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Some Considerations on the Cut Locus of a Riemannian Manifold

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§ 0. Introduction

Let (M, g) be a compact connected Riemannian manifold of dimension n and fix a point p of M. Let γ_X be a geodesic emanating from p with the unit initial direction $X \in T_pM$. We define the cut point of p along γ_X as the last point on γ_X to which the geodesic minimizes the distance. The locus $C(p)$ of all cut points of p is called the cut locus of p.

By the above definition M is obtained from $C(p)$ by attaching an n-cell and the cut locus contains the essential informations on the topology of M. Now the problem of determining the structure of the cut locus is interesting in connection with the singularity theory. Recently in case of analytic Riemannian structures or in generic case much progress has been made by M. Buchner ([2] [3] [4]). But since their works appeal to the powerful general theory (Hironaka's or Mather's theory), the concrete structure of the cut locus is not given explicitly.

On the other hand the above problem is answered for the 2-dimensional analytic case by S.B. Myers ([8]), symmetric spaces and Berger's spheres by T. Sakai ([11] [12]) and M. Takeuchi ([13]). But with respect to an arbitrary metric, the cut locus may be very complicated, for example, H. Gluck, D. Singer ([5]) showed that there exists a metric on any manifold whose cut locus is not triangurable.

The main purpose of the present paper is to study the relation between the cut locus and the union of all unstable manifolds of critical points with positive index of some Morse functions.

Firstly in Section 1 we approximate the distance function from p by Morse function with respect to C^0-topology and define the set $C^1(p)$ as the limit set of all unstable manifolds of critical points with positive index of Morse functions. Then $C^1(p)$ is contained in the cut locus of p and inherits the essence of the topology of M under some conditions. We call $C^1(p)$ the essential cut locus. In general it seems that the structure of

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$C^1(p)$ is complicated, but in case of analytic surfaces $C^1(p)$ may be determined explicitly.

In Sections 2, 3 we consider the problem of the construction of Riemannian metric when its cut locus is preassigned. H. Gluck, D. Singer constructed a Riemannian metric so that a non triangurable set (infinitely many arcs sharing a common end point) becomes the cut locus in ([5]). See also L. Berard Bergery ([1]) when the preassigned cut locus is a submanifold. Here we consider the following problem; "For a given Morse function with some conditions on a compact manifold construct a Riemannian metric and find a point p such that $C(p)$ coincides with the union of all unstable manifolds of critical points with positive index of the given Morse function."

In Section 2 we will use the handle decomposition of any surface M under the given Morse function and introduce adequate metrics on the handles. Then we attach them to construct a desired metric on M with the aid of Weinstein's technique ([15]).

In Section 3 we will be concerned with the following problem which is firstly mentioned by Rauch ([9]); "For any point p in a compact simply connected Riemannian manifold M, do the first conjugate locus and cut locus of p have a common point?" This is true for M homeomorphic to S^2. Then A. Weinstein solved the conjecture negatively as follows; "For any compact manifold except for S^2, there exists a Riemannian metric on M and a point $p \in M$ whose first conjugate locus and cut locus are disjoint." Although Wienstein's construction of such a metric is valid for an arbitrary manifold except for S^2, the cut locus of the point was not explicitly given. In Section 3 we want to construct a metric on S^3 and choose a point in S^3 whose cut locus can be explicitly seen and disjoint from the first conjugate locus by the same spirit and method as in Section 2.

I would like to express my sincere thanks to Prof. T. Sakai for his kind advice.

§ 1. The essential cut locus

Let ρ be the distance function from p. Since ρ is not differentiable at p, we modify ρ to be smooth at p as follows; We take a normal coordinate system $(x_i, B_\varepsilon(o_p))$ around p. Then $\rho(x)=\sqrt{\Sigma x_i^2}$. Take $0<\varepsilon' <\varepsilon$. Now let $\psi(r)$ be a C^∞-function such that

$$\psi(r)=\begin{cases}1 & r\geq\varepsilon' \\ 0 & r\leq 0\end{cases}$$

$$\psi'(r)>0 \qquad 0<r<\varepsilon'$$

We define a C^∞-function $\phi(r)$ by

$$\phi(r):=\int_0^r \psi(t)dt+\varepsilon'-\int_0^{\varepsilon'}\psi(t)dt.$$

Note that $\phi(r)=r$ if $r\geq\varepsilon$, and $\phi(0)>0$. Now we define $\tilde{\rho}(y)$ by

$$\tilde{\rho}(y)=\begin{cases}\rho(y) & \text{if } \rho(y)>\varepsilon\\ \phi(\sqrt{\Sigma y_i^2}) & \text{if } \rho(y)\leq\varepsilon\,.\end{cases}$$

$\tilde{\rho}$ is smooth at the origin, which is a critical point of index 0. We have $\|\operatorname{grad}\tilde{\rho}\|=1$ outside a neighborhood of origin and the cut locus of p. The trajectories of grad $\tilde{\rho}$ are geodesics from p. From now on we call this $\tilde{\rho}$ the modified distance function.

We take a sequence of positive numbers $\{\varepsilon_j\}_{j=1}^\infty$ with $\varepsilon_j\to 0$ (as $j\to\infty$). We define the ε_j-neighborhoods N_j of $C(p)$, by

$$N_j:=\{\operatorname{Exp}(tX)\,|\,t>s(X)-\varepsilon_j,\ X\in U_pM\}$$

where $s(X)$ is the distance from p to the cut point along γ_X. We put

$$F_j:=\left\{\text{Morse function } f \text{ on } M\ \middle|\ \begin{array}{l}|f-\tilde{\rho}|_0<\varepsilon_j\\ f=\tilde{\rho}\ \text{ outside } N_j\end{array}\right\}$$

Lemma 1.1. $F_j\neq\phi$.

Proof. Take a C^∞-function ϕ on M such that

$$\phi=\begin{cases}0 & \text{on } \overline{N_{j+1}}\\ 1 & \text{on } CN_j\end{cases}$$

and a positive number N such that $\|\operatorname{grad}\phi\|<N$. We first construct a C^∞-function g on M which equals $\tilde{\rho}$ on CN_{j+1} and is contained in a neighborhood of $\tilde{\rho}$ with respect to C^0-topology (i.e. $|g-\tilde{\rho}|_0<1/(8N)$). For the construction take a C^∞-function ϕ_1 on M

$$\phi_1=\begin{cases}0 & \text{on } \overline{N_{j+2}}\\ 1 & \text{on } CN_{j+1}\end{cases}$$

with $0\leq\phi_1\leq 1$. Take a C^∞-function u with $|\tilde{\rho}-u|_0<1/(8N)$. We define desired g as follows.

$$g:=\begin{cases}u & \text{on } N_{j+2}\\ (1-\phi_1)u+\phi_1\tilde{\rho} & \text{on } N_{j+1}\backslash N_{j+2}\\ \tilde{\rho} & \text{on } CN_{j+1}.\end{cases}$$

Next choose a Morse function g_1 on M with

$$|g_1-g|_0<\frac{1}{8N}, \qquad \|\text{grad}\, g_1-\text{grad}\, g\|<\frac{1}{4},$$

which is enough closed to g_1 with respect to C^1-topology. We define f as follows;

$$f:=\begin{cases}\tilde{\rho} & \text{on } CN_j\\ (1-\phi)g_1+\phi\tilde{\rho} & \text{on } N_j\backslash N_{j+1}\\ g_1 & \text{on } N_{j+1}.\end{cases}$$

It is not difficult to see that f has no critical points on $N_j\backslash N_{j+1}$ and $f\in F_j$.
In fact on $N_j\backslash N_{j+1}$, we have

$$\begin{aligned}\text{grad}\, f&=(1-\phi)\,\text{grad}\, g_1+\phi\,\text{grad}\,\tilde{\rho}+(\tilde{\rho}-g_1)\,\text{grad}\,\phi\\ &=(1-\phi)\,\text{grad}\,\tilde{\rho}+(1-\phi)(\text{grad}\, g_1-\text{grad}\, g)\\ &\quad+\phi\,\text{grad}\,\tilde{\rho}+(\tilde{\rho}-g_1)\,\text{grad}\,\phi.\end{aligned}$$

If $\text{grad}\, f=0$ at a point on $N_j\backslash N_{j+1}$, then

$$-\text{grad}\,\tilde{\rho}=(1-\phi)(\text{grad}\, g_1-\text{grad}\, g)+(\tilde{\rho}-g_1)\,\text{grad}\,\phi.$$

Take sufficiently large j, then

$$\|\text{grad}\,\tilde{\rho}\|=1$$

$$|1-\phi|_0\|\text{grad}\, g_1-\text{grad}\, g\|<\frac{1}{4}$$

$$|\tilde{\rho}-g_1|_0\|\text{grad}\,\phi\|\leq\{|\tilde{\rho}-g|_0+|g-g_1|_0\}\cdot\|\text{grad}\,\phi\|\leq\frac{1}{4}.$$

These derive a contradiction. q.e.d.

For any j we take out a Morse function f_j in F_j which has the least number of critical points. Then we have a sequence of Morse functions $\{f_j\}$ which converges to ρ with respect to C^0-topology.

Let ϕ_t^j be a 1-parameter group of diffeomorphisms of $\text{grad}\, f_j$. We define the union $C_j(p)$ of unstable manifolds of critical points of f_j with positive index, i.e.

$$C_j(p):=\left\{x\in M\;\middle|\;\begin{array}{l}\phi_{-\infty}^j(x)\ (:=\lim_{t\to-\infty}\phi_t^j(x))=\text{a critical point of}\\ f_j \text{ with index}\geq 1\end{array}\right\}.$$

Lemma 1.2. $C_j(p) \subset N_j$

Proof. Recall that $f_j = \tilde{\rho}$ on $M \backslash N_j$. To the contrary assume that there exists a point $x \in (M \backslash N_j) \cap C_j(p)$, then $\phi_t^j(x) \in M \backslash N_j$ for $t \leq 0$ and $\phi_t^j(x)$ is on a geodesic from p to x. Hence $\phi_{-\infty}^j(x)$ is a critical point of $\tilde{\rho}$ in $M \backslash N_j$ and equals p which is a critical point of index 0. This contradicts the fact $x \in C_j(p)$. q.e.d.

We define $C^1(p)$ by

$$C^1(p) := \{x \in M \mid x \text{ is a cluster point of a sequence } \{y_j\},\ y_j \in C_j(p)\}.$$

From Lemma 1.2 $C^1(p)$ is contained in $C(p)$. Set $C^2(p) := C(p) \backslash C^1(p)$. Let N_j^1 be the ε_j-neighborhood of $C^1(p)$.

Lemma 1.3. *For any j there exists $i \geq j$ such that critical points of f_i with index ≥ 1 are in N_j^1.*

Proof. Suppose not, then there is a subsequence $\{f_{j(k)} \in F_{j(k)}\}$ and critical points $x_{j(k)}$ of $f_{j(k)}$ with index ≥ 1 such that $x_{j(k)} \in M \backslash N_j^1$. Since M is compact, we can choose a subsequence of $x_{j(k)}$ which converges to a point $x_0 \in M \backslash N_j^1$. On the other hand, $x_{j(k)} \in C_{j(k)}(p)$ and $x_0 \in C^1(p)$, which is contradiction. q.e.d.

Lemma 1.4. *For any j there exists a continuous map $F: M \times I \to M$ such that $F(x, 0) = x$ and $F(C(p), 1) \subset N_j^1$*

Proof. For j take i in Lemma 1.3. There is a positive number δ such that $\phi_\delta^i(M \backslash p) = M \backslash B_{\varepsilon'}(o_p)$. For any point x in $M \in B_{\varepsilon'}(o_p)$, there exists a positive number T_x such that $\phi_{T_x}^i(x)$ in N_j^1. As $M \backslash B_{\varepsilon'}(o_p)$ is compact, there is a positive number T such that $\phi_T^i(M \backslash B_{\varepsilon'}(o_p)) \subset N_j^1$. Finally we define a continuous map F by $F(x, s) := \phi_{sT}^i(x)$. q.e.d.

Summing up we have

Theorem 1.5. *The set $C^1(p)$ is contained in the cut locus and for any neighborhood N of $C^1(p)$ there exists a continuous map $F: M \times I \to M$ such that $F(x, 0) = x$ and $F(C(p), 1) \subset N$.*

Corollary 1.6. *If there is a deformation retractable neighborhood N of $C^1(p)$ such that $t \to F(t, x)$ intersects ∂N transversally for $x \in M \backslash N$, then there exists a continuous map $G: C(p) \times I \to C(p)$ such that $G(x, 0) = x$, $G(C(p), 1) = C^1(p)$ and $G(y, s) = y$ for $y \in C^1(p)$, $s \in I$.*

Proof. We take a deformation retraction $R: N \times I \to N$ such that $R(x, 0) = x$, $R(N, 1) = C^1(p)$ and $R(y, s) = y$ for $y \in C^1(p)$, $s \in I$. For any

$x \in C(p) \backslash N$, we take s_x as the first value of t such that $F(x, t) \in \partial N$. Then s_x is continuous and we put $s_0 := \mathrm{Max}_{x \in (C(p) \backslash N)} s_x$. We define a continuous map $H\colon C(p) \times I \to M \backslash \{p\}$ as follows;

for $x \in C(p) \backslash N$

$$H(x, s) := \begin{cases} F(x, 2s_0 s) & 0 \leq s \leq \dfrac{s_x}{2s_0} \\ F(x, s_x) & \dfrac{s_x}{2s_0} < s \leq \dfrac{1}{2} \\ R(F(x, s_x), 2s-1) & \dfrac{1}{2} < s \leq 1 \end{cases}$$

for $x \in C(p) \cap N$

$$H(x, s) := \begin{cases} x & 0 \leq s \leq \dfrac{1}{2} \\ R(x, 2s-1) & \dfrac{1}{2} < s \leq 1. \end{cases}$$

Next we take a continuous map $\psi\colon M \backslash \{p\} \to C(p)$ under which $x \in M \backslash (\{p\} \cup C(p))$ corresponds to the cut point of the unique minimizing geodesic emanating from p and passing through x, and $x \in C(p)$ to itself. Finally we define a continuous map $G\colon C(p) \times I \to C(p)$ by $G := \Psi \circ H$ which satisfies the conclusion of Corollary 1.6. q.e.d.

Corollary 1.7. *Under the same assumption as Corollary* 1.6. *We have*

$$\pi_*(C^1(p), y) \cong \pi_*(C(p), y) \cong \pi_*(M, y) \quad y \in C^1(p) \quad 0 < * < n-1$$

and

$$H_*(C^1(p)) \cong H_*(C(p)) \cong H_*(M) \quad 0 < * \underset{(=)}{<} n-1.$$

(=; *when M is orientable*)

Remark. Roughly speaking $C^1(p)$ contains topologically essential part of $C(p)$, although the structure of $C^1(p)$ is not so clear because we take a limit. Moreover our definition of $C^1(p)$ depends on the choice of N_j and $f_j \in F_j$.

But for some class of metrics, $C^1(p)$ seems to be defined intrinsically independent of the choice of N_j and $f_j \in F_j$. We give an example; Let (S^2, g) be an analytic metric on the 2-sphere. The structure of $C(p)$ in this case has been studied very explicitly by S. B. Myers. Namely he showed that $C(p)$ is a (finite) tree in this case. By an end point of $C(p)$ we shall mean a point x of $C(p)$ from which there issues one and only one 1-cell of $C(p)$. The number of minimal geodesics joining p to a cut point x is called the order of the cut point x. Then the order of a cut point always equals its order as a vertex of the tree $C(p)$.

We call $x \in S^2$ a limit point of critical points of index 2 (resp. index 1) if and only if for any neighborhood N of x there is a positive integer k such that for any $j \geq k$ there exists $x_j \in N$ which is a critical point of f_j of index 2 (resp. index 1). We call $x \in (C(p)\backslash\{\text{all end points}\})$ a minimal (resp. maximal) point of $\rho|_{C(p)}$ if and only if there exists a neighborhood U of x in $C(p)$ such that $\rho(y) > \rho(x)$ (resp. $\rho(y) < \rho(x)$) for any $y \in U\backslash\{x\}$.

Then we get the following results.

1. Let x be a cut point of order n ($n > 2$). Then among n 1-cells which issue from x, there is at most one 1-cell on which $\rho|_{C(p)}$ is increasing.

2. The number of maximal points and minimal points of $\rho|_{C(p)}$ is finite.

3. A point $x \in C(p)$ is maximal (resp. minimal) point of $\rho|_{C(p)}$ if and only if $x \in C(p)$ is a limit point of critical points of index 2 (resp. index 1).

4. For any minimal point there issue just two branches of $C(p)$ along each of which ρ increases till the next maximal point. Then $C^1(p)$ is the union of all such branches from minimal points to the next maximal points.

Remark. It seems that these results in this section may hold good for non compact complete Riemannian manifolds.

§ 2. The construction of cut locus

For a Morse function f on a compact manifold we denote by C_f the union of all unstable manifolds of critical points of f with positive index.

Let M be a surface. We fix a Riemannian metric on M. We consider Morse functions f on M of distance function type, namely which satisfy the following condition $(*)$

$$(*) \quad \begin{cases} (1) & \text{There is only one critical point of index 0.} \\ (2) & \text{There is no saddle connection.} \end{cases}$$

(2) means that there is no integral curve of grad f which connects two critical points of index 1.

Remark. There are enough many Morse functions which satisfy the condition $(*)$.

Theorem 2.1. *For any Morse function f on M satisfying the condition $(*)$, there exist a point $p \in M$ and a Riemannian metric g such that $C(p)$ coincides with C_f.*

We will use the handle decomposition of M under the given Morse function f and introduce an adequate metric on the handles. For that purpose we prepare the following lemma.

Lemma 2.2. *Let $a, b>0$ be given. Then for any $s \in [0, a)$, there exist $0<d<e<1$ and a C^∞-function $F(t)$ on $[0, 1]$ such that*

$$F(t)=s \ (0\leq t\leq d), \quad F'(t)\geq 0, \quad F''(t)\geq 0,$$
$$F'(e)=b, \quad F^{(n)}(e)=0 \ (n>2), \quad F(e)=(1-e)\cdot a.$$

Proof. First take an e with $(a-s)/(a+b)<e<(a-s)/a$. Then since $0<a\cdot(1-e)-s<b\cdot e$, we can choose $0<d<e$ and a C^∞-function ϕ on $[0, 1]$ such that

$$(1) \qquad \phi(t)=\begin{cases} 0 & (0\leq t\leq d) \\ b & (e\leq t<1), \end{cases} \qquad 0\leq\phi(t)\leq b, \quad \phi'(t)\geq 0$$

$$(2) \qquad \int_0^e \phi(t)dt=a\cdot(1-e)-s.$$

Finally put $F(t)=s+\int_0^t \phi(t)dt.$ q.e.d.

Let q^0 be the critical point of index 0 and $f(q^0)=a$. Choose $a'>a$ such that $\mathscr{D}: =f^{-1}[a, a']$ does not contain any critical points except q^0. Then $\partial\mathscr{D}=f^{-1}(a')$ is a circle. We put $K: =\overline{M-\mathscr{D}}$. Let q_j^1 (resp. q_i^2) be a critical point of index 1 (resp. index 2). In this section the suffix j (resp. i) is always corresponding to critical point q_j^1 (resp. q_i^2) of index 1 (resp. index 2); $j=1, 2, \cdots$, the number of critical points of index 1 and $i=1, 2, \cdots$, the number of critical points of index 2.

We put $St_j: =\{\text{the stable manifold of } q_j^1\}\cup\{q^0\}$. St_j is a circle in M from $(*)$. We take a closed tubular neighborhood N_j of St_j with the following properties;

(1) We put $H_j^1: =N_j^1\cap K$. For any j, H_j^1 does not contain any critical points except q_j^1.

(2) For any j the boundary of H_j^1 consists of the two segments in the circle $\partial\mathscr{D}$ and two C^∞-curves α_1^j, α_2^j on $K\cap\partial N_j$ such that $\alpha_k^j\cap\partial\mathscr{D}=$ {two end points}.

(3) Each H_j^1 does not intersect each other.

(4) The C^∞-curve α_k^j $(k=1, 2)$ transversally intersects the integral curves of grad f except 2 end points and at end points α_k^j is tangent to integral curves of grad f.

Each connected component of $K\backslash(\bigcup_j H_j^1)$ contains exactly one critical point q_i^2 of index 2. Let H_i^2 be the closure of this connected component. H_i^2 is homeomorphic to a disk. For any i we denote by $m(i)$ the number of integral curves of grad f connecting critical points of index 1 and q_i^2. The boundary of H_i^2 consists of $m(i)$ C^∞-curves on the circle $\partial\mathscr{D}$ and $m(i)$ C^∞-curves α_k^j which are determined by the Morse function f. Still more each curve α_k^j intersects one of the above integral curves in H_i^2. We put $C_i^2 := C_f \cap H_i^2$, $C_j^1 := C_f \cap H_j^1$, then C_i^2 consists of q_i^2 and the union of the above integral curves in H_i^2.

Now corresponding to each critical point q_j^1 of index 1 we take a rectangle $\mathscr{H}_j^1$ and segments a_k^j, $c_k^j(k=1, 2)$ as indicated in Figure 2.1. The length of a_1^j and a_2^j equals $2s$ where s is enough small. We denote by $\mathscr{C}_j^1$ the segment joining the middle points of a_1^j, a_2^j.

We take a C^∞-diffeomorphism ϕ_j^1 from some neighborhood of $\mathscr{H}_j^1$ into M such that

$$\phi_j^1(\mathscr{H}_j^1)=H_j^1, \qquad \phi_j^1(\mathscr{C}_j^1)=C_j^1,$$
$$\phi_j^1(a_k^j)=\alpha_k^j, \qquad \phi_j^1(c_k^j)\subset\partial\mathscr{D}.$$

Secondly corresponding to each critical point q_i^2 of index 2, we take a 2-handle $\mathscr{H}_i^2$ as indicated in Figure 2.2 when $m(i)\geq 2$, as in Figure 2.3 when $m(i)=1$. Moreover we take C^∞-curves $c_{k,k+1}^i$ and segments b_k^i, d_k^i (k; mod $m(i)+1$) as indicated in the figures. $\bigcup_k d_k^i$ of $\mathscr{H}_i^2$ which corresponds to C_i^2 is denoted by $\mathscr{C}_i^2$.

The diagram $\mathscr{H}_i^2(d_k^i, b_k^i, c_{k,k+1}^i)$ should have the following properties;

(1) The angle $\measuredangle(d_k^i, d_{k+1}^i)$ at o is equal to that of the corresponding curves at q_i^2 in C_i^2.

(2) The end points of d_k^i are o and the middle point of the segment b_k^i. For any i and k the length of b_k^i equals $2s$.

(3) For any sufficiently small number $\delta>0$, $c_{k,k+1}^i[0, \delta]$ (resp. $c_{k,k+1}^i[1-\delta, 1]$) is parallel to d_k^i (resp. d_{k+1}^i).

(4) Rays emanating perpendicularly from points of $c_{k,k+1}^i$ into $\mathscr{H}_i^2$ minimize the distance from $\bigcup_k c_{k,k+1}^i$ just until they intersect $\mathscr{C}_i^2$ for the first time.

(5) There exists a C^∞-diffeomorphism ϕ_i^2 from some neighborhood

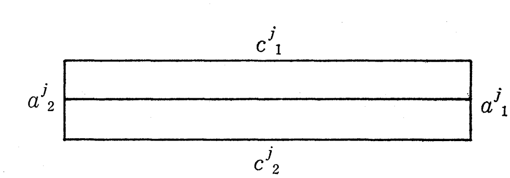


Fig. 2.1. $\mathscr{H}_j^1$.

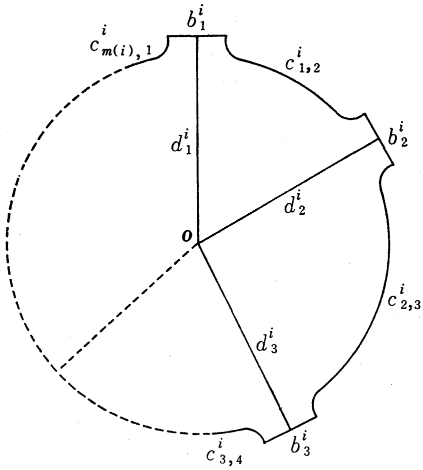


Fig. 2.2. $\mathscr{H}_i^2$ (as $m(i)\geq 2$).

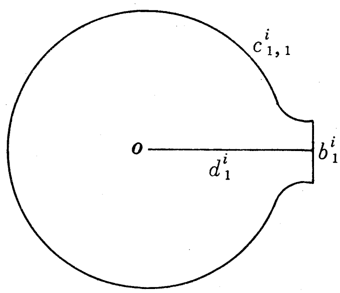


Fig. 2.3. $\mathscr{H}_i^2$ (as $m(i)=1$).

of $\mathscr{H}_i^2$ into M with following properties;

(i) $\phi_i^2(\mathscr{H}_i^2)=H_i^2$, $\phi_i^2(\mathscr{C}_i^2)=C_i^2$.

(ii) For any $m\leq m(i)$ there exist j and k such that $\phi_i^2(b_m^i)=\alpha_k^j$, $(\phi_i^2)^{-1}\circ\phi_j^1|_{a_k^j}$ is an isometry from a_k^j to b_m^i.

(iii) $\phi_1^2(c_{k,k+1}^i)\subset\partial\mathscr{D}$.

This is possible from Lemma 2.2, Munkres' lemma [7, Lem. 6.1.] and easy consideration.

Now we identify a_k^j and b_m^i such that $\phi_i^2(b_m^i)=\phi_j^1(a_k^j)$ by the isometrry $(\phi_i^2)^{-1}\circ\phi_j^1\colon a_k^j\to b_m^i$. We get $\mathscr{K}$ which is made from $\mathscr{H}_i^2$ and $\mathscr{H}_j^1$ by the above identifications and endow $\mathscr{K}$ with that flat metric. We put $\mathscr{C}\colon=(\bigcup_i\mathscr{C}_i^2)\cup(\bigcup_j\mathscr{C}_j^1)$. We take a C^∞-diffeomorphism $\Phi\colon\mathring{\mathscr{K}}\to\mathring{K}$ with the following properties;

(1) For sufficiently small ε we put $\mathscr{H}_i^{2'}\colon=\mathscr{H}_i^2\backslash\{\varepsilon\text{-neighborhood of } \bigcup_k b_k^i\}$ and $\mathscr{H}_j^{1'}\colon=\mathscr{H}_j^1\backslash\{\varepsilon\text{-neighborhood of } (a_1^j\cup a_2^j)\}$. For any i and j

$\Phi|_{\mathring{\mathscr{H}}_i^{2\prime}}=\phi_i^2$, $\Phi|_{\mathring{\mathscr{H}}_j^{1\prime}}=\phi_j^1$.

(2) $\Phi(\mathscr{C})=C_f$.

This is possible from easy consideration using Munkres' lemma [7, Lem. 6.1]. We induce a metric on K by Φ from $\mathscr{K}$.

We exchange s for $s'=(9/10)s$, then we get $\mathscr{K}'(\subset\mathscr{K})$ by the same procedure. Thus $\mathscr{C}$ is the cut locus of $\partial\mathscr{K}'$ by the property (4) of $\mathscr{H}_i^2$. We put $K':=\Phi(\mathscr{K}')$. Let D be $M\backslash K'$ then it seems that D is an embedded disk in M and the cut locus of ∂D in K' is C_f.

We get a new metric on M and a point p in M whose cut locus is C_f from the following Proposition 2.3.

Proposition 2.3. *Let D be an n-disk embedded in a C^∞ manifold M. For any Riemannian metric on $M\backslash\mathring{D}$, there is a Riemannian metric on M agreeing with the original metric on $M\backslash\mathring{D}$ such that for some point p in D* Exp_p *is a diffeomorphism of unit disk about the origin in T_pM onto D.*

Proposition 2.3 is proved by the same method of Weinstein's proposition [15, Prop. C].

Remark. In this case the intersection of the first conjugate locus and cut locus of p coincides with the set of critical points of index 2.

Remark. It seems that the similar result may hold good for the higher dimensional case. In this case the conditions for Morse functions of distance function type are following;

(1) There is only one critical point of index 0.

(2) There is no integral curve from a critical point of index i to a critical point of index $\leq i$.

§ 3. Some example on S^3

Our main result of this section is as follows.

Theorem 3.1. *There exist a Riemannian metric g on S^3 and a point p in S^3 whose first conjugate locus and cut locus are disjoint, where the cut locus is given in Figure* 3.1.

This figure of cut locus is slightly deformed from so called dunce hat in R^3.

In Section 2 for a given Morse function on a compact surface, we have constructed a metric and chosen a point p, so that the cut locus $C(p)$ of p is nothing but the union of unstable manifolds of critical points with positive index.

Here although we don't give a Morse function explicitly, our construction of a metric is done by the same spirit, namely the handle body

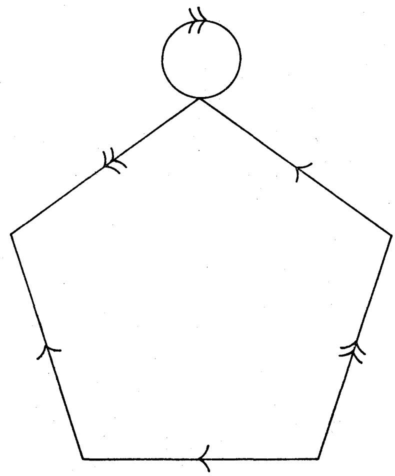


Fig. 3.1. Identify the same arrows.

decomposition. To begin with we prepare handle bodies $\mathscr{H}(\mathrm{III})$ (3-handle), $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$ (2-handles), $\mathscr{H}(\mathrm{I}_1)$, $\mathscr{H}(\mathrm{I}_2)$ (1-handles), $\mathscr{H}(0)$ (0-handle) which should correspond critical points of index 3, 2, 1 and 0 of a (not explicitly given) Morse function. Next we attach the 2-handles $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$ to the 3-handle $\mathscr{H}(\mathrm{III})$, then the 1-handles $\mathscr{H}(\mathrm{I}_1)$, $\mathscr{H}(\mathrm{I}_2)$ to $\mathscr{H}(\mathrm{III})\cup\mathscr{H}(\mathrm{II}_1)\cup\mathscr{H}(\mathrm{II}_2)$, at last the 0-handle $\mathscr{H}(0)$ to $\mathscr{H}(\mathrm{III})\cup\mathscr{H}(\mathrm{II}_1)\cup\mathscr{H}(\mathrm{II}_2)\cup\mathscr{H}(\mathrm{I}_1)\cup\mathscr{H}(\mathrm{I}_2)$. In general when topologists construct a manifold from handle bodies, they firstly attach 1-handles to a 0-handle, etc. But in our case we reverse this order of attaching process (See [10]).

At the start we will construct a 3-handle $\mathscr{H}(\mathrm{III})$ in R^3 and give the canonical metric on $\mathscr{H}(\mathrm{III})$ induced from R^3. Take a regular tetrahedron $\varDelta$ with vertices a_k, faces H_k (corresponding to a_k) ($k=1, 2, 3, 4$) whose edges are of length 1. Let o be the center of $\varDelta$, h_k the center of H_k and e_k the segment connecting o and h_k. Let $L_{i,j}$ ($i<j$) be the quadrirateral whose vertices are o, h_i, h_j and the middle point $m_{i,j}$ of the edge $H_i\cap H_j$. Let $\varDelta_k$ be the domain of $\varDelta$ bounded by $L_{i,j}$'s ($k\neq i,j$) and containing a_k. We take a point a_j^i on the segment $\overline{h_i m_{i,j}}$ so that $\overline{h_i a_j^i}=10^{-1}$ ($i, j=1, 2, 3, 4$). Put $e_j^i:=\overline{h^i a_j^i}$. We take a point $a_{j,k}^i\in\varDelta_k$ ($k\neq i,j$) on the straight line in H_i which passes the point a_j^i orthogonally to e_j^i so that $\overline{a_j^i a_{j,k}^i}=10^{-2}$. Let b_j^i be the segment connecting $a_{j,k}^i$ and $a_{j,k'}^i$ ($k, k'\neq i,j$).

We construct C^∞-curves $c_{i,j}$ ($i<j$): $[0, 1]\to L_{i,j}$ with the following properties;

(1) $c_{i,j}(0)=a_j^i$, $c_{i,j}(1)=a_i^j$.

(2) $c_{i,j}|_{[0,1/10]}$ (resp. $c_{i,j}|_{[9/10,1]}$) is a segment parallel to e_i (resp. e_j) whose length is greater than 10^{-2}.

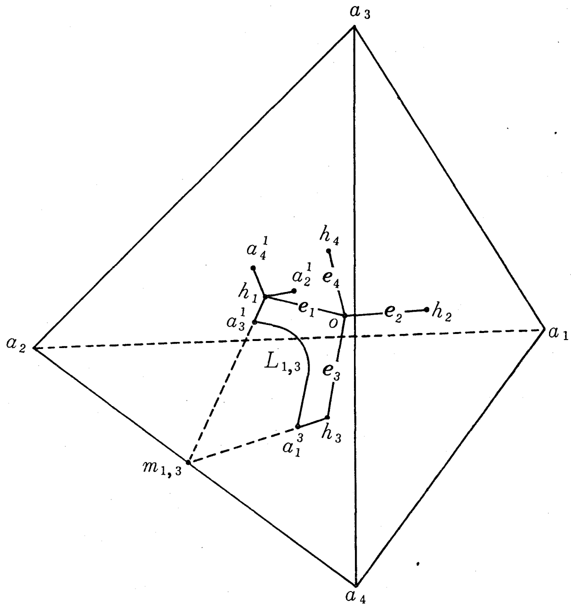


Fig. 3.2.

(3) $c_{ij}|_{(0,1)}$ is contained in the interior of $L_{i,j}$.
This is possible from Lemma 2.1. Let $\tilde{L}_{i,j}$ be the domain bounded by $c_{i,j}$, e_i, e_j, e_j^i and e_i^j. Then we get a diagram $C(\text{III})\colon =\bigcup_{i,j} \tilde{L}_{i,j}$ contained in Δ as in Figure 3.3.

Now we construct a 3-handle $\mathscr{H}(\text{III})$ by giving a body to $\mathscr{C}(\text{III})$. For the purpose it suffices to construct the boundary surfaces (with corners) of $\mathscr{H}(\text{III})$.

Firstly we take domains $H_i(\text{III})$ in H_i respectively ($i=1, 2, 3, 4$) as in Figure 3.4. We take C^∞-curves $d_{j,k}^i$ ($j, k\neq i$; $j<k$): $[0, 1]\to H_i$ with the following properties;

(1) $d_{j,k}^i$ are congruent each other.

(2) $d_{j,k}^i(0)=a_{j,l}^i$; $d_{j,k}^i(1)=a_{k,l}^i$ ($l\neq i, j, k$)

(3) $d_{j,k}^i|_{[0,1/10]}$ (resp. $d_{j,k}^i|_{[9/10,1]}$) is a segment parallel to e_j^i (resp. e_k^i) whose length is greater than 10^{-2}.

(4) Rays emanating perpendicularly from points of $\bigcup_{j,k\neq i} d_{j,k}^i$ into $H_k(\text{III})$ minimize the distance from $\bigcup_{j,k\neq i} d_{j,k}^i$ just until they intersect $\bigcup_i e_i^k$ for the first time.

Secondly we take surfaces $C_{i,j}$ ($i<j$) generated by the family of the straight lines such that each of them passes through a point of $c_{i,j}$ orthogonally to $L_{i,j}$. Put

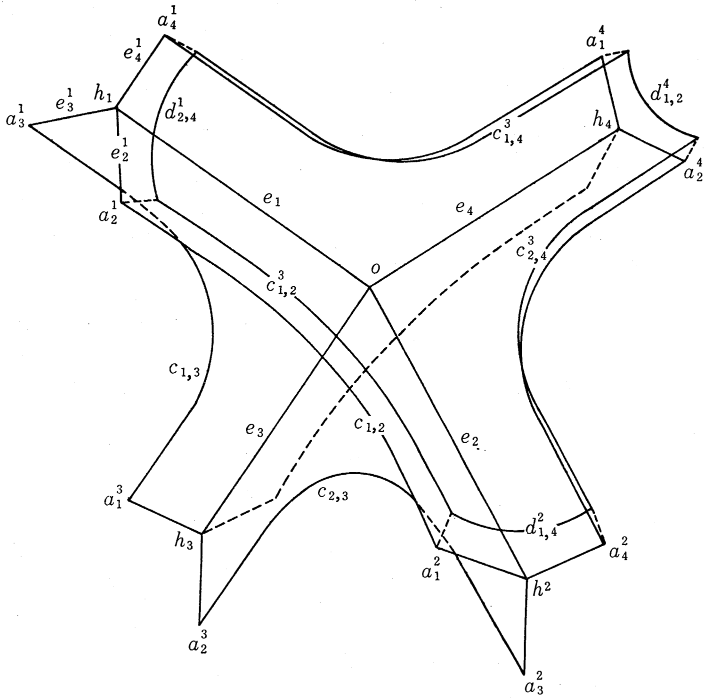


Fig. 3.3. Diagram $\mathscr{C}$(III) and surface S_3(III).

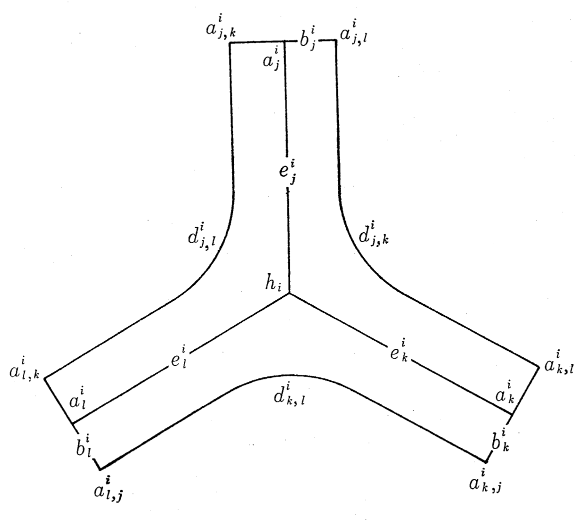


Fig. 3.4. H_i(III) (i, j, k, l are different each other).

$$C_{ij}(\mathrm{III}) := \{x \in C_{i,j} \mid d(x, c_{i,j}) \leq 10^{-2}\}$$

and define curves $c_{i,j}^{k}$ $(k \neq i, j)$ by

$$c_{i,j}^{k} := \{x \in C_{i,j} \cap \Delta_k \mid d(x, c_{i,j}) = 10^{-2}\}.$$

Thirdly we take C^∞-surfaces $S_k(\mathrm{III})$ $(k=1, 2, 3, 4)$ with corners in the domains Δ_k with the following properties;

(1) $S_k(\mathrm{III})$ are congruent each other.

(2) The boundary of $S_k(\mathrm{III})$ consists of the curves $d_{i,j}^{l}$ $(i, j, l \neq k;$ $i, j \neq l;$ $i<j)$ and $c_{i,j}^{k}(i, j \neq k)$.

(3) The 10^{-2}-neighborhood of $c_{i,j}^{k}$ in $S_k(\mathrm{III})$ is contained in a plane parallel to $L_{i,j}$.

(4) The 10^{-2}-neighborhood of $d_{i,j}^{l}$ in $S_k(\mathrm{III})$ is isometric to $d_{i,j}^{l} \times [0, 10^{-2}]$.

(5) Rays emanating perpendicularly from points of $\bigcup_k S_k(\mathrm{III})$ into $\mathscr{H}(\mathrm{III})$ minimize the distance from $\bigcup_k S_k(\mathrm{III})$ just unitl they intersect $\mathscr{C}(\mathrm{III})$ for the first time. This is possible from Lemma 2.2 and easy consideration.

Next we take a 2-handle $\mathscr{H}(\mathrm{II})$ which is the product of $H_4(\mathrm{III})$ and $[0, 1]$, where $H_4(\mathrm{III})$ is given in the above. We endow $\mathscr{H}(\mathrm{II})$ with the canonical metric induced from R^3. We make two copies $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$ of the above $\mathscr{H}(\mathrm{II})$. Put

$$\begin{array}{lll}
H_1(\mathrm{II}_i) := H_4(\mathrm{III}) \times \{0\} & \text{in } \mathscr{H}(\mathrm{II}_i) & (i=1, 2) \\
H_2(\mathrm{II}_i) := H_4(\mathrm{III}) \times \{1\} & \text{in } \mathscr{H}(\mathrm{II}_i) & \\
f_{j,1}^{i} := e_j^4 \times \{0\} & \text{in } \mathscr{H}(\mathrm{II}_i) & (j=1, 2, 3) \\
f_{j,2}^{i} := e_j^4 \times \{1\} & \text{in } \mathscr{H}(\mathrm{II}_i) & \\
C(\mathrm{II}_i) := \bigcup_{j=1}^{3} e_j^4 \times [0, 1] & \text{in } \mathscr{H}(\mathrm{II}_i) & \\
B_j(\mathrm{II}_i) := b_j^4 \times [0, 1] & \text{in } \mathscr{H}(\mathrm{II}_i). &
\end{array}$$

Finally we take $\mathscr{H}(\mathrm{I}) := D^2 \times [-10^{-2}, 10^{-2}]$ as a 1-handle, where D^2 is a 2-disk, and denote by $\mathscr{C}(\mathrm{I})$ the subset $D^2 \times \{0\}$ in $\mathscr{H}(\mathrm{I})$. We prepare two copies $\mathscr{H}(\mathrm{I}_1)$, $\mathscr{H}(\mathrm{I}_2)$ (resp. $\mathscr{C}(\mathrm{I}_1)$, $\mathscr{C}(\mathrm{I}_2)$) of $\mathscr{H}(\mathrm{I})$ (resp. $\mathscr{C}(\mathrm{I})$).

In the following we shall get D^3 from previously constructed handles $\mathscr{H}(\mathrm{I}_1)$, $\mathscr{H}(\mathrm{I}_2)$, $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$, $\mathscr{H}(\mathrm{III})$ by considering adequate attachments. Now we take isometric mappings

$$\begin{array}{ll}
\Phi_1 \colon H_1(\mathrm{II}_2) \longrightarrow H_1(\mathrm{III}), & \Phi_2 \colon H_1(\mathrm{II}_1) \longrightarrow H_2(\mathrm{III}) \\
\Phi_3 \colon H_2(\mathrm{II}_1) \longrightarrow H_3(\mathrm{III}), & \Phi_4 \colon H_2(\mathrm{II}_2) \longrightarrow H_4(\mathrm{III})
\end{array}$$

such that

$$\begin{array}{lll}
\Phi_1(f^2_{1,1})=e^1_2, & \Phi_1(f^2_{2,1})=e^1_3, & \Phi_1(f^2_{3,1})=e^1_4 \\
\Phi_2(f^1_{1,1})=e^2_4, & \Phi_2(f^1_{2,1})=e^2_3, & \Phi_2(f^1_{3,1})=e^2_1 \\
\Phi_3(f^1_{1,2})=e^3_4, & \Phi_3(f^1_{2,2})=e^3_2, & \Phi_3(f^1_{3,2})=e^3_1 \\
\Phi_4(f^2_{1,2})=e^4_1, & \Phi_4(f^2_{2,2})=e^4_2, & \Phi_4(f^2_{3,2})=e^4_3.
\end{array}$$

We attach $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$ to $\mathscr{H}(\mathrm{III})$ by Φ_1, Φ_2, Φ_3, Φ_4 and get a diagram $\mathscr{H}(\mathrm{II\ III})$. We denote $\mathscr{C}(\mathrm{II}_1)\cup\mathscr{C}(\mathrm{II}_2)\cup\mathscr{C}(\mathrm{III})$ by $\mathscr{C}(\mathrm{II\ III})$. Then the metric of $\mathscr{H}(\mathrm{II}_1)$, $\mathscr{H}(\mathrm{II}_2)$ and $\mathscr{H}(\mathrm{III})$ are smoothly connected by the attaching process and induce the metric on $\mathscr{H}(\mathrm{II\ III})$. Put

$$\begin{aligned}
\mathscr{B}^1 &:= C_{2,3}(\mathrm{III})\cup B_2(\mathrm{II}_1) \\
\mathscr{B}^2 &:= C_{1,4}(\mathrm{III})\cup B_1(\mathrm{II}_2)\cup C_{1,2}(\mathrm{III})\cup B_3(\mathrm{II}_1)\cup C_{1,3}(\mathrm{III})\cup B_2(\mathrm{II}_2) \\
&\qquad \cup C_{2,4}(\mathrm{III})\cup B_1(\mathrm{II}_1)\cup C_{3,4}(\mathrm{III})\cup B_3(\mathrm{II}_2),
\end{aligned}$$

which are annuli and contained in the boundary of $\mathscr{H}(\mathrm{II\ III})$. The 10^{-2}-neighborhood N of $\mathscr{B}^1$ (resp. $\mathscr{B}^2$) in $\mathscr{H}(\mathrm{II\ III})$ is a product of the 10^{-2}-neighborhood A of a circle $\mathscr{B}^1\cap\mathscr{C}(\mathrm{II\ III})$ (resp. $\mathscr{B}^2\cap\mathscr{C}(\mathrm{II\ III})$) in $\mathscr{C}(\mathrm{II\ III})$ and an interval $[-10^{-2}, 10^{-2}]$, and the induced metric on N is a product metric of the induced metric on A and the canonical metric on $[-10^{-2}, 10^{-2}]$.

Next take any C^∞-diffeomorphism $\psi\colon \partial D^2\to S^1$ and define $\Psi_i\colon \partial D^2\times[-10^{-2}, 10^{-2}]\subset\mathscr{H}(\mathrm{I}_i)\to\mathscr{B}^i(\simeq S^1\times[10^{-2}, 10^{-2}])\subset\mathscr{H}(\mathrm{II\ III})$ by $\Psi_i(x, t):=(\psi(x), t)$ $(i=1, 2)$. We attach $\mathscr{H}(\mathrm{I}_1)$ to $\mathscr{H}(\mathrm{II\ III})$ by Ψ_1 and denote the diagram by $\mathscr{H}(\mathrm{I}_1\ \mathrm{II\ III})$. We attach $\mathscr{H}(\mathrm{I}_2)$ to $\mathscr{H}(\mathrm{I}_1\ \mathrm{II\ III})$ by Ψ_2 and denote the diagram by $\mathscr{H}(\mathrm{I\ II\ III})$. We introduce a Riemannian metric on $\mathscr{C}(\mathrm{I}_i):=D^2\times\{0\}$ by Proposition 2.3 which is smoothly connected to the metric on $\mathscr{C}(\mathrm{II\ III})$. On $\mathscr{H}(\mathrm{I}_i)$ we introduce the product metric of its metric on $\mathscr{C}(\mathrm{I}_i)$ and the canonical one on $[-10^{-2}, 10^{-2}]$, then this metric is smoothly connected to the metric on $\mathscr{H}(\mathrm{II\ III})$.

$\mathscr{H}(\mathrm{I\ II\ III})$ is diffeomorphic to D^3. It is trivial that $(\mathscr{H}(\mathrm{I}_1), \mathscr{H}(\mathrm{II}_1))$ is a cancelling pair. Moreover $(\mathscr{H}(\mathrm{I}_2)), \mathscr{H}(\mathrm{II}_2))$ is a cancelling pair, too. In fact $\mathscr{H}(\mathrm{I}_1\ \mathrm{II\ III})$ and the attaching circle $(\partial D^2\times\{0\})$ of $\mathscr{H}(\mathrm{I}_2)$ are as in Figure 3.5. This attaching circle is isotopic to a circle which intersects a circle $\partial H_1(\mathrm{II}_2)$ in $\partial\mathscr{H}(\mathrm{II}_2)$ transversally at a single point (see Figure 3.5). Thus $(\mathscr{H}(\mathrm{I}_2), \mathscr{H}(\mathrm{II}_2))$ is a cancelling pair by the first cancellation theorem in [6].

The cut locus of the boundary of $\mathscr{H}(\mathrm{I\ II\ III})$ is $\mathscr{C}(\mathrm{I\ II\ III}):=\mathscr{C}(\mathrm{I}_1)\cup\mathscr{C}(\mathrm{I}_2)\cup\mathscr{C}(\mathrm{II\ III})$ with respect to the previously introduced metric and the diagram $\mathscr{C}(\mathrm{I\ II\ III})$ is as Figure 3.1.

We attach a new 3-dimensional disk $\mathscr{H}(0)$ to $\mathscr{H}(\mathrm{I\,II\,III})$ by an arbitrary diffeomorphism from $\partial\mathscr{H}(0)$ to $\partial\mathscr{H}(\mathrm{I\,II\,III})$ and get S^3. We apply Proposition 2.3 and get a new metric on S^3 and a point p in S^3

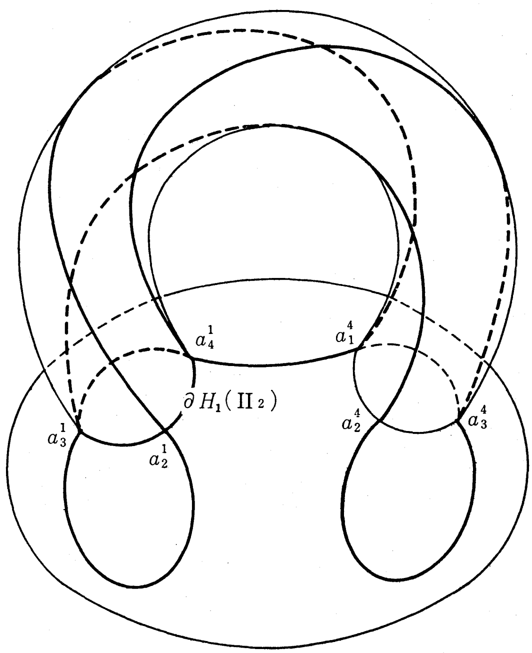
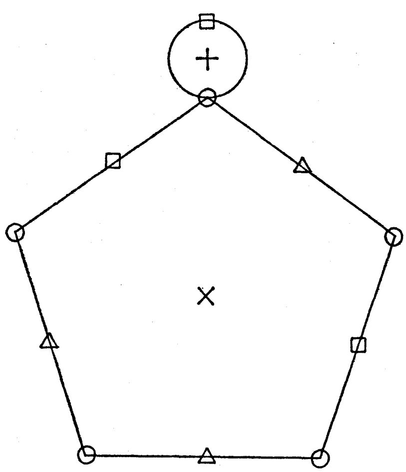


Fig. 3.5. $\mathscr{H}(\mathrm{I}_1\,\mathrm{II}\,\mathrm{III})$, attaching circle of $\mathscr{H}(\mathrm{I}_2)$ and $\partial H_1(\mathrm{II}_2)$.



×:	critical point of index	1
+:	〃	1
△:	〃	2
□:	〃	2
○:	〃	3

Fig. 3.6.

whose cut locus is $\mathscr{C}$(I II III). From the construction of the metric there is no conjugate point of p in a neighborhood of the cut locus of p. $\mathscr{C}$(I II III) is nothing but Figure 3.1 if we carefully check the identification under the above attaching procedure. Hence the metric satisfies our conditions and Theorem 3.1 is proved.

We don't give explicitly a Morse function corresponding to the above handle body decomposition, but we show the position of its critical points. The critical point of index 0 is the previously defined point p. The critical points with positive index exist on the cut locus of p as in Figure 3.6.

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