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Higher Residues Associated with an Isolated Hypersurface Singularity

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§ 0. Introduction

The aim of this note is, as a preliminary to Prof. Saito's article in this volume, to give a brief introduction to the theory of primitive integrals associated with isolated hypersurface singularities which he is now developing extensively. The whole theory, though still incomplete, can be found in [8] and a summary in [7].

The central idea of this theory is an intimate connection between geometry and transcendental functions with the theory of elliptic integrals as a motivated example. In his case the geometric object is a (so far) local isolated hypersurface singularity, with which he associates a specific differential form called the primitive form. This primitive form enables us to study the singularity with analytic methods. Unfortunately, the existence of the primitive form has not yet been established in general except for a small but significant number of cases from explicit computations ([6]). The integrals of this form would give a new type of transcendental functions generalizing elliptic integrals of the first kind.

In order to state the fundamental properties of primitive forms which relate to geometric properties of singularity, we need to introduce what we call higher residue pairings (cf. [7]), whose definition is the goal of this note. Among many notions introduced by Saito with higher residues the exponent is one of the most important and is discussed in his article [9].

§ 1. A Hamiltonian system of an isolated hypersurface singularity

In this section we introduce what we call a Hamiltonian system which is the object of our study. In fact we treat a special kind of a Hamiltonian system: one which is associated with a universal unfolding of an isolated hypersurface singularity.

Definition (1.1). A *Hamiltonian system* $(X \xrightarrow{\phi} S \xrightarrow{\pi} T, \delta_1)$ is a collection of the following data: i) X, S, T are the germs of manifolds of

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respective dimensions $m+n$, m, $m-1$; ii) $\phi: X\to S$, $\pi: S\to T$ are the germs of holomorphic maps; iii) δ_1 is a holomorphic vector field on S. They satisfy the following conditions: a) $\pi: S\to T$ is a submersion with $\pi^{-1}(\mathcal{O}_T)=\{g \in \mathcal{O}_S: \delta_1 g=0\}$ where $\mathcal{O}_S$ and $\mathcal{O}_T$ denote the sheaves of the germs of holomorphic functions on S and T respectively (hence there is a local coordinate system $(t_1, \cdots, t_m)=(t_1, t')$ (resp. $(t_2, \cdots, t_m)=(t')$) of S (resp. T) such that π is the projection $(t_1, t')\to(t')$ and $\delta_1=\partial/\partial t_1$); *b*) $q=\pi\circ\phi$ is smooth (hence there is a local coordinate system $(x_0, \cdots, x_n, t_2, \cdots, t_m)=(x, t')$ of X such that $\phi(x, t')=(F_1(x, t'), t_2, \cdots, t_m)$).

Notation (1.2). Taking the fibre product we have the following commutative diagram:

$$\begin{array}{ccc} Z=X\times_T S & \xrightarrow{\tilde{\pi}} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow{\pi} & T. \end{array}$$

Then Z has a canonical system of local coordinates (x, t) such that $p: (x, t)\to(t)$, $\tilde{\pi}: (x, t)\to(x, t')$ and the graph of $\phi: X\to S$ in Z is defined by $F=t_1-F_1(x, t')=0$.

(1.3). Let $\mathcal{D}er_S$ denote the sheaf of germs of holomorphic vector fields on S. We define

$$\mathcal{G}=\{\delta \in \pi_* \mathcal{D}er_S: [\delta_1, \delta]=0\},$$

which is an $\mathcal{O}_T$-free Module of rank m. Moreover we obtain an exact sequence of $\mathcal{O}_T$-Lie algebras,

$$0\longrightarrow\mathcal{O}_T\delta_1\longrightarrow\mathcal{G}\longrightarrow\mathcal{D}er_T\longrightarrow 0.$$

Assumption (1.4). The function $f(x)=F_1(x, 0)$ has an isolated critical point at $x=0$.

In this note we consider only Hamiltonian systems satisfying the above assumption. Then ϕ is an unfolding of f. The dimension of $\mathcal{O}_{q^{-1}(0),0}/\mathcal{J}_0$, $\mathcal{J}_0=(\partial f/\partial x_0, \cdots, \partial f/\partial x_n)$, as a $\boldsymbol{C}$-vector space is the Milnor number of f.

Definition (1.5). The map ϕ is a *universal unfolding* if the map

$$\begin{array}{ccc} T_{S,0} & \longrightarrow & \mathcal{O}_{q^{-1}(0),0}/\mathcal{J}_0 \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \delta & \longrightarrow & (\tilde{\delta}F)|_{t'=0} \end{array}$$

is bijective where $T_{S,0}$ is the tangent space of S at 0 and $\tilde{\delta}$ is an arbitrary lifting of δ to Z by p.

All the unfoldings can be constructed canonically (and locally) from the universal unfolding.

Therefore, in this note we always *assume* for simplicity that ϕ is universal. In particular, we have $m=\mu$.

Example (1.6). $f=x^n$.

$$F=t_1-(x^n+t_2x+t_3x^2+\cdots+t_{n-1}x^{n-2}).$$

Definition (1.7). i) Let C be the subvariety of X defined by the ideal $\mathscr{I}=(\partial F_1/\partial x_0, \cdots, \partial F_1/\partial x_n)$. It is the critical set of the map ϕ and is a complete intersection under the assumption (1.4).

ii) Put $D=\phi(C)\subset S$ which is called the *discriminant* of ϕ.

Remark (1.8). i) Moreover, the critical set C is smooth when ϕ is universal, and then $q|_C$: $C\to T$ is a branched covering of degree μ and $q_*\mathcal{O}_C$ is a free $\mathcal{O}_T$-Module of rank μ.

ii) Let $A(t')\in \mathrm{End}_{\mathcal{O}_T}(q_*\mathcal{O}_C)$ be the multiplication by t_1 in $q_*\mathcal{O}_C$, and put $\Delta=\det(t_1I-A(t'))$. Then D is nothing but the divisor defined by $\Delta=0$.

Example (1.9). In the example (1.6) Δ is the usual discriminant of F up to scalar multiplication when we consider F as a polynomial in x.

§ 2. Gauss-Manin connection

(2.1) In this section we consider X, S, T in (1.1) as local manifolds defined by

$$X=\{(x, t'); \|x\|<\varepsilon, \|t'\|<\delta', |F_1|<\delta\},$$
$$S=\{(t); |t_1|<\delta, \|t'\|<\delta'\},$$
$$T=\{(t'); \|t'\|<\delta'\}$$

with $1\gg\varepsilon\gg\delta\gg\delta'>0$.

Fact (2.2) (Milnor [4]). The map ϕ: $X-\phi^{-1}(D)\to S-D$ is locally trivial as a differentiable map whose general fibre $X_t=\phi^{-1}(t)$ has a homotopy type of a bouquet of μ times S^n.

Hence

$$H=\bigcup_{t\in S-D} H^n(X_t, Z)\longrightarrow S-D$$

forms a local system on $S-D$ determined by the monodromy representation

$$M\colon \pi_1(S-D, t_0) \longrightarrow \mathrm{Aut}\,(H^n(X_{t_0}, \boldsymbol{Z}))$$

and

$$\mathscr{H} = H \otimes_{\boldsymbol{Z}} \mathcal{O}_{S-D}$$

is a locally free sheaf of rank μ.

Fact (2.3) (Griffiths, also see Brieskorn [1], Katz-Oda [2]).

The sheaf $\mathscr{H}$ admits a holomorphic connection ∇ called Gauss-Manin connection whose horizontal sections are precisely $H \otimes_{\boldsymbol{Z}} \boldsymbol{C}$. Here a connection is a map

$$\begin{array}{ccc} \nabla\colon \mathscr{D}er_{S-D} \times \mathscr{H} & \longrightarrow & \mathscr{H} \\ \cup\!\!\!\!\!\!\;\;\;\;\;\; & & \cup\!\!\!\!\!\! \\ (\delta, \omega) & & \nabla_\delta \omega \end{array}$$

such that for $f \in \mathcal{O}_{S-D}$, we have

$$\nabla_\delta(f\omega) = \delta(f)\omega + f\nabla_\delta \omega.$$

To give a connection on $\mathscr{H}$ is equivalent to give a $\mathscr{D}_{S-D}$-Module structure on $\mathscr{H}$ where $\mathscr{D}_{S-D}$ is the ring of germs of holomorphic differential operators on $S-D$.

Now we can state our main problem explicitly.

Problem (2.4). Extend $\mathscr{H}$ to the whole of S and study the behaviour of ∇ near D and its relation to the geometry of ϕ.

We can give several answers to this problem, but they are in fact related to each other.

Definition (2.5).

$$\begin{aligned} \mathscr{H}^{(0)} &:= \phi_* \Omega^{n+1}_{X/T} / dF_1 \wedge d\phi_* \Omega^{n-1}_{X/T}, \\ \mathscr{H}^{(-1)} &:= \phi_* \Omega^{n}_{X/T} / dF_1 \wedge \phi_* \Omega^{n-1}_{X/T} + d\phi_* \Omega^{n-1}_{X/T}, \\ \mathscr{H}^{(-2)} &:= \boldsymbol{R}^n \phi_* (\Omega^{\cdot}_{X/S}) \\ &= \frac{\mathrm{Ker}\left(\phi_* \Omega^n_{X/S} \xrightarrow{\;d\;} \phi_* \Omega^{n+1}_{X/S}\right)}{\mathrm{Im}\left(\phi_* \Omega^{n-1}_{X/S} \xrightarrow{\;d\;} \phi_* \Omega^n_{X/S}\right)}. \end{aligned}$$

where

$$\Omega^p_{X/T}=\Omega^p_X/q^*(\Omega^1_T)\wedge\Omega^{p-1}_X,$$
$$\Omega^p_{X/S}=\Omega^p_X/\phi^*(\Omega^1_S)\wedge\Omega^{p-1}_X$$
$$\cong\Omega^p_{X/T}/dF_1\wedge\Omega^{p-1}_{X/T}$$

are the Kähler differential forms and the former is locally free for $0\leq p\leq n+1$.

We denote $\Omega^{n+1}_{X/S}$ by Ω_F. Note that the support of Ω_F is C. Then these three sheaves are related as follows.

Proposition (2.6). *We have two exact sequences:*

i) $0\to\mathscr{H}^{(-1)}\xrightarrow{\wedge dF_1}\mathscr{H}^{(0)}\xrightarrow{r}\phi_*\Omega_F\to 0,$

ii) $0\to\mathscr{H}^{(-2)}\xrightarrow{i}\mathscr{H}^{(-1)}\xrightarrow{d}\phi_*\Omega_F\to 0,$

where r and i are the canonical morphisms coming from the definition.

Proof. i) Only the injectively of $\wedge dF_1$: $\mathscr{H}^{(-1)}\to\mathscr{H}^{(0)}$ is non-trivial. If $\omega\in\phi_*\Omega^n_{X/T}$ satisfies $dF_1\wedge\omega=dF_1\wedge d\eta$ for $\eta\in\phi_*\Omega^{n-1}_{X/T}$, i.e. $dF_1\wedge(\omega-d\eta)=0$, then there is $\xi\in\phi_*\Omega^{n-1}_{X/T}$ with $\omega-d\eta=dF_1\wedge\xi$ by the division lemma ((3.5) below).

ii) Note that $\mathscr{H}^{(-1)}\cong\phi_*\Omega^n_{X/S}/d\phi_*\Omega^{n-1}_{X/S}$. Then surjectivity of d: $\mathscr{H}^{(-1)}\to\phi_*\Omega_F$ follows from the De Rham lemma for $\Omega^{\cdot}_X$, and the rest of the exactness is evident.

Theorem (2.7) (Sebastiani [11], Malgrange [3]). $\mathscr{H}^{(i)}$, $i=0,-1,-2$, *are locally free sheaves of rank* μ.

The proof obtained so far is very difficult in spite of its apparent simplicity. A new proof would give us another insight in the nature of these sheaves.

Proposition (2.8). *Let* $\mathscr{H}^{(i)}$ *be as in* (2.5). *They can be considered as extensions of* $\mathscr{H}$ *by the following identification.*

i) *For* $t\in S-D$, *we have*

$$\mathscr{H}^{(-2)}_{|X_t}=\mathscr{H}^{(-1)}_{|X_t}=\Omega^n_{X_t}/d\Omega^{n-1}_{X_t},$$

which is identified with $H^n(X_t, \mathbf{C})$ *by De Rham theorem, or explicitly, letting* $\omega\in\Omega^n_{X/T}$ *correspond to the dual map*

$$\begin{array}{ccc} H_n(X_t, \mathbf{Z}) & \longrightarrow & \mathbf{C} \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \gamma & \longrightarrow & \int_\gamma \omega_{|X_t}. \end{array}$$

ii) *For* $\omega\in\Omega^{n+1}_{X/T}$ *we define an element in* $H^n(X_t, \mathbf{C})$ *by*

$$\begin{array}{ccc} H_n(X_t, \boldsymbol{Z}) & \longrightarrow & \boldsymbol{C} \\ \cup\!\!\!\!\text{\small w} & & \cup\!\!\!\!\text{\small w} \\ \gamma & \longrightarrow & \int_\gamma \mathrm{Res}_{X_t}\left(\frac{\omega}{t_1 - F_1}\right), \end{array}$$

which gives the identification of $\mathscr{H}^{(0)}_{|X_t}$ *with* $H^n(X_t, \boldsymbol{C})$.

Definition (2.9). The *Gauss-Manin connection* is the integrable covariant differentiation

$$\begin{array}{ccc} \nabla : \mathscr{D}er_S \times \mathscr{H}^{(-1)} & \longrightarrow & \mathscr{H}^{(0)} \\ (\delta, \omega) & \longrightarrow & \nabla_\delta \omega \end{array}$$

defined by

$$\nabla_{\partial/\partial t_i}[\zeta] = (-1)^{i+1}[(dt_2 \wedge \cdots \wedge dt_m)^{-1} dt_1 \wedge \cdots \wedge d\hat{t}_i \wedge dt_m \wedge d\zeta]$$

where $[\zeta]$ denotes the class in $\mathscr{H}^{(-1)}$ corresponding to $\zeta \in \Omega^n_{X/T}$.

Remark (2.10). Another definition of the Gauss-Manin connection is given by

$$\begin{array}{ccc} \nabla : \mathscr{H}^{(-2)} & \longrightarrow & \Omega^1_S \otimes \mathscr{H}^{(-1)} \\ \omega & \longrightarrow & d\omega = dF_1 \wedge \omega_1 + \sum_{i=2}^m dt_i \wedge \omega_i. \end{array}$$

Then $\nabla_{\partial/\partial t_1}$ has a particular importance because of the following property.

Proposition (2.11). ∇_{δ_1} *defines isomorphisms of* $\mathcal{O}_T$*-Modules as*

$$\nabla_{\delta_1} : \mathscr{H}^{(-2)} \xrightarrow{\sim} \mathscr{H}^{(-1)},$$

$$\nabla_{\delta_1} = d : \mathscr{H}^{(-1)} \xrightarrow{\sim} \mathscr{H}^{(0)},$$

both of which are compatible with the exact sequences in (2.6), *i.e. we have a commutative diagram*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{H}^{(-1)} & \xrightarrow{dF_1} & \mathscr{H}^{(0)} & \longrightarrow & \phi_* \Omega_F & \longrightarrow & 0 \\ & & \nabla_{\delta_1} \uparrow \wr & & \nabla_{\delta_1} = d \uparrow \wr & & \| & & \\ 0 & \longrightarrow & \mathscr{H}^{(-2)} & \longrightarrow & \mathscr{H}^{(-1)} & \xrightarrow{d} & \phi_* \Omega_F & \longrightarrow & 0. \end{array}$$

The compatibility of ∇_δ *with the exact sequences holds more generally for* $\delta \in \mathscr{G}$ (1.2).

These properties of $\mathscr{H}^{(i)}$, $i=0, -1, -2$, and ∇_{δ_1} now enable us to embed $\mathscr{H}^{(i)}$ into an infinite sequence of $\mathscr{H}^{(-k)}$, $k=0, 1, 2, \cdots$.

Definition-Proposition (2.12). Put

$$\mathscr{H}^{(-k)}=\{\omega \in \mathscr{H}^{(-1)};\ (\nabla_{\delta_1})^{k-1}\omega \in \mathscr{H}^{(-1)}\}.$$

This coincides with the previous definition of $\mathscr{H}^{(-2)}$ and we have the exact sequences

$$0 \longrightarrow \mathscr{H}^{(-k-1)} \longrightarrow \mathscr{H}^{(-k)} \xrightarrow{r^{(k)}} \phi_* \Omega_F \longrightarrow 0,$$
$$k=1, 2, \cdots$$

and isomorphisms

$$\nabla_{\delta_1}: \mathscr{H}^{(-k-1)} \xrightarrow{\sim} \mathscr{H}^{(-k)}$$

compatible with the above exact sequences.

They form a filtration of $\mathscr{H}^{(0)}$, whose (completed) graded ring is, roughly speaking, Taylor expansions along D.

Definition (2.13). We define

$$\begin{aligned}\mathscr{D}er_S(-\log C) &= \operatorname{Ker}(\mathscr{D}er_S \longrightarrow \mathcal{O}_C) \\ & \qquad\qquad\quad \delta \longmapsto \delta F_{|C} \\ &= \{\delta \in \mathscr{D}er_S:\ \delta\Delta \in (\Delta)\},\end{aligned}$$

which is a Lie subalgebra of $\mathscr{D}er_S$, and let

$$\Omega_S^1(\log D)=\mathcal{O}_S\frac{d\Delta}{\Delta}+\Omega_S^1,$$

which is naturally dual to $\mathscr{D}er_S(-\log C)$. The sections of $\Omega_S^1(\log D)$ are called *logarithmic differential* 1-*forms* (along D) (cf. (1.7)).

Proposition (2.14). *We have*

$$\mathscr{D}er_S(-\log C)=\{\delta \in \mathscr{D}er_S;\ \nabla_\delta \mathscr{H}^{(-1)} \subset \mathscr{H}^{(-1)}\}$$

or equivalently

$$\nabla: \mathscr{H}^{(-1)} \longrightarrow \Omega_S^1(\log D) \otimes \mathscr{H}^{(-1)},$$

that is, the connection ∇ *has only a logarithmic pole along D. This property is the regular singularity of* ∇.

§ 3. Higher residue pairings

In this section we state the following result which is the goal of this note and prove the most part of it.

Main Theorem (3.1). *There exists an infinite sequence of $\mathcal{O}_T$-bilinear forms*

$$K^{(k)}: \pi_*\mathcal{H}^{(0)} \times \pi_*\mathcal{H}^{(0)} \longrightarrow \mathcal{O}_T, \qquad k=0, 1, 2, \cdots$$

such that

i) $K^{(k)}$ *is symmetric or skew-symmetric respectively when* k *is even or odd;*

ii) $K^{(0)}([\psi_1 dx], [\psi_2 dx]) = \mathrm{Res}_{X/T}\left[\dfrac{\psi_1\psi_2 dx}{\partial F/\partial x_0 \cdots \partial F/\partial x_n}\right]$ *for* $[\psi_i dx] \in \pi_*\mathcal{H}^{(0)}$;

iii) $K^{(k)}(\omega_1, \omega_2) = K^{(k-1)}(\nabla_{\delta_1}\omega_1, \omega_2)$, *for* $\omega_1 \in \pi_*\mathcal{H}^{(-1)}$, $\omega_2 \in \pi_*\mathcal{H}^{(0)}$;

iv) $K^{(k)}(t_1\omega_1, \omega_2) - K^{(k)}(\omega_1, t_1\omega_2) = (n+k)K^{(k-1)}(\omega_1, \omega_2)$, *for* $\omega_i \in \pi_*\mathcal{H}^{(0)}$;

v) $K^{(k)}(\omega_1, \omega_2) = K^{(k)}(\nabla_\delta\, \omega_1, \omega_2) + K^{(k)}(\omega_1, \nabla_\delta\, \omega_2)$ *for* $\omega_i \in \mathcal{H}^{(-1)}$, $\delta \in \mathcal{G}$.

A very important application of this theorem, the theory of exponent is given in Prof. Saito's article [9] in this volume. In particular, the meaning of the property iv) becomes clear there.

Here we exhibit his latest simplified proof in [10], though it does not differ in essence from the preceding ones.

For the proof he introduces a new complex called *quantization*, with which we embed (the completion of) $\pi_*\mathcal{H}^{(0)}$ into a bigger $\mathcal{O}_T$-Module on which ∇_{δ_1} has no poles (i.e. $\mathcal{D}_S$ acts) and we define residue pairings there.

Notation (3.2). We denote by $\boldsymbol{C}[[\delta_1^{-1}]]$ the formal power series ring in δ_1^{-1} and by $\boldsymbol{C}((\delta_1^{-1}))$ its quotient field, whose elements are formal Laurent series. A natural filtration is given by

$$F^k = \left\{\sum_{p \le k} \delta_1^p \in \boldsymbol{C}((\delta_1^{-1}))\right\}, \qquad k \in \boldsymbol{Z}.$$

In our case it would be better to consider $\boldsymbol{C}((\delta_1^{-1}))$ as the completion of the localization of $\boldsymbol{C}[\delta_1]$ at the ideal $I=(\delta_1)$.

Note that in $\boldsymbol{C}((\delta_1^{-1}))$ the multiplication by δ_1 is invertible and induces

$$\delta_1: F^k \xrightarrow{\sim} F^{k+1}.$$

Definition (3.3) (*Quantization*). We consider the complex

$$(Q^\cdot) = (\Omega^\cdot_{X/T}((\delta_1^{-1})), \hat{d})$$

where $\hat{d}=\delta_1^{-1}d-dF_1\wedge$, and its filtration

$$F^kQ^p=\Omega^p_{X/T}\otimes F^k$$

which $\hat{d}$ preserves.

Proposition (3.4). i) *The complex* $(Q^{\cdot})$ *is purely* $(n+1)$*-codimensional,*

$$\mathscr{H}^p(Q^{\cdot})=0,\ p\neq n+1.$$

The same is true of $(F^kQ^{\cdot})$ *and* $(Q^{\cdot}/F^kQ)$.

ii) *The submanifold* C *is the common support of* $\mathscr{H}^{n+1}(Q^{\cdot})$, $\mathscr{H}^{n+1}(F^kQ^{\cdot})$, *and* $\mathscr{H}^{n+1}(Q^{\cdot}/F^kQ^{\cdot})$.

iii) *The canonical injection*

$$i^{(k)}:\mathscr{H}^{n+1}(F^kQ^{\cdot})\longrightarrow\mathscr{H}^{n+1}(Q^{\cdot})$$

induced from the inclusion: $F^k(Q^{\cdot})\to(Q^{\cdot})$ *defines a filtration* $\{F^k\mathscr{H}^{n+1}(Q^{\cdot})=\mathrm{Im}\, i^{(k)}\}$ *on* $\mathscr{H}^{n+1}(Q^{\cdot})$ *which is complete and*

$$F^k\mathscr{H}^{n+1}(Q^{\cdot})/F^{k-1}\mathscr{H}^{n+1}(Q^{\cdot})=\Omega_F.$$

iv) *The multiplication by* δ_1 (3.2) *induces isomorphisms*

$$\delta_1\colon F^k\mathscr{H}^{n+1}(Q^{\cdot})\xrightarrow{\sim}F^{k+1}\mathscr{H}^{n+1}(Q^{\cdot}),$$

hence, naturally,

$$\mathscr{H}^{n+1}(Q^{\cdot})\xrightarrow{\sim}F^k\mathscr{H}^{n+1}(Q^{\cdot})[\delta_1].$$

The key fact needed for the proof is the so-called division lemma.

Theorem (3.5) ([5]). *The sequence*

$$0\longrightarrow\mathcal{O}_{X/T}\xrightarrow{\wedge dF_1}\Omega^1_{X/T}\longrightarrow\cdots\longrightarrow\Omega^n_{X/T}\xrightarrow{\wedge dF_1}\Omega^{n+1}_{X/T}\longrightarrow\Omega_F\longrightarrow 0.$$

is exact.

Then, noting that the complex $(F^k(Q^{\cdot})/F^{k-1}(Q^{\cdot}),\hat{d})$, $k\in\mathbf{Z}$, is nothing but the above $(\Omega^{\cdot}_{X/T},\wedge dF_1)$, the proposition follows directly by the next technical lemma.

Lemma (3.6). *Let* $\{F^kQ^{\cdot}\}_k$ *be a (descending) filtered complex.*

i) *Suppose that* $F^kQ^{\cdot}$ *is complete with respect to the filtration, i.e.*

$$F^kQ^p\simeq\varprojlim_{k'>k}(F^kQ^p/F^kQ^p).$$

For a fixed $p \in \mathbf{Z}$, *if*

$$\mathscr{H}^p(F^{k'}Q^{\cdot}/F^{k'+1}Q^{\cdot})=0 \qquad \textit{for all } k' \geq k,$$

then

$$\mathscr{H}^p(F^kQ^{\cdot})=0.$$

ii) *Suppose that the filtration is exhausted, i.e.* $Q^p=\bigcup_k F^kQ^p$. *For* $p \in \mathbf{Z}$, *if*

$$\mathscr{H}^p(F^{k'}Q^{\cdot}/F^{k'+1}Q^{\cdot})=0 \qquad \textit{for all } k'<k$$

then

$$\mathscr{H}^p(Q^{\cdot}/F^kQ^{\cdot})=0.$$

(3.7) The natural inclusion $\Omega^{n+1}_{X/T}\to\Omega^{n+1}_{X/T}[[\delta_1^{-1}]]=F^0Q^{n+1}$ induces a map

$$i^{(0)}: \pi_*\mathscr{H}^{(0)}\longrightarrow\mathscr{H}^{n+1}(q_*F^{(0)}Q^{\cdot})\subset\mathscr{H}^{n+1}(q_*Q^{\cdot}),$$

which is compatible with the filtration, i.e.

$$i^{(k)}: \pi_*\mathscr{H}^{(-k)}\longrightarrow\mathscr{H}^{n+1}(q_*F^{(-k)}Q^{\cdot}).$$

Proposition (3.8). *For* $k\geq 0$ *we have a commutative diagram whose vertical sequences are* (*already known to be*) *exact*:

$$\begin{array}{ccc}
 & 0 & & 0 \\
 & \downarrow & & \downarrow \\
i^{(k)}: & \pi_*\mathscr{H}^{(-k)} & \longrightarrow & \mathscr{H}^{n+1}(q_*F^{(-k)}Q^{\cdot}) \\
 & \downarrow & & \downarrow \\
i^{(k-1)}: & \pi_*\mathscr{H}^{(-k+1)} & \longrightarrow & \mathscr{H}^{n+1}(q_*F^{(-k+1)}Q^{\cdot}) \\
 & \downarrow & & \downarrow \\
 & q_*\Omega_F & = & q_*\Omega_F \\
 & \downarrow & & \downarrow \\
 & 0 & & 0.
\end{array}$$

Corollary (3.9). *The filtered completion of* $\pi_*\mathscr{H}^{(-k)}$ *is isomorphic to* $\mathscr{H}^{n+1}(q_*F^{(-k)}Q^{\cdot})$.

Proposition (3.10). *The inclusion* $i^{(k)}$ *is a* $\mathscr{G}[\delta_1^{-1}]$-*homomorphism.*

Remark (3.11). The sheaf $\mathscr{D}_T((\delta_1^{-1}))$ acts on $\mathscr{H}^{n+1}(q_*Q^{\cdot})$ and $\mathscr{D}_T[[\delta_1^{-1}]]^0 =\{\sum D_m\delta^{-m};\ D_m \in \mathscr{D}_T,\ \deg D_m\leq m\}$ on $\pi_*\mathscr{H}^{(-k)}$ and on $\mathscr{H}^{n+1}(q_*F^{(-k)}Q^{\cdot})$. Therefore we can consider the first action as an extension of the second.

(3.12) For a Stein open set $U\subset T$ we take a Stein covering $\mathfrak{U}=\{U_0, \cdots, U_n\}$ of $(X-C)\cap q^{-1}(U)$ defined by $U_i=\{x \in X\cap q^{-1}(U);\ (\partial F/\partial x_i)(x)\neq 0\}$.

Consider the double Čech complex

$$(C^{\cdot}Q^{\cdot}) \qquad \Omega^{p,q}=C^{p}(\mathfrak{U}, \Omega^{q}_{X/T})((\delta_1^{-1})), \quad 0\le p\le n, \quad 0\le q\le n+1$$

where $\partial: \Omega^{p,q}\to\Omega^{p+1,q}$ denotes the Čech coboundary, and $\hat{d}: \Omega^{p,q}\to\Omega^{p,q+1}$ is as in (3.3). The total coboundary operator is then defined by

$$\hat{\partial}=\partial+(-1)^{n+p+q}\hat{d}.$$

Note that $H^p_{\partial}(\Omega^{p,q})=H^p(X\cap q^{-1}(U), \Omega^q_{X/T})((\delta_1^{-1}))\cong H^{p+1}_C(X\cap q^{-1}(U), \Omega^q_{X/T})((\delta_1^{-1}))$ which vanishes for $p\neq n$.

Proposition (3.13). *The single complex associated with* $(C^{\cdot}Q^{\cdot})$ *is acyclic with respect to* $\hat{\partial}$.

Proof. Clear from (3.4) i) and ii).

Corollary (3.14). *We can define a map*

$$L: \mathscr{H}^{n+1}(q_*Q^{\cdot})\longrightarrow q_*\mathscr{H}^{n+1}_C(\mathcal{O}_X)((\delta_1^{-1}))$$

which preserves the filtration, i.e.

$$L(\mathscr{H}^{n+1}(q_*F^kQ^{\cdot}))\subset q_*\mathscr{H}^{n+1}_C(\mathcal{O}_X)\otimes F^k.$$

Proof. The proof of this is just a general nonsense.

The image of the inclusion $\Gamma(U, q_*\Omega^{n+1}_{X/T})((\delta_1^{-1}))\to C^0(\mathfrak{U}, \Omega^{n+1}_{X/T})((\delta_1^{-1}))$ is contained in Ker $\hat{\partial}=$ Im $\hat{\partial}$. In other words for $\omega\in q_*\Omega^{n+1}_{X/T}((\delta_1^{-1}))$ there is a $c=(c^i)\in\bigoplus_{i=0}^n\Omega^{i,n-i}$ with $\hat{\partial}c=\omega$. Then we define

$$L([\omega])=\text{the cohomology class of } c^n.$$

The following zigzag process gives a more elementary illustration of this map:

$$\begin{array}{llllll}
\Gamma(\Omega^{n+1}) \longrightarrow & C^0(\Omega^{n+1}) & & & & \\
\quad \cup\!\!\!\!\!\!\shortmid & \uparrow \hat{d} & & & & \\
\quad \omega & & & & & \\
 & C^0(\Omega^n) \xrightarrow{\ \partial\ } & C^1(\Omega^n) & & & \\
 & \quad \cup\!\!\!\!\!\!\shortmid & \uparrow \hat{d} & & & \\
 & \quad c^0 & & & & \\
 & & C^1(\Omega^{n-1}) \longrightarrow & & & \\
 & & \quad \cup\!\!\!\!\!\!\shortmid & \ddots & & \\
 & & \quad c^1 & & \longrightarrow & C^n(\Omega^1) \\
 & & & & & \uparrow \hat{d} \\
 & & & & c^n \in & C^n(\Omega^0).
\end{array}$$

Remark (3.15). In view of (3.11) the multiplication by δ_1^{-1} is compatible with L but the one by δ_1 is *not*. The compatibility holds if one defines the dual sequence of $\mathcal{O}_T$-Modules $\check{\mathscr{H}}^{(k)}$ ($\check{\mathscr{H}}^{(k)} \to \check{\mathscr{H}}^{(k+1)}$) and the dual connection ∇_{δ_1}: $\check{\mathscr{H}}^{(k)} \rightleftarrows \check{\mathscr{H}}^{(k+1)}$ (cf. [10]).

(3.16). For $\omega \in \mathscr{H}^{n+1}(\Omega^{\cdot})$ we expand $L(\omega)$ into Laurent series as

$$L(\omega)=\sum_{k\leq k_0} D^{(-k)}(\omega)\delta_1^k.$$

Note that

$$D^{(-k)}(\omega\delta_1)=D^{(-k+1)}(\omega).$$

Definition (3.7).

$$\check{\mathscr{H}}^{(0)}=\{\omega \in q_*\mathscr{H}_C^{n+1}(\mathcal{O}_X);\ dF_1\wedge d\omega=0 \text{ in } q_*\mathscr{H}_C^{n+1}(\Omega_{X/T}^2)\}.$$

Theorem (3.18). *The image of $D^{(-k)}$ is contained in $\check{\mathscr{H}}^{(0)}$ and we have an exact sequence*

$$0\longrightarrow \mathscr{H}^{n+1}(q_*F^{(-k)}\Omega^{\cdot})\longrightarrow \mathscr{H}^{n+1}(q_*\Omega^{\cdot})\xrightarrow{D^{(k-1)}}\check{\mathscr{H}}^{(0)}.$$

We omit the proof here. See [10].

Remark (3.19). Roughly speaking, to apply the map $D^{(k-1)}$ to $\sum_{i\leq k_0} a_i\delta^i \in \mathscr{H}^{n+1}(q_*\Omega^{\cdot})$ means to forget the part $\sum_{i\leq -k} a_i\delta^i$.

Definition (3.20). For $\omega, \omega' \in \pi_*\check{\mathscr{H}}^{(0)}$ we define

$$K^{(k)}(\omega, \omega')=\mathrm{Res}_{X/T}\,[(\nabla^k\omega)\omega'] \in \mathcal{O}_T, \qquad k=0, 1, 2, \cdots$$

where $\nabla^k=D^{(k)}\circ i^{(0)}$, and $\mathrm{Res}_{X/T}$: $q_*\mathscr{H}_C^{n+1}(\Omega_{X/T}^{n+1})\to q_*\mathcal{O}_C\xrightarrow{Tr}\mathcal{O}_T$ is the canonical residue map.

Example (3.21). For $\omega=[\phi dx] \in \pi_*\check{\mathscr{H}}^{(0)}$ we have $\omega=\partial(c^i)$ with

$$c^0\equiv\left(\frac{\phi}{\partial F_1/\partial x_k}dx_0\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\right)_{k=0,\cdots,n,} \quad (\text{mod } \delta_1^{-1}),$$

$$c^1\equiv\left(\frac{\phi}{\partial F/\partial x_j\cdot\partial F_1/\partial x_k}dx_0\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\right)_{0\leq j<k\leq n}$$

$$(\text{mod } \delta_1^{-1})$$

$$\vdots$$

$$c^n\equiv\left(\frac{\phi}{\prod(\partial F_1/\partial x_i)}\right) \quad (\text{mod } \delta_1^{-1}).$$

Hence for $\omega=[\phi dx]$, $\omega'=[\psi dx] \in \pi_*\mathscr{H}^{(0)}$

$$K^{(0)}(\omega, \omega')=\mathrm{Res}_{X/T}\left[\frac{\phi\psi dx}{\partial F_1/\partial a_0\cdots\partial F_1/\partial x_n}\right].$$

(3.22). For the complete proof of the main theorem (3.1) we refer the reader to [8]. It is rather involved though elementary. Note that we have already given the definition of $K^{(k)}$ and a proof of the properties ii), iii). For a proof of the property i) compare also [7].

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