfrak{u}(n))=C[F_j^{(i)} \mid 1 \leqq j \leqq i \leqq n] \text{ and}$$
$$\Lambda(\mathfrak{so}(n))=C[F_{2j}^{(i)} \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n].$$

In the following section, we shall show that these generators are functionally independent.

Remark. Though we do not use in the present paper, we have the following:

Proposition 2.3. *$\Lambda(\mathfrak{u}(n))$ and $\Lambda(\mathfrak{so}(n))$ are maximal commutative sub-*

algebras of the Poisson algebras $C[\mathfrak{u}(n)]$ *and* $C[\mathfrak{so}(n)]$, *respectively.*

For the symplectic group $Sp(n)$, we can easily construct a commutative subalgebra $\Lambda(\mathfrak{sp}(n))$ in the same manner as above. However, the fact corresponding to Proposition 2.3 does not hold.

§ 3. Eigenvalues and eigen-polynomials

For each $a=(a_1, \cdots, a_{n-1})$, $a_1 \geqq a_2 \geqq \cdots \geqq a_{n-1}$, $t \in \boldsymbol{R}$ and $z=(z_1, \cdots, z_{n-1}) \in \boldsymbol{C}^{n-1}$, define a Hermitian matrix $X=X(a, t, z)$ by

$$(3.1) \qquad X=\begin{bmatrix} a_1 & & 0 & \bar{z}_1 \\ & \ddots & & \vdots \\ 0 & & a_{n-1} & \bar{z}_{n-1} \\ z_1 & \cdots & z_{n-1} & t \end{bmatrix}.$$

Then the eigen-polynomial F_X of X is given by

$$(3.2) \qquad \begin{aligned} F_X(\lambda) &= \det(\lambda E_n - X) \\ &= (\lambda - t) \prod_{i=1}^{n-1} (\lambda - a_i) - \sum_{i=1}^{n-1} z_i \bar{z}_i \prod_{1 \leqq j \leqq n-1, j \neq i} (\lambda - a_j). \end{aligned}$$

Proposition 3.1. *Let* $b_1 \geqq b_2 \geqq \cdots \geqq b_n$ *be the eigenvalues of* X. *Then the following relations hold*:

$$(3.3) \qquad b_1 \geqq a_1 \geqq b_2 \geqq a_2 \geqq \cdots \geqq a_{n-1} \geqq b_n.$$

Conversely, for any $a_1, \cdots, a_{n-1}, b_1, \cdots, b_n$ *which satisfy* (3.3), *there exist* $t \in \boldsymbol{R}$ *and* $z=(z_1, \cdots, z_{n-1}) \in \boldsymbol{C}^{n-1}$ *such that*

$$(3.4) \qquad \det(\lambda E_n - X(a, t, z)) = \prod_{i=1}^{n} (\lambda - b_i).$$

Proof. To prove the first half, it is enough to show for the case $a_1 > a_2 > \cdots > a_{n-1}$ and $z_1 z_2 \cdots z_{n-1} \neq 0$, since $b_1, \cdots, b_n$ are continuous functions of a, t and z. In this case, we have

$$(3.5) \qquad (-1)^i F_X(a_i) = (-1)^{i+1} z_i \bar{z}_i \prod_{1 \leqq j \leqq n-1, j \neq i} (a_i - a_j) > 0$$

for $1 \leqq i \leqq n-1$. From this, we have immediately $b_1 > a_1 > b_2 > \cdots > a_{n-1} > b_n$. To prove the latter half, we consider the following two cases: (i) The case of $a_1 > a_2 > \cdots > a_{n-1}$. For any $b_1, \cdots, b_n$ which satisfy (3.3), let define $t \in \boldsymbol{R}$ and $z=(z_1, \cdots, z_{n-1})$ by $t=\sum_{j=1}^{n} b_j - \sum_{i=1}^{n-1} a_i$ and $z_i = (-(A_i)^{-1} B_i)^{1/2}$, where $A_i = \prod_{1 \leqq j \leqq n-1, j \neq i} (a_i - a_j)$ and $B_i = \prod_{j=1}^{n} (b_j - a_i)$.

Note that $A_i \neq 0$ and $-(A_i)^{-1}B_i \geqq 0$. Then (3.4) is satisfied for these t and z. (ii) General case. Assume that $a=(a_1, \cdots, a_{n-1})$ and $b_1, \cdots, b_n$ satisfy (3.3). Define $a'_1 > \cdots > a'_{r-1}$, $d(k)$ and $e(k)$, $1 \leqq k \leqq r-1$, by $a_i = a'_k$ for $d(k-1) < i \leqq d(k)$ $(d(0)=0)$ and $e(k)=d(k)-d(k-1)$. Furthermore, define $b'_1 \geqq \cdots \geqq b'_r$ by

$$\prod_{i=1}^{n} (\lambda - b_i) = \prod_{j=1}^{r-1} (\lambda - a'_j)^{e(j)-1} \prod_{k=1}^{r} (\lambda - b'_k).$$

Then, since $b'_1 \geqq a'_1 \geqq b'_2 \geqq \cdots \geqq a'_{r-1} \geqq b'_r$, we can choose, as in (i), z'_i, $1 \leqq i \leqq r-1$, and $t \in \boldsymbol{R}$ such that

$$(3.6) \qquad \prod_{k=1}^{r} (\lambda - b'_k) = (\lambda - t) \prod_{i=1}^{r-1} (\lambda - a'_i) - \sum_{i=1}^{r-1} z'_i \bar{z}'_i \prod_{1 \leqq j \leqq r-1, j \neq i} (\lambda - a'_j).$$

Let define $z=(z_1, \cdots, z_{n-1})$ by $z_{d(k)} = z'_k$ and $z_i = 0$ for $i \neq d(k)$, $1 \leqq k \leqq r-1$. Multiplying $\prod_{j=1}^{r-1} (\lambda - a'_j)^{e(j)-1}$ to the both hand sides of (3.6), we obtain (3.4) for $X = X(a, t, z)$. q.e.d.

For each $a=(a_1, \cdots, a_{n-1})$, $a_1 \geqq \cdots \geqq a_{n-1}$, let define real-valued functions $F_{a,1}, \cdots, F_{a,n}$ and $A_{a,1} \geqq A_{a,2} \geqq \cdots \geqq A_{a,n}$ on $\boldsymbol{R}^{2n-1} = \{(t, x, y) \mid x=(x_1, \cdots, x_{n-1}), y=(y_1, \cdots, y_{n-1})\}$ by

$$F_X(\lambda) = \lambda^n + \sum_{i=1}^{n} (-1)^i F_{a,i} \lambda^{n-i} = \prod_{i=1}^{n} (\lambda - A_{a,i}),$$

where $X = X(a, t, z)$, $z=(z_1, \cdots, z_{n-1})$ and $z_i = x_i + (-1)^{1/2} y_i$. $F_{a,i}$ is a polynomial function and $A_{a,i}$ is a continuous function. Furthermore, let define a subset L_a of $\boldsymbol{R}^{2n-1}$ by

$$L_a = \left\{ p=(t, x, y) \in \boldsymbol{R}^{2n-1} \mid \prod_{i=1}^{n-1} (a_i - A_{a,i}(p))(a_i - A_{a,i+1}(p)) = 0 \right\}$$

and the Jacobi matrix J_a by

$$J_a = \partial(F_{a,1}, \cdots, F_{a,n}) / \partial(t, x_1, \cdots, x_{n-1}, y_1, \cdots, y_{n-1}).$$

Then we have

Proposition 3.2.

$$L_a = \{p=(t, x, y) \in \boldsymbol{R}^{2n-1} \mid \operatorname{rank}(J_a(p)) \leqq n-1\}.$$

Proof. Let F_a be a polynomial of the variables (t, x, y, λ) defined by $F_a(t, x, y, \lambda) = F_{X(a,t,z)}(\lambda)$, and $f_a, f_{a,i}$, $1 \leqq i \leqq n-1$, be polynomials of λ defined by

$$f_a(\lambda)=\prod_{i=1}^{n-1}(\lambda-a_i), \qquad f_{a,i}(\lambda)=\prod_{1\leqq j\leqq n-1, j\neq i}(\lambda-a_j).$$

Then we have

$$\begin{cases} \partial F_a(t, x, y, \lambda)/\partial t=-f_a(\lambda), \\ \partial F_a(t, x, y, \lambda)/\partial x_i=-2x_i f_{a,i}(\lambda), \\ \partial F_a(t, x, y, \lambda)/\partial y_i=-2y_i f_{a,i}(\lambda). \end{cases} \tag{3.7}$$

n polynomials f_a and $f_{a,i}$, $1\leqq i\leqq n-1$, are linearly independent in the ring $\boldsymbol{R}[\lambda]$ of polynomials if and only if

$$\prod_{1\leqq i<j\leqq n-1}(a_i-a_j)\neq 0.$$

From this and (3.7), it follows that $\operatorname{rank}(J_a(p))=n$ if and only if $a_1>\cdots>a_{n-1}$ and $\prod_{i=1}^{n-1}(x_i^2+y_i^2)\neq 0$ for $p=(t, x, y)$. On the other hand, by the definition of $A_{a,i}$, we have that $(a_i-A_{a,i}(p))(a_i-A_{a,i+1}(p))\neq 0$ if and only if $a_{i-1}<a_i<a_{i+1}$ and $z_i\neq 0$, where $a_0=-\infty$, $a_n=+\infty$. From these, our assertion follows. q.e.d.

For a Hermitian matrix C of degree $n-1$, $t\in\boldsymbol{R}$ and $z=(z_1, \cdots, z_{n-1})\in\boldsymbol{C}^{n-1}$, define a Hermitian matrix $X_C=X_C(t, z)$ of degree n by

$$X_C=\begin{bmatrix} & & \bar{z}_1 \\ & C & \vdots \\ & & \bar{z}_{n-1} \\ z_1 & \cdots\ z_{n-1} & t \end{bmatrix}.$$

Let $a_1\geqq a_2\geqq\cdots\geqq a_{n-1}$ be the eigenvalues of C. Then, by Proposition 3.1, we have

Proposition 3.3. *The eigenvalues $b_1\geqq\cdots\geqq b_n$ of X_C satisfy*

$$b_1\geqq a_1\geqq b_2\geqq a_2\geqq\cdots\geqq a_{n-1}\geqq b_n. \tag{3.8}$$

Conversely, for any $b_1, \cdots, b_n$ which satisfy (3.8), *there exist $t\in\boldsymbol{R}$ and $z\in\boldsymbol{C}^{n-1}$ such that the eigenvalues of $X_C(t, z)$ are $b_1, \cdots, b_n$.*

We put $z_i=x_i+(-1)^{1/2}y_i$, $x_i, y_i\in\boldsymbol{R}$, and consider $b_1, \cdots, b_n$ to be functions of t, $x=(x_1, \cdots, x_{n-1})$ and $y=(y_1, \cdots, y_{n-1})$. Let $F_{C,i}$, $1\leqq i\leqq n$, be the i-th fundamental symmetric polynomial of $b_1, \cdots, b_n$. We consider $F_{C,i}$ to be a polynomial function on $\boldsymbol{R}^{2n-1}$. Define the Jacobi matrix J_C and a subset L_C of $\boldsymbol{R}^{2n-1}$ by

$$J_C=\partial(F_{C,1}, \cdots, F_{C,n})/\partial(t, x_1, \cdots, x_{n-1}, y_1, \cdots, y_{n-1})$$

and

$$L_C=\left\{p=(t, x, y) \in \boldsymbol{R}^{2n-1} \mid \prod_{i=1}^{n-1}(a_i-b_i(p))\,(a_i-b_{i+1}(p))=0\right\}.$$

Then, by Proposition 3.2, we have

Proposition 3.4.

$$L_C=\{p=(t, x, y) \in \boldsymbol{R}^{2n-1} \mid \operatorname{rank}(J_C(p)) \leqq n-1\}.$$

For any skew-Hermitian matrix X of degree n, i.e., $X \in \mathfrak{u}(n)$, the matrix consisting of the first i raws and columns of X is denoted by X_i, $1 \leqq i \leqq n$. Note that $X_i \in \mathfrak{u}(i)$. Let $F_j^{(i)}$ and $A_j^{(i)}$, $1 \leqq j \leqq i \leqq n$, be functions on $\mathfrak{u}(n)$ defined by

$$\det(E_i-(-1)^{1/2}X_i\lambda)=1+\sum_{j=1}^{i} F_j^{(i)}(X)\lambda^j=\prod_{j=1}^{i}(1+A_j^{(i)}(X)\lambda), \quad \text{and}$$

$$A_1^{(i)} \geqq \cdots \geqq A_i^{(i)}.$$

Then $F_j^{(i)}$ is a polynomial function and $A_j^{(i)}$ is a continuous function. Le. $d=n(n+1)/2$ and $(x_1^{(n)}, \cdots, x_n^{(n)}; x_1^{(n-1)}, \cdots, x_{n-1}^{(n-1)}; \cdots; x_1^{(1)})$ be the standard coordinates of $\boldsymbol{R}^d=\boldsymbol{R}^n \times \boldsymbol{R}^{n-1} \times \cdots \times \boldsymbol{R}^1$. We define maps $A: \mathfrak{u}(n) \to \boldsymbol{R}^d$ $F: \mathfrak{u}(n) \to \boldsymbol{R}^d$ and $S: \boldsymbol{R}^d \to \boldsymbol{R}^d$ and a subset D of $\boldsymbol{R}^d$ as follows:

$$A(X)=(A_j^{(i)}(X) \mid 1 \leqq j \leqq i \leqq n),$$
$$F(X)=(F_j^{(i)}(X) \mid 1 \leqq j \leqq i \leqq n),$$
$$S(x_j^{(i)} \mid 1 \leqq j \leqq i \leqq n)=(y_j^{(i)} \mid 1 \leqq j \leqq i \leqq n),$$

where $y_j^{(i)}$ are determined by $\prod_{j=1}^{i}(1+x_j^{(i)}\lambda)=1+\sum_{j=1}^{i} y_j^{(i)}\lambda^j$, and

$$D=\{(x_j^{(i)}) \in \boldsymbol{R}^d \mid x_{j-1}^{(i)} \geqq x_{j-1}^{(i-1)} \geqq x_j^{(i)},\ 2 \leqq j \leqq i \leqq n\}. \tag{3.9}$$

F and S are polynomial maps and A is a continuous map. The following theorem is obtained directly from Propositions 3.3 and 3.4.

Theorem 3.5. *We have* $S \circ A=F$ *and* $D=\operatorname{Image}(A)$. *The map* $S|_D: D \to \operatorname{Image}(F)$ *is a homeomorphism, and hence* Image(F) *contains a non-empty open subset of* $\boldsymbol{R}^d$. *Moreover, the set of degenerate points of the differential* dF *of* F *coincides with the inverse image* $A^{-1}(\partial D)$ *of the boundary* ∂D *of* D.

Corollary 3.6. *The polynomial functions* $\{F_j^{(i)} \mid 1 \leqq j \leqq i \leqq n\}$ *on* $\mathfrak{u}(n)$ *are functionally (and hence algebraically) independent. Therefore, the subring* $\mathcal{A}(\mathfrak{u}(n))$ *defined in Section 2 is isomorphic to a ring of polynomials of* $n(n+1)/2$*-variables.*

Proof. Let B be a negative-definite symmetric bilinear form on $\mathfrak{u}(n)$ defined by $B(X, Y)=\mathrm{tr}(XY)$. Then the projection $\pi_i: \mathfrak{g}_n^* \to \mathfrak{g}_i^*$ defined in Section 2 is identified with the orthogonal projection $\pi_i': \mathfrak{g}_n \to \mathfrak{g}_i$, $1 \leqq i \leqq n$. Then our assertion follows directly from Theorem 3.5. q.e.d.

As to the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group $SO(n)$ of degree n, the similar arguments as above also hold: For any $X \in \mathfrak{so}(n)$, X_i denotes the matrix consisting of the first i rows and columns of X, $1 \leqq i \leqq n$. Let $F_{2j}^{(i)}$ and $A_j^{(i)}$, $1 \leqq j \leqq [i/2]$, $2 \leqq i \leqq n$, be polynomial functions and continuous functions on $\mathfrak{so}(n)$, respectively, defined by

$$\begin{aligned}\det(E_i + X_i\lambda) &= 1 + \sum_{j=1}^{[i/2]-1} F_{2j}^{(i)}(X)\lambda^{2j} + \hat{F}_{2[i/2]}^{(i)}(X)\lambda^{2[i/2]} \\ &= \prod_{j=1}^{[i/2]} (1 + (A_j^{(i)}(X))^2\lambda^2),\end{aligned}$$

and for odd i, $F_{2[i/2]}^{(i)} = \hat{F}_{2[i/2]}^{(i)}$, $A_1^{(i)} \geqq A_2^{(i)} \geqq \cdots \geqq A_{[i/2]}^{(i)} \geqq 0$, for even i, $(F_{2[i/2]}^{(i)})^2 = \hat{F}_{2[i/2]}^{(i)}$, $A_1^{(i)} \geqq \cdots \geqq A_{[i/2]-1}^{(i)} \geqq |A_{[i/2]}^{(i)}|$, $F_{2[i/2]}^{(i)} = A_1^{(i)} A_2^{(i)}, \cdots, A_{[i/2]}^{(i)}$. Let $d = \sum_{i=2}^n [i/2] = (n(n-1)/2 + [n/2])/2$ and $(x_1^{(n)}, \cdots, x_{[n/2]}^{(n)}; x_1^{(n-1)}, \cdots, x_{[(n-1)/2]}^{(n-1)}; \cdots; x_1^{(2)})$ be the standard coordinates of $\boldsymbol{R}^d$. We define maps $A: \mathfrak{so}(n) \to \boldsymbol{R}^d$, $F: \mathfrak{so}(n) \to \boldsymbol{R}^d$ and $S: \boldsymbol{R}^d \to \boldsymbol{R}^d$ and a subset D of $\boldsymbol{R}^d$ as follows:

$$\begin{aligned}&A(X) = (A_j^{(i)}(X) \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n),\\ &F(X) = (F_{2j}^{(i)}(X) \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n),\\ &S(x_j^{(i)} \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n) = (y_j^{(i)} \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n),\end{aligned}$$

where $y_j^{(i)}$ is the j-th fundamental symmetric polynomial of $(x_1^{(i)})^2, \cdots, (x_{[i/2]}^{(i)})^2$ for $1 \leqq j < i/2$ and $y_{i/2}^{(i)} = x_1^{(i)} x_2^{(i)} \cdots x_{i/2}^{(i)}$ for even i, and

$$\begin{aligned}D = \{(x_j^{(i)}) \in \boldsymbol{R}^d \mid\ & x_1^{(i)} \geqq x_1^{(i-1)} \geqq x_2^{(i)} \geqq x_2^{(i-1)} \geqq \cdots \geqq x_{[(i-1)/2]}^{(i)} \geqq |x_{[i/2]}^{(i-1)}| \\ & \text{for odd } i, \text{ and } x_1^{(i)} \geqq x_1^{(i-1)} \geqq x_2^{(i)} \geqq x_2^{(i-1)} \geqq \cdots \geqq x_{[(i-1)/2]}^{(i-1)} \geqq |x_{i/2}^{(i)}| \\ & \text{for even } i,\ 2 \leqq i \leqq n\}.\end{aligned}$$

Theorem 3.7. *We have $S \circ A = F$ and $D = \mathrm{Image}(A)$. The map $S|_D$: $D \to \mathrm{Image}(F)$ is a homeomorphism, and hence* Image (F) *contains a non-empty open subset of $\boldsymbol{R}^d$. Moreover, the set of degenerate points of the differential dF of F coincides with the inverse image $A^{-1}(\partial D)$ of the boundary ∂D of D.*

Corollary 3.8. *The polynomial functions $\{F_{2j}^{(i)} \mid 1 \leqq j \leqq [i/2],\ 2 \leqq i \leqq n\}$ on $\mathfrak{so}(n)$ are functionally (and hence algebraically) independent. Therefore,* *the subring $\Lambda(\mathfrak{so}(n))$ defined in Section 2 is isomorphic to a ring of polynomials of $(n(n-1)/2+[n/2])/2$-variables.*

§ 4. Symmetric spaces

In the present section, we shall consider, for every symmetric spaces M obtained as the quotient spaces of the groups $U(n)$, $SU(n)$ and $SO(n)$, the following:

(1) a realization of M in a Euclidean space,

(2) the moment map μ: TM $(\cong T^*M)\to\mathfrak{g}$ $(\cong\mathfrak{g}^*)$,

(3) the image of the map $A\circ\mu$: $TM\to D$, and

(4) a complete system of first integrals in involution for the geodesic flow of M.

For the sake of simplicity, we give the proofs for only a few cases.

(I) Complex Grassmannian manifold

$$M=G_{p,q}(C)=U(p+q)/U(p)\times U(q) \qquad (1\leqq p\leqq q).$$

We consider M to be the set of all orthogonal projections of rank p in C^n, $n=p+q$. M is realized in the space $(C)_n$ of all complex matrices of degree n as follows:

$$M=\{X\in(C)_n\,|\,X^*=X,\ X^2=X,\ \mathrm{tr}\,X=p\}.$$

The tangent bundle TM is realized in $(C)_n\times(C)_n$ as follows:

$$TM=\{(X,Y)\in M\times(C)_n\,|\,Y^*=Y,\ XY+YX=Y\}.$$

The action φ: $U(n)\times M\to M$ of $U(n)$ on M is given by $\varphi(g,X)=gXg^{-1}$. For each $X\in M$, the linear map ψ_X: $\mathfrak{u}(n)\to T_XM$ defined by $\psi_X(Z)=\rho_\varphi(Z)|_X$ is given by $\psi_X(Z)=[Z,X]$. By Proposition 1.5, the restriction of the moment map to T_XM coincides with the dual map of ψ_X. If we define a Riemannian metric g on M and an Ad $U(n)$-invariant, negative-definite symmetric bilinear form B on $\mathfrak{u}(n)$ by

$$g_X(Y_1,Y_2)=\mathrm{tr}\,(Y_1Y_2) \qquad \text{for } (X,Y_i)\in TM,\ i=1,2,$$

and

$$B(Z_1,Z_2)=\mathrm{tr}\,(Z_1Z_2) \qquad \text{for } Z_i\in u(n),\ i=1,2,$$

respectively, then we have

$$g_X(\psi_X(Z),Y)=\mathrm{tr}\,([Z,X]Y)=\mathrm{tr}\,(Z[X,Y])=B(Z,[X,Y])$$

for $(X, Y) \in TM$, $Z \in \mathfrak{u}(n)$. It follows that the moment map μ: $TM \to \mathfrak{u}(n)$ is given by

$$\mu(X, Y) = [X, Y].$$

Proposition 4.1.

$$\text{Image}\,(A \circ \mu) = \{(x_j^{(i)}) \in D \mid x_i^{(n)} + x_{n-i+1}^{(n)} = x_j^{(n)} = 0,\quad 1 \leqq i \leqq p,\ p < j \leqq n-p\}.$$

Proof. For $(X, Y) \in TM$, we have $\dim(XC^n) = p$, $\dim((E_n - X)C^n) = q$, $(XC^n) \perp (E_n - X)C^n$, $XY^2 = (Y - YX)Y = Y(Y - XY) = Y^2X$ and $Y^2XC^n = XY^2C^n \subset XC^n$. Since Y is a Hermitian matrix, the eigenvalues of Y^2 are all non-negative. Let a_i^2, $1 \leqq i \leqq p$, be the eigenvalues of Y^2 in XC^n, where $a_1 \geqq a_2 \geqq \cdots \geqq a_r > a_{r+1} = \cdots = a_p = 0$, and α_i, $1 \leqq i \leqq p$, be the corresponding orthonormal eigenvectors. Since $YXC^n = (Y - XY)C^n = (E_n - X)YC^n \subset (E_n - X)C^n$, we have $Y\alpha_i \in (E_n - X)C^n = (XC^n)^\perp$ for $1 \leqq i \leqq p$. $\beta_i = (a_i)^{-1}Y\alpha_i$, $1 \leqq i \leqq r$, are orthonormal and $Y\alpha_i = 0$ for $i \geqq r+1$. Extend β_i, $1 \leqq i \leqq r$, to an orthonormal basis $\{\beta_i \mid 1 \leqq i \leqq q\}$ of $(E_n - X)C^n$. Then we have $Y\alpha_i = a_i\beta_i$, $Y\beta_i = a_i\alpha_i$ for $1 \leqq i \leqq p$ and $Y\beta_i = 0$ for $p < i \leqq q$. Let $Y_1 = \mu(X, Y) = [X, Y]$. Then we have $Y_1\alpha_i = -a_i\beta_i$, $Y_1\beta_i = a_i\alpha_i$ for $1 \leqq i \leqq p$ and $Y_1\beta_j = 0$ for $p < j \leqq q$. It follows that the eigenvalues of $-(-1)^{1/2}Y_1$ are $a_1 \geqq \cdots \geqq a_p \geqq 0 = \cdots = 0 \geqq -a_p \geqq \cdots \geqq -a_1$. Conversely, for any $a_1 \geqq a_2 \geqq \cdots \geqq a_p \geqq 0$ and any orthonormal basis $\{e_i\}$ of C^n, define $X, Y \in (C)_n$ by $Xe_i = e_i$, $1 \leqq i \leqq p$, $Xe_j = 0$, $p < j \leqq n$, $Ye_i = a_ie_{p+i}$, $1 \leqq i \leqq p$, $Ye_{p+i} = a_ie_i$, $1 \leqq i \leqq p$ and $Ye_j = 0$, $2p < j \leqq n$. Then $(X, Y) \in TM$ and the eigenvalues of $-(-1)^{1/2}\mu(X, Y)$ are $a_1 \geqq \cdots \geqq a_p \geqq 0 = \cdots = 0 \geqq -a_p \geqq \cdots \geqq -a_1$. As $\{e_i\}$ runs over all orthonormal basis of C^n, $\mu(X, Y)$ runs over all elements of $\mathfrak{u}(n)$ which have the eigenvalues as above. From this, our assertion follows easily. q.e.d.

Let $F_j^{(i)}$, $1 \leqq j \leqq i \leqq n$, be the functions defined in Section 3, and $\mu_j^{(i)} = \mu^*(F_j^{(i)})$ be the pull-back of $F_j^{(i)}$ by the moment map μ.

Theorem 4.2. *The following functions on $TG_{p,q}(C)$ are functionally independent*:

$$\mu_{2j}^{(n)},\ 1 \leqq j \leqq p,$$
$$\mu_j^{(i)},\ 2p \leqq i < n,\ 1 \leqq j \leqq 2p \text{ and}$$
$$\mu_j^{(i)},\ 1 \leqq i < 2p,\ 1 \leqq j \leqq i.$$

The number of these functions is

$$p + 2p(q-p) + ((2p-1) + (2p-2) + \cdots + 2 + 1) = 2pq = \dim_R G_{p,q}(C).$$

The other $\mu_j^{(i)}$*'s vanish identically.* $-2\mu_2^{(n)}$ *is the function assigning to each tangent vector the square of its length. Thus, the above functions provide a complete system of first integrals in involution for the geodesic flow of* $G_{p,q}(\boldsymbol{C})$.

Proof. From Proposition 4.1 and the definition of D in Section 3, we have that $x_j^{(i)}=0$ for $2p\leqq i\leqq n$, $p<j\leqq i-p$. Our assertion follows easily from this. q.e.d.

In the following, $(\mu_1^{(n)})^2-2\mu_2^{(n)}$ is always the function assigning to each tangent vector the square of its length, so we shall not state it one by one.

(II) $\tilde{M}=U(n)/O(n)$, $M=SU(n)/SO(n)$

Let J be the complex structure of the vector space $\boldsymbol{C}^n\cong\boldsymbol{R}^{2n}$. Then we have $J^2=-E_{2n}$ and ${}^tJ=-J$. $\tilde{M}$ is realized in $(\boldsymbol{R})_{2n}$ as follows:

$$\tilde{M}=\{P\in(\boldsymbol{R})_{2n}\,|\,PJ+JP=0,\ P^2=E_{2n},\ {}^tP=P\}.$$

Hence, $T\tilde{M}$ is realized in $\tilde{M}\times(\boldsymbol{R})_{2n}$ as follows:

$$T\tilde{M}=\{(P,X)\in\tilde{M}\times(\boldsymbol{R})_{2n}\,|\,XJ+JX=0,\ PX+XP=0,\ {}^tX=X\}.$$

The moment map μ: $T\tilde{M}\to\mathfrak{u}(n)$ is given by $\mu(P,X)=[P,X]$.

Proposition 4.3.

$$\operatorname{Image}(A\circ\mu)=D.$$

Theorem 4.4. *The functions* $\mu_j^{(i)}=\mu^*(F_j^{(i)})$, $1\leqq j\leqq i\leqq n$, *on* $T\tilde{M}$ *are functionally independent. The number of these functions is* $d=n(n+1)/2=\dim\tilde{M}$. *On* TM, $\mu_1^{(n)}$ *vanishes identically, and the other* $\mu_j^{(i)}$*'s are independent. Thus, the geodesic flows of* $\tilde{M}$ *and* M *are completely integrable.*

(III) $\tilde{M}=U(2n)/Sp(n)$, $M=SU(2n)/Sp(n)$

Let J be the complex structure of the vector space $\boldsymbol{C}^{2n}=\boldsymbol{R}^{4n}$. Then we have that $J^2=-E_{4n}$ and ${}^tJ=-J$. $\tilde{M}$ and $T\tilde{M}$ are realized as follows:

$$\tilde{M}=\{P\in(\boldsymbol{R})_{4n}\,|\,{}^tP+P=JP+PJ=0,\ P^2=-E_{4n}\},$$
$$T\tilde{M}=\{(P,X)\in\tilde{M}\times(\boldsymbol{R})_{4n}\,|\,{}^tX+X=JX+XJ=PX+XP=0\}.$$

The moment map μ: $T\tilde{M}\to\mathfrak{u}(n)$ is given by $\mu(P,X)=PX$.

Proposition 4.5.

$$\operatorname{Image}(A\circ\mu)=\{(x_j^{(i)})\in D\,|\,x_{2j-1}^{(2n)}=x_{2j}^{(2n)},\ 1\leqq j\leqq n\}.$$

Proof. For each $(P, X) \in T\tilde{M}$, let $A=-JPX$. Then A is a symmetric matrix; ${}^tA=XPJ=A$. It follows that the eigenvalues of A are real numbers. Let a_1 be the maximal eigenvalue of A and $v_1 \in \boldsymbol{R}^{4n}$, $\|v_1\|=1$, be the corresponding eigenvector. Then $\{v_1, Jv_1, Pv_1, JPv_1\}$ are orthonormal and span a vector space L over $\boldsymbol{R}$. Since A commutes with J and P, we have $Av=a_1v$ for all $v \in L$, i.e., the eigenvalues of A on L are a_1, a_1, a_1, a_1. If we consider L to be a 2-dimensional complex vector space with respect to the action of $\boldsymbol{C}=\boldsymbol{R}[J]$ and A to be a matrix of complex coefficients, then the eigenvalues of A are a_1, a_1. The orthogonal complement $L^{\perp}$ of L is invariant under the actions of J, P and A. We can apply a similar arguments as above to $L^{\perp}$. Consequently, we have that the eigen-polynomial of A is given by $(\lambda-a_1)^2(\lambda-a_2)^2\cdots(\lambda-a_n)^2$. Conversely, for any $a_i \in \boldsymbol{R}$, $1\leqq i\leqq n$, and any orthonormal basis $\{e_i \mid 1\leqq i\leqq 2n\}$ of $\boldsymbol{C}^{2n}$ $(\cong\boldsymbol{R}^{4n})$, define $P, X \in (\boldsymbol{R})_{4n}$ by

$$\begin{aligned} Pe_{2i-1}&=e_{2i}, & Xe_{2i-1}&=a_iJe_{2i},\\ Pe_{2i}&=-e_{2i-1}, & Xe_{2i}&=-a_iJe_{2i-1},\\ PJe_{2i-1}&=-Je_{2i}, & XJe_{2i-1}&=a_ie_{2i},\\ PJe_{2i}&=Je_{2i-1}, & XJe_{2i}&=-a_ie_{2i-1}, \end{aligned}$$

for $1\leqq i\leqq n$. Here note that $\{e_i, Je_i \mid 1\leqq i\leqq 2n\}$ is an orthonormal basis of $\boldsymbol{R}^{4n}$. Then we have $(P, X) \in T\tilde{M}$ and

$$\det(\lambda E_{2n}+(-1)^{1/2}B)=\prod_{i=1}^{n}(\lambda-a_i)^2,$$

where $B=\mu(P, X)=PX$. q.e.d.

Lemma 1. *Let $\boldsymbol{C}[x_1, \cdots, x_n]$ be the ring of polynomials of variables $x_1, \cdots, x_n$. If we define $F_i, G_j \in \boldsymbol{C}[x_1, \cdots, x_n]$, $1\leqq i\leqq 2n$, $1\leqq j\leqq n$, by*

$$1+\sum_{i=1}^{2n} F_i\lambda^i=\prod_{i=1}^{n}(1+x_i\lambda)^2=A$$

and

$$1+\sum_{i=1}^{n} G_i\lambda^i=\prod_{i=1}^{n}(1+x_i\lambda)=B,$$

then we have

$$\boldsymbol{C}[F_1, \cdots, F_{2n}]=\boldsymbol{C}[F_1, \cdots, F_n]=\boldsymbol{C}[G_1, \cdots, G_n]$$

as subrings of $\boldsymbol{C}[x_1, \cdots, x_n]$.

Proof. Compairing the coefficients of A and B^2, we have

$$F_i=\sum_{p+q=i,p,q\geqq 0} G_pG_q, \qquad 1\leqq i\leqq 2n, \tag{4.1}$$

where $G_0=1$, $G_p=0$ if $p>n$. It follows that $F_i\in C[G_1,\cdots,G_n]$, $1\leqq i\leqq 2n$. Next, by the induction with respect to i, we show that $G_i\in C[F_1,\cdots,F_n]$, $1\leqq i\leqq n$. For $i=1$, we have $G_1=F_1/2$. Fix i, $1<i\leqq n$, and assume that $G_j\in C[F_1,\cdots,F_n]$ for all $1\leqq j\leqq i-1$. Then, by (4.1), we have

$$G_i=\frac{1}{2}\left(F_i-\sum_{p=1}^{i-1} G_pG_{i-p}\right)\in C[F_1,\cdots,F_n].$$

q.e.d.

The following lemma is proved similarly as above.

Lemma 2. *Let $C[x_1,\cdots,x_n,y_1,\cdots,y_m]$ be the ring of polynomials of variables $x_1,\cdots,x_n,y_1,\cdots,y_m$. If we define $F_i,G_j,H_k\in C[x_1,\cdots,x_n,y_1,\cdots,y_m]$, $1\leqq i\leqq n$, $1\leqq j\leqq m$, $1\leqq k\leqq n+m$, by*

$$1+\sum_{i=1}^{n} F_i\lambda^i=\prod_{i=1}^{n}(1+x_i\lambda),$$

$$1+\sum_{i=1}^{m} G_i\lambda^i=\prod_{i=1}^{m}(1+y_i\lambda) \quad \textit{and}$$

$$1+\sum_{i=1}^{n+m} H_i\lambda^i=\prod_{i=1}^{n}(1+x_i\lambda)\prod_{j=1}^{m}(1+y_j\lambda),$$

then we have

$$C[F_1,\cdots,F_n,G_1,\cdots,G_m]=C[F_1,\cdots,F_n,H_1,\cdots,H_{n+m}]$$

as subrings of $C[x_1,\cdots,x_n,y_1,\cdots,y_m]$.

Theorem 4.6. *The following functions on $T\tilde{M}$ are functionally independent:*

$$\mu_i^{(2n)},\ 1\leqq i\leqq n,$$
$$\mu_j^{(2n-1)},\ 1\leqq j\leqq n-1 \quad \textit{and}$$
$$\mu_j^{(i)},\ 1\leqq i\leqq 2n-2,\ 1\leqq j\leqq i.$$

The number of these functions is

$$n+(n-1)+((2n-2)+(2n-3)+\cdots+2+1)=n(2n-1)=\dim\tilde{M}.$$

The other $\mu_j^{(i)}$'s are polynomial functions of the above functions. On TM,

$\mu_1^{(2n)}$ *vanishes identically, and the other functions stated above are independent.*

Proof. We identify $\mu_j^{(i)}=\mu^*(F_j^{(i)})$ with a function on $\hat{D}=A\circ\mu(T\tilde{M})\subset D$. The restrictions of the coordinate functions $x_j^{(i)}$ of D onto $\hat{D}$ are also denoted by $x_j^{(i)}$. By proposition 4.5, we have $x_1^{(2n)}=x_1^{(2n-1)}=x_2^{(2n)}\geqq x_2^{(2n-1)}\geqq x_3^{(2n)}=x_3^{(2n-1)}=x_4^{(2n)}\geqq\cdots\geqq x_{2n-2}^{(2n-1)}\geqq x_{2n-1}^{(2n)}=x_{2n-1}^{(2n-1)}=x_{2n}^{(2n)}$. Hence, we can choose $x_{2i-1}^{(2n)}$, $1\leqq i\leqq n$, $x_{2j}^{(2n-1)}$, $1\leqq j\leqq n-1$, and $x_j^{(i)}$, $1\leqq i\leqq 2n-2$, $1\leqq j\leqq i$, as a system of (independent) coordinate functions on $\hat{D}$. If we define functions h_i, $1\leqq i\leqq n$, and g_j, $1\leqq j\leqq n-1$, on $\hat{D}$ by

$$1+\sum_{i=1}^{n} h_i\lambda^i=\prod_{i=1}^{n}(1+x_{2i-1}^{(2n)}\lambda)$$

and

$$1+\sum_{j=1}^{n-1} g_j\lambda^j=\prod_{j=1}^{n-1}(1+x_{2j}^{(2n-1)}\lambda),$$

then h_i, $1\leqq i\leqq n$, g_j, $1\leqq j\leqq n-1$, and $\mu_j^{(i)}$, $1\leqq j\leqq i\leqq n-2$, are functionally independent of $\hat{D}$. Now, by Lemmas 1 and 2, we have $C[h_1,\cdots,h_n,g_1,\cdots,g_{n-1},\mu_j^{(i)}(1\leqq j\leqq i\leqq n-2)]=C[\mu_1^{(2n)},\cdots,\mu_n^{(2n)},\mu_1^{(2n-1)},\cdots,\mu_{n-1}^{(2n-1)},\mu_j^{(i)}(1\leqq j\leqq i\leqq n-2)]$. Then our assertion follows. q.e.d.

(IV) $M=G_{p,q}(\boldsymbol{R})=SO(n)/S(O(p)\times O(q))$ $(1\leqq p\leqq q,\ n=p+q)$

$$M=\{X\in(\boldsymbol{R})_n\mid {}^tX=X,\ X^2=X,\ \operatorname{tr}X=p\},$$
$$TM=\{(X,Y)\in M\times(\boldsymbol{R})_n\mid {}^tY=Y,\ XY+YX=Y\}.$$

The moment map μ: $TM\to\mathfrak{so}(n)$ is given by $\mu(X,Y)=[X,Y]$.

Proposition 4.7.

$$\operatorname{Image}(A\circ\mu)=\{(x_j^{(i)})\in D\mid x_j^{(n)}=0,\ p<j\leqq[n/2]\}.$$

Theorem 4.8. *The following functions on $TG_{p,q}(\boldsymbol{R})$ are functionally independent:*

$$\mu_{2j}^{(i)},\ 2p\leqq i\leqq n,\ 1\leqq j\leqq p \text{ and}$$
$$\mu_{2j}^{(i)},\ 2\leqq i<2p,\ 1\leqq j\leqq[i/2].$$

The number of these functions is

$$p(q-p+1)+2((p-1)+(p-2)+\cdots+2+1)=pq=\dim G_{p,q}(\boldsymbol{R}).$$

The other $\mu_j^{(i)}$'s vanish identically.

(V) $\tilde{M}=O(2n)/U(n)$, $M=SO(2n)/U(n)$
$\tilde{M}$ is realized in $(\boldsymbol{R})_{2n}$ as follows:

$$\tilde{M}=\{P \in (\boldsymbol{R})_{2n} \mid {}^tP+P=0,\ P^2=-E_{2n}\}.$$

M is a connected component of $\tilde{M}$. Hence, we have

$$TM=\{(P, X) \in M\times(\boldsymbol{R})_{2n} \mid {}^tX+X=0,\ PX+XP=0\}.$$

The moment map μ: $TM\to\mathfrak{so}(2n)$ is given by $\mu(P, X)=PX$.

Proposition 4.9.

$$\mathrm{Image}(A\circ\mu)=\begin{cases}\{(x_j^{(i)}) \in D \mid x_{2j-1}^{(2n)}=x_{2j}^{(2n)},\ 1\leqq j\leqq n/2\} & (n;\ \mathit{even})\\ \{(x_j^{(i)}) \in D \mid x_{2j-1}^{(2n)}=x_{2j}^{(2n)},\ 1\leqq j\leqq [n/2],\ x_n^{(2n)}=0\} & (n;\ \mathit{odd})\end{cases}$$

Theorem 4.10. *The following functions on TM are functionally independent*:

$$\mu_{2j}^{(2n)},\ 1\leqq j\leqq [n/2],$$
$$\mu_{2j}^{(2n-1)},\ 1\leqq j\leqq [(n-1)/2] \quad \mathit{and}$$
$$\mu_{2j}^{(i)},\ 2\leqq i\leqq 2n-2,\ 1\leqq j\leqq [i/2].$$

The number of these functions is

$$[n/2]+[(n-1)/2]+n-1+2((n-2)+(n-3)+\cdots+2+1)$$
$$=n(n-1)=\dim M.$$

(VI) $\tilde{M}=U(n)$, $M=SU(n)$

$$\tilde{M}=\{X \in (\boldsymbol{C})_n \mid XX^*=E_n\},$$
$$T\tilde{M}=\{(X, Y) \in \tilde{M}\times(\boldsymbol{C})_n \mid XY^*+YX^*=0\}.$$

If we define an action φ of the group $U(n)\times U(n)$ on $\tilde{M}$ by $\varphi((X_1, X_2), X_3)=X_1X_3X_2^{-1}$. Then the moment map μ: $T\tilde{M}\to\mathfrak{u}(n)\times\mathfrak{u}(n)$ is given by $\mu(X, Y)=(YX^{-1}, -X^{-1}Y)$.

Proposition 4.11.

$$\mathrm{Image}\ ((A\times A)\circ\mu\colon T\tilde{M}\longrightarrow D\times D)$$
$$=\{(x_j^{(i)}, y_j^{(i)}) \in D\times D \mid x_j^{(n)}+y_{n-j+1}^{(n)}=0,\ 1\leqq j\leqq n\}.$$

Let define μ_r: $T\tilde{M}\to\mathfrak{u}(n)$, $r=1, 2$, by $\mu(X, Y)=(\mu_1(X, Y), \mu_2(X, Y))$ and put $\mu_{r,j}^{(i)}=\mu_r^*(F_j^{(i)})$.

Theorem 4.12. *The following functions on $T\tilde{M}$ are functionally independent*:

$$\mu_{1,j}^{(n)},\ 1 \leqq j \leqq n \quad \textit{and}$$
$$\mu_{r,j}^{(i)},\ 1 \leqq i \leqq n-1,\ 1 \leqq j \leqq i,\ r=1,\ 2.$$

The number of these functions is

$$n+2((n-1)+(n-2)+\cdots+2+1)=n^2=\dim \tilde{M}.$$

On TM, $\mu_{1,1}^{(n)}$ vanishes identically, and the others stated above are independent.

(VII) $M=SO(n)$

Since the arguments are quite similar as in (VI), we state only the conclusions.

Proposition 4.13.

$$\mathrm{Image}\,((A\times A)\circ\mu\colon TM \longrightarrow D\times D)$$
$$=\begin{cases}\{(x_j^{(i)}, y_j^{(i)}) \in D\times D \mid x_j^{(n)}=y_j^{(n)},\ 1\leqq j\leqq [n/2]\} & (n;\ \textit{odd})\\ \{(x_j^{(i)}, y_j^{(i)}) \in D\times D \mid x_j^{(n)}=y_j^{(n)},\ 1\leqq j< n/2,\ x_{n/2}^{(n)}=-y_{n/2}^{(n)}\} & (n;\ \textit{even})\end{cases}$$

Theorem 4.14. *The following functions on TM are functionally independent*:

$$\mu_{1,2j}^{(n)},\ 1 \leqq j \leqq [n/2] \quad \textit{and}$$
$$\mu_{r,2j}^{(i)},\ 2 \leqq i < n,\ 1 \leqq j \leqq [i/2],\ r=1,\ 2.$$

The number of these functions is

$$[n/2]+2([(n-1)/2]+[(n-2)/2]+\cdots+[2/2])=n(n-1)/2=\dim M.$$

Appendix. The convex polyhedral domain D and joint representations of G_i

We note certain relations which are seen between the image of the map $A\circ\mu\colon TM\to\mathfrak{g}\to D$ studied in Section 4 and joint representations of $G_n, G_{n-1}, \cdots, G_1$ on $C^\infty(M)$. We denote G_n^* the set of finite-dimensional irreducible representations of G_n. For the sake of simplicity, we state only for the case $G_n=U(n)$. The complexification $\mathfrak{g}_n^C=\mathfrak{gl}(n, \boldsymbol{C})$ of $\mathfrak{u}(n)$ is identified with the set $(\boldsymbol{C})_n$ of all complex matrices of degree n. Define $H_i \in \mathfrak{g}_n$, $1\leqq i\leqq n$, and subalgebras $\mathscr{A}$ and $\mathfrak{A}$ of $\mathfrak{g}$ by $H_i=E_{ii}$, $\mathscr{A}=\sum_{i=1}^n \boldsymbol{C}H_i$ and $\mathfrak{A}=\sum_{1\leqq i<j\leqq n}\boldsymbol{C}E_{ij}$, where E_{ij} is the matrix unit of $(\boldsymbol{C})_n$. Then $\mathscr{A}+\mathfrak{A}$ is a maximal solvable subalgebra of $\mathfrak{g}_n$. For any $\rho\in G_n^*$, V_ρ denotes a

module of the irreducible representation ρ. The highest weight subspace is given by $V_\rho^{\mathfrak{A}}=\{v \in V|X\cdot v=0$ for all $X \in \mathfrak{A}\}$. Note that $\dim V_\rho^{\mathfrak{A}}=1$. Define a multi-index $k_i=k_i^{(n)}(\rho)$ of ρ by $H_i v=k_i v$ for all $v \in V_\rho^{\mathfrak{A}}$. The following is well-known.

Proposition A.1. *$k_1, \cdots, k_n$ are integers and*

$$k_1 \geqq k_2 \geqq \cdots \geqq k_n. \tag{A.1}$$

Conversely, for any integers $k_1, \cdots, k_n$ which satisfy (A.1), *there exists a unique $\rho \in G_n^*$ such that*

$$k_i^{(n)}(\rho)=k_i,\ 1\leqq i \leqq n.$$

For $\rho \in G_n^*$ and $\rho' \in G_{n-1}^*$, the multiplicity of ρ' in ρ is denoted by $[\rho: \rho']$.

Proposition A.2 (cf. [7]). *For any $\rho \in G_n^*$ and $\rho' \in G_{n-1}^*$, we have $[\rho: \rho'] \leqq 1$. Moreover, $[\rho: \rho']=1$ if and only if*

$$k_1^{(n)}(\rho) \geqq k_1^{(n-1)}(\rho') \geqq k_2^{(n)}(\rho) \geqq k_2^{(n-1)}(\rho') \geqq \cdots \geqq k_{n-1}^{(n-1)}(\rho') \geqq k_n^{(n)}(\rho). \tag{A.2}$$

The condition (A.2) is similar to the condition in the definition of the convex polyhedral domain D in (3.9). If we define a subset $\tilde{D}$ in $\Gamma=G_n^* \times G_{n-1}^* \times \cdots \times G_1^*$ by

$$\tilde{D}=\{(\rho_n, \cdots, \rho_1) \in \Gamma | [\rho_i: \rho_{i-1}]=1,\ 2\leqq i \leqq n\},$$

then, by Proposition A.2, we have

Proposition A.3.

$$\tilde{D}=\{(\rho_*) \in \Gamma | k_1^{(i)}(\rho_i) \geqq k_1^{(i-1)}(\rho_{i-1}) \geqq k_2^{(i)}(\rho_i) \geqq k_2^{(i-1)}(\rho_{i-1}) \geqq \cdots \\ \geqq k_{i-1}^{(i-1)}(\rho_{i-1}) \geqq k_i^{(i)}(\rho_i),\ 2 \leqq i \leqq n\}.$$

Proposition A.3 suggests that there exists a natural inclusion of $\tilde{D}$ into D. This conjecture becomes more detailed if we consider a decomposition of $C^\infty(M)$ under the action of $G_n, \cdots, G_1$ into one-dimensional subspaces. As an example, we consider the case of $M=G_{p,q}(C)$. For any $(\rho_*)=(\rho_n, \cdots, \rho_1) \in \tilde{D}$, $C^\infty_{(\rho_*)}(M)$ denotes the subspace of all functions $f \in C^\infty(M)$ such that $G_i \circ f$ is isomorphic to V_{ρ_i} as a G_i-module, $1 \leqq i \leqq n$, where $G_i \circ f$ is a G_i-module generated by f under the action of G_i.

Theorem A.4. *For any $(\rho_*) \in \tilde{D}$, $\dim C^\infty_{(\rho_*)}(M) \leqq 1$, and $L^2(M)$ is orthogonally decomposed into the sum of these subspaces. Moreover, $\dim C^\infty_{(\rho_*)}(M)=1$ if and only if*

$$k_i^{(n)}(\rho_n)+k_{n-i+1}^{(n)}(\rho_n)=k_j^{(n)}(\rho_n)=0,\ 1\leqq i\leqq p,\ p<j\leqq n-p.$$

If we compair Theorem A.4 with Proposition 4.1, then it is easily seen that there exists a correspondence between a lattice in the image of $A\circ\mu$ and the one-dimensional subspaces $C^{\infty}_{(\rho*)}(M)$. Similar correspondences are observed for all symmetric spaces stated in Section 4.

References

[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, 2-nd ed., Benjamin/Cummings, 1978.

[2] V. Guillemin and S. Sternberg, The moment map and collective motion, Ann. Physics., **127** (1980), 220–253.

[3] A. S. Mishchenko and A. T. Fomenko, Euler equations on finite-dimensional Lie groups, Math. USSR-Izv., **12** (1978), 371–389.

[4] A. Thimm, Über die Integrabilität geodätischer Flüsse auf homogenen Räumen, Dissertation, Bonn, 1980.

[5] A. Thimm, Integrable geodesic flows on homogeneous spaces, Ergod. Theory and Dynam. Sys., **1** (1981), 495–517.

[6] A. Weinstein, Quasi-classical mechanics on spheres, Symposia Math., **14** (1974), 25–32.

[7] H. Weyl, Zur Darstellungstheorie und Invariantenabzählung der projektiven, der Komplex- und der Drehungsgruppe, Acta Math., **48** (1926), 255–278.

Department of Mathematics,
Yamagata University,
Yamagata, Japan