Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 87–91

Notes on the Kolmogorov's Remark Concerning Classical Dynamical Systems on Closed Surfaces

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§ 1. Introduction

It is well-known that the geodesic flow on a closed surface with a negative curvature metric is ergodic. The geodesic flow can be regarded as the dynamical system describing the inertial motion of a particle on the surface with the metric. Since a closed surface with a negative curvature metric cannot be isometrically embedded in the three dimensional Euclidean space E^3, this type of ergodic motion itself cannot be realized on a surface in E^3. On the other hand, in the address of A.N. Kolmogorov [1] (see also Ya. G. Sinai [2]), he remarked that, around a closed surface of genus greater than one in E^3, one can distribute a finite number of centers of attraction and repulsion such that the motion of a particle on M under these external forces is equivalent to the inertial motion in a negative curvature metric on M. As this remark was stated intuitively and without proof, one can only guess the details. The purpose of this note is to set up the situation in more detail and to prove a proposition which seems to be closely related to the remark.

In Section 2, we prepare some concepts in the theory of classical dynamical systems and state our results. In Section 3, the proposition in Section 2 is proved, where only standard methods are used. In Section 4, relations between our results and the Kolmogolov's remark are discussed.

The author wishes to express his hearty thanks to Professor I. Kubo for suggesting this problem and to Professors K. Shiohama, Y. Tashiro and H. Totoki for their helpful advice.

§ 2. Preliminaries and results

All manifolds and functions are assumed to be smooth in this paper. Let M be a manifold and TM (resp. T^*M) the tangent (resp. cotangent) bundle of M. Let $x=(x^i)$ be local coordinates on M and $(x, p)=(x^i, p_i)$

Received January 29, 1983.
Revised May 30, 1983.

the canonical local coordinates on T^*M. Let g be a metric on M and U a real-valued function on M. In this paper, we deal with only classical dynamical systems which are described as Hamiltonian systems whose Hamiltonian functions are of the following type. A real-valued function H on T^*M is said to be the Hamiltonian function relative to (g, U) if it is defined by

$$H(x, p) := \frac{1}{2} g^{jk} p_j p_k + U(x),$$

where g^{jk} are the components of the inverse matrix of $g = (g_{jk})$. The function U is said to be the potential. The Hamiltonian flow on T^*M is defined by the so-called Hamiltonian equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} = g^{ij} p_j,$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} = -\frac{1}{2}\left(\frac{\partial}{\partial x^i} g^{jk}\right) p_j p_k - \frac{\partial}{\partial x^i} U(x).$$

By setting $v^i = g^{ij} p_j$ in this equation, the corresponding equations on TM are

$$\frac{dx^i}{dt} = v^i,$$
$$\frac{dv^i}{dt} = -\left\{ {i \atop j\,k} \right\} v^j v^k - g^{ij} \frac{\partial}{\partial x^j} U(x),$$

where $\left\{ {i \atop j\,k} \right\}$ are the Christoffel's symbols. Note that the obtained flow on TM is the geodesic flow if U is constant over M. Since the Hamiltonian function is constant on each orbit of the Hamiltonian flow, this constant value is said to be the energy of the orbit.

Now we state

Proposition. *Let M be a closed surface with a metric g on M. Assume that the Euler-Poincaré characteristic $\chi(M)$ is negative. Then there exist a function U on M and a negative curvature metric $\bar{g}$ on M such that the Hamiltonian flow orbits relative to (g, U) with energy* 0 *coincide with those relative to* $(\bar{g}, 0)$ *with energy* 1/2.

Corollary. *Let A be a C^2-dense subset of all smooth functions on M. Then an element of A can be chosen as the function U in the above proposition.*

Assuming that $\chi(M) > 0$ (resp. $= 0$) in Proposition, we get a positive (resp. vanishing) curvature metric g instead of the negative curvature metric in the result.

§ 3. Proof of Proposition

We firstly prepare two lemmas. Let M be a manifold with a metric g, U and $\bar{U}$ real-valued functions on M.

Lemma 1. *Let E (resp. $\bar{E}$) be a constant real number satisfying $E > U(x)$ (resp. $\bar{E} > \bar{U}(x)$) and $\bar{g} = e^{2\rho}g$, where $e^{2\rho(x)} := (E - U(x))/(\bar{E} - \bar{U}(x))$. Then the Hamiltonian flow orbits relative to (g, U) with energy E coincide with those relative to $(\bar{g}, \bar{U})$ with energy $\bar{E}$.*

Proof. For a solution $(x(t), p(t))$ of the Hamiltonian equations relative to (g, U), reparametrize it by $(\bar{x}(s), \bar{p}(s)) = (x(t), p(t))$ and $s = \varphi(t)$, where $\varphi(t)$ satisfies $\dot{\varphi}(t) := d\varphi(t)/dt = e^{2\rho(x(t))}$. Then the Hamiltonian equations relative to (g, U) imply

$$\frac{d\bar{x}^i}{ds} = \dot{\varphi}^{-1}\frac{dx^i}{dt} = e^{-2\rho}g^{ij}p_j = \bar{g}^{ij}\bar{p}_j,$$

$$\begin{aligned}\frac{d\bar{p}_i}{ds} = \dot{\varphi}^{-1}\frac{dp_i}{dt} &= -\frac{1}{2}e^{-2\rho}\Big(\frac{\partial}{\partial x^i}g^{jk}\Big)p_jp_k - e^{-2\rho}\frac{\partial}{\partial x^i}U \\ &= -\frac{1}{2}\Big(\frac{\partial}{\partial \bar{x}^i}\bar{g}^{jk}\Big)\bar{p}_j\bar{p}_k - e^{-2\rho}\Big(\Big(\frac{\partial}{\partial x^i}\rho\Big)g^{jk}p_jp_k + \frac{\partial}{\partial x^i}U\Big) \\ &= -\frac{1}{2}\Big(\frac{\partial}{\partial \bar{x}^i}\bar{g}^{jk}\Big)\bar{p}_j\bar{p}_k - \frac{\partial}{\partial \bar{x}^i}\bar{U} \\ &\quad + 2e^{-2\rho}\Big(\frac{\partial}{\partial x^i}\rho\Big)\Big(\frac{1}{2}g^{jk}p_jp_k + U - E\Big).\end{aligned}$$

The last term vanishes if the energy of the orbit $(x(t), p(t))$ equals to E.

Let M be a closed surface with a metric g. Then we get

Lemma 2. *Assume that $\chi(M) < 0$ (resp. > 0, $= 0$). Then there exists a function ρ on M such that $\bar{K} < 0$ (resp. > 0, $= 0$), where $\bar{K}$ is the Gaussian curvature of the metric $\bar{g} := e^{2\rho}g$.*

Proof. Take the orientable double covering for a non-orientable M. Let K be the Gaussian curvature of g. By applying the Gauss-Bonnet formula, we have

$$\int_M (K - \hat{K}) * 1 = 0, \tag{1}$$

where $*1$ is the volume element with respect to g and $\hat{K}:=2\pi\chi(M)/\mathrm{Vol}(M)$. Then there exists a function ρ which satisfies the following equation

$$(2) \qquad \Delta\rho=K-\hat{K},$$

where Δ is the Laplace-Beltrami operator on functions. Set $\bar{g}=e^{2\rho}g$, and we have

$$(3) \qquad \bar{K}=e^{-2\rho}(K-\Delta\rho)=e^{-2\rho}\hat{K}.$$

Note that the existence of the solution ρ of the equation (2) with the condition (1) is essentially due to the Riesz theorem. Referring to the well-known Hodge decomposition theorem, however, we can easily find the solution.

Put $U=-(1/2)e^{2\rho}$, $\bar{U}=0$ and $\bar{E}=1/2$ in Lemma 1, where ρ is the function obtained in Lemma 2. Then we get the function and metric desired in Proposition.

For the proof of Corollary, we assume that $\chi(M)$ is not equal to zero. From the formula (3), we can prove that there exists $\varepsilon(>0)$ such that

$$|e^{2\rho'}K'-\hat{K}|_0<|\hat{K}|/2 \quad \text{and} \quad U'<0 \quad \text{for} \quad |U-U'|_2<\varepsilon,$$

where $\rho'=(1/2)\log(-2U')$ and K' is the curvature of the metric $e^{2\rho'}g$. Thus K' and $\hat{K}$ have the same signature.

§ 4. Discussion

The types of the external forces seem to be not clear in the Kolmogorov's statement [1]. We can guess that they have central potential functions and the resultant force has a potential function in the form

$$(1) \qquad V(q)=\sum_{i=1}^{I} V_i(|q-q_i|),$$

where V_i is a central potential and $q_i \in E^3$ for each i. Moreover we may assume that all central potential functions are of the same type, i.e., there exists a real valued function W on the set of all positive real numbers such that each V_i is represented by $c_i W$ for a real constant number $c_i \in R$. Then the function (1) is written in the form

$$(2) \qquad V(q)=\sum_{i=1}^{I} c_i W(|q-q_i|).$$

A natural example of such a potential function is the Coulomb potential caused by finite points $q_i \in E^3$ with charges c_i, that is,

$$V(q)=\sum_{i=1}^{I}-\frac{1}{4\pi}\frac{c_i}{|q-q_i|}. \tag{3}$$

As another example in physics, we have the Yukawa potential.

Let M be a closed surface embedded in E^3 and $C^\infty(M)$ the space of all real-valued smooth functions on M. For potential functions V in the form (2), we define a subspace

$$A:=\{U \in C^\infty(M);\ U=V|_M \text{ for } c_i \in R,\ q_i \in E^3\backslash M\}.$$

If A is C^2-dense in $C^\infty(M)$, then the corollary to Proposition assures that the Kolmogorov's remark holds for potentials of this type. In the case of Coulomb potentials, for the function U which appeared in Proposition, there exists a smooth function c on E^3 with the compact support such that

$$U(x)=-\frac{1}{4\pi}\int_{E^3}\frac{c(q)}{|x-q|}dq.$$

We do not know if one can find a function which is in the form (3) and sufficiently near to U in C^2-topology on $C^\infty(M)$.

References

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