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On the Number of Closed Geodesics and Isomertry-Invariant Geodesics

Minoru Tanaka

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Introduction

This article is a survey of the Gromoll-Meyer theorem [13] on the number of closed geodesics and some theorems related to their theorem. The following is the theorem Gromoll and Meyer proved in 1969.

Theorem. *Any compact* 1*-connected riemannian manifold* M *has infinitely many closed geodesics if the sequence of the Betti numbers for the free loop space of* M *is unbounded.*

Though there is a long and rich history on closed geodesics on a compact riemannian manifold since Poincaré [38], Lusternik and Schnirelmann [30], etc., our survey covers an only small portion of the history. However the author believes that it is worth while introducing their method of proving the Gromoll-Meyer theorem and how the theorem has given influence to some theorems of closed geodesics. Note that no symmetric spaces of rank one satisfy the hypothesis on the Betti numbers. But there are many manifolds satisfying the assumption. Note also that the assumption is a topological one. It would be interesting to estimate the number of closed geodesics on a compact riemannian manifold, the quantity of differential geometry in terms of topological properties of manifolds only. From this point of view, it should be referred that Lusternik and Schnirelmann proved in 1929 that there exist at least three

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closed geodesics without self-intersections on a 1-connected compact surface [30]. And it should be referred also that Fet and Lusternik proved in 1951 that there exists at least one closed geodesic on a compact riemannian manifold [10]. Klingenberg claims that the hypothesis of the Betti numbers in the Gromoll-Meyer theorem can be removed, and he gave a proof of the claim in his lecture note [24]. But the proof is still incomplete. In Finsler manifolds case this hypothesis can not be removed. In fact Katok [23] constructed a Finsler metric on a 2-sphere which has only two closed geodesics. In 1980 Matthias proved that any Finsler manifold M has infinitely many closed geodesics if the sequence of the Betti numbers for the free loop space of M is unbounded [31]. In [12] Gromoll and Meyer defined local homological invariants, characteristic invariants and characteristic submanifolds for isolated critical points and they applied these invariants to prove their theorem [13]. It is difficult to estimate the number of closed geodesics in case closed geodesics lie on degenerate critical orbits. In their proof the above invariants are very useful to handle degenerate critical orbits. A related but more general theory than that of closed geodesics is the one of isometry-invariant geodesics developed by Grove [15], [16]. A non-constant geodesic $c: \boldsymbol{R} \to M$ is said to be invariant under an isometry A on M if $Ac(t)=c(t+1)$ for all $t \in \boldsymbol{R}$. Thus closed geodesics of period 1 are invariant under id_M, the identity map on M. Grove and Tanaka extended the Gromoll-Meyer theorem by means of isometry-invariant geodesics [18], [42]. Local homological invariants play an important role to estimate the number of isometry-invariant geodesics. From chapter I to III this article is written for those who are not familiar with infinite dimensional manifolds. Chapter I is occupied by theorems from functional analysis necessary for calculus on infinite dimensional manifolds. Chapter II is devoted to Morse theory on Hilbert manifolds developed by Palais [37]. In Chapter III various path-spaces are introduced and it is given the structures of Hilbert riemannian manifolds to the path-spaces. As an application it will be given an outline of a proof of Fet-Lusternik theorem mentioned above. In Chapter IV, it will be stated an outline of the proof of the Gromoll-Meyer theorem and an outline of the proof of the generalized theorem by Grove and Tanaka. Some theorems related to the Gromoll-Meyer theorem are stated without proof. Our reference is not complete, cf. [24] for a complete reference. [11] and [24] are good textbooks to understand infinite dimensional manifolds.

Chapter I. Reviews from Analysis

In this chapter all the theorems and propositions will be stated without proofs. The proofs can be found in [9], [29], [43].

§ 1. The open map theorem and the spectral theory

Let $E_1, E_2, \cdots, E_r$ and F be Banach spaces and let $L(E_1, \cdots, E_r; F)$ be the set of all bounded r-multilinear maps from $E_1 \times \cdots \times E_r$ into F. In particular when $E_1 = \cdots = E_r = E$, $L^r(E, F)$ will be used instead of $L(E, \cdots, E; F)$. The operator norm $\|T\|$ of $T \in L(E_1, \cdots, E_r; F)$ is defined by

$$\|T\| = \sup \{\|T(x_1, \cdots, x_r)\| \mid \|x_i\| \leqslant 1 \text{ for all } i,\ 1 \leqslant i \leqslant r\}.$$

Then $L(E_1, \cdots, E_r; F)$ becomes a complete normed space by the norm, i.e. a Banach space.

Theorem 1.1 (*open map theorem*). *If $f \in L(E, F)$ is surjective then f is an open map.*

Theorem 1.2. *Let H be a real (resp. complex) Hilbert space with an inner product $\langle\ ,\ \rangle$. For each bounded real (resp. complex) valued linear map f, there exists a unique element $a \in H$ such that $f(x) = \langle x, a\rangle$ for all $x \in H$.*

As corollaries we have

Corollary 1.3. *If A is an element of $L(H, H)$, there exists a unique $A^* \in L(H, H)$ satisfying $\|A^*\| = \|A\|$ and $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in H$.*

Corollary 1.4. *Let A, B be bounded linear maps on H and let z be a complex number. Then the following hold.*

(i) $(A+B)^* = A^* + B^*$
(ii) $(zA)^* = \bar{z}A^*$, where $\bar{z}$ denotes the complex conjugate of z.
(iii) $(AB)^* = B^*A^*$
(iv) $(A^*)^* = A$
(v) $\|A \circ A^*\| = \|A\|^2$.

If a bounded linear map A satisfies $A^* = A$ then A is called *self adjoint*. A projection P is defined as the bounded linear map on H satisfying $P^2 = P = P^*$. Actually a projection P is the orthogonal projection onto $P(H)$. Let $L_s^2(H, R)$ be the Banach space of all symmetric bilinear maps in $L^2(H, R)$.

Proposition 1.5. *There exists a canonical linear isomorphism between*

$L_s^2(H, R)$ and $\{A \in L(H, H) \mid A^* = A\}$.

Let A be a bounded linear map on a Hilbert space H. The *spectrum* of A is defined by

$$\sigma(A) = \{\lambda \in C \mid A - \lambda \,\mathrm{Id} \text{ does not have a continuous inverse}\}.$$

Theorem 1.6. $\sup\{|\lambda| \mid \lambda \in \sigma(A)\} \leqslant \|A\|$.

Let $F(A)$ be the set of all complex-valued functions which are holomorphic on a neighborhood of $\sigma(A)$. For each $f \in F(A)$, $f(A)$ is defined by the Dunford's integral (p. 225 in [43]).

Theorem 1.7. *If A is self adjoint then the spectrum of A is closed subset of R.*

Theorem 1.8. *If $f, g \in F(A)$ then $\alpha f + \beta g \in F(A)$ for any complex numbers α, β, $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$, $fg \in F(A)$ and $(fg)(A) = f(A)g(A)$.*

§ 2. Frechét derivatives and integrals

Let U be an open subset of a Banach space E and let f be a map from U into a Banach space F. If there exists a $T \in L(E, F)$ satisfying

$$\lim_{x \to 0} \|f(p+x) - f(p) - T(x)\| / \|x\| = 0$$

then f is said to be differentiable at $p \in U$. If T exists, it is unique. Hence T will be denoted by df_p. Let I be an open interval of R. If $f\colon I \to F$ is differentiable at $t \in I$, then $df_t \in L(R, F)$ and $\dot{f}(t)$ will denote $df_t(1)$.

Proposition 2.1. *If $f \in L(E_1, E_2; F)$ then f is differentiable at each point of $E_1 \times E_2$ and $df_{(x,y)}(s, t) = f(x, t) + f(s, y)$.*

Proposition 2.2. *Let A (resp. B) be an open subset of a Banach space E(resp. F). If a map $f\colon A \to F$ is differentiable at $x_0 \in A$ and a map $g\colon B \to G$ is differentiable at $f(x_0) \in B$, then $g \circ f$ is differentiable at x_0 and $d(g \circ f)_{x_0} = dg_{f(x_0)} df_{x_0}$.*

Suppose that a map $f\colon A \to F$ is differentiable at each point of the open subset A of a Banach space E. Then df is a map from A into $L(E, F)$. If df is continuous on A, then f is called of class C^1. If df is differentiable at a point $p \in A$ then $d^2 f_p = d(df)_p$ is an element of $L(E, L(E, F))$. If we identify $L^{k+1}(E, F)$ and $L(E, L^k(E, F))$ for each positive integer k, then d^2f is a map from A into $L^2(E, F)$. Inductively suppose that d^kf: $A \to L^k(E, F)$ exists and differentiable at $p \in A$. Then $d^{k+1}f_p$ is defined by $d^{k+1}f_p = d(d^kf)_p$. If $d^{k+1}f_p$ exists at each point $p \in A$ and $d^{k+1}f$: $A \to L^{k+1}(E, F)$ is continuous then f is said to be of class C^{k+1}. If f is of class C^k for every positive integer k, f is said to be of class C^∞. A continuous map will be called of class C^0.

Proposition 2.3. *Suppose that f: $A \to F$ is of C^2. Then d^2f: $A \to L^2(E, F)$ is symmetric at each point of A.*

Proposition 2.4. *Suppose that f: $A \to F$ is of class C^1 and that A is convex. Then the inequality*

$$\|f(z)-f(x)\| \leqslant \|z-x\| \sup_{0 \leqslant t \leqslant 1} \|\dot{f}(x+t(z-x))\|$$

holds for each $x, z \in A$.

Theorem 2.5 (*Inverse function theorem*). *Suppose that f: $A \to F$ is of class C^k ($k \geqslant 1$) and that df_{x_0} is a linear homeomorphism for a point $x_0 \in A$. Then there exists an open neighborhood V of x_0 such that the map $f|V$: $V \to f(V)$ is a C^k-diffeomorphism.*

Proposition 2.6. *Let U_i ($i=1, 2, \cdots, n$) be an open subset of a Banach space E_i and let f be a continuous map from $U_1 \times \cdots \times U_n$ into F. f is of class C^k if and only if d_if: $U_1 \times \cdots \times U_n \to L(E_i, F)$ is of class C^{k-1} for any $1 \leqslant i \leqslant n$. Here d_if denotes the partial derivative with respect to the i^{th} component.*

Proposition 2.7. *Let E and F be Banach spaces which are linear isomorphic. If $GL(E, F)$ denotes the set of all linear isomorphisms of E onto F, then $GL(E, F)$ is open in $L(E, F)$ and a map $u \in GL(E, F) \to u^{-1} \in GL(F, E)$ is of class C^∞.*

Let I be an interval of R whose end points a, b may be $+\infty$ or $-\infty$. A map f from I into a Banach space F is called a step function if there exist finite points $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ such that f is constant on each open interval (x_i, x_{i+1}) $(0 \leqslant i \leqslant n-1)$. A map f: $I \to F$ is called a regulated function if $f(x+0)$ $(x \neq b)$ or $f(x-0)$ $(x \neq a)$ exists for any $x \in I$.

Proposition 2.8. *Suppose that I is a compact interval $[a, b]$. Then a map f: $I \to F$ is a regulated function if and only if f can be uniformly approximated by step functions.*

A continuous map g: $I \to F$ is called a *primitive function* of a map f from I into F if $\dot{g} = f$ except for a countable subset of I.

Proposition 2.9. *Let I be an interval of R. If a map f of I into F is regulated, then f has a primitive function.*

Definition of integrals. Let f be a regulated function from an interval $[\alpha, \beta]$ into F and let g be a primitive function of f. Then the integral of f from α to β is defined by

$$\int_{\alpha}^{\beta} f(t)dt = g(\beta) - g(\alpha) .$$

From Proposition 2.4 $g(\beta) - g(\alpha)$ is independent of the choice of primitive functions.

Proposition 2.10. *Suppose that a map f is a regulated function of $[\alpha, \beta]$ into E and that u is a bounded linear map of E into F. Then*

$$\int_{\alpha}^{\beta} u(f(t))dt = u\left(\int_{\alpha}^{\beta} f(t)dt\right)$$

holds.

Proposition 2.11. *Let f be a regulated function of $[\alpha, \beta]$ into F. Then*

$$\left\|\int_{\alpha}^{\beta} f(t)dt\right\| \leqslant \int_{\alpha}^{\beta} \|f(t)\|dt \leqslant (\beta-\alpha) \sup_{\alpha \leqslant t \leqslant \beta} \|f(t)\|$$

holds.

Proposition 2.12. *Let A be an open subset of E. If f: $[\alpha, \beta] \times A \to F$ is continuous, then $g(z) = \int_{\alpha}^{\beta} f(t, z)dt$ is a continuous map from A into F.*

Proposition 2.13. *Under the same assumption as the above proposition, the function g is of class C^1 and $dg_z = \int_{\alpha}^{\beta} d_2 f_{(t,z)} dt$ if $d_2 f$ exists and it is continuous on $[\alpha, \beta] \times A$.*

Proposition 2.14. *Let U be a convex open subset of E. If a map f of U into F is of class C^p, then*

$$f(x+y) = f(y) + df_x(y)/1! + \cdots + d^{p-1}f_x(y^{p-1})/(p-1)!$$
$$+ \int_0^1 (1-t)^{p-1} d^p f_{x+ty}(y^p)dt/(p-1)!$$

where y^k stands for $(y, \cdots, y)$ (k times).

§ 3. The existence and uniqueness theorems for differential equations on a Banach space

Let U be an open subset of a Banach space E and let f be a $C^p(p\geqslant 0)$ map from $J\times U$ into E, where J is an open interval containing 0 in R. By a local flow for f at a point $x_0 \in U$, we mean a mapping

$$\alpha\colon J_0\times U_0 \longrightarrow U$$

where J_0 is an open subinterval of J containing 0 and where U_0 is an open subset of U containing x_0 such that for each x in U_0

$$\alpha_x(t)=\alpha(t, x)$$

is an integral curve for f with initial condition x. Here an integral curve for f with initial condition x is a mapping β of an open subinterval J_0 of J containing 0 into U such that

$$\dot{\beta}(t)=f(t, \beta(t)), \qquad \beta(0)=x.$$

Theorem 3.1. *Let J be an open interval of R containing* 0 *and let U be an open subset of a Banach space E. Suppose that a map f from $J\times U$ into E is of class C^p $(p\geqslant 1)$. Then there exists a unique local flow, which is of class C^{p+1}, for f at each point $x_0 \in U$.*

Definition of a Banach manifold. Let X be a topological set. An atlas on X is a collection of pairs (U_i, φ_i), $i \in I$, satisfying the following condition:

1. Each U_i is an open subset of X and U_i cover X.
2. Each φ_i is a homeomorphism of U_i onto an open subset $\varphi_i(U_i)$ of a Banach space E_i.
3. The map $\varphi_j\circ\varphi_i^{-1}\colon \varphi_i(U_i\cap U_j)\to\varphi_j(U_i\cap U_j)$ is of class C^∞ for each pair of $i, j \in I$.

A Hausdorff space X with a maximal atlas will be called a *Banach manifold.*

Each element (U_i, φ_i) of the atlas will be called a chart and U_i (resp. φ_i) will be called a coordinate neighborhood (resp. a coordinate function). E_i will be called the target of φ_i. On each connected component of X we may assume that the E_i's are some fixed E, because the differential of $\varphi_j\circ\varphi_i^{-1}$ gives a topological linear isomorphism between E_i and E_j when U_i and U_j have a common point. Let I be the set of all coordinate functions at a point p in M. Let E denote the set $\prod_{\varphi\in I} E_\varphi$ where E_φ denotes the target of φ, and let π_φ be the natural projection of E onto E_φ. Then *the tangent space M_p* at p to M is defined by

$$\{v \in E \,|\, d(\varphi \circ \psi^{-1})_{\psi(p)}(\pi_\psi v) = \pi_\varphi v \text{ for any } \varphi, \psi \in I\}.$$

M_p becomes a Banach space induced from the map $\pi_\varphi | M_p$. Each element of M_p will be called a tangent vector at p. Let M and N be Banach manifolds and let f be a continuous map from M into N. If $\psi \circ f \circ \varphi^{-1}$ is smooth (C^∞) for any chart (U, φ) for M and any chart (V, ψ) for N with $f(U) \subset V$, then f will be called *a smooth map*. In case f is smooth *the differential map* df_p at p is defined by

$$df_p(v) = (d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}(\pi_\varphi v))_{\psi \in J}$$

for any $v \in M_p$, where J denotes the set of all coordinate functions at $f(p)$. Then df_p is a linear map of M_p into $N_{f(p)}$. *The tangent bundle* TM of M is defined by the Banach manifold $TM = \bigcup_{p \in M} M_p$ with the atlas induced from the atlas for M. Let X be a smooth vector field on M. *An integral curve of* X is defined by a smooth map σ of an open interval into M such that $d\sigma_t(1) = X_{\sigma(t)}$. In what follows $d\sigma_t(1)$ will be denoted by $\dot\sigma(t)$ or by $\sigma'(t)$.

Theorem 3.2. *For each $p \in M$ there is an integral curve σ_p of X with $\sigma_p(0) = p$ such that every integral curve of X with initial condition p is a restriction of σ_p. The integral curve σ_p will be called the (maximal) integral curve of X with initial condition p.*

The above theorem is proved by means of Theorem 3.1.

Chapter II. Morse Theory on Hilbert riemannian Manifolds

§ 1. The generalized Morse lemma

Let M be a Banach manifold and let f be a smooth function on M. If $df_p \neq 0$ then p is called a *regular point* of f and otherwise p is called *a critical point* of f. For a critical point p of f, a bilinear form $H(f)_p$ on M_p, which is called the Hessian of f at p, is defined by

$$H(f)_p(v, w) = d^2(f, \varphi^{-1})_{\varphi(p)}(v_\varphi, w_\varphi),$$

where φ is a coordinate function at p. This definition is independent of the choice of φ and $H(f)_p$ is continuous and symmetric [Proposition 2.3 in Chapter I]. If $H(f)_p$ is *non-degenerate*, i.e. the map $v \in M_p \mapsto H(f)_p(v, \cdot) \in L(M_p, R)$ is a linear isomorphism, p is called *non-degenerate*. Otherwise p is called *degenerate*. We define *the index* of p to be the supremum of the dimensions of subspaces W of M_p on which $H(f)_p$ is negative definite. *The null space* of p is defined by the vector space of all tangent vectors v satisfying $H(f)_p(v, w)=0$ for any $w \in M_p$. *The nullity* of p is defined by the dimension of the null space. Let F be a smooth function on a Hilbert space $(H, \langle\ ,\ \rangle)$. If the origin 0 of H is a critical point of F, a self adjoint operator A on H is determined by

$$\langle Ax, y\rangle = H(F)_0(x, y)$$

[Proposition 1.5 in Chapter I]. F will be called a Fredholm function at 0 when A-Id is a compact operator. Hence the index of 0 is finite if F is a Fredholm function at 0. The following lemma was proved by Palais [37] and Gromoll, Meyer [12].

Lemma 1.1. *Let f be a Fredholm function at the origin defined in a convex neighborhood U of the origin in a Hilbert space $(H, \langle\ ,\ \rangle)$. Then there exist an origin preseving diffeomorphism Φ of some neighborhood of* 0 *in H into H and an origin preserving smooth map h defined in some neighborhood of* 0 *in $N=\ker H(f)_0$ into $E=N^{\perp}$, the orthogonal complement of N, such that*

$$f\circ\Phi(x, y)=\|Px\|^2-\|(\mathrm{Id}-P)x\|^2+f(h(y), y)$$

with an orthogonal projection $P\colon E\to E$.

Proof. Define $\varphi\colon E\oplus N\to E\oplus N$ by $\varphi(x, y)=(P_1(\nabla f_{(x,y)}), y)$ where $P_1\colon H\to E$ denotes the orthogonal projection to E and where $\nabla f_{(x,y)}$ is the gradient vector of f at (x, y), i.e. it is characterized by the property that $\langle \nabla f_{(x,y)}, v\rangle = df_{(x,y)}(v)$ for any $v \in H$ [Theorem 1.2 in Chapter I]. Since $d(P_1(\nabla f))=P_1A$ at the origin, the differential of φ at the origin has the form $d\varphi_0=P_1A\oplus \mathrm{Id}_N$. Here A denotes the self adjoint bounded linear operator determined from the Hessian of f at the origin. Since $A|E$ is a linear isomorphism of E onto itself, $d\varphi_0$ has a continuous inverse [Theorem 1.1 in Chapter I]. From the inverse function theorem [Theorem 2.5 in Chapter I], φ is locally invertible in a neighborhood of 0 in H and the equation $\varphi^{-1}(0, y)=(h(y), y)$ defines a function $h\colon U\to E$ in some neighborhood U of 0 in N, $h(0)=0$. Observing $(0, y)=\varphi(h(y), y)$ we obtain $P_1\nabla f_{(h(y), y)}=0$, i.e. $(h(y), y)$ is a critical point of $f|E\oplus y$. Set $g(x, y)= f(x, y)-f(h(y), y)$ and $\psi(x, y)=(x-h(y), y)$. ψ is a diffeomorphism, $g\circ\psi^{-1}(x, y)=f(x+h(y), y)-f(h(y), y)$, $g\circ\psi^{-1}(0, y)=0$. Furthermore $(0, y)$ is a critical point of $g\circ\psi^{-1}|E\oplus y$. Let d_E denote the partial differential for functions on H with respect to the E-component. For $(x, y) \in E\oplus N$, we define a continuous bilinear form B_{xy} on E by

$$B_{xy}=\int_0^1 (1-t)d_E^2(g\circ\psi^{-1})_{(tx,y)}dt$$

and an operator $A_{xy}: E \to E$ by $\langle A_{xy}x_1, x_2\rangle = B_{xy}(x_1, x_2)$, clearly $2A_{00} = P_1 A | E$ is invertible. From Proposition 2.14 in Chapter I, we have

$$g \circ \psi^{-1}(x, y) = B_{xy}(x, x) = \langle A_{xy}x, x\rangle.$$

Define $D_{xy} = A_{xy}^{-1}A_{00}$. Since the inversion is a smooth map of the open set of the invertible operators onto itself [Proposition 2.7 in Chapter I], D is a smooth map of some neighborhood of the origin into $L(E, E)$ and each D_{xy} is invertible. Now $D_{00} = \mathrm{Id}_E$ and since a square root function is defined in a neighborhood of Id_E by a convergent power series with real coefficients we can define a smooth map $C: U \to L(E, E)$ with each C_{xy} invertible, if U is taken sufficiently small, by $C_{xy} = (D_{xy})^{1/2}$. Since A_{00} and A_{xy} are self adjoint we see easily from the definition of D_{xy} that $D_{xy}^* A_{xy} = A_{xy}D_{xy} = A_{00}$ and clearly the same relation then holds for any polynomial in D_{xy} hence for C_{xy} which is a limit of such polynomials. Thus $C_{xy}^* A_{xy} C_{xy} = A_{xy}(C_{xy})^2 = A_{xy}D_{xy} = A_{00}$. If we write $\Psi(x, y) = (C_{xy}^{-1}x, y)$ then Ψ is a local diffeomorphism, because $d\Psi_0 = C_{00}^{-1} \oplus \mathrm{Id}_N = \mathrm{Id}_H$. Hence we have $g \circ \psi^{-1} \circ \Psi^{-1}(x, y) = \langle A_{00}x, x\rangle$ and $f \circ \psi^{-1} \circ \Psi^{-1}(x, y) = \langle A_{00}x, x\rangle + f(h(y), y)$. Let $T = |A_{00}|^{-1/2}$ so that $A_{00}T^2 = P - (\mathrm{Id} - P)$ where $P = \chi(A_{00})$ and χ the characteristic function of $[0, \infty)$ [Theorem 1.8 in Chapter I]. Then Φ defined by $\Phi(x, y) = \psi^{-1} \circ \Psi^{-1}(Tx, y)$ is the desired local diffeomorphism.

Remark. In case 0 is a non-degenerate critical point of f the above claim holds without the assumption on A [37].

Corollary 1.2. *The index of f at the origin is the dimension of the range of* $\mathrm{Id} - P$.

Proof. The proof is easy to note that the Hessian of $f(h(y), y)$ at the origin is completely vanishing and that $\mathrm{Id} - P$ is injective on any subspace on which the Hessian of f is negative definite.

§ 2. Hilbert riemannian manifolds

Let M be a *Hilbert manifold*, i.e. M is a Banach manifold whose targets are separable Hilbert spaces and let $(H, \langle\ ,\ \rangle)$ be a target of M. For each $p \in M$ let $\langle\ ,\ \rangle_p$ be an admissible inner product in M_p, i.e. a positive definite, symmetric, bilinear form on M_p such that the norm $\|v\| = \langle v, v\rangle_p^{1/2}$ defines the topology of M_p. We will call the map $p \to \langle\ ,\ \rangle_p$ a *riemannian structure* for M if for any chart $(D(\varphi), \varphi)$ the function $G^\varphi(x)$, which is defined by $\langle G^\varphi(x)u, v\rangle = \langle d\varphi_x^{-1}(u), d\varphi_x^{-1}(v)\rangle_x$ for $u, v \in H$, is smooth on $D(\varphi)$. A Hilbert manifold with a riemmannian structure will be called a *Hilbert riemannian manifold.*

If σ: $[a, b] \to M$ is a piecewise C^1-map then

$$L(\sigma)=\int_a^b \|\dot{\sigma}(t)\|\, dt$$

is well defined and is called *the length* of σ. We can define a metric ρ in each connected component of M by defining $\rho(x, y)$ to be the infimum of the lengths of all piecewise C^1-paths joining x and y. The topology given by this metric is the given topology of M [37]. Let f be a smooth real valued function on a Hilbert riemannian manifold M. Given $p \in M$, df_p is a continuous linear functional on M_p, hence there is a unique tangent vector $\nabla f_p \in M_p$ such that $df_p(v)=\langle v, \nabla f_p\rangle_p$ for all $v \in M_p$ [Theorem 1.2 in Chapter I]. ∇f: $p \in M \to \nabla f_p \in TM$ is called *the gradient vector field* of f. Note that ∇f is smooth [p. 313 in 37].

Infinite dimensional manifolds are not locally compact. Therefore in order to develope Morse theory on such a manifold, we need assume the following condition which is called Palais-Smale Condition (C).

(C) If S is any subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from zero, then there is a critical point of f adherent to S.

With Condition (C), analogous claims to the case of finite dimensional manifolds [33] can be concluded. Refer to [37] on the proofs of the claims in this section and the next one.

Proposition 2.1. *Suppose that f satisfies Condition* (C) *and that any critical point p of f with $a \leqslant f(p) \leqslant b$ are non-degenerate for real numbers a, b. Then there are at most a finite number of critical points p of f with $a \leqslant f(p) \leqslant b$.*

Proposition 2.2. *Suppose that M is complete and that f satisfies Condition* (C). *If there is no critical point in the closed interval* $[a, b]$, *then $M^a=\{x \in M \mid f(x) \leqslant a\}$ and M^b are diffeomorphic.*

If a smooth function f on M satisfies Condition (C) and all critical points of f are non-degenerate, then f will be called a *Morse function.*

Theorem 2.3. *Let M be a complete Hilbert riemannian manifold, f a Morse function, c a critical value of f, $p_1, \cdots, p_n$ the critical points of finite index on $f^{-1}(c)$, and let d_i be the index of p_i. If c is the only critical value of f in a closed interval* $[a, b]$, *then M^b has the same homotopy type as M^a attached with cells of dimension $d_1, \cdots, d_n$.*

Remark. The critical points of infinite index on the level c is homotopically invisible since the unit sphere is a strong deformation retract of the unit disc in a separable Hilbert space of infinite dimension.

§ 3. Additional results and critical submanifolds

In this section M will denote a Hilbert riemannian manifold and f will denote a smooth function on M satisfying Condition (C). Let K be the set of critical points of f and let $\dot{K}$ be the frontier of K.

Theorem 3.1. *$f|\dot{K}$ is proper,* i.e. *for given real numbers a, b, $\dot{K}\cap f^{-1}[a, b]$ is compact.*

Theorem 3.2. *If f is bounded below on a connected component M_0 of a complete manifold M, then $f|M_0$ assumes its greatest lower bound.*

Corollary 3.3. *If K has no interior point and if f is bounded below on M then f assumes its greatest lower bound.*

Let W be a connected submanifold of M. If each point of W is a critical point of f, then W is called a *critical submanifold* [7]. Moreover if the null space of the Hessian of f at each point p of W is included in W_p, then W is called *non-degenerate.* The indexes of any points of W are constant if W is a non-degenerate critical submanifold [32]. In [7], [32] Morse theory for f, whose critical sets decompose non-degenerate critical submanifolds, were developed by Bott and Meyer. Morse theory for such a function will be useful to estimate the number of closed geodesics on finite dimensional riemannian manifolds. In fact closed geodesics will be characterized by nonzero valued critical points of the energy function on a certain path-space and such a critical point lies in a 1-dimensional compact critical submanifold of the path-space.

Chapter III. Various Path-Spaces

§ 1. Preliminaries

A map σ of the unit interval I into R^n is called absolutely continuous if either and hence both of the following two conditions are satisfied:

(1) Given $\varepsilon>0$ there exists $\delta>0$ such that if $0\leqslant t_0<\cdots<t_{2k+1}\leqslant 1$ and $\sum_{i=0}^{k}|t_{2i+1}-t_{2i}|<\delta$ then

$$\sum_{i=0}^{k}\|\sigma(t_{2i+1})-\sigma(t_{2i})\|<\varepsilon.$$

(2) There is a $g\in L^1(I, R^n)$ (the set of summable functions of I into R^n) such that

$$\sigma(t)=\sigma(0)+\int_0^t g(s)ds.$$

The equivalence of these two conditions is a classical theorem of Lebesgue. It follows from (2) that $\sigma'(t)$ exists for almost all $t \in I$ and that $\sigma' \in L^1(I, R^n)$ and

$$\sigma(t)=\sigma(0)+\int_0^t \sigma'(t)dt.$$

Let $H_0(I, R^n)$ be the set of measurable functions σ of I into R^n such that

$$\langle \sigma, \sigma \rangle_0=\int_0^1 \|\sigma(t)\|^2 dt < \infty.$$

Then $H_0(I, R^n)$ is a Hilbert space with the inner product $\langle\ ,\ \rangle_0$. We shall denote by $H_1(I, R^n)$ the set of absolutely continuous maps $\sigma\colon I \to R^n$ such that $\sigma' \in H_0(I, R^n)$. Then $H_1(I, R^n)$ is a Hilbert space under the inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ defined by $\langle\!\langle \sigma, \rho \rangle\!\rangle=\langle \sigma(0), \rho(0) \rangle+\langle \sigma', \rho' \rangle_0$. The proofs of the claims without proofs in this section will be found in [37].

Lemma 1.1. *If $\sigma \in H_1(I, R^n)$ then*

$$\|\sigma(t)-\sigma(s)\| \leqslant |t-s|^{1/2} \|\sigma'\|_0.$$

We shall denote the set of continuous maps of I into R^n by $C^0(I, R^n)$, considered as a Banach space with norm $\|\cdot\|_\infty$ defined by

$$\|\sigma\|_\infty=\max\{\|\sigma(t)\| \mid t \in I\}.$$

Corollary 1.2. *If $\sigma \in H_1(I, R^n)$ then*

$$\|\sigma\|_\infty \leqslant 2|||\sigma|||=2\sqrt{\langle\!\langle \sigma, \sigma \rangle\!\rangle}.$$

Corollary 1.3. *The inclusion maps of $H_1(I, R^n)$ into $C^0(I, R^n)$ and $H_0(I, R^n)$ are compact.*

Theorem 1.4. *If $\varphi\colon R^n \to R^p$ is a C^{k+2}-map then $\sigma \mapsto \varphi\sigma$ is a C^k-map $\bar{\varphi}\colon H_1(I, R^n) \to H_1(I, R^p)$. Moreover if $1 \leqslant m \leqslant k$ then*

$$d^m\bar{\varphi}_\sigma(\lambda_1, \cdots, \lambda_m)(t)=d^m\varphi_{\sigma(t)}(\lambda_1(t), \cdots, \lambda_m(t)).$$

§ 2. Hilbert manifolds of curves on manifolds

If V is a finite dimensional manifold we define $H_1(I, V)$ to be the set of continuous maps σ of I into V such that $\varphi \circ \sigma$ is absolutely continuous and $\|(\varphi \circ \sigma)'\|$ locally square summable for each coordinate function φ for V. For each $\sigma \in H_1(I, V)$ we define $H_1(I, V)_\sigma = \{X \in H_1(I, TV) \mid X(t) \in V_{\sigma(t)}$ for all $t \in I\}$. If N is a submanifold $V \times V$ we define $\Omega_N(V) = \{\sigma \in H_1(I, V) \mid (\sigma(0), \sigma(1)) \in N\}$ and if $\sigma \in \Omega_N(V)$ we define $\Omega_N(V)_\sigma = \{X \in H_1(I, V)_\sigma \mid (X(0), X(1)) \in N_{(\sigma(0), \sigma(1))}\}$.

Theorem 2.1. *If V is a closed submanifold of R^n and if N is a closed totally geodesic submanifold of $V \times V$, then $\Omega_N(V)$ consists of all $\sigma \in H_1(I, R^n)$ such that $\sigma(I) \subset V$ and $(\sigma(0), \sigma(1)) \in N$ and is a closed submanifold of the Hilbert space $H_1(I, R^n)$. If $\sigma \in \Omega_N(V)$ then the tangent space to $\Omega_N(V)$ at σ (as a submanifold of $H_1(I, R^n)$) is just $\Omega_N(V)_\sigma$.*

Proof. The first claim is trivial. Since V and N are closed in R^n, $R^n \times R^n$ respectively, it follows that $\Omega_N(V)$ is closed in $C^0(I, R^n)$, hence in $H_1(I, R^n)$ by Corollary 1.3. In the same way we see that $\Omega_N(V)_\sigma$ is a closed subspace of $H_1(I, R^n)$. Since V is a closed submanifold of R^n, we can construct a riemannian metric for R^n such that V is a totally geodesic submanifold. Then let $E\colon R^n \times R^n \to R^n$ denote the corresponding exponential map. For each $\sigma \in \Omega_N(V)$ define $\exp_\sigma\colon H_1(I, R^n) \to H_1(I, R^n)$ by $\exp_\sigma(X) = \bar{E}(\sigma, X)$. Then by Theorem 1.4, $\exp_\sigma$ is smooth and $\exp_\sigma(\mathbf{0}) = \sigma$. Moreover $d(\exp_\sigma)_{\mathbf{0}}(X)(t) = d_2E_{\mathbf{0}}(\sigma(t), X(t)) = X(t)$ by Theorem 1.4. From the inverse function theorem (Theorem 2.5 in Chapter I) $\exp_\sigma$ maps a neighborhood of zero in $H_1(I, R^n)$ diffeomorphically onto a neighborhood of σ in $H_1(I, R^n)$. Since V and N are totally geodesic it follows that for X near zero in $H_1(I, R^n)$ $\exp_\sigma X \in \Omega_N(V)$ if and only if $X \in \Omega_N(V)_\sigma$. Consequently $\exp_\sigma^{-1}$ restricted to a neighborhood of σ in $\Omega_N(V)$ is a coordinate function in $\Omega_N(V)$ which is the restriction of a coordinate function for $H_1(I, R^n)$, so $\Omega_N(V)$ is a closed submanifold of $H_1(I, R^n)$ and its tangent space at σ is $\Omega_N(V)_\sigma$.

Remark. $\Omega_N(V)$ is a closed submanifold of $H_1(I, V) = \Omega_{V \times V}(V)$ for any closed submanifold N of $V \times V$, because $\Omega_N(V)$ is the inverse image of the submersion defined by $\sigma \in H_1(I, V) \mapsto (\sigma(0), \sigma(1)) \in V \times V$. But the restriction of $\exp_\sigma^{-1}$ to $\Omega_N(V)$ cannot be a coordinate function unless N is totally geodesic.

Theorem 2.2. *Let V and W be closed submanifolds of R^n and of R^m respectively and let φ be a smooth map of V into W. Then $\bar{\varphi}\colon H_1(I, V) \to H_1(I, W)$ defined by $\bar{\varphi}(\sigma) = \varphi \circ \sigma$ is a smooth map of $H_1(I, V)$ into $H_1(I, W)$. Moreover $d\bar{\varphi}_\sigma\colon H_1(I, V)_\sigma \to H_1(I, W)_{\bar{\varphi}(\sigma)}$ is given by $d\bar{\varphi}_\sigma(\lambda)(t) = d\varphi_{\sigma(t)}(\lambda(t))$.*

Proof. Extend φ to a smooth map of R^n into R^m. Then Theorem

2.2 follows from Theorems 1.4 and 2.1.

Let (M, g) be a finite dimensional riemannian manifold and let j be an imbedding of M as a closed submanifold of a Euclidean space R^n. Then from Theorem 2.2 the differential structure induced on $H_1(I, M)$ from the manifold $H_1(I, j(M))$ is independent of j. Moreover for a closed submanifold K of $M \times M$, the differential structures induced on $\Omega_K(M)$ from $\Omega_{j\times j(K)}(j(M))$ is independent of j. Let $\bar{d}$ denote the distance function on $j(M)$. For each $\sigma \in H_1(I, j(M))$ and a positive number ε we define

$$\{\bar{c} \in H_1(I, j(M)) \mid \bar{d}(\bar{c}(t), \bar{\sigma}(t)) < \varepsilon \text{ for any } t \in I\}$$
$$(\text{resp. } \{X \in H_1(I, j(M))_{\bar{\sigma}} \mid |||X||| < \varepsilon \text{ for any } t \in I\})$$

by $B_\varepsilon^\infty(\bar{\sigma})$ (resp. $\tilde{B}_\varepsilon^\infty(\mathbf{0}_{\bar{\sigma}})$). By exp and $\overline{\exp}$ we define the exponential maps of M and $j(M)$ with the induced riemannian metric from R^n respectively.

Proposition 2.3. *For any $\bar{\sigma} \in H_1(I, j(M))$ $(B_\varepsilon^\infty(\bar{\sigma}), \overline{\exp}_{\bar{\sigma}}^{-1})$ is a chart for a sufficiently small positive ε. Particularly $(B_\varepsilon^\infty(\sigma), \exp_\sigma^{-1})$ is a chart for $H_1(I, M)$.*

Proof. It follows from Corollary 1.2 that $B_\varepsilon^\infty(\sigma)$ is open in $H_1(I, j(M))$. Let U_t be an open neighborhood of $\sigma(t)$ such that its closure is contained in a normal convex neighborhood. Since $\sigma(I)$ is compact, there exist a finite covering $U_{t_1}, \cdots, U_{t_k}$ of $\sigma(I)$. From the construction of U_t, there exists a positive ε such that if $d(p, q) < \varepsilon$ and if $p \in U_{t_i}$ for some i, $1 \leqslant i \leqslant k$, then p and q are in a normal convex neighborhood. For such an ε, $\exp_\sigma^{-1}$ maps $B_\varepsilon^\infty(\sigma)$ homeomorphically, thus diffeomorphically onto $\tilde{B}_\varepsilon^\infty(0_\sigma)$. If we take j as an isometric imbedding, then $\exp_\tau^{-1} \circ \exp_\sigma$ is smooth. Note that the map $\overline{dj}$ of $H_1(I, M)_\sigma$ onto $H_1(I, j(M))_{j\sigma}$ is a continuous linear isomorphism, i.e. smooth.

A canonical riemannian structure $\langle\ ,\ \rangle_1$ of $H_1(I, M)$ is defined by means of the riemannian metric tensor g of M [11]. If we identify the tangent space to $H_1(I, M)$ at σ with $H_1(I, M)_\sigma$, then $\langle\ ,\ \rangle_1$ is defined by

$$\langle X, Y\rangle_1 = \langle X, Y\rangle_0 + \langle X', Y'\rangle_0$$

where X', Y' denote the covariant derivatives along σ and where $\langle X, Y\rangle_0 = \int_0^1 g(X(t), Y(t))dt$. From now on we will define $\Omega_N(M)$ with the riemannian structure $\langle\ ,\ \rangle_1$ by $\Lambda_N M$. $\|\cdot\|_1$ (resp. $\|\cdot\|$) will denote the norm induced from the inner product $\langle\ ,\ \rangle_1$ (resp. g). We define d_∞ by

$$d_\infty(\sigma, \tau) = \max d(\sigma(t), \tau(t)), \qquad \sigma, \tau \in H_1(I, M).$$

Lemma 2.4. *Let σ, τ be elements of $\Lambda_{M\times M}(M)$. Then*

$$d_\infty(\sigma, \tau) \leqslant \sqrt{2}\, d_1(\sigma, \tau)$$

where d_1 denotes the distance function on $\Lambda_{M\times M}(M)$.

Proof. Let φ be any piecewise C^1 curve connecting σ and τ in $\Lambda_{M\times M}(M)$. Then we can consider φ as a map $\tilde{\varphi}$ of $I\times I$ into M by $\tilde{\varphi}(t, s) = \varphi(s)(t)$. Choose $t_0 \in I$ such that $d_\infty(\sigma, \tau) = d(\sigma(t_0), \tau(t_0))$. Then

$$\begin{aligned} d_\infty^2(\sigma, \tau) &= d^2(\sigma(t_0), \tau(t_0)) \leqslant \left(\int_0^1 \left\| \frac{\partial \tilde{\varphi}}{\partial s}(t_0, s) \right\| ds\right)^2 \\ &\leqslant \left(\int_0^1 \max_t \left\| \frac{\partial \tilde{\varphi}}{\partial s}(t, s) \right\| ds\right)^2 \leqslant 2\left(\int_0^1 \left\| \frac{\partial \varphi}{\partial s}(s) \right\|_1 ds\right)^2 \\ &= 2L(\varphi)^2 \end{aligned}$$

for any piecewise smooth curve φ. Note that it follows from Lemma 1.1 that

$$\max_t \left\| \frac{\partial \tilde{\varphi}}{\partial s}(t, s) \right\| \leqslant \sqrt{2} \left\| \frac{\partial \varphi}{\partial s}(s) \right\|_1$$

for each fixed s. The above inequalities imply $d_\infty(\sigma, \tau) \leqslant \sqrt{2}\, d_1(\sigma, \tau)$.

Theorem 2.5. *If M is a complete riemannian manifold and N is a closed submanifold of $M\times M$, then $\Lambda_N(M)$ is a complete Hilbert riemannian manifold.*

Proof. It is sufficient to prove the case when $N = M\times M$, since $\Lambda_N(M)$ is a closed submanifold of $\Lambda_{M\times M}(M)$. Let $\{c_n\}$ be a Cauchy sequence in $\Lambda_{M\times M}(M)$. From Lemma 2.4 $\{c_n\}$ is a Cauchy sequence in $(C^0(I, M), d_\infty)$. Thus $\{c_n\}$ converges uniformly to $c_\infty \in C^0(I, M)$. Since $\Lambda_N(M)$ is dense in $C^0(I, M)$, we can assume that for sufficiently large m, c_m is contained in a fixed natural chart $(B_\varepsilon^\infty(c), \exp_c^{-1})$. Put $X_m = \exp_c^{-1}(c_m)$ and let $P_1(t), \cdots, P_k(t)$ be an orthogonal parallel basis along c. Then we get $X_m(t) = \sum_i X_m^i(t) P_i(t)$ and $x_m = (X_m^1, \cdots, X_m^k) \in H_1(I, R^k)$. The sequence $\{x_m\}$ is a Cauchy sequence in $(H_1(I, R^k), |||\cdot|||)$, since

$$\langle\sigma, \sigma\rangle_1/3 \leqslant \langle\!\langle\sigma, \sigma\rangle\!\rangle \leqslant 3\langle\sigma, \sigma\rangle_1.$$

Thus x_m converges to $(f^1, \cdots, f^k)$ and X_m converges to $\sum_i f^i P_i$.

Let E be the energy function on $\Lambda_N(M)$, i.e.

$$E(\sigma) = 1/2 \int_0^1 g(\dot{\sigma}, \dot{\sigma})\, dt.$$

To check that E is smooth, embed M isometrically into a Euclidean space R^n [34]. Let J be the bounded symmetric bilinear form on $H_1(I, R^n)$ defined by

$$J(c)=1/2\int_0^1 \|\dot{c}(t)\|^2 dt.$$

Clearly J is smooth, because

$$dJ_c(X)=\int_0^1 \langle \dot{c}(t), \dot{X}(t)\rangle dt,$$

$$d^2J_c(X, Y)=\int_0^1 \langle \dot{X}(t), \dot{Y}(t)\rangle dt, \quad d^kJ_c=0\ (k\geqq 3).$$

If we denote the isometric embedding by j, $E=J\circ\bar{j}$. Thus E is smooth.

Proposition 2.6. *For each* $c\in \Lambda_N(M)$,

$$dE_c(X)=\int_0^1 g(\dot{c}(t), X'(t))dt, \quad X\in \Lambda_N(M)_c.$$

Proof. Let φ be a smooth curve in $\Lambda_N(M)$ with $\varphi(0)=c$, $\dot{\varphi}(0)=X$. We have

$$\begin{aligned} dE_c(X)&=\left(\frac{d}{ds}\right)_{s=0} E(\varphi(s)) \\ &=1/2\frac{d}{ds}\left(\int_0^1 g\left(\frac{\partial\tilde{\varphi}}{\partial t}(t, s), \frac{\partial\tilde{\varphi}}{\partial t}(t, s)\right)ds\right)_{s=0} \\ &=\int_0^1 g\left(\frac{\partial\tilde{\varphi}}{\partial t}(t, 0), \frac{\nabla}{\partial s}\frac{\partial\tilde{\varphi}}{\partial t}(t, 0)\right)dt \\ &=\int_0^1 g(\dot{c}(t), X'(t))dt, \end{aligned}$$

where $\tilde{\varphi}(t, s)=\varphi(s)(t)$ and $\nabla/\partial s$ denotes the covariant derivative along the curve $\tilde{\varphi}(t, s)$, $t=$const.

Let Δ be the diagonal of M. By ΛM, we denote the manifold, whose differential structure is induced from that of $\Lambda_\Delta M$, of all maps σ from R into M with $\sigma|[0, 1]\in \Lambda_\Delta M$ and with $\sigma(t)=\sigma(t+1)$.

Theorem 2.7. $c\in \Lambda M$ *is a critical point of the energy function if and only if* $c\in \Lambda M$ *is either a constant map or a periodic* (*or closed*) *geodesic.*

Proof. Only if part is trivial. Suppose that c is a critical point.

Let Z be the parallel vector field in ΛM_c satisfying $Z(1)=A(1)$, where $A(t)$ is the unique solution of the differential equation defined by

$$A'(t)=\dot{c}(t), \quad A(0)=0.$$

Since $\dot{c}$ is square summable, the solution $A(t)$ belongs to $H_1(I, M)_c$. Put $B(t)=A(t)-tZ(t)$. Then $B(0)=0_{c(0)}$, $B(1)=0_{c(1)}$ and $B'(t)=\dot{c}(t)-Z(t)$. Hence $0=dE_c(B)=\langle \dot{c}, B'\rangle_0=\langle \dot{c}, \dot{c}-Z\rangle_0$ and $\langle Z, \dot{c}-Z\rangle_0=\langle z, B'\rangle_0=\int_0^1 d/dt \langle Z, B\rangle dt=0$. It follows from these two equations that $\dot{c}=Z$. This implies that c is either a geodesic or a constant map. Thus for any $X \in \Lambda M_c$,

$$0=dE_c(X)=\langle \dot{c}, X'\rangle_0=\langle \dot{c}(1)-\dot{c}(0), X(0)\rangle.$$

Since $X(0)$ can be taken arbitrary, $\dot{c}(0)=\dot{c}(1)$. Therefore c is periodic.

The following two theorems may be proven by the similar way as above.

Theorem 2.8. *Let K and L denote closed submanifolds of M. Then $c \in \Lambda_{K\times L}(M)$ is a critical point of the energy function if and only if c is a constant map in $K\cap L$ or a geodesic which intersects orthogonally to K at $c(0)$ and to L at $c(1)$.*

Let A be an isometry on M and let $G(A)$ be the graph of A. We shall define $\{\sigma: R\to M \mid \sigma|[0, 1] \in \Lambda_{G(A)}(M),\ A\sigma(t)=\sigma(t+1)\}$ by $\Lambda(M, A)$. $\Lambda(M, A)$ can be introduced the differential structure and the riemannian one by $\Lambda_{G(A)}(M)$. A geodesic c of R into M is called *A-invariant* if $Ac(t)=c(t+1)$ [15], [40]. Note that a closed geodesic is an id_M-invariant geodesic.

Theorem 2.9. *$c \in \Lambda(M, A)$ is a critical point of the energy function if and only if c is a constant map in the fixed point set of A or an A-invariant geodesic.*

If M is non-compact, the energy function on $\Lambda_N M$ does not satisfy Palais-Smale Condition (C) in general. But the energy function in the following case satisfies Condition (C).

Theorem 2.10. *Suppose that M is a complete riemannian manifold and that N is a closed submanifold of $M\times M$. If $p_1(N)$ or $p_2(N)$, where p_1 (resp. p_2) denotes the projection of $M\times M$ to the first (resp. the second) component, is compact, then the energy function on $\Lambda_N M$ satisfies Condition (C).*

The above proof is given in [15]. A short proof in case N is totally geodesic is given in [22].

Let $C^0_N(I, M)$ be the space with the compact-open topology of all continuous maps x of I into M with $(x(0), x(1)) \in N$. Then the following theorem is proved in [11], [15].

Theorem 2.11. *The inclusion map of $\Lambda_N M$ into $C^0_N(I, M)$ has a homotopy inverse. Hence $\Lambda_N M$ and $C^0_N(I, M)$ are the same homotopy type.*

In what follows we will denote the energy function on $\Lambda(M, A)$ by E^A, where A is an isometry on M.

Theorem 2.12. *For any critical point c of E^A, the Hessian of E^A at c is given by*

$$D^2E^A_c(X, Y) = \int_0^1 (g(X', Y') - g(R(X, \dot{c})\dot{c}, Y))dt$$

for $X, Y \in \Lambda(M, A)_c$, where R denotes the curvature tensor of (M, g).

Proof. Since $D^2E^A_c$ is symmetric bilinear form [Proposition 2.3 in Chapter I], it is sufficient to prove the case where $X = Y$. Notice the equation

$$2D^2E^A_c(X, Y) = D^2E^A_c(X+Y, X+Y) - D^2E^A_c(X, X) - D^2E^A_c(Y, Y).$$

Let φ be a smooth curve of $(-\varepsilon, \varepsilon)$ into $\Lambda(M, A)$ with $\varphi(0) = c$, $\dot{\varphi}(0) = X$. Then

$$\begin{aligned} D^2E^A_c(X, X) &= \frac{\partial^2}{\partial \tau^2} E^A(\varphi(\tau))\Big|_{\tau=0} \\ &= \frac{\partial}{\partial \tau} \int_0^1 g\left(\frac{\nabla}{\partial \tau} \frac{\partial \tilde{\varphi}}{\partial t}(t, \tau), \frac{\partial \tilde{\varphi}}{\partial t}(t, \tau)\right)dt\Big|_{\tau=0} \\ &= \int_0^1 (g(X', X') - g(R(X, \dot{c})\dot{c}, X))dt. \end{aligned}$$

For the above detail calculation, see [33].

$\Lambda(M, A)$ has a natural R^1-action, i.e. the translation of parameters of the curves in $\Lambda(M, A)$. Clearly the action preserves the energy and it is an isometry. The following theorem was proved by Grove [16].

Theorem 2.13. *The R^1 -action on $\Lambda(M, A)$ is continuous.*

Theorem 2.14. *E^A is a Fredholm function at each critical point of E^A.*

Proof. Suppose c is a critical point of E^A. Let S denote the operator defined by $\langle SX, Y\rangle_1 = D^2E^A_c(X, Y)$. Then $\langle (S - \mathrm{Id})X, Y\rangle_1 = -\langle X, Y\rangle_0$

$-\langle R(X, \dot{c})\dot{c}, Y\rangle_0$. Thus there exists a positive constant K such that $\|(S-\mathrm{Id})X\|_1 \leqslant K\|X\|_0$ for any $X \in \Lambda(M, A)_c$. Suppose that $\{X_n\}$ is a bounded sequence in $\Lambda(M, A)_c$. It follows from Corollary 1.3 in III that the sequence has a convergent subsequence. Note that we can imbed $\Lambda(M, A)_c$ isometrically as a closed subspace of $H_1(I, R^n)$, $n=\dim M$, by means of orthonormal parallel vector fields along c. Thus $\{(S-\mathrm{Id})X_n\}$ has a Cauchy subsequence, hence a convergent subsequence in $\Lambda(M, A)_c$.

Lemma 2.15. *Let U be an open subset of R^p and let φ be a piecewise smooth map of $U \times I$ into R^n in the following sense. There exist finite numbers $0=t_0<t_1<\cdots<t_k=1$ such that φ is continuous on $U\times I$ and smooth on $U\times[t_i, t_{i+1}]$ for each i, $0\leqslant i\leqslant k-1$. Then the induced map $\tilde{\varphi}$ of U into $H_1(I, R^n)$ defined by $\tilde{\varphi}(x)(t)=\varphi(x, t)$ is smooth. Furthermore the differential map of $\tilde{\varphi}$ at x is given by*

$$d\varphi_x(h)(t)=\sum_i h_i \frac{\partial \varphi}{\partial x_i}(x, t).$$

Proof. Since $\partial\varphi/\partial x_i$ is piecewise smooth for each i, the map $\widetilde{\partial\varphi/\partial x_i}$ from U into $H_1(I, R^n)$ is defined. For a fixed $x \in U$, let T denote the bounded linear map of R^p into $H_1(I, R^n)$, $T(h)=\sum_i h_i(\widetilde{\partial\varphi/\partial x_i})(x)$. Since $\|\varphi(x+h)-\varphi(x)-T(h)\|_1/\|h\|$ tends to zero at $\|h\|$ goes to zero, $\tilde{\varphi}$ is differentiable at x and its derivative $d\tilde{\varphi}_x$ is T. From the following inequality

$$\|d\tilde{\varphi}_x - d\tilde{\varphi}_y\|^2 \leqslant \max_{1\leqslant i\leqslant p}\left\{\int_0^1 \|\varphi_i(x, t)-\varphi_i(y, t)\|^2 dt + \int_0^1 \|\varphi_{it}(x, t)-\varphi_{it}(y, t)\|^2 dt\right\},$$

where $\varphi_i=\partial\varphi/\partial x_i$, $\varphi_{it}=\partial^2\varphi/\partial x_i\partial t$, $d\tilde{\varphi}_x$ is continuous and hence $\tilde{\varphi}$ is C^1. Suppose that $\tilde{\varphi}$ is $C^{k-1}(k>1)$ for any piecewise smooth map φ. Since $d\tilde{\varphi}_x(h)=\sum_i h_i\tilde{\varphi}_i(x)$, $d\tilde{\varphi}_x$ is C^{k-1} and hence $\tilde{\varphi}$ is C^k. This implies $\tilde{\varphi}$ is a smooth map of U into $H_1(I, R^n)$.

As in [33] or in [4], ΛM can be approximated by finite dimensional manifolds. $\Omega(t_0, \cdots, t_k)$ denotes the set of all curves $c \in \Lambda M$ such that $c|[t_i, t_{i+1}]$ is a minimizing geodesic for each $0\leqslant i\leqslant k-1$. If we choose a sufficiently fine subdivision of $[0, 1]$ for each real number α, $\Omega^\alpha(t_0, \cdots, t_k) = E^{-1}[0, \alpha)\cap\Omega(t_0, \cdots, t_k)$ can be realized as an open submanifold of M^k. We obtain the following theorem from the above lemma.

Theorem 2.16. *The inclusion map of $\Omega^\alpha(t_0, \cdots, t_k)$ into ΛM is an embedding map.*

Remark. V. Bangert kindly gave the author another proof of the above theorem.

§ 3. Applications

Theorem 3.1. *If M is a compact, connected and non-simply connected riemannian manifold, then M has a closed geodesic.*

Proof. Since the connected component of $C^0_\Delta(I, M)$ corresponds to the conjugate classes of $\pi_1(M)$, the fundamental group of M, $C^0_\Delta(I, M)$ is not connected. Thus it follows from Theorem 2.11 that ΛM is not connected. From Theorem 2.10, the energy function $E=E^{\mathrm{id}_M}$ satisfies Condition (C). Hence it follows from Theorem 3.2 in II that E assumes a positive minimum on a connected component Λ_0 of ΛM which has no common points with the set of all constant curves. The point which gives the minimum of $E|\Lambda_0$ is a positive E-valued critical point, i.e. a closed geodesic.

Any 1-connected compact riemannian manifolds also have closed geodesics. The proof of this case is more difficult. The following lemma is useful to prove it [15].

Lemma 3.2. *Suppose that A is an isometry on a compact riemannian manifold with fixed points. Then* $\mathrm{Fix}(A)=(E^A)^{-1}(0)$ *is a union of non-degenerate critical submanifolds of $\Lambda(M, A)$ and there exists a positive ε such that* $\mathrm{Fix}(A)$ *is a strong deformation retract of* $\Lambda(M, A)^\varepsilon=(E^A)^{-1}[0, \varepsilon]$.

Theorem 3.3. *Any* 1*-connected and compact riemannian manifolds have closed geodesics.*

The above theorem is generalized by means of isometry-invariant geodesics [15]:

Theorem 3.4. *If an isometry A on a* 1*-connected compact riemannian manifold M is homotopic to* id_M, *then there exists an A-invariant geodesic on M.*

Chapter IV. The Gromoll-Meyer Theorem on Closed Geodesics and Some Theorems Related to their Theorem

§ 1. An outline of the proof of the Gromoll-Meyer theorem and some related theorems about closed geodesics

Among various problems about closed geodesics, one of the most prominent questions is whether or not there exist infinitely many closed geodesics on an arbitrary compact riemannian manifold. The main interest lies in the question of whether it is possible to estimate the number of closed geodesics in terms of topological properties of the manifold only. In 1969 Gromoll and Meyer succeeded to find such a criterion [13]. They obtained the following result:

Theorem 1.1. *Let M be a 1-connected and compact riemannian manifold. Then M has infinitely many closed geodesics if the Betti numbers for the space $C^0(S^1, M)$ $(=C^0_\Delta(I, M))$ are unbounded.*

Let us note that any symmetric space of rank one does not satisfy the above topological condition about the Betti numbers. In 1977, Sullivan and Vigué-Poirrier found the necessary and sufficient for ΛM to have a unbounded sequence of rational Betti numbers [39].

Theorem 1.2. *Let M be a compact and 1-connected manifold. The sequence of the rational Betti numbers for $C^0(S^1, M)$ is unbounded if and only if the number of generators for rational cohomology of M is not less than two.*

As a corollary we get,

Corollary 1.3. *If a compact riemannian manifold M has the same homotopy type as the product manifold of two 1-connected compact manifolds, then M has infinitely many closed geodesics.*

In 1977, Ziller calculated the Betti numbers for the free loop spaces of all compact symmetric spaces and he obtained [44]:

Theorem 1.4. *Let M be a compact and 1-connected riemannian manifold which has the same homotopy type as a compact symmetric space of non rank one. Then the sequence of the Betti numbers for $C^0(S^1, M)$ is unbounded. Hence M has infinitely many closed geodesics.*

The R^1-action on ΛM induces S^1-action, because $c(t+1)=c(t)$ for any $c \in \Lambda M$. Here $S^1=[0, 1]/\{0, 1\}$. If $S^1 \cdot c$ is a non-degenerate critical submanifold (cf. p. 12) for any non-constant critical point $c \in \Lambda M$, then the proof of Theorem 1.1 is much simpler. When $S^1 \cdot c$ is non-degenerate, the closed geodesic c will be called non-degenerate. In such a case Gromov proved the growth estimate for $N(t)$, the number of geometrical different closed geodesics of length $\leqslant t$ [14]. This result was improved by Ballmann and Ziller [5]. The following theorem gives another proof of Theorem 1.1 in the case where all closed geodesics are non-degenerate.

Theorem 1.5. *Let M be a compact and 1-connected riemannian mani-*

fold all of whose closed geodesics are non-degenerate. Then there exist positive constants α, β depending on the riemannian metric such that

$$N(t) \geqq \alpha \max_{1 \leqslant k \leqslant \beta t} \dim H_k(\Lambda M, R)$$

for any principal ideal domain R and all t sufficiently large.

In Finsler metric case the assumption on the Betti numbers in Theorem 1.1 can not be removed; Katok found an example of a 2-sphere with Finsler metric which has only two closed geodesics [23], [45].

A Katok example. Let S^2 be the unit 2-sphere in R^3 with center at the origin. Define φ_t by the rotation of angle t around z-axis. Then φ_t is a 1-parameter of isometries such that $\varphi_t = \varphi_{t+2\pi}$. Let V be the vector field generated by φ_t. Define H_0, H_1: $T^*S^2 \to R$ by $H_0(x) = \|x\|_*$ and $H_1(x) = x(V)$ where $\|\cdot\|_*$ is the dual norm of the riemannian metric of S^2. Let $H_\alpha = H_0 + \alpha H_1$ for a real α. Then $N_\alpha = H_\alpha L_\alpha^{-1}$ defines a Finsler metric on S^2, where L_α is the Legendre transformation of $1/2\ H_\alpha^2$.

Theorem 1.6. *If α is irrational and sufficiently small then the closed geodesics of N_α are $\gamma_+(t) = \gamma((1+\alpha)^2 t)$ and $\gamma_-(t) = \gamma(-(1-\alpha)^2 t)$. The periods of γ_+ and γ_- are $2\pi/(1+\alpha)^2$, $2\pi/(1-\alpha)^2$ respectively and their lengths are $2\pi/1+\alpha$, $2\pi/1-\alpha$ respectively. Here* $\gamma(t) = (\cos t, \sin t, 0)$.

Remark. The geodesics of N_α are given by $\varphi_{\alpha t} \circ \gamma_x(t/\|x\|_*)$, where γ_x is the great circle, i.e. the geodesic of the unit 2-sphere, with $\dot{\gamma}_x(0) =$ the dual tangent vector of x.

In 1980 Matthias extended Theorem 1.1 by means of Finsler manifolds [31].

Theorem 1.7. *Let M be a compact and* 1*-connected Finsler manifold. Then there exist infinitely many closed geodesics if the sequence of the Betti numbers for $C^0(S^1, M)$ is unbounded.*

Morse theory for the energy function E on ΛM is used to prove Theorem 1.1. Non-constant critical points of E correspond to closed geodesics on M, but the correspondence is not injective. Closed geodesics on M correspond to towers of critical orbits injectively. This makes it difficult to estimate the number of closed geodesics. Formulas for indexes and nullities of iterated closed geodesics are crucial to overcome the difficulty. Let c be a non-constant closed geodesic and c^m be the m times iterated closed geodesic of c, i.e. $c^m(t) = c(mt)$ for $t \in R$. Then the following formulas of indexes $\lambda(c^m)$ and nullities $\nu(c^m)$ of $S^1 \cdot c^m$ were found by Bott [7].

Theorem 1.8. *There exist nonnegative integer valued functions* $N(z)$, $\Lambda(z)$ on $\{z \in C \mid \|z\|=1\}$ *such that*

$$\lambda(c^m)=\sum_{z^m=1}\Lambda(z), \qquad \nu(c^m)=\sum_{z^m=1}N(z)$$

for any positive integers m.

From the above formulas we obtain

Lemma 1.9. *If* $\lambda(c^m)\neq 0$ *for some positive* m, *then there exist positive* ε, a *such that*

$$\lambda(c^{m_1})-\lambda(c^{m_2})\geqq(m_1-m_2)\varepsilon-a$$

for any positive integers $m_1\geqq m_2>0$.

In [12] Gromoll and Meyer defined a characteristic invariant for an isolated critical point, $\mathscr{H}^0(c)$, which has a very useful property.

Lemma 1.10. *Suppose that* $\nu(c^m)=\nu(c)$ *for some positive integer* m *and that* $S^1\cdot c^m$ *is an isolated critical orbit. Then* $\mathscr{H}^0(c)$ *is isomorphic to* $\mathscr{H}^0(c^m)$.

From Lemma 1.8 we can guarantee the assumption of the nullities in the above lemma:

Lemma 1.11. *There exist positive integers* $k_1, \cdots, k_s$ *and sequences* $\{m_j^i\}_{i>0}$, $j=1, \cdots, s$ *of positive integers such that any positive integer* m *has a unique decomposition* $m=m_j^ik_j$ *and*

$$\nu(c^m)=\nu(c^{k_j}).$$

Combining Lemmas 1.9, 1.10, 1.11 and Morse inequalities, we can prove the boundedness of dim $H_k(\Lambda M)$ for any $k\geqq 2$ dim M if M has only finitely many closed geodesics.

§ 2. Isometry-invariant geodesics

In 1976 Grove and Tanaka proved the following theorem [18].

Theorem 2.1. *Let* M *be a compact,* 1-*connected riemannian manifold and let* f: $M\to M$ *be an isometry of finite order. Then there are infinitely many* f-*invariant geodesics on* M *if the sequence of the Betti numbers for the space* $\Lambda(M, f)$ *is unbounded.*

Note. Any f-invariant geodesics are closed geodesics. The case $f=\mathrm{id}_M$ in the above theorem is the Gromoll-Meyer theorem.

In 1981, Tanaka extended Theorem 2.1 [42] which is optimal.

Theorem 2.2. *Let M be a compact, 1-connected riemannian manifold and let $A: M \to M$ be an arbitrary isometry. Then there are infinitely many A-invariant geodesics on M if the sequence of the Betti numbers for the space $\Lambda(M, A)$ is unbounded.*

In [19], a necessary and sufficient condition on A and on M for $\Lambda(M, A)$ to have an unbounded sequence of rational Betti numbers was obtained by Grove, Halperin and Vigué-Poirrier. Recently Grove and Halperin found a sufficient condition on M for $\Lambda(M, A)$ to have an unbounded sequence of rational Betti numbers for any isometry A [20].

Theorem 2.3. *Let M be a 1-connected and compact riemannian manifold. Then the sequence of rational Betti numbers for $\Lambda(M, A)$ is unbounded for any isometry A if M is rationally hyperbolic. Thus there are infinitely many A-invariant geodesics for any isometry A on M.*

Note. M is called rationally hyperbolic if the integers $\hat{\rho}_p = \sum_{q \leqslant p} \dim \pi_q(M) \otimes \boldsymbol{Q}$ grow exponentially in p.

Let us sketch out a proof of Theorem 2.2. Suppose that A has only finitely many invariant geodesics. Then all A-invariant geodesics are closed. This fact was proved by Grove [16]. Let c be a nonconstant critical point for E^A. Let $\alpha(\geqq 1)$ denote the least period of c. Then for any nonnegative integers m, $c^{m\alpha+1}$ also is a critical point for E^A, and the critical point lies in a critical orbit, orb $(c^{m\alpha+1}) = \{T_u(c^{m\alpha+1}) \mid u \in R\}$. Here $T_u(c)(t) = c(t+u)$. Let $\lambda(c^{m\alpha+1})$ and $\nu(c^{m\alpha+1})$ denote the index and the nullity of orb $(c^{m\alpha+1})$ respectively. Then we obtain the following formulas like Theorem 1.8.

Proposition 2.4. *There exist finitely many nonnegative integer valued functions $\Lambda^i(\rho)$, $N^i(\rho)$ defined on the unit circle $\{\rho \in \boldsymbol{C} \mid |\rho| = 1\}$ and finitely many complex numbers z_i of absolute value 1, $i = 1, \cdots, p$ such that*

$$\lambda(c^{m\alpha+1}) = \sum_{i=1}^{p} \sum_{\rho^m = z_i} \Lambda^i(\rho)$$

$$\nu(c^{m\alpha+1}) = \sum_{i=1}^{p} \sum_{\rho^m = z_i} N^i(\rho).$$

From Proposition 2.4 we have

Lemma 2.5. *If $\lambda(c^{m\alpha+1}) \neq 0$ for a nonnegative integer m, then there exist positive numbers ε, a such that*

$$\lambda(c^{m_1\alpha+1})-\lambda(c^{m_2\alpha+1})\geqq(m_1-m_2)\varepsilon-a$$

for any integers $m_1\geqq m_2\geqq 0$.

In case where A is of finite order, i.e. $A^k=\mathrm{id}_M$ for some positive integer k, the iteration maps can be defined of $\Lambda(M, A)$ into $\Lambda(M, A^m)$. There are only finitely many path-spaces $\Lambda(M, A^m)$, $m\in \boldsymbol{Z}$. In this case we can compare the characteristic invariants of $c^{m\alpha+1}$ by the iteration maps like in the case of the Gromoll-Meyer theorem. Hence the proof of our theorem in the case is similar to that of Gromoll-Meyer theorem. However we do not have any suitable iteration maps any more in an arbitrary isometry case. The action of A on an invariant closed geodesic is very complicated. If the least period of c is rational (resp. irrational) then we say A acts rationally (resp. irrationally) on c. $\{c^{m\alpha+1}\,|\,m\in \boldsymbol{Z}_+\}$ will be called the rational tower (resp. irrational tower) of c if A acts rationally (resp. irrationally) on c. These terminologies were suggested by Grove. The characteristic invariants of a rational tower are reduced to those restricted to a totally geodesic submanifold Fix (A^k). Hence the following lemma is proven by reducing to the case of a finite order isometry.

Lemma 2.6. *Let* $c^{m\alpha+1}$ *be a rational tower. Then there are only finitely many characteristic invariants among all characteristic invariants of the tower if* orb $(c^{m\alpha+1})$, $m\in \boldsymbol{Z}_+$ *are isolated critical orbits.*

In an irrational tower case we use a complete different method to prove that there are only finitely many characteristic invariants among all of them of an irrational tower. Roughly speaking the proof is done by approximations of E^A by other energy functions in a topological sense. The nullity formulas in Proposition 2.4 tell us which energy functions are suitable approximations of E^A. The following property of Jacobi fields are crucial in the approximations.

Lemma 2.7. *There exists a constant* m_0 *such that any Jacobi fields* Y *along* c *with* $A_*Y(t)=Y(t+m\alpha+1)$ *are periodic if* m *is a greater integer than* m_0.

Lemma 2.8. *Let* $c^{m\alpha+1}$, $m\in \boldsymbol{Z}_+$ *be an irrational tower. Then there are only finitely many characteristic invariants of the tower if* orb $(c^{m\alpha+1})$ *are isolated critical orbits.*

Combining Lemmas 2.5, 2.6, 2.7, 2.8 and Morse inequalities we can conclude the boundedness of dim $H_k(\Lambda(M, A))$ if A has only finitely many invariant geodesics.

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Tokai University
Hiratsuka, Kanagawa
259–12 Japan