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On Fano 3-Folds with $B_2 \geq 2$

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This article is an introduction to the classification of Fano 3-folds, i.e., 3-dimensional smooth projective varieties with ample anticanonical bundles, whose second Betti numbers are not less than two. We shall show, with proof or by an example, the principle of "how to classify them" so that one will be able to do it. The complete classification will be published elsewhere.

After stating our main results in § 1, we summarize the results of Iskovskih, Šokurov and Mori which are indispensable to our classification in §§ 2 and 3. § 5 is devoted to the classification of Fano 3-folds with $B_2 = 2$, especially, imprimitive ones. In § 6, we investigate the properties of Fano conic bundles which play an essential role in the classification of imprimitive Fano 3-folds with $B_2 \geqq 3$ (§ 7), in the proof of Theorem 1.6 (§ 8) and in the proof of Theorem 1.2. (§ 9).

§ 0. Del Pezzo surface

We review here the theory of del Pezzo surfaces because it is useful for understanding the outline of the classification of Fano 3-folds.

Let S be a del Pezzo surface, i.e., a smooth surface with negative canonical bundle. The positive integer $(-K_S)^2$ is called the degree of S. By Serre duality, Kodaira's vanishing theorem and the Riemann-Roch theorem, we have

$$(0.1) \qquad q = p_g = 0, \quad \rho(S) = B_2(S), \quad \dim |-K_S| = (-K_S)^2.$$

By Noether's formula $c_1^2 + c_2 = 12\chi(\mathcal{O}_S)$, We have

$$(0.2) \qquad (-K_S)^2 + \rho(S) = 10.$$

In particular, we have

Proposition 0.3. *$(-K_S)^2 \leq 9$ and $\rho(S) \leq 9$ for every del Pezzo surface S.*

In the case of Fano 3-folds, we have $(-K_X)^3 \leq 64$ and $\rho(X) \leq 10$. But

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the proof is not so easy as the proposition above.

About the anticanonical system $|-K_S|$, we have

Proposition 0.4. (1) *If $(-K_S)^2 \geq 2$, then $|-K_S|$ is free, i.e., free from fixed components and base points.*

(2) *If $(-K_S)^2 \geq 3$, then $|-K_S|$ is very ample.*

In the case $d=(-K_S)^2 \geq 3$, the surface S_d of degree d embedded into $\boldsymbol{P}^d$ by the morphism φ attached to $|-K_S|$ is called the anticanonical model of S.

If $\rho(S)=1$, then by the Poincaré duality, there is an ample divisor D with $(D^2)=1$. Since $(-K_S)^2=9$, $-K_S$ is numerically equivalent to $3D$. It is easy to see that $\dim |D|=2$ and the map associated to $|D|$ is an isomorphism. Hence we have

Proposition 0.5. *If $\rho(S)=1$, then $S \cong \boldsymbol{P}^2$.*

(0.6) Extremal ray of a del Pezzo surface S (Theorem 2.1 [6])

Let $NE(S) \subset (\{\text{divisors}\}/\approx) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ be the cone generated by effective divisors on S modulo numerical equivalence. This cone is a closed polyhedral cone. If $\rho(S) \geq 2$, then an irreducible reduced curve C such that the equivalence class $[C]$ lies on the edge of $NE(S)$ satisfies one of the following:

1) C is an exceptional curve of the first kind.

2) $\rho(S)=2$, S has a $\boldsymbol{P}^1$-bundle structure and C is its fibre.

Proposition 0.7. *If S is minimal, then $\rho(S)=1$ or $S \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$.*

Proof. Assume that $\rho(S)>1$. Since S has no exceptional curve of the first kind, $\rho(S)=2$ and S has a $\boldsymbol{P}^1$-bundle structure by (0.6). Since $NE(S)$ has two edge, S has two $\boldsymbol{P}^1$-bundle structures. Hence S is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. q.e.d.

Proposition 0.8. *Let E be an exceptional curve of the first kind on S and $\alpha: S \rightarrow S'$ the blowing down of E. Then S' is a del Pezzo surface.*

By these propositions, a del Pezzo surface S satisfies one of the following:

	$d=(-K_S)^2$	$\rho(S)$	S
1)	9	1	$\boldsymbol{P}^2$
2)	8	2	$\boldsymbol{P}^1 \times \boldsymbol{P}^1$
3)	8	2	$\boldsymbol{F}_1$
4)	$1 \leq d \leq 7$	$10-d$	the blow-up of $\boldsymbol{P}^2$ at $9-d$ points in "general position".

The meaning of "general position" in 4) is as follows.

Proposition 0.9. ([5]) *The blow-up of $\boldsymbol{P}^2$ at n points $x_1, \cdots, x_n$ $(n \leq 8)$ is a del Pezzo surface if and only if no* 3 *of them lie on a line, no* 6 *on a conic, and for $n=8$ all eight do not lie on a cubic which is singular at one of* $x_1, \cdots, x_8$.

By these propositions, we can conclude that the del Pezzo surfaces have exactly 10 deformation types.

§ 1. Main results

Our final result on Fano 3-folds is the following:

Theorem 1.1. *There are exactly* 87 *types of Fano* 3*-folds with $B_2 \geq 2$ up to deformations.* (*See* [7] *for the description of the* 87 *types.*)

Theorem 1.2. *A Fano* 3*-fold with $B_2 \geq 6$ is isomorphic to $\boldsymbol{P}^1 \times S_{11-B_2}$, where S_d is a del Pezzo surface of degree d. In particular, every Fano* 3*-fold has $B_2 \leq 10$.*

The number $N(b)$ of Fano 3-folds with $B_2 = b$ up to deformations is as follows:

b	2	3	4	5	6	7	8	9	10	≥ 11
$N(b)$	36	31	12	3	1	1	1	1	1	0

We refer the reader to [4] for Fano 3-folds with $B_2 = 1$ which are called Fano 3-folds of the first species.

Definition 1.3. A Fano 3-fold is *imprimitive* if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve. A Fano 3-fold is *primitive* if it is not imprimitive.

Example 1.4. A Fano 3-fold with $B_2 = 1$ is primitive. $\boldsymbol{P}^1 \times \boldsymbol{P}^2$, $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and a divisor $W_6 \subset \boldsymbol{P}^2 \times \boldsymbol{P}^2$ of bidegree (1, 1) are primitive Fano 3-folds. If a Fano 3-fold is a double cover of another primitive Fano 3-fold with ample branch locus, then it is also primitive.

Example 1.5. Let C be a disjoint union of n smooth irreducible curves in $\boldsymbol{P}^3$ and assume that C is a scheme theoretic intersection of cubics. Then the blow-up X of $\boldsymbol{P}^3$ along C is an imprimtive Fano 3-fold with $B_2 = n+1$.

Proof. It suffices to show that $-K_X$ is ample. Let $\alpha \colon X \to \boldsymbol{P}^3$ be the blowing up and $D = \alpha^{-1}(C)$ the exceptional divisor. Then we have $-K_X$

$\sim\alpha^*(-K_{P^3})-D\sim 4\alpha^*H-D$ for a plane H in P^3. The linear system $|3\alpha^*H-D|$ is free by our assumption and $|\alpha^*H|$ is also free. Hence $|-K_X|=|(3\alpha^*H-D)+\alpha^*H|$ is free. By Proposition 4.6. [2] it suffices to show that $(-K_X\cdot Z)>0$ for every irreducible reduced curve Z on X.

Case in which $\alpha(Z)$ is a point. Z is an exceptional line and $(D\cdot Z)=-1$. Hence $(-K_X\cdot Z)=(4\alpha^*H\cdot Z)-(D\cdot Z)=4(H\cdot\alpha_*Z)+1=1$.

Case in which $\alpha(Z)$ is not a point. Since $|3\alpha^*H-D|$ is free and H is ample, we have

$$(-K_X\cdot Z)=(\alpha^*H\cdot Z)+(3\alpha^*H-D\cdot Z)\geq(H\cdot\alpha^*Z)>0 \qquad \text{q.e.d.}$$

The following is the first step of our classification and will be proved in § 8.

Theorem 1.6. *Let X be a primitive Fano 3-fold. Then we have*

(1) $B_2(X)\leq 3$,

(2) *if $B_2(X)=2$, then X is a conic bundle over P^2 and*

(3) *if $B_2(X)=3$, then X is a conic bundle over $P^1\times P^1$ and has either a divisor $D\cong P^1\times P^1$ such that $\mathcal{O}_D(D)\cong\mathcal{O}(-1,-1)$ or another conic bundle structure over $P^1\times P^1$.*

By (3) of the theorem, it is not hard, though we omit it here, to see that a primitive Fano 3-fold with $B_2=3$ satisfies one of the following:

(a) $(-K_X)^3=12$. X is a double cover of $P^1\times P^1\times P^1$ whose branch locus is a smooth divisor of tridegree (2, 2, 2).

(b) $(-K_X)^3=14$. X is a smooth member of $|L^{\otimes 2}\otimes\mathcal{O}_{P^1\times P^1}\mathcal{O}(2,3)|$ on the P^2-bundle $P(\mathcal{O}\oplus\mathcal{O}(-1,-1)^{\oplus 2})$ over $P^1\times P^1$ such that $X\cap Y$ is irreducible, where L is the tautological line bundle and Y is the unique member of $|L|$.

(c) $(-K_X)^3=48$. X is isomorphic to $P^1\times P^1\times P^1$.

(d) $(-K_X)^3=52$. X is isomorphic to the P^1-bundle $P(\mathcal{O}\oplus\mathcal{O}(1,1))$ over $P^1\times P^1$.

The primitive Fano 3-folds with $B_2=2$ cannot be cassified only by (2) of Theorem 1.6. The following will be proved in § 5.

Theorem 1.7. *The primitive Fano 3-folds with $B_2=2$ have the following 9 deformation types*:

	$(-K_X)^3$	X	*types of extremal rays*
1)	6	*a double cover of $P^1\times P^2$ whose branch locus is a divisor of bidegree* (2.4)	C_1-D_1
2)	12	*a double cover of W_6, see* 6), *whose branch locus is a member of $|-K_{W_6}|$ or a smooth divisor on $P^2\times P^2$ of bidegree* (2.2)	C_1-C_1

3)	14	*a double cover of* V_7, *see* 8), *whose branch locus is a member of* $\lvert -K_{V_7}\rvert$	C_1-E_3 *or* E_4
4)	24	*a double cover of* $\boldsymbol{P}^1\times\boldsymbol{P}^2$ *whose branch locus is a divisor of bidegree* (2.2)	C_1-D_2
5)	30	*a smooth divisor on* $\boldsymbol{P}^2\times\boldsymbol{P}^2$ *of bidegree* (1.2)	C_1-C_2
6)	48	W_6, *a smooth divisor on* $\boldsymbol{P}^2\times\boldsymbol{P}^2$ *of bidegree* (1.1)	C_2-C_2
7)	54	$\boldsymbol{P}^1\times\boldsymbol{P}^2$	C_2-D_3
8)	56	$V_7=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(1))$	C_2-E_2
9)	62	$\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(2))$	C_2-E_5

§ 2. Known results

First we recall some known results on Fano 3-folds which are necessary for our classification.

O. Elementary facts. Let X be a Fano 3-fold. Then by Serre duality, the Riemann-Roch theorem and Kodaira's vanishing theorem, we have

i) $h^i(\mathcal{O}_X)=0$ for every $i>0$,

ii) $(-K_X\cdot c_2(X))=24\chi(\mathcal{O}_X)=24$,

iii) $\dim\lvert -K_X\rvert=(1/2)(-K_X)^3+2$ and

iv) Pic X is torsion-free.

In particular, the Picard number $\rho(X)$ is equal to the second Betti number $B_2(X)$ and is a topological invariant of X.

I. Fano 3-folds with index ≥ 2 ([3]). The largest integer r which divides $-K_X$ in Pic X is called the *index* of X. Fano 3-folds with index ≥ 2 are classified as follows:

index of X	$B_2(X)$	$(-K_X)^3$	X
4	1	64	$\boldsymbol{P}^3$
3	1	54	a smooth quadric Q in $\boldsymbol{P}^4$
2	1	$8d$	V_d, $1\leq d\leq 5$
	2	48	W_6
	3	48	$\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$
	2	56	V_7, see Theorem 1.7,

where 1) V_1 is a double cover of the Veronese cone $W_4\subset\boldsymbol{P}^6$ whose branch locus is a smooth intersection of W_4 and a cubic hypersurface not passing through the vertex of the cone,

2) V_2 is a double cover of $\boldsymbol{P}^3$ whose branch locus is a smooth quartic hypersurface,

3) V_3 is a smooth cubic hypersurface of $\boldsymbol{P}^4$,

4) V_4 is a complete intersection of two quadrics in $\boldsymbol{P}^5$,

5) V_5 is a complete intersection of a linear subspace $\boldsymbol{P}^6$ in $\boldsymbol{P}^9$ and the Grassmann variety Grass $(\boldsymbol{P}^4 \supset \boldsymbol{P}^1)$ embedded in $\boldsymbol{P}^9$ by the Plücker embedding, and

6) W_6 is a smooth divisor on $\boldsymbol{P}^2 \times \boldsymbol{P}^2$ of bidegree (1.1). W_6 is isomorphic to the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(T_{\boldsymbol{P}^2})$ over $\boldsymbol{P}^2$.

II. Fano 3-folds whose anticanonical systems are not very ample ([3]). Let X be a Fano 3-fold of index 1. Then we have

1) the anticanonical system $|-K_X|$ is free (i.e., free from fixed components and base points) except for the following two cases

(i) X is a isomorphic to the blow-up of V_1 along a smooth elliptic curve which is a complete intersection of two members of $|-(1/2)K_{V_1}|$. One has $(-K_X)^3=4$.

(ii) X is isomorphic to $\boldsymbol{P}^1 \times S_1$, where S_1 is a del Pezzo surface of degree 1. One has $(-K_X)^3=6$.

2) Assume that $|-K_X|$ is free but not very ample. Such Fano 3-folds are called hyperelliptic and classified as follows:

$B_2(X)$	$(-K_X)^3$	X
1	2	a double cover of $\boldsymbol{P}^3$ whose branch locus is a smooth sextic.
1	4	a double cover of a quadric hypersurface $Q \subset \boldsymbol{P}^4$ whose branch locus is a smooth intersection of Q and a quartic hypersurface
2	6	a double cover of $\boldsymbol{P}^1 \times \boldsymbol{P}^2$ whose branch locus is a smooth divisor on $\boldsymbol{P}^1 \times \boldsymbol{P}^2$ of bidegree (2.4)
2	8	the blow-up of V_2 along an elliptic curve which is a complete intersection of two members of $|-(1/2)K_{V_2}|$
9	12	$\boldsymbol{P}^1 \times S_2$, where S_2 is a del Pezzo surface of degree 2.

III. Existence of lines ([8], [9]). Let X be a Fano 3-fold whose anticanonical system is very ample. Its anticanonical model, i.e., the image of X by the morphism attached to $|-K_X|$, is a subvariety of degree $2g-2$ in $\boldsymbol{P}^{g+1}$, where $g=(1/2)(-K_X)^3+1$.

Theorem (Šokurov). *If the index of X is equal to* 1 *and if $X \not\cong \boldsymbol{P}^1 \times \boldsymbol{P}^2$, then there exists a* 1*-dimensional family of lines on the anticanonical model $X_{2g-2} \subset \boldsymbol{P}^{g+1}$.*

§ 3. Extremal rays of Fano 3-folds

We apply the theory in [6] to Fano 3-folds. Let X be a Fano 3-fold with $B_2 \geq 2$. Let $N_Z(X)$ (resp. $NE_Z(X)$) be the set of numerically equivalence classes of 1-cycles (resp. effective 1-cycles) on X. Let $N(X)=N_Z(X) \otimes_Z \boldsymbol{R}$ and $NE(X)$ the cone in $N(X)$ generated by $NE_Z(X)$. By Theorem 1.2 [6], we have

(3.1) $NE(X)$ is a closed polyhedral cone.

Let R be an extremal ray of X, i.e., a half line which is an edge of the polyhedral cone $NE(X)$. There exists a morphism $f: X \to Y$ to a normal projective variety such that (i) $f_* \mathcal{O}_X \cong \mathcal{O}_Y$ and (ii) for any irreducible reduced curve C on X, $[C] \in R$ if and only if $f(C)$ is a point. Such an f is unique up to an isomorphism, called the contraction of R and denoted by $\mathrm{cont}_R: X \to Y$ (Theorem 3.1 [6]).

Put $\mu(R)=\min\{(-K_X \cdot Z) \mid Z$ is a rational curve such that $[Z] \in R\}$ and let $l=l_R$ be a rational curve such that $[l] \in R$ and $(-K_X \cdot l)=\mu(R)$. Then there exists an exact sequence

(3.2) $$0 \longrightarrow \mathrm{Pic}\, Y \xrightarrow{f^*} \mathrm{Pic}\, X \xrightarrow{(\ \cdot l)} Z \longrightarrow 0,$$

where $(\ \cdot l)(D)=(D \cdot l)$ for $D \in \mathrm{Pic}\, X$. (Theorem 1.2 [6]. The surjectivity of $(\ \cdot l)$ is a consequence of classification of R and Corollary 3.6 below and not true in general if X is not a Fano 3-fold.) In particular, we have $\rho(X) =\rho(Y)+1$.

R and $f=\mathrm{cont}_R$ are classified as follows:

Case $\dim Y=3$: There exists an irreducible reduced divisor D of X such that $f|_{X-D}$ is an isomorphism and $\dim f(D) \leq 1$. Such D is uniquely determined by R and called the exceptional divisor of R. Moreover f is the blowing-up of Y by the ideal defining $f(D)$ (given the reduced structure). f and D satisfy one of the following ([6] Theorem 3.3 and Corollary 3.4)

type of R	f and D	$\mu(R)$	l
E_1	$f(D)$ is a smooth curve, Y is smooth and $f\|_D: D \to \phi(D)$ is a $\boldsymbol{P}^1$-bundle.	1	exceptional line
E_2	$f(D)$ is a point, Y is smooth, $D \cong \boldsymbol{P}^2$ and $\mathcal{O}_D(D) \cong \mathcal{O}_P(-1)$.	2	line in $D \cong \boldsymbol{P}^2$
E_3	$f(D)$ is an ordinary double point. $D \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$ and $s \times \boldsymbol{P}^1$ and $\boldsymbol{P}^1 \times t$ are numerically equivalent for every $s, t \in \boldsymbol{P}^1$.	1	$s \times \boldsymbol{P}^1$ or $\boldsymbol{P}^1 \times t$ in D

E_4	$f(D)$ is a double point. D is an irreducible reduced singular quadric surface in P^3, $\mathcal{O}_D(D)\cong\mathcal{O}_D\otimes\mathcal{O}_P(-1)$.	1	generator of the cone D
E_5	$f(D)$ is a quadruple point of Y, $D\cong P^2$, and $\mathcal{O}_D(D)\cong\mathcal{O}_P(-2)$.	1	line in $D\cong P^2$

If R is of E_1-type, then either (1) Y is a Fano 3-fold or (2) $D\cong P^1\times P^1$ and $\mathcal{O}_D(D)\cong\mathcal{O}(-1, -1)$ (Proposition 4.5).

R is called of $E_{1,a}$-type in the case (1) and of $E_{1,b}$-type in the case (2). If R is of $E_{1,b}$-type, then a horizontal section of $f|_D$: $D\to C$ is not numerically equivalent to a fibre of $f|_D$ (because Y is projective), and belongs to another extremal ray of $E_{1,b}$-type. If $B_2(X)=2$, then Y is always a Fano 3-fold because Y is projective and $\rho(Y)=1$. Moreover, Y is of index≥ 2 (Proposition 4.10).

In the case R is of E_2-type, Y is a Fano 3-fold. The proof is similar to and easier than Proposition 4.5.

If the ray R is of type E_2, E_3, E_4 or E_5, then the exceptional divisor has the following property:

(3.3) Both $\mathcal{O}_D(D)$ and ω_D are negative, i.e., their inverses are ample. Every curve in D can move in D.

(3.4) D is mapped to a point by every morphism g from X onto a curve if there is any.

Proof. If R is of type E_2, E_4 or E_5, then D has no nontrivial morphism onto a curve. If R is of type E_3, then $D\cong P^1\times P^1$. Hence either $g(s\times P^1)$ or $g(P^1\times t)$ is a point, or equivalently, the intersection number $(s\times P^1\cdot g^{-1}(\text{point}))$ or $(P^1\times t\cdot g^{-1}(\text{point}))$ is zero. Since $s\times P^1$ and $P^1\times t$ are numerically equivalent for every s, $t\in P^1$ by definition, both $g(s\times P^1)$ and $g(P^1\times t)$ are points. It follows that $g(D)$ is a point. q.e.d.

Case dim $Y=2$: Y is a smooth projective surface, f: $X\to Y$ is a conic bundle and $f^{-1}(C)$ is irreducible for every irreducible curve C on Y (Proposition 6.3). We have the following two cases:

type of R	f	$\mu(R)$	l
C_1	f has a singular fibre	1	an irreducible component of a reducible fibre or a reduced part of a multiple fibre
C_2	f is a P^1-bundle	2	a fibre

Proposition 3.5. *Y is rational.*

Proof. Since $q(X)=0$ and f is surjective, $q(Y)=0$. By the formula

$-4K_Y \approx f_*(-K_X)^2 + \Delta_f$ (Proposition 6.2, (4)), $(K_Y \cdot A)$ is negative for every ample divisor A on Y. Hence all the plurigenera $P_m(Y)$ vanish. Therefore Y is rational by Castelnuovo's criterion. q.e.d.

Corollary 3.6. *If R is of C_2-type, then f is locally trivial for Zariski topology.*

Case $\dim Y=1$: Y is a smooth curve and $\rho(X)=\rho(Y)+1=2$. Every fibre of f is irreducible and reduced and the generic fibre X_η is a del Pezzo surface. We have the following three cases:

type of R	f	$\mu(R)$
D_1	X_η is a del Pezzo surface of degree d, $1 \leq d \leq 6$.	1
D_2	f is a quadric bundle, i.e., every fibre is isomorphic to a normal quadric surface in $\boldsymbol{P}^3$.	2
D_3	f is a $\boldsymbol{P}^2$-bundle.	3

Since $q(X)=0$ and f is surjective, $q(Y)=0$. If follows that $Y \cong \boldsymbol{P}^1$.

§ 4. Blowing-up and blowing-down of Fano 3-folds

Let $f: X \to Y$ be the blowing-up of a smooth 3-dimensional variety Y along a smooth irreducible curve C on Y. We will keep the meaning of these symbols in this section. The following are easily verified:

(4.1) $-K_X \sim f^*(-K_Y) - D$ for the exceptional divisor D of f.

(4.2) $D \cong \boldsymbol{P}(N^*_{C/Y})$ and $\mathcal{O}_D(-D)$ is the tautological line bundle, where $N_{C/Y}$ is the normal bundle of C and $N^*_{C/Y}$ is its dual vector bundle.

(4.3)
$$(D^3) = -\deg N_{C/Y}, \qquad (D^2 \cdot -K_X) = 2p_a(C) - 2,$$
$$(D \cdot (-K_X)^2) = (-K_Y \cdot C) + 2 - 2p_a(C) \quad \text{and}$$
$$(-K_X)^3 = (-K_Y)^3 - 2\{(-K_Y \cdot C) - p_a(C) + 1\}.$$

Lemma 4.4. *Assume that X is a Fano 3-fold. Then we have*
(1) $(-K_Y \cdot C) > 2p_a(C) - 2$,
(2) $(-K_Y \cdot C) \geq 0$ *if C is rational. The equality holds if and only if $N_{C/Y} \cong \mathcal{O}(-1)^{\oplus 2}$ or equivalently $D \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$, and*
(3) $(-K_Y \cdot C) > p_a(C) - 1$ *and* $(-K_X)^3 < (-K_Y)^3$.

Proof. Since $-K_X$ is ample, $((-K_X)^2 \cdot D)$ is positive. Hence (1) follows from (4.3). Since $-K_X$ is ample, so is $\mathcal{O}_D(-K_X)$. Since $\mathcal{O}_D(-K_X)$ is a tautological line bundle of $f|_D: D \to C$, the direct image $F = (f|_D)_* \mathcal{O}_D(-K_X)$ is ample. If $C \cong \boldsymbol{P}^1$, every vector bundle on C is a direct sum of line bundles. Hence $((-K_X)^2 \cdot D) = \deg F \geq 2$ and the equality holds if and only if $F \cong \mathcal{O}(1)^{\oplus 2}$. (2) follows from this and (4.3) because $F \cong N_{C/Y} \otimes \omega_C^{-1}$. (3) is an easy consequence of (1), (2) and (4.3). q.e.d.

Proposition 4.5. *If X is a Fano 3-fold, then one of the following holds:*

(1) *Y is a Fano 3-fold, and*

(2) $C \cong P^1$ *and* $N_{C/Y} \cong \mathcal{O}(-1)^{\oplus 2}$ *or equivalently* $D \cong P^1 \times P^1$ *and* $\mathcal{O}_D(D) \cong \mathcal{O}(-1, -1)$. *Even in the case* (2), $|-nK_Y|$ *is free for some* $n > 0$.

Proof. First we show

Claim: $|-nK_Y|$ is free for some $n > 0$.

$|-mK_X|$ is free for some $m > 0$. Therefore by (4.1), $|-mK_Y|$ has no fixed components or no base points outside of C. By Kodaira's vanishing theorem, the restriction map $H^0(\mathcal{O}_Y(-K_Y)) \cong H^0(\mathcal{O}_X(-K_X + D)) \to H^0(\mathcal{O}_D(-K_X + D)) \cong H^0(\mathcal{O}_C(-K_Y))$ is surjective. Since $h^0(\mathcal{O}_C(-K_Y)) \geq \chi(\mathcal{O}_C(-K_Y)) = (-K_Y \cdot C) - p_a(C) + 1 > 0$ by Lemma 4.4, $|-K_Y|$ has a member not containing C. Hence $|-mK_Y|$ has only a finite number of base points. By Zariski's Theorem, $|-m'mK_Y|$ is free for some $m' > 0$.

$(-K_Y \cdot Z) > 0$ for every irreducible curve $Z \neq C$ on Y, because $(-K_Y \cdot Z) = (f^*(-K_Y) \cdot Z') = (-K_X \cdot Z') + (D \cdot Z') > 0$, where Z' is the strict transform of Z. Hence, if $(-K_Y \cdot C) > 0$, then Y is a Fano 3-fold by Proposition 4.6, [2]. If $(-K_Y \cdot C) = 0$, then C and D satisfy (2) by Lemma 4.4. q.e.d.

Corollary 4.6. *Let V be the blow-up of a Fano 3-fold W along a disjoint union of two smooth curves* C_1 *and* C_2 *on W. If V is a Fano 3-fold, then the blow-up* V_i *of W along* C_i *is a Fano 3-fold for each* $i = 1, 2$.

Next we consider a necessary condition for X to be a Fano 3-fold. The following is used very often in our classification.

Proposition 4.7. *If X is a Fano 3-fold, then C does not meet any curve Z with* $(-K_Y \cdot Z) = 1$ *in a zero-dimensional set.*

Proof. Assume that C meets a curve Z with $(-K_Y \cdot Z) = 1$ in a zero dimensional set. Let Z' be the strict transform of Z by f. Then, by (4.1), we have $(-K_X \cdot Z') = (f^*(-K_Y) \cdot Z') - (D \cdot Z') = (-K_Y \cdot f_* Z') - (D \cdot Z') = 1 - (D \cdot Z')$. By our assumption $(D \cdot Z')$ is positive. Hence $(-K_X \cdot Z)$ is nonpositive and $-K_X$ is not ample. q.e.d.

Corollary 4.8. *Assume that Y is isomorphic to the blow-up of Y'. If X is a Fano 3-fold, then either C is disjoint from the exceptional divisor of Y/Y' or C is an exceptional line of Y/Y' (i.e., an irreducible reduced curve on Y which is mapped to a point of Y').*

Proposition 4.9. *Assume that* $C \cong \boldsymbol{P}^1$ *and* $(-K_Y \cdot C)=1$. *If* X *is a Fano* 3*-fold, then* $N_{C/Y} \cong \mathcal{O}_{\boldsymbol{P}} \oplus \mathcal{O}_{\boldsymbol{P}}(-1)$.

Proof. Since $\deg N_{C/Y}=(-K_Y \cdot C)+\deg K_C=-1$, $N_{C/Y} \cong \mathcal{O}_{\boldsymbol{P}}(n) \oplus \mathcal{O}(-1-n)$ for some $n \geq 0$. Let s be the section of $\boldsymbol{P}^1$-bundle $D=\boldsymbol{P}(N^*_{C/Y}) \to C$ corresponding to the exact sequence

$$0 \longrightarrow \mathcal{O}(n+1) \longrightarrow N^*_{C/Y} \longrightarrow \mathcal{O}(-n) \longrightarrow 0.$$

By (4.1) and (4.2), we have $(-K_X \cdot s)=(f^*(-K_Y) \cdot s)-(D \cdot s)=(-K_Y \cdot f_* s)+(\mathcal{O}_D(-D) \cdot s)_D=1-n$. Therefore if X is a Fano 3-fold, we have $n=0$.

q.e.d.

The following proposition shows that, in order to classify imprimitive Fano 3-folds, it is not necessary to consider the blowing-up of Fano 3-folds of the first species. Its proof is heavily due to Šokurov's result.

Proposition 4.10. *If* Y *is a Fano* 3*-fold of the first species, i.e., with* $B_2=1$ *and of index* 1*, then* X *is not a Fano* 3*-fold.*

Proof. Case 1. $|-K_Y|$ is not very ample. By II of § 2, $(-K_Y)^3=2$ or 4. If X were a Fano 3-fold, $(-K_X)^3$ would be equal to 2 by (4.3). Since X is of index 1 and $B_2(X) \geq 2$, $|-K_X|$ would be very ample through II of § 2 and hence $X \cong \boldsymbol{P}^3$ by the anticanonical map, which is a contradiction.

Case 2. $|-K_Y|$ is very ample. By III of § 2, Y has a 1-dimensional family of rational curves Z with $(-K_Y \cdot Z)=1$. Let S be the union of such Z's. Since $\rho(Y)=1$, S is ample. Hence C meets a rational curve Z with $(-K_Y \cdot Z)=1$. If $C \cap Z$ is 0-dimensional, then X is not a Fano 3-fold by Proposition 4.7. If $C=Z$ and $N_{Z/Y} \not\cong \mathcal{O} \oplus \mathcal{O}(-1)$, then X is not a Fano 3-fold by Proposition 4.9. If $C=Z$ and $N_{Z/Y} \cong \mathcal{O} \oplus \mathcal{O}(-1)$, then we can show that C meets other $d+1$ rational curves Z' (counted with multiplicity) with $(-K_Y \cdot Z')=1$, where d is the degree of the surface which is the union of all the deformations of Z. Hence X is not a Fano 3-fold by Proposition 4.7.

§ 5. Classification of Fano 3-folds with $B_2=2$

Let X be a Fano 3-fold with $B_2=2$. Since $N(X) \cong \boldsymbol{R}^2$, $NE(X)$ has just two edges, i.e., extremal rays R_1 and R_2 by (3.1). Set $f_i=\mathrm{cont}_{R_i}\colon X \to Y_i$, $\mu_i=\mu(R_i)$ and $l_i=l_{R_i}$ for $i=1, 2$. Let L_i be an ample generator of $\mathrm{Pic}\, Y_i$ and put $H_i=f_i^* L_i$. The proof of the following is one of the essential parts of the classification of Fano 3-folds with $B_2=2$.

Theorem 5.1. (1) *Pic X is the direct sum of f_1^*Pic Y_1 and f_2^*Pic Y_2. $\{H_1, H_2\}$ is a basis of Pic X and $\{l_2, l_1\}$ is the dual basis of $N_Z(X)$. Moreover, $-K_X \approx \mu_2 H_1 + \mu_1 H_2$.*

(2) *If R_2 is of type E_2, E_3, E_4 or E_5, then $(D_2 \cdot l_1)=1$, where D_2 is the exceptional divisor of R_2.*

Proof of (1) (mainly in the case X is primitive).

If R_1 or R_2 is of D_3-type, then X is a Fano $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ and we know that X is isomorphic to $\boldsymbol{P}^1 \times \boldsymbol{P}^2$ or the blow-up of $\boldsymbol{P}^3$ along a line. For these X, (1) is easily checked. Hence we may assume that neither R_1 nor R_2 is of D_3-type.

By (3.2), we have two exact sequences

$$0 \longrightarrow \text{Pic } Y_i \xrightarrow{f_i^*} \text{Pic } X \xrightarrow{(\ \cdot l_i)} Z \longrightarrow 0 \qquad i=1,\ 2.$$

By the definition of $f_i = \text{cont}_{R_i}$, it is obvious that f_1^* Pic $Y_1 \cap f_2^*$ Pic $Y_2 = \{0\}$. Let a be the order of the quotient group Pic $X/\sum_{i=1,2} f_i^*$ Pic Y_i. By the above exact sequences, both $(H_1 \cdot l_2)$ and $(H_2 \cdot l_1)$ are equal to a. Since $(H_i \cdot l_i)=0$ and $(-K_X \cdot l_i)=\mu_i$ for $i=1, 2$, we have

$$a(-K_X) \approx \mu_2 H_1 + \mu_1 H_2. \tag{5.2}$$

Hence it suffices to show that $a=1$. We will deduce it from the equality

$$24a = a(-K_X \cdot c_2(X)) = \mu_2(H_1 \cdot c_2(X)) + \mu_1(H_2 \cdot c(X)) \tag{5.3}$$

and an estimation of $(c_2(X) \cdot H_i)$.

By the Riemann-Roch Theorem, we have

Lemma 5.4. *Let D be an effective divisor on X, then $(c_2(X) \cdot D) = 6\chi(\mathcal{O}_D) + 6\chi(\mathcal{O}_D(D)) - 2(D^3) - ((-K_X)^2 \cdot D)$.*

By the lemma, we have

(5.5)

type of R	$E_1=E_{1,a}$	E_2	E_3 or E_4	E_5
$(c_2(X) \cdot H)$	$24/r+$ deg C	$24/r$	$24/r$	$45/r$

C_1	C_2	D_1	D_2	D_3
deg $\Delta+6$	6	$12-(K_{X_\eta})^2$	4	3

,

where r is the largest number which devides $-K_X+D$ (resp. $-K_X+2D$, $2(-K_X)+D$), D being the exceptional divisor of R, in Pic X if R is of type E_1 or E_3 or E_4 (resp. E_2, E_5), C is the center of the blowing-up f and deg C is $(L \cdot C)$ if R is of E_1-type, Δ is the discriminant locus $\{y \in Y | X_y$ is not smooth} of f if R is of C_1-type, where $Y \cong \boldsymbol{P}^2$ by Proposition 3.7, in the case R is of type C_1 or C_2, and X_η is the generic fibre of f if f is of D_1-type.

For example, in the case R is of type E_3 or E_4, applying the lemma to the exceptional divisor D, we have $(c_2(X) \cdot D) = 6+0-4-2=0$. Since $-K_X + D \approx rH$, $(c_2(X) \cdot H)$ is equal to $(1/r)\{(c_2(X) \cdot -K_X) + (c_2(X) \cdot D)\} = 24/r$.

If R is of type C_1, then $Y \cong \boldsymbol{P}^2$ and by the formula $\Delta \approx -4K_{P^2} - f_*(-K_X)^2$ (Proposition 6.2), we have $\deg \Delta < 12$. It follows that

$$(5.6) \qquad 7 \le (c_2(X) \cdot H) \le 17 \qquad \text{if } R \text{ is of } C_1\text{-type.}$$

If R is of type D_1, then $1 \le (K_{X_\eta})^2 \le 6$ by the classification of extremal rays. Hence we have

$$(5.7) \qquad 6 \le (c_2(X) \cdot H) \le 11 \qquad \text{if } R \text{ is of } D_1\text{-type.}$$

We consider the case in which X is primitive. By Theorem 1.6, at least one of R_1 and R_2 (say R_1) is of type C_1 or C_2. R_2 is not E_1-type since X is primitive. Hence by (5.5), (5.6) and (5.7), we have

$$(5.8) \qquad (c_2(X) \cdot H_2) \le 24 \quad \text{or} \quad = 45.$$

Case 1. R_1 is of C_1-type

i) $\mu_2 = 1$. $a(-K_X) \approx H_1 + H_2$. If $(c_2(X) \cdot H_2) = 45$, then we have by (5.3) and (5.6), $51 = 7+45 \le 24a \le 17+45 = 62$, which is a contradiction. Hence $(c_2(X) \cdot H_2) \le 24$ by (5.8) and we have $24a = (c_2(X) \cdot H_1) + (c_2(X) \cdot H_2) \le 17+24 = 41$. It follows that $a=1$.

ii) $\mu_2 = 2$. $a(-K_X) \approx 2H_1 + H_2$. If a were even, then H_2 would be divisible by 2 in Pic X, which contradicts to our choice of H_2. Hence a is odd. Since R_2 is of type E_2 or C_2 or D_2 (see § 3), $(c_2(X) \cdot H_2) \le 24$ by (5.5). Hence we have $24a \le 2 \cdot 17 + 24 = 58$ by (5.3) and (5.6). It follows that $a=1$.

Case 2. R_1 is of C_2-type

i) $\mu_2 = 1$. $a(-K_X) \approx H_1 + 2H_2$. By the same reason as above, a is odd. By (5.3) and (5.5), we have $12a = 3 + (c_2(X) \cdot H_2)$. By (5.8), we have $a=1$.

ii) $\mu_2 = 2$. $a(-K_X) \approx 2H_1 + 2H_2$. By (5.3), $12a = 6 + (c_2(X) \cdot H_2)$. In the case R_2 is of E_2-type, from the equality $12a = 6 + (24/r)$, we obtain $a=1$ and $r=4$. If R_2 is not of E_2-type, then $(c_2(X) \cdot H_2) \le 6$ and we have $a=1$.

In the case X is imprimitive, (1) can be proved in a way quite similar to the primitive case by Lemma 5.9 below. Since X is imprimitive, R_1 or R_2 is of E_1-type.

Lemma 5.9. *If R_1 is of E_1-type, we have*

(1) $\deg C_1 \leq (r_1-(\mu_2/a))^2 d_1$, *where* C_1 *is the center of* $f_1 = \mathrm{cont}_{R_1}$ *and* $d_1=(L_1^3)$.

(2) $(c_2(X)\cdot H_1) \leq 31$.

(3) $(c_2(X)\cdot H_1) \leq 25$ *if* $a \leq 3$, ≤ 23 *if* $a \leq 2$ *and* ≤ 17 *if* $a=1$.

(4) $(c_2(X)\cdot H_1) \leq 20$ *if* $a \leq 3$ *and* $\mu_2=2$.

Proof. (1) Since both H_1 and H_2 are numerically effective, $(H_1\cdot H_2^2) \geq 0$ (cf. [2]). On the other hand, by (4.2) and $-K_X \approx r_1H_1-D_1$, we have

$$\begin{aligned} 0 \leq \mu_1^2(H_1\cdot H_2^2) &= (H_1\cdot(a(-K_X)-\mu_2H_1)^2) \\ &= (H_1\cdot((ar_1-\mu_2)H_1-aD_1)^2) \\ &= (ar_1-\mu_2)^2(H_1^3)+a^2(H_1\cdot D_1^2). \end{aligned}$$

Since $(H_1\cdot D_1^2) = -\deg C_1$ and $(H_1^3)=(L_1^3)=d_1$, we have our assertion.

(2) By Proposition 4.10, the index r_1 of the Fano 3-fold Y_1 is greater than 1. Hence by I in § 2, the possibility of the pair (r_1, d_1) is limited to (4, 1), (3, 2) and $(2, d_1)$, $1 \leq d_1 \leq 5$. Hence by (1), we have $(c_2(X)\cdot H_1) = (24/r_1)+\deg C_1 \leq \max_{(r_1,d_1)}[(24/r_1)+(r_1-(1/a))^2 d_1]=31$, where [ ] is the Gauss symbol. This shows (2). (3) and (4) are proved in a similar way.

q.e.d.

Proof of (2) *in Theorem* 5.1. Here we only prove (2) in the case X is primitive.

Case 1. R_1 is of C_1-type. By (1) of Theorem 5.1, $-K_X \approx \mu_2H_1+H_2$. Since $H_2\cdot D_2 \approx 0$ and $H_1^2 \approx 2l_1$, we have $2(l_1\cdot D_2)=(H_1^2\cdot D_2)=(1/\mu_2^2)((-K_X)^2\cdot D_2)=1$ if R_2 is of type E_2 or E_5, and $=2$ if R_2 is of type E_3 or E_4. Hence R_2 is of type E_3 or E_4 and $(l_1\cdot D_2)=1$.

Case 2. R_1 is of type C_2. By (1) of Theorem 5.1, $-K_X \approx \mu_2H_1+2H_2$. R_2 is not of type E_3 or E_4, because $24=(c_2(X)\cdot H_1)+2(c_2(X)\cdot H_2)=6+(48/r)$ has no integral solution. Hence R_2 is of type E_2 or E_5 and we have $(l_1\cdot D_2)=(H_1^2\cdot D_2)=(1/\mu_2^2)((-K_X)^2\cdot D_2)=1$. q.e.d.

Now we classify the primitive Fano 3-folds with $B_2=2$ by using Theorem 5.1. We denote by $(*-**)$ the case in which the ray R_1 is of $*$-type and R_2 of $**$-type. We may assume that R_1 is of C_1 or C_2-type.

Case 1. R_2 is also of C_1 or C_2-type. Both f_1 and f_2 are conic bundles over $\boldsymbol{P}^2$. Consider the morphism $f=(f_1, f_2)\colon X \to \boldsymbol{P}^2\times\boldsymbol{P}^2$. By the definition of f_i, f is finite.

Claim. f_*X is a divisor of bidegree $(2/\mu_2, 2/\mu_1)$.

Proof. Put $M_i=\pi_i^*\mathcal{O}_{P^2}(1)$ for $i=1, 2$, where π_i: $\boldsymbol{P}^2\times\boldsymbol{P}^2\to\boldsymbol{P}^2$ is the projection onto the i-th factor. Since $f^*M_i\approx H_i$ and $H_i^2\approx(2/\mu_i)l_i$, we have

$$(f_*X\cdot\text{point}\times\text{line})=(f_*X\cdot M_1^2\cdot M_2)=(f^*M_1^2\cdot f^*M_2)$$
$$=(H_1^2\cdot H_2)=\frac{2}{\mu_1}(l_1\cdot H_2)=\frac{2}{\mu_1}$$

by Theorem 5.1. It follows that $f_*X\approx(2/\mu_2)M_1+(2/\mu_1)M_2$. q.e.d.

Claim. If the natural morphism α: $X\to f(X)$ is birational, then it is an isomorphism.

Proof. Assume that α is birational. Since α is finite, we have $-K_X\sim\alpha^*(-K_{f(X)})+\Delta$ by the residue formula, where Δ is the zeros of the conductor ideal. Since $f(X)$ is of bidegree $(2/\mu_2, 2/\mu_1)$, we have

$$\alpha^*(-K_{f(X)})\approx f^*\Big(3-\frac{2}{\mu_2}\Big)M_1+f^*\Big(3-\frac{2}{\mu_1}\Big)M_2\approx\mu_2H_1+\mu_1H_2.$$

Hence by Theorem 5.1, Δ is empty, that is, $X\to f(X)$ is an isomorphism. q.e.d.

(C_1-C_1) f is generically one-to-one or two-to-one. In the former case, f is an embedding and the image is a divisor of bidegree (2, 2). In the latter case, X is double cover of $f(X)$. $f(X)\subset\boldsymbol{P}^2\times\boldsymbol{P}^2$ is of bidegree (1, 1) and smooth because both $\pi_1|_{f(X)}$ and $\pi_2|_{f(X)}$ are equidimensional (type 2) in Theorem 1.7).

(C_1-C_2) X is isomorphic to a smooth divisor on $\boldsymbol{P}^2\times\boldsymbol{P}^2$ of bidegree (1, 2) (type 5) in Theorem 1.7).

(C_2-C_2) X is isomorphic to W_6, a smooth divisor on $\boldsymbol{P}^2\times\boldsymbol{P}^2$ of bidegree (1, 1) (type 6) in Theorem 1. 7).

Case 2. R_2 is of D_1, D_2 or D_3-type. Consider the morphism $f=(f_1, f_2)$: $X\to\boldsymbol{P}^2\times\boldsymbol{P}^1$. f is finite and surjective. By Theorem 5.1, we have $\deg f=(H_1^2\cdot H_2)=(2/\mu_1)(l_1\cdot H_2)=2/\mu_1$. Therefore X is a double cover of $\boldsymbol{P}^2\times\boldsymbol{P}^1$ or isomorphic to $\boldsymbol{P}^2\times\boldsymbol{P}^1$ (type 1), 4) and 7) in Theorem 1.7).

Case 3. R_2 is of E_2, E_3, E_4 or E_5-type. By (2) of Theorem 5.1, the degree of $f_1|_{D_2}$: $D_2\to\boldsymbol{P}^2$ is equal to $(H_1^2\cdot D_2)=(2/\mu_1)(l_1\cdot D_2)=2/\mu_1=1$ or 2. Hence if $D_2\cong\boldsymbol{P}^2$, then $\mu_1=2$ and $\deg f_1|_{D_2}=1$, that is, if R_2 is of E_2 or E_5-type, then f_1 is a $\boldsymbol{P}^1$-bundle and D_2 is a section of f_1. Therefore X is isomorphic to $\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(1))$ or $\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(2))$ according as R_2 is of E_2-type or E_5-type (type 8) or 9) in Theorem 1.7).

In the case R_2 is of type E_3 or E_4-type, then $\mu_1=1$ and $\deg f_1|_{D_2}=2$. Since $\mathcal{O}_F(-K_X-D_2)$ is trivial for every fibre F of f_1, it is isomorphic to

$f_1^*\mathcal{O}_P(n)$ for some integer n. Since $\mathcal{O}_{D_2}(-K_X-D_2)\cong\mathcal{O}_{D_2}(2)$, $n=2$. Since $\mathcal{O}_{D_2}(-K_X)\cong\mathcal{O}_{D_2}(1)$, $f_{1,*}\mathcal{O}_{D_2}(-K_X)\cong\mathcal{O}_P\oplus\mathcal{O}_P(1)$. Hence by the exact sequence $0\to\mathcal{O}_X(-K_X-D_2)\to\mathcal{O}_X(-K_X)\to\mathcal{O}_{D_2}(-K_X)\to 0$, we have $0\to\mathcal{O}_P(2)\to f_{1,*}\mathcal{O}_X(-K_X)\to\mathcal{O}_P\oplus\mathcal{O}_P(1)\to 0$. Hence $f_{1,*}\mathcal{O}_X(-K_X)=\mathcal{O}_P\oplus\mathcal{O}_P(1)\oplus\mathcal{O}_P(2)$. Since f_2 is conic bundle, X is a divisor of $\boldsymbol{P}(\mathcal{O}_P\oplus\mathcal{O}_P(1)\oplus\mathcal{O}_P(2))$. Since $-K_X\sim L|_X$, X is linearly equivalent to $2L$, where L is the tautological line bundle. Let S be the section of $\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))$ corresponding to the exact sequence $0\to\mathcal{O}(1)\oplus\mathcal{O}(2)\to\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)\to\mathcal{O}\to 0$. Since $L|_S$ is trivial and X is Fano, $X\cap S=\phi$. $\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2))-S$ is isomorphic to the line bundle $Z=\boldsymbol{V}(\mathcal{O}_{V_7}((-1/2)K_{V_7}))$ over V_7. It is easy to see the restriction of $Z\to V_7$ to X is a double covering. (type 3) in Theorem 1.7)

So we have proved that every primitive Fano 3-fold with $B_2=2$ belongs to a class in Theorem 1.7. It is easily verified that every class in Theorem 1.7 is a family of Fano 3-folds parametrized by an irreducible variety. Moreover every two different classes cannot be deformed to each other because they have different values of $(-K_X)^3$. Therefore the primitive Fano 3-folds with $B_2=2$ have exactly 9 deformation types as described in Theorem 1.7.

The imprimitive Fano 3-folds with $B_2=2$ are also classified by using Theorem 5.1. For example, we consider the case (E_1-E_1). In this case, X is the graph of birational map between Y_1 and Y_2. Let C_i be the center of the blowing-up f_i, r_i the index of Y_i and put $d_i=(L_i^3)$.

(5.11)
$$\deg C_1=(r_1-1)^2d_1-(r_2-1)d_2$$
$$(-K_X)^3=(3r_1-2)d_1+(3r_2-2)d_2.$$

Proof. $-K_X\sim f_i^*(-K_{Y_i})-D_i\sim r_iH_i-D_i$ for $i=1, 2$. On the other hand, $-K_X\approx H_1+H_2$ by Theorem 5.1. Hence $D_1\approx(r_1-1)H_1-H_2$ and $D_2\approx(r_2-1)H_2-H_1$. Since $(H_i^2\cdot D_i)=0$ for both $i=1, 2$, we have $(H_1^2\cdot H_2)=(r_1-1)(H_1^3)=(r_1-1)d_1$ and $(H_1\cdot H_2^2)=(r_2-1)d_2$. Therefore

$$\begin{aligned}\deg C_1&=-(H_1\cdot D_1^2)=-(H_1\cdot((r_1-1)H_1-H_2)^2)\\&=-(r_1-1)^2(H_1^3)+2(r_1-1)(H_1^2\cdot H_2)-(H_1\cdot H_2^2)\\&=-(r_1-1)^2d_1+2(r_1-1)^2d_1-(r_2-1)d_2\\&=(r_1-1)^2d_1-(r_2-1)d_2.\end{aligned}$$

The second equality is obtained in the same way. q.e.d.

By Proposition 4.10, both r_1 and r_2 are greater than 1. By (5.11),

$$\deg C_1+\deg C_2=(r_1-1)(r_1-2)d_1+(r_2-1)(r_2-2)d_2.$$

Hence either r_1 or r_2 is greater than 2. By I of § 2, we can show that Fano

3-folds with $B_2=2$ of type (E_1-E_1) have the following 6 deformation types:

	$(-K_X)^3$	Y_1	$\deg C_1$	$p_a(C_1)$	Y_2	$\deg C_2$	$p_a(C_2)$
1)	20	$\boldsymbol{P}^3$	6	3	$\boldsymbol{P}^3$	6	3
2)	24	$\boldsymbol{P}^3$	5	1	Q	5	1
3)	26	$\boldsymbol{P}^3$	5	2	V_4	1	0
4)	28	Q	4	0	Q	4	0
5)	30	$\boldsymbol{P}^3$	4	0	V_5	2	0
6)	34	Q	3	0	V_5	1	0

Here $p_a(C_1)$ can be computed by (4.3).

As a corollary of the classification, we have

Proposition 5.12. *An imprimitive Fano 3-fold X with $B_2=2$ satisfies one of the following*:

(a) *X is isomorphic to the blow-up of $\boldsymbol{P}^3$ along a smooth irreducible curve which is a scheme-theoretic intersection of cubics.*

(b) *X is isomorphic to the blow-up of $Q\subset\boldsymbol{P}^4$ along a smooth irreducible curve which is a scheme-theoretic intersection of members of $|\mathcal{O}_Q(2)|$.*

(c) *X is isomorphic to the blow-up of V_d, $1\leq d\leq 5$, along an elliptic or rational curve which is a scheme-theoretic intersection of members of $|-(1/2)K_{V_d}|$.*

§ 6. Fano conic bundle

First we recall some general properties of a conic bundle $f\colon X\to S$.

Definition 6.1. A morphism $f\colon X\to S$ from a smooth variety X onto a smooth surface S is a *conic bundle* if every fibre is isomorphic to a conic, i.e., a scheme of zeros of a nonzero homogeneous form of degree 2 on $\boldsymbol{P}^2$. The set $\{s\in S\,|\,f^{-1}(s)$ is not smooth$\}$ is called the *discriminant locus* of f and denoted by Δ_f.

Proposition 6.2. (1) *f is flat and $f_*\omega_X^{-1}$ is a vector bundle of rank* 3 *and the natural map $X\to\boldsymbol{P}(f_*\omega_X^{-1})$ is an embedding. In particular, X is projective if S is projective.*

(2) *If Δ_f is non-empty, then it is a curve with only ordinary double points and Sing $\Delta_f=\{s\in S\,|\,f^{-1}(s)$ is non-reduced$\}$.*

(3) *If a smooth rational curve C is a connected component of Δ_f, then $f^{-1}(C)$ is reducible.*

(4) $\Delta_f\approx -f_*K^2_{X/S}\approx -f_*K^2_X-4K_S$.

Proof. For (1) and (2) see Chapter I [1]. Let C be a smooth rational curve which is a connected component of Δ_f. By (2), $f^{-1}(s)$ is a union of two smooth rational curves, for every $s \in C$. Let $\tilde{C} \subset \mathrm{Hilb}_X$ be the parametrizing space of those rational curves. Since $C \cong \boldsymbol{P}^1$ and $\tilde{C}$ is isomorphic to an étale double cover of C, $\tilde{C}$ is disconnected. Hence $f^{-1}(C)$ is reducible. This shows (3). Let C be a smooth curve on S intersecting Δ_f transversally and such that the surface $f^{-1}(C)$ is smooth. It is easy to see that rational equivalence classes of such curves generate Pic S. $Y := f^{-1}(C)$ is isomorphic to the blow-up of a $\boldsymbol{P}^1$-bundle Y_0 over C. Since $(K_{Y_0/C})^2 \sim 0$, $-K^2_{Y/C}$ is rationally equivalent to the sum of the singular points of fibres with all coefficients 1. It follows that $-f_* K^2_{Y/C} \sim \Delta_f \cdot C$. Since $\omega_{Y/C}$ is canonically isomorphic to $\omega_{X/S}|_Y$, we have $\Delta_f \cdot C \sim -f_* K^2_{Y/C} \sim -f_*(K^2_{X/S} \cdot Y) \sim -f_*(K^2_{X/S} \cdot f^* C) \sim -(f_* K^2_{X/S}) \cdot C$. In particular, $(\Delta_f \cdot C) = -(f_* K^2_{X/S} \cdot C)$, that is, $\Delta_f \approx -f_* K^2_{X/S}$. On the other hand, since $f_* K_X \sim -2S$, we have $-f_* K^2_{X/S} \sim -f_*(K_X - f^* K_S)^2 \sim -f_* K^2_X - 4K_S$, which shows (4). q.e.d.

Proposition 6.3. *Let $f: X \to S$ be a conic bundle over a projective surface S. Then we have*

(1) $\rho(X) - \rho(S) = 1$ *if and only if $f^{-1}(C)$ is irreducible for every irreducible curve C on S.*

(2) *Assume that $f^{-1}(C)$ is reducible for an irreducible curve C on S. Then* i) *C is smooth,* ii) *$f^{-1}(C)$ is a union of E_1 and E_2 such that $f|_{E_i}: E_i \to C$ is a $\boldsymbol{P}^1$-bundle for $i=1, 2$ and* iii) *there are a conic bundle $g_i: Y_i \to S$ and a morphism $\alpha_i: X \to Y_i$ which is the contraction of all fibres of $f|_{E_i}$, such that $g_i \circ \alpha_i = f$ for both $i=1, 2$. Moreover $\Delta_{g_1} = \Delta_{g_2}$, $\rho(Y_1) = \rho(Y_2)$, $\Delta_f = \Delta_{g_i} \coprod C$ and $\rho(X) = \rho(Y_i) + 1$ for $i=1, 2$.*

By an induction on $\rho(X) - \rho(S)$, we have

Corollary 6.4. *Δ_f has n connected components $C_1, \cdots, C_n$ such that C_i is smooth and $f^{-1}(C_i)$ is reducible for $i=1, \cdots, n$, where $n = \rho(X) - \rho(S) - 1$.*

Proposition 6.5. *Let the situation be the same as in* (2) *of Proposition* 6.3 *and assume, in addition, that X is a Fano* 3*-fold. Then we have*

(1) *If Y_1 is not a Fano* 3*-fold, then $C \cong \boldsymbol{P}^1$, $E_1 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $\mathcal{O}_{E_1}(E_1) \cong \mathcal{O}(-1, -1)$.*

(2) *Either Y_1 or Y_2 is a Fano* 3*-fold.*

Proof. (1) is an immediate consequence of Proposition 4.5. Assume that neither Y_1 nor Y_2 is a Fano 3-fold. Since every fibre of f is connected $s = E_1 \cap E_2$ is not empty. Since $\mathcal{O}_{E_1}(E_1)$ is negative by (1), $(s.E_1) = (s \cdot E_1|_{E_1})_{E_1}$ is negative. On the other hand, since s moves in $E_2 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $(s \cdot E_1)$ is non-negative, which is a contradiction. q.e.d.

The following is another important property of Fano conic bundles.

Proposition 6.6. *Let f: $X \to S$ be a conic bundle and assume that X is a Fano 3-fold. Let E be an irreducible reduced curve on S such that $(E^2)<0$ and $f^{-1}(E)$ is irreducible. Then we have*

(1) *E is an exceptional curve of the first kind.*

(2) *$f|_{f^{-1}(E)}$, is a trivial $\boldsymbol{P}^1$-bundle over E, and*

(3) *there are a Fano conic bundle f': $X' \to S'$ and A: $X \to X'$, the contraction of the horizontal fibres of $f|_{f^{-1}(E)}$ such that $f' \circ A = \alpha \circ f$, where α: $S \to S'$ is the blow down of E.*

Proof. Let Z be a curve on X such that $f(Z)=E$. By (3.1), there are irreducible reduced curves C_i, $1 \leq i \leq n$, such that $[C_i]$ is on an edge of $NE(X)$ and that $Z \approx \sum_{i=1}^{n} a_i C_i$ for some positive numbers a_i. Since $(f_* Z \cdot E)$ is negative, $(f_* C_i \cdot E)$ is negative for some i. It follows that there is an irreducible reduced curve C such that $[C]$ belongs to an extremal ray R of X and $f(C)=E$. Since $(C \cdot f^{-1}(E))$ is equal to $(f_* C \cdot E)$ and negative, R is of type E_1, E_2, E_3, E_4 or E_5 and $f^{-1}(E)$ is the exceptional divisor of R by the classification of extremal rays. Since $f^{-1}(E)$ has a morphism $\varphi = f|_{f^{-1}(E)}$ onto $E \cong \boldsymbol{P}^1$ which does not contract the curve C belonging to the ray R, and since C and the fibre of φ are not numerically equivalent, R is of E_1-type. Hence $f^{-1}(E)$ is smooth and has a $\boldsymbol{P}^1$-bundle structure ψ: $f^{-1}(E) \to T$ over a smooth curve T which contracts C to a point. It is easy to see that the morphism (φ, ψ): $f^{-1}(E) \to E \times T$ is an isomorphism and both E and T are rational, which shows (2). By (2), C is a section of φ and we have $(E^2)=(f_* C \cdot E)=(C \cdot f^{-1}(E))=-1$ from which (1) follows. Since $N_{f^{-1}(E)/X}$ is isomorphic to $\varphi^* N_{E/S}$ and is not negative, the X' in $A = \mathrm{cont}_R$: $X \to X'$, is a Fano 3-fold by Proposition 4.5. It will be clear that the morphism f': $X' \to S$ is well defined and is a conic bundle. q.e.d.

Corollary 6.7. *Let f: $X \to S$ be a Fano conic bundle. Then we have*

(1) *every irreducible curve E on S with $(E^2)<0$ is an exceptional curve of the first kind, and*

(2) *if E is an exceptional curve of the first kind, then Δ_f is disjoint from E or contains E as a connected component.*

Proof. (1) follows from (1) of Proposition 6.6, (2) of Proposition 6.3 and (2) of Proposition 6.5. If $f^{-1}(E)$ is reducible then Δ_f contains E as a connected component by Proposition 6.3. If $f^{-1}(E)$ is irreducible, then Δ_f is disjoint from E by Proposition 6.6. Hence we have (2). q.e.d.

For the classification of imprimitive Fano 3-folds with $B_2 \geq 3$, it is necessary to classify the curves C on a Fano conic bundle Y such that the blow-up of Y along C is a Fano 3-fold. Propositions 6.8 and 6.10 give strong necessary conditions on $C \subset Y$.

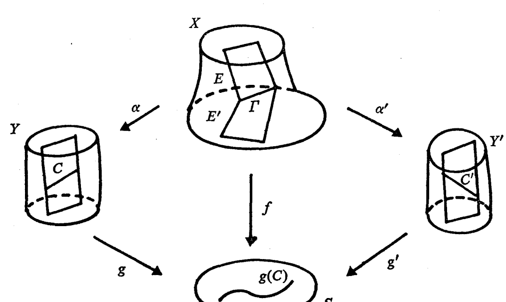
Proposition 6.8. *Let g: $Y \to S$ be a Fano conic bundle and C a smooth irreducible curve on Y. Assume that the blow-up X of Y along C is a Fano 3-fold. Then we have*

(1) *C does not meet any singular fibre of g; and* (2) *C is either* (i) *a smooth fibre of g or* (ii) *a subsection of g, i.e., $g|_C$ is an embedding $C \hookrightarrow S$. In the case* (i), *X is a conic bundle over S', the blow-up of S at $g(C)$. In the case* (ii), *$f = g \circ \alpha$ is a conic bundle such that $\varDelta_f = \varDelta_g \coprod g(C)$, where α: $X \to Y$ is the blowing-up along C.*

Proof. (1) Assume that C meets a singular fibre. Then C meets l, an irreducible component of a reducible fibre or the reduced part of a multiple fibre. In both cases, $(-K_Y \cdot l) = 1$. Hence if $C \neq l$, then X is not a Fano 3-fold by Proposition 4.7. If $C = l$ and l is an irreducible component of a reducible fibre, C meets another component of the reducible fibre and hence X is not a Fano 3-fold. If $C = l$ and l is the reduced part of a multiple fibre, then $N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$ and hence X is not a Fano 3-fold by Proposition 4.9.

(2) If $g(C)$ is a point, then by (1), C is a smooth fibre of g. Assume that $g(C)$ is not a point and X is a Fano 3-fold. Let Z be the proper transform of a fibre $g^{-1}(s)$. Then we have $(D \cdot Z) = (\alpha^*(-K_Y) + K_X \cdot Z) = (-K_Y \cdot \alpha_* Z) - (-K_X \cdot Z) < (-K_Y \cdot g^{-1}(s)) = 2$, where D is the exceptional divisor of α. Hence, for every $s \in S$, $g^{-1}(s)$ is disjoint from C or intersect C transversally at one point. Therefore $g|_C$ is an embedding. The latter part of (2) is almost clear. q.e.d.

If C is a subsection of a conic bundle g: $Y \to S$ such that $g(C) \cap \varDelta g = \phi$ and X is the blow-up of Y along C, then there are a conic bundle g': $Y' \to S$ (the elementary transform of g along C) and a morphism α': $X \to Y'$ such that $\varDelta_{g'} = \varDelta_g$, $\rho(Y) = \rho(Y')$, satisfying the two conditions (a) $g \circ \alpha = g' \circ \alpha'$ and (b) α' is birational and an irreducible reduced curve Z on X is contracted to a point by α' if and only if Z is the proper transform of a smooth fibre of g meeting C (Proposition 6.3).



(6.9) $$(-K_{Y'})^3=(-K_Y)^3+2(g(C)^2)_S-4(-K_{Y/S}\cdot C).$$

Proof. Applying (4.3) to α and α', we have

$$\begin{aligned}(-K_X)^3&=(-K_Y)^3-2\{(-K_Y\cdot C)-p_a(C)+1\}\\&=(-K_{Y'})^3-2\{(-K_{Y'}\cdot C')-p_a(C')+1\},\end{aligned}$$

where C' is the center of the blowing-up $\alpha'\colon X\to Y'$. Since $C\cong C'\cong g(C)$ and $(g^*K_S\cdot C)=(g'^*K_S\cdot C')$, we have

$$(-K_{Y'})^3=(-K_Y)^3-2(-K_{Y/S}\cdot C)+2(-K_{Y'/S}\cdot C').$$

Since C and C' are subsections, $(-K_{Y/S}\cdot C)$ and $(-K_{Y'/S}\cdot C')$ are equal to the self-intersection numbers $(C^2)_{g^{-1}g(C)}$ and $(C'^2)_{g'^{-1}g'(C')}$, respectively. Hence our assertion follows from:

Claim. $(C^2)_{g^{-1}g(C)}+(C'^2)_{g'^{-1}g'(C)}=(g(C)^2)_S$.

Let E(resp. E') be the exceptional divisor of α (resp. α') and Γ the intersection of E and E'. Since E' is the strict transform of $g^{-1}(C)$, $(C^2)_{g^{-1}g(C)}$ is equal to $(\Gamma^2)_{E'}$. In the same way, we have $(C'^2)_{g'^{-1}g'(C)}=(\Gamma^2)_E$. On the other hand, we have $(\Gamma^2)_E+(\Gamma^2)_{E'}=(E'^2\cdot E)+(E^2\cdot E')=(E+E'\cdot E\cdot E')=(f^*f_*\Gamma\cdot\Gamma)=(f_*\Gamma\cdot f_*\Gamma)_S=(g(C)^2)_S$, which shows the claim. q.e.d.

Proposition 6.10. *Let $g\colon Y\to S$ be a Fano conic bundle and C a smooth irreducible subsection of g. If the blow-up X of Y along C is a Fano 3-fold, then the elementary transform $g'\colon Y'\to S$ satisfies one of the following*:

(1) *Y' is a Fano 3-fold, and*

(2) *$C\cong \boldsymbol{P}^1$, $g|_{g^{-1}g(C)}$ is a trivial $\boldsymbol{P}^1$-bundle and $(-K_{Y/S}\cdot C)=2\{(g(C)^2)_S+1\}$.*

Proof. Applying Proposition 4.5 for α', we have $E'\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $(-K_{Y'}\cdot C')=0$ if Y' is not a Fano 3-fold. Hence $C\cong\boldsymbol{P}^1$ and $g^{-1}g(C)$ is a trivial $\boldsymbol{P}^1$-bundle. By the claim in the proof of (6.9), $(-K_{Y/S}\cdot C)+(-K_{Y'/S}\cdot C')=(g(C)^2)_S$. Hence $(-K_{Y/S}\cdot C)=(g(C)^2)_S-(-K_{Y'/S}\cdot C')=(g(C)^2)-(g'^*K_S\cdot C')-(-K_{Y'}\cdot C')=(g(C)^2)-(K_S\cdot g(C))=2\{(g(C)^2)_S+1\}$. q.e.d.

§ 7. Classification of imprimitive Fano 3-folds with $B_2\geq 3$

We show how to classify the imprimitive Fano 3-folds with $B_2\geq 3$.

Proposition 7.1. *An imprimitive Fano 3-fold X with $B_2=3$ satisfies one of the following*:

(1) *X is isomorphic to the blow-up of a Fano conic bundle over $\boldsymbol{P}^2$, and*

(2) *X is isomorphic to the blow-up of $\boldsymbol{P}^3$ or Q along a disjoint union of two smooth irreducible curves on it.*

Proof. Since X is imprimitive, X is isomorphic to the blow-up of a Fano 3-fold Y with $B_2=2$ along a smooth irreducible curve C on Y. If Y has a conic bundle structure, then X satisfies (1). Hence by Theorem 1.6, we may assume that Y is imprimitive. By Proposition 5.12 we have the following 3 cases:

Case in which Y is a blow-up of $\boldsymbol{P}^3$: Let α: $Y \to \boldsymbol{P}^3$ be the blow-up. If C is disjoint from the exceptional divisor of α, then X satisfies (2). If C meets the exceptional divisor, then C is an exceptional line of α by Corollary 4.8. X is isomorphic to the blow-up of V_7 by the proper transform of the center of α by β, where β: $V_7 \to \boldsymbol{P}^3$ is the blow-up at the point $\alpha(C)$. Since V_7 is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$, X satisfies (1).

Case in which Y is a blow-up of Q: Let α: $Y \to Q$ be the blow-up. If C is disjoint from the exceptional divisor of α, then X satisfies (2). If C is an exceptional line of α, then X is a blow-up of $\tilde{Q}$ along a curve, where $\tilde{Q}$ is the blow up of Q at $\alpha(C)$. Since $\tilde{Q}$ is isomorphic to the blow-up of $\boldsymbol{P}^3$ along a conic, X satisfies (1) or (2) by the consideration in the above case.

Case in which Y satisfies (c) *in Proposition* 5.12: We show that no blow-up of Y along a smooth irreducible curve is a Fano 3-fold. Let α: $Y \to V_d$ be the blow-up. Y has a del Pezzo fibering f: $Y \to \boldsymbol{P}^1$, and a one-dimensional family of curves l with $(-K_Y \cdot l)=1$ and contained in fibres of f. Let S be the union of such l's.

Assume that $C \cap S \neq \phi$. Then C meets an l with the above property. If $C \neq l$, then we have $(-K_X \cdot l') \leq 0$ for the proper transform l' and X is not a Fano 3-fold. If $C=l$ and C is contained in a smooth fibre of f, then C meets another l' with $(-K_Y \cdot l')=1$ and contained in the same fibre because the degree of the del Pezzo surface is equal to d and ≤ 5. Hence X is not a Fano 3-fold. In the case $C=l$ and C is contained in a singular fibre, we have a deformation of C whose general members C' are contained in smooth fibres. Since the blow-up X' along C' is not a Fano 3-fold, X is not a Fano 3-fold because the ampleness of $-K_X$ is an open condition.

Assume that $C \cap S = \phi$. Since S is not contained in the exceptional divisor of α and $\rho(V_d)=1$, $\alpha(S)$ is an ample divisor of V_d. Hence C is an exceptional line of α. X is isomorphic to the blow-up of V'_d by the proper transform Z' of Z by β, where β: $V'_d \to V_d$ is the blowing up at $\alpha(C)$.

Since V_d is of index 2 and $-K_{V'_d} \sim \beta^*(-K_{V_d})-2D$ for the exceptional divisor D of β, $-K_{V'_d}$ is divisible by 2 in Pic V'_d. By I of § 2, V'_d is not a Fano 3-fold. Since $Z' \cong Z$ is an elliptic curve and irrational X is not a Fano 3-fold by Proposition 4.5. q.e.d.

X satisfying (1) in Proposition 7.1 are classified by Proposition 4.7, 6.8, 6.9 and 6.10. X satisfying (2) are classified by Corollary 4.6 and Proposition 4.7.

Example 7.2. Let Y be a Fano 3-fold in 2) of Theorem 1.7. Then Y is *terminal*, i.e., the blow up of Y along no smooth irreducible curve is a Fano 3-fold.

Proof. Y has two conic bundle structures π and τ with ample discriminant locus. By Proposition 6.8, if the blow-up X of Y along C is a Fano 3-fold, then both $\pi(C)$ and $\tau(C)$ would be a point. But such a curve C does not exist. Hence Y is terminal. q.e.d.

Example 7.3. Let g be the $\boldsymbol{P}^1$-bundle $Y=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(2))\rightarrow\boldsymbol{P}^2$ and D the negative section of f. If the blow-up X of Y along a smooth irreducible curve C is a Fano 3-fold, then we have

(1) C is a subsection disjoint from D, and

(2) $\deg g(C)\leq 4$

Proof. Assume that X is a Fano 3-fold. Since $(-K_Y\cdot l)=1$ for every line l in D, C is disjoint from D. Hence we have (1) by Proposition 6.8. Put $m=\deg g(C)$. Since $-K_{Y/\boldsymbol{P}^2}\sim 2D+g^*\mathcal{O}_{\boldsymbol{P}}(2)$, $(-K_{Y/\boldsymbol{P}^2}\cdot C)$ is equal to $2m$ by (1). Since $(-K_{Y/\boldsymbol{P}^2}\cdot C)=2m\neq 2(m^2+1)$, the elementary transform $g'\colon Y'\rightarrow\boldsymbol{P}^2$ is a Fano $\boldsymbol{P}^1$-bundle by Proposition 6.10 and $(-K_{Y'})^3=(-K_Y)^3+2m^2-8m=62+2m^2-8m$ by (6.9). On the other hand by the classification of Fano 3-folds with $B_2=2$, for a Fano $\boldsymbol{P}^1$-bundle Z over $\boldsymbol{P}^2$ we have $(-K_Z)^3=62$, 56, 54, 48, 46, 38 or 30 (cf. the forthcoming paper). Therefore we have $m\leq 4$. q.e.d.

For every C satisfying (1) and (2) of Example 7.3, the blow-up of Y along C is a Fano 3-fold and we have

$\deg g(C)$	1	2	3	4
$(-K_X)^3$	50	40	32	26
Y'	V_7	$\boldsymbol{P}^1\times\boldsymbol{P}^2$	V_7	$\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(2))$

All Fano 3-folds with $B_2\geq 4$ are imprimitive. They can be classified by Propositions 4.7 and 6.8.

§ 8. Proof of Theorem 1.6.

Let X be a primitive Fano 3–fold with $B_2 \geq 2$. By definition, X has no extremal ray of $E_{1,a}$-type.

(8.1) X has an extremal ray of type C_1, C_2, D_1, D_2 or D_3.

Proof. Let $R_1, \cdots, R_n$ be the extremal rays of X of type $E_{1,b}$, E_2, E_3, E_4 or E_5 and $D_1, \cdots, D_n$ their exceptional divisors.

Claim: $D_i \cap D_j = \phi$ if $D_i \neq D_j$.

Assume that $s = D_i \cap D_j$ is not empty. Then $(s \cdot D_i) = (s \cdot D_i|_{D_i})_{D_i}$ is negative because $\mathcal{O}_{D_i}(D_i)$ is negative and $(s \cdot D_i)$ is nonnegative because s moves in D_j, which is a contradiction. Hence $D_i \cap D_j = \phi$.

Let Γ be the subcone of $NE(X)$ generated by $R_1, \cdots, R_n$.

Claim: $(Z \cdot D_i) \leq 0$ for every $Z \in \Gamma$ and i.

We may assume that Z belongs to R_j for some j. If $D_j = D_i$, then $(Z \cdot D_i) = (Z \cdot D_j) < 0$. If $D_j \neq D_i$, then $(Z \cdot D_i) = 0$ because $D_i \cap D_j = \phi$.

By the claim, we have $\Gamma \neq NE(X)$, e.g. $(-K_X)^2 \notin \Gamma$. Therefore, by (3.1), $NE(X)$ has an extremal ray not of type $E_{1,b}$, E_2, E_3, E_4 or E_5. q.e.d.

(8.2) If X has an extremal ray R of type D_1, D_2 or D_3, then X has another one of type C_1 or C_2.

Proof. By definition of D–type, $\rho(X) = 2$. Let R' be another extremal ray of X.

Claim: R' is not of type E_2, E_3, E_4 or E_5.

Assume to the contrary. By (3.4), the exceptional divisor D of R' is mapped to a point by $\mathrm{cont}_R: X \to \boldsymbol{P}^1$. Since every fibre of cont_R is irreducible, D is a fibre of cont_R, which contradicts (3.3).

Claim: R' is not of type D_1, D_2 or D_3.

Assume to the contrary. Since $R \neq R'$, the morphism $(\mathrm{cont}_R, \mathrm{cont}_{R'}): X \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is surjective, which contradicts $\rho(X) = 2$.

By the above two claims, R' is of type C_1 or C_2. q.e.d.

By (8.1) and (8.2), X has an extremal ray R of C_1 or C_2-type. By definition of C-type, the conic bundle $f = \mathrm{cont}_R: X \to S$ satisfies

(8.3) $\rho(X) = \rho(S) + 1$ and $f^{-1}(C)$ is irreducible for every irreducible curve C on S.

(8.4) $$S \cong \boldsymbol{P}^2 \quad \text{or} \quad \boldsymbol{P}^1 \times \boldsymbol{P}^1$$

Proof. By Proposition 3.5 and (3) of Proposition 6.6, S is a minimal rational surface. Hence S is isomorphic to $\boldsymbol{P}^2$ or $\boldsymbol{P}^1$-bundle $\boldsymbol{F}_n=\boldsymbol{P}(\mathcal{O}\oplus\mathcal{O}(n))$ over $\boldsymbol{P}^1(n\neq\pm 1)$. By (1) of Proposition 6.6, S is isomorphic to $\boldsymbol{P}^2$ or $\boldsymbol{F}_0\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. q.e.d.

(8.5) If $B_2=3$, X has an extremal ray $R'\neq R$ of type $E_{1,b}$, C_1, or C_2. Moreover if R' is of type C_1 or C_2, then $\mathrm{cont}_{R'}$ is a conic bundle over $\boldsymbol{P}^1\times\boldsymbol{P}^1$.

Proof. Let R' be an extremal ray $\neq R$. It suffices to show that R' is not of type E_2, E_3, E_4 or E_5. Assume to the contrary and let D be the exceptional divisor of R'. By (3.4), D is mapped to a point by $f\colon X\to\boldsymbol{P}^1\times\boldsymbol{P}^1$, which is a contradiction. The latter part follows from (8.4). q.e.d.

Theorem 1.6 follows from (8.3), (8.4) and (8.5).

§ 9. Fano 3-folds with $B_2\geq 6$

In this section, we prove Theorem 1.2. The following is the first step of the proof.

(9.1) An arbitrary Fano 3-fold Y with $B_2=3$ satisfies one of the following:

i) Y has a conic bundle structure.

ii) Y is isomorphic to the blow-up of $\boldsymbol{P}^3$ along a disjoint union of a line and a conic.

Proof (Outline). By Propositions 7.1 and 6.8, we may assume that Y is isomorphic to the blow-up of $V=\boldsymbol{P}^3$ or Q along the disjoint union of two smooth curves C_1 and C_2. Let Y_i be the blow-up of V along C_i for $i=1, 2$. Since $\rho(Y_i)=2$, Y_i has a unique extremal ray whose contraction f_i is not the above map $Y_i\to V$. We may assume that f_i is not a conic bundle by Proposition 6.8. If both f_1 and f_2 are morphisms onto $\boldsymbol{P}^1$, then Y has a morphism onto $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and hence Y is a conic bundle over $\boldsymbol{P}^1\times\boldsymbol{P}^1$. Hence we may assume that f_1 is a birational morphism. Let D be the exceptional divisor of f_1. If D is not of E_2-type, then D is covered by curves Z with $(-K_{Y_1}\cdot Z)=1$. On the other hand, since the image $\bar{D}\subset V$ is ample, $(D\cdot C_2')>0$ for the proper transform C_2' of C_2. Hence X is not a Fano 3-fold by Proposition 4.7, which contradicts our assumption. Hence D is of E_2-type. Then we have $(D\cdot C_2')=1$. For otherwise, there is a line in D which intersects C_2' in at least 2 points. For the proper transform Z of the line, we have $(Z\cdot -K_X)\leq 0$ which contradicts our assumption. Hence V is isomorphic to $\boldsymbol{P}^3$, the image $\bar{D}$ is a plane in V and C_2' is the proper transform of a line C_2 in V. Since $\mathcal{O}_D(D)\cong\mathcal{O}_D(-1)$ and $\mathcal{O}_{\bar{D}}(\bar{D})\cong\mathcal{O}_{\bar{D}}(1)$, C_1 is a conic in $\bar{D}$. q.e.d.

By the following fact, we can apply the results in § 6 to prove Theorem 1.2.

(9.2) A Fano 3-fold X with $B_2=4$ has a conic bundle structure.

Proof. By Theorem 1.6, X is imprimitive. Hence X is isomorphic to the blow-up of a Fano 3-fold Y with $B_2=3$ along a smooth irreducible curve C. If Y satisfies i) of (9.1), then X has a conic bundle structure by (2) of Proposition 6.8. Assume that Y satisfies ii) of (9.1) and let α: $Y \to \boldsymbol{P}^3$ the blowing up along a disjoint union of a line L and a conic Q.

Claim: C is an exceptional line of α.

Assume to the contrary. Then by Corollary 4.8, C is disjoint from the exceptional divisor of α. Let Z be a line which meets L and $\alpha(C)$ and is contained in the plane containing Q. Since Z meets Q at two points counted with multiplicity, we have $(-K_X \cdot Z')=(\beta^*\mathcal{O}_P(4)\cdot Z')-(\beta^{-1}(Q)\cdot Z')-(\beta^{-1}(L)\cdot Z')-(\beta^{-1}(\alpha(C))\cdot Z')\leq 4-2-1-1=0$ for the proper transform Z' of Z by β, where β: $X \to \boldsymbol{P}^3$ is the blowing-up along $Q \coprod L \coprod \alpha(C)$. This contradicts our assumption that X is a Fano 3-fold.

By the claim, X is isomorphic to the blow-up of V_7, the blow-up of $\boldsymbol{P}^3$ at a point, along the proper transform of $L \coprod Q$. Since V_7 is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^2$, X has a conic bundle structure by Proposition 6.8. q.e.d.

We prepare the following two lemmas for the proof of (9.5), a stronger version of (9.2).

Lemma 9.3. *Let f: $X \to S$ be a $\boldsymbol{P}^1$-bundle over S such that S has a $\boldsymbol{P}^1$-bundle structure π: $S \to \boldsymbol{P}^1$. If $f|_{f^{-1}(l)}$ is trivial for every fibre l of π, then X is isomorphic to $Z \times_{\boldsymbol{P}^1} S$ for a $\boldsymbol{P}^1$-bundle Z over $\boldsymbol{P}^1$.*

Proof. Let E be a vector bundle such that $\boldsymbol{P}(E) \cong X$. By assumption, twisting E by some line bundle, we may assume that $E|_l \cong \mathcal{O}_l^{\oplus 2}$ for every fibre of π. By the base change theorem, we have $E \cong \pi^*\pi_*E$. It follows that $X \cong \boldsymbol{P}(\pi_*E) \times_{\boldsymbol{P}^1} S$. q.e.d.

Lemma 9.4. *Let f: $X \to S$ be a Fano $\boldsymbol{P}^1$-bundle over S which has a $\boldsymbol{P}^1$-bundle structure π: $S \to \boldsymbol{P}^1$. Then one of the following holds*:

(a) *$f^{-1}(l) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ for every fibre l of π, and*

(b) *$f^{-1}(l) \cong \boldsymbol{F}_1$ for every fibre l of π.*

Proof. The $\boldsymbol{P}^1$-bundle $f^{-1}(l)$ over l is a fibre of $\pi \circ f$: $X \to \boldsymbol{P}^1$ and hence a del Pezzo surface. Therefore $f^{-1}(l) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ or $\boldsymbol{F}_1$. Since $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $\boldsymbol{F}_1$ cannot be deformed to each other, we have either (a) or (b). q.e.d.

(9.5) An arbitrary Fano 3-fold with $B_2=4$ has a conic bundle structure $f\colon X\to S$ satisfying one of the following:

(1) $S\not\cong \boldsymbol{P}^1\times\boldsymbol{P}^1$, and
(2) $S\cong \boldsymbol{P}^1\times\boldsymbol{P}^1$ and Δ_f is ample.

Proof. By (9.2), we may assume that X has a conic bundle structure $f\colon X\to\boldsymbol{P}^1\times\boldsymbol{P}^1$ with Δ_f not ample. Δ_f is a disjoint union $\coprod_{i=1}^n s_i\times\boldsymbol{P}^1$ (or $\coprod_{i=1}^n \boldsymbol{P}^1\times s_i$) for $s_1,\cdots,s_n\in\boldsymbol{P}^1$. By (3) of Proposition 6.2 and Proposition 6.3, X is isomorphic to the blow-up of a Fano $\boldsymbol{P}^1$-bundle $g\colon Y\to\boldsymbol{P}^1\times\boldsymbol{P}^1$ along a disjoint union of subsections C_i over $s_i\times\boldsymbol{P}^1$, $i=1,\cdots,n$. Since $B_2(X)=4$, we have $n=1$. By Lemma 9.4, we have either (a) $g^{-1}(\boldsymbol{P}^1\times t)\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ for every $t\in\boldsymbol{P}^1$ or (b) $g^{-1}(\boldsymbol{P}^1\times t)\cong\boldsymbol{F}_1$ for every $t\in\boldsymbol{P}^1$. C_1 meets $g^{-1}(\boldsymbol{P}^1\times t)$ transversally at one point. Since X is a Fano 3-fold, the point does not lie on the exceptional curve of the first kind on $\boldsymbol{F}_1$ in the case (b). Let $g'\colon Y'\to\boldsymbol{P}^1\times\boldsymbol{P}^1$ be the elementary transform of g with center C_1. $g'^{-1}(\boldsymbol{P}^1\times t)$ is isomorphic to the elementary transform of $g^{-1}(\boldsymbol{P}^1\times t)$ with center $C_1\cap g^{-1}(\boldsymbol{P}^1\times t)$. Therefore in the case (b), $g'^{-1}(\boldsymbol{P}^1\times t)\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ for every $t\in\boldsymbol{P}^1$. Therefore by Lemma 9.3, $Y\cong\boldsymbol{P}^1\times Z$ in the case (a) and $Y'\cong\boldsymbol{P}^1\times Z$ in the case (b) for a $\boldsymbol{P}^1$-bundle Z over $\boldsymbol{P}^1$. In both cases, X is isomorphic to the blow-up of $\boldsymbol{P}^1\times Z$ along a smooth irreducible curve C.

Case $Z\not\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. $\boldsymbol{P}^1\times Z$ is a $\boldsymbol{P}^1$-bundle over Z. Hence X is isomorphic to a conic bundle over Z or the blow-up of Z at a point. Hence X satisfies (1).

Case $Z\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. X is isomorphic to the blow-up of $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ along C. $\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1$ has three $\boldsymbol{P}^1$-bundle structure $\pi_1,\pi_2,\pi_3\colon\boldsymbol{P}^1\times\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1\times\boldsymbol{P}^1$. It is easily seen that $\pi_i(C)$ is a point or an ample divisor for some $i=1$, 2 or 3. By Proposition 6.8, if $\pi_i(C)$ is a point, then X satisfies (1) and if $\pi_i(C)$ is ample, then X satisfies (2). q.e.d.

Since a Fano 3-fold with $B_2\geq 5$ is imprimitive, we have by Proposition 6.8 and (9.5):

(9.6) A Fano 3-fold X with $B_2\geq 5$ has a conic bundle structure $f\colon X\to S$ such that $S\not\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$.

Let $f\colon X\to S$ be an arbitrary Fano conic bundle. By (1) of Corollary 6.7 and the classification of relatively minimal rational surface, we have

(9.7) $\quad S\cong\boldsymbol{P}^2, \boldsymbol{F}_1$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$ if $\rho(S)\leq 2$ and

(9.8) there is a morphism α: $S \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ which is a blowing-up at $\rho(S)-2$ points on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ if $\rho(S) \geq 3$.

Lemma 9.9. *If $\rho(S) \geq 3$, then f is a $\boldsymbol{P}^1$-bundle.*

Proof. Let $z_1=(x_1, y_1), \cdots, z_n=(x_n, y_n) \in \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the center of α. Put $E_i=\alpha^{-1}(z_i)$ and let L_i and M_i be the proper transforms of $x_i \times \boldsymbol{P}^1$ and $\boldsymbol{P}^1 \times y_i$ by α, respectively, for $i=1, \cdots, n$. Both L_i and M_i are exceptional curves of the first kind by (1) of Corollary 6.7.

Claim: Δ_f is disjoint from E_i, L_i and M_i for every i.

Assume that Δ_f meets E_i. Then by (2) of Corollary 6.7, Δ_f contains E_i as a connected component. Hence Δ_f meets L_i. But L_i is a connected component of neither E_i nor $\Delta_f - E_i$, which contradicts Corollary 6.7. It follows that $\Delta_f \cap E_i = \phi$. In the cases of L_i and M_i, the proof is the same.

By the calim, Δ_f is contained in $S - \cup E_i - \cup L_i - \cup M_i = \boldsymbol{P}^1 \times \boldsymbol{P}^1 - \cup x_i \times \boldsymbol{P}^1 - \cup \boldsymbol{P}^1 \times y_i$. Since this surface is affine and Δ_f is complete, Δ_f is empty, which shows the lemma. q.e.d.

The following is a key to the proof of Theorem 1.2.

Proposition 9.10. *If f: $X \to S$ is a Fano conic bundle and $\rho(S) \geq 3$, then f is a trivial $\boldsymbol{P}^1$-bundle.*

Proof. By Lemma 9.9 and (3) of Proposition 6.6, X is isomorphic to $Y \times_\alpha S$ for a Fano $\boldsymbol{P}^1$-bundle g: $Y \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Since both $f|_{f^{-1}(L_1)}$ and $f|_{f^{-1}(M_1)}$ are trivial $\boldsymbol{P}^1$-bundles, so are both $g|_{g^{-1}(x_1 \times \boldsymbol{P}^1)}$ and $g|_{f^{-1}(\boldsymbol{P}^1 \times y_1)}$. By Lemma 9.4, $g|_{g^{-1}(x \times \boldsymbol{P}^1)}$ is trivial for every $x \in \boldsymbol{P}^1$. By Lemma 9.3, $Y \cong Z \times \boldsymbol{P}^1$ for a $\boldsymbol{P}^1$-bundle Z over $\boldsymbol{P}^1$. Since $Z \cong g^{-1}(\boldsymbol{P}^1 \times y_1) \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, g is trivial. It follows that f is a trivial $\boldsymbol{P}^1$-bundle. q.e.d.

Proof of Theorem 1.2. Let X be a Fano 3-fold with $B_2 \geq 6$. By (9.6), X has a conic bundle structure f: $X \to S$ with $S \not\cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$.

Claim: $\rho(S) \geq 3$.

Assume to the contrary. Then, by Corollary 6.4, the number of connected components of Δ_f is not less than $\rho(X)-\rho(S)-1 \geq 3$. Since every curve on $\boldsymbol{P}^2$ is connected, $S \not\cong \boldsymbol{P}^2$. Since every curve on $\boldsymbol{F}_1$ disjoint from the exceptional curve of the first kind is connected, $S \not\cong \boldsymbol{F}_1$ by Corollary 6.7. This contradicts (9.7).

By the claim and Proposition 9.10, f is a trivial $\boldsymbol{P}^1$-bundle. Hence X is isomorphic to $\boldsymbol{P}^1 \times S$. Since X is a Fano 3-fold, S is a del Pezzo surface, which completes the proof of Theorem 1.2.

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