CHAPTER 13

Density Ratio of a Maximal Invariant

13.1. Density ratio as a ratio of integrals over the group. There are statistical problems, especially testing problems, where the object of interest is the density ratio of two distributions of a maximal invariant $T = t(X)$, say P_1^T and P_2^T, rather than the individual distributions P_i^T themselves. If Theorem 8.6 applies, and if for $i = 1, 2$, P_i is a distribution on $\mathfrak{X}$ with density p_i with respect to a χ-relatively invariant measure λ as in (8.11), then (8.12) shows that

$$\frac{dP_2^T}{dP_1^T}(t) = \frac{\int p_2(gs(t))\chi(g)\mu_G(dg)}{\int p_1(gs(t))\chi(g)\mu_G(dg)}. \tag{13.1.1}$$

It is seen that the measure $\mu_{\mathfrak{T}}$ in (8.12) drops out, so that it is unnecessary to deal with the factorization (8.10). We can go a step further. In (13.1.1) replace g by gg_1 with any fixed $g_1 \in G$, and observe $\chi(gg_1) = \chi(g)\chi(g_1)$ and $\mu_G(dgg_1) = \Delta_r(g_1)\mu_G(dg)$, in which Δ_r is the right-hand modulus of G (Section 7.1). Since $\chi(g_1)\Delta_r(g_1)$ can be taken outside the integrals in (13.1.1), it drops out of the ratio. The result is that on the right-hand side of (13.1.1) $s(t)$ is replaced by $g_1 s(t)$. Since $g_1 \in G$ is arbitrary, $g_1 s(t)$ is an arbitrary point on the G-orbit of $s(t)$. Replacing this point by x, (13.1.1) can be written

$$\frac{dP_2^T}{dP_1^T}(t(x)) = \frac{\int p_2(gx)\chi(g)\mu_G(dg)}{\int p_1(gx)\chi(g)\mu_G(dg)} \tag{13.1.2}$$

for any $x \in \mathfrak{X}$. Thus, it is not necessary to construct a function s with range a global cross section $\mathfrak{Z}$. The right-hand side of (13.1.2) does not even contain the maximal invariant function t and may be regarded as the density ratio at $x \in \mathcal{X}$ of the two distributions P_1 and P_2 restricted to the measurable subsets of the abstract space $\mathcal{X}/G$. However, in practice a statistician will usually want to represent $\mathcal{X}/G$ by a more concrete space $\mathcal{T}$, which does entail explicit construction of a maximal invariant function t. In such cases it is necessary to ascertain that $\mathcal{T}$ is a homeomorphic image of $\mathcal{X}/G$. We shall say then that $t : \mathcal{X} \to \mathcal{T}$ is a **topological maximal invariant**. On the other hand. there are problems in which the group G is not completely specified but only known to belong to a certain family. In that case one does not have an explicit maximal invariant, and the possibility of writing the density ratio in the form (13.1.2) is the only handle one has on the problem. For an application see Wijsman (1967a).

The simplicity of (13.1.2) suggests that it is valid under weaker assumptions than those of Assumption 8.2. This is indeed the case and will be treated in some detail in Sections 13.2 and 13.3. The idea of obtaining a density ratio of a maximal invariant T as a ratio of two integrals over the group was first proposed by Stein (1956a,b). An early application was made by Giri (1964). More recent applications include Andersson and Perlman (1984), Sinha and Sarkar (1984), Kariya and Sinha (1985), and Szkutnik (1988). A proof of (13.1.2) was first given by Wijsman (1967b) for $\mathcal{X}$ a *linear Cartan G-space* and using as tool a *local cross section* (these concepts will be defined in Section 13.2). Extension to affine transformations of a certain kind was made in Wijsman (1972, Section 7). Andersson (1982) proved (13.1.2) under the condition that the action of G on $\mathcal{X}$ be *proper*. Proofs of (13.1.2) have also been given under weaker conditions, by assuming only measurability rather than continuity, by Bondar (1976) and by Farrell (1976, 1985). There is also a proof by Chang, Galvin, and Rukhin (1989) assuming measurability and finiteness of the denominator in (13.1.2). Since in typical applications there tends to be a great amount of regularity, in particular continuity, we shall present below only the method of local cross section (Section 7.2) and the method of proper action (Section 7.3).

13.2. Method of local cross section. Cartan G-space. Let $T = t(X)$ be the maximal invariant of Chapter 8, with values in $\mathcal{T}$, and let t be an arbitrary point of $\mathcal{T}$ (as before, we use the symbol t both for a function and for a point). For the value of dP_2^T/dP_1^T at t one only needs to know the distributions P_i^T $(i = 1, 2)$ restricted to an open neighborhood of t. This corresponds to an open invariant neighborhood of a single G-orbit in $\mathcal{X}$. Even if there is a global cross section $\mathcal{Z}$ (e.g., if Assumption 8.2 is satisfied), one only uses a small piece of $\mathcal{Z}$, namely any open neighborhood of $z = s(t)$ in $\mathcal{Z}$. This is called a *local cross section.* More precisely, one defines first a (local) *slice*, and then a local cross section is a special kind of slice.

13.2.1. DEFINITION. *A* **slice** *at $x \in \mathcal{X}$ is a set $S \subset \mathcal{X}$ such that (i) $x \in S$; (ii) GS is open in $\mathcal{X}$; and (iii) there exists a continuous equivariant function $f : GS \to G/G_x$ such that $f^{-1}(G_x) = S$.*

If a slice S has the further property that $GS = \mathcal{X}$, then S is called a *global slice.* This was already defined in Remark 11.5.

13.2.2. DEFINITION. *A* **local cross section** *S at x is a slice at x such that if both $s \in S$ and $gs \in S$ for some $g \in G$, then $gs = s$.*

Thus, if S is a local cross section, it has at most one point in common with each orbit. (Compare this with a global cross section, which has exactly one point in common with each orbit.) A special case of a local cross section was defined in Section 5.8, with a Lie group G and a closed Lie subgroup H there taking the place of $\mathcal{X}$ and G here.

It is proved in Wijsman (1966) Lemma 3, that if S is a local cross section at x, then $G_s = G_x$ for every $s \in S$. Thus, a local cross section S shares with the global cross section $\mathcal{Z}$ of Chapter 8 the property that the isotropy subgroup is the same at every point of S. One of the consequences is that $G/G_x \times S$ and GS are in 1-1 correspondence through the bijection $(gG_x, s) \to gs$, and this correspondence is a homeomorphism (Wijsman, 1966). One can conclude from this that there is a 1-1 correspondence between the open invariant subsets of GS and the subsets of S that are open in S, and that S is l.c. in this relative topology if $\mathcal{X}$ is l.c. This leads on GS to a factorization (8.10), provided G_x is compact, with $\mu_{\mathcal{T}}$ replaced by a measure on S, say μ_S; then to (8.12) with μ_S instead of $\mu_{\mathcal{T}}$; and finally to (13.1.2), still only on a neighborhood of a single G-orbit. If there is a local cross section for each orbit, then (13.1.2) has been established everywhere. Since the existence of a local cross section with compact isotropy subgroup at one point of an orbit implies the same at every point of that orbit, we may equivalently require that there be a local cross section with compact isotropy subgroup at every $x \in \mathcal{X}$.

The existence of a local cross section with compact isotropy subgroup at each $x \in \mathcal{X}$ is by no means guaranteed. For instance, if $\mathcal{X} = R^n$ with $n \geq 2$, and $G = GL(n)$ acting on $\mathcal{X}$ by linear transformations, then G_x is noncompact at every $x \in \mathcal{X}$. A second example is furnished by the irrational flow on the torus (Chapter 1) where $G_x = \{e\}$ (therefore compact) at every x, but there is no local cross section at any x. The reason for this is that the action of G on $\mathcal{X}$ is unpleasant. However, even if the action of the group on the space is nice, there will often be points at which there is no local cross section. An example of this is given in Wijsman (1966), with $\mathcal{X} = R^2 - \{0\}$ and G consisting of all matrices $\text{diag}(c, c)$ and $\text{diag}(c, -c)$, $c > 0$. Then points of the form $x = (x_1, 0)$ have no local cross section but all other points do. In general, if the set of points x at which there is no local cross section with compact G_x has λ-measure 0, then we can remove this set from $\mathcal{X}$ and on the remainder the formula (13.1.2) is valid. (Note that the exceptional set is always invariant.) This serves the purpose equally well since the density ratio is defined only up to λ-null sets anyway. In the third example above the exceptional set is indeed of λ-measure (= 2-dimensional Lebesgue measure) 0, but in the first two examples it is not.

In order to arrive at a useful sufficient condition for the existence of a local cross section with compact G_x at every x we shall restrict ourselves to $\mathcal{X}$ an invariant subset of R^n under the action of translations and linear transformations. More precisely, we make the following assumption:

13.2.3. ASSUMPTION. *Let F be a linear subspace of R^n with $0 \leq \dim F < n$ and let the group H be F under vector addition, acting on R^n by $x \to x + b$, $b \in F$. Let G be a closed Lie subgroup of $GL(n)$ acting on R^n by linear transformations such that G transforms F into itself. Let $\mathcal{X}$ be an open subset of R^n, invariant under G and under H.*

It will be shown first that under Assumption 13.2.3 the translations can be dealt with immediately and that the only problem resides in the linear transformations.

13.2.4. LEMMA. *Under Assumption 13.2.3, $K \equiv GH$ $(= HG)$ is a group and H is normal in K. Furthermore, if E is any linear complement of F in R^n, then the action of G on $\mathcal{X}$ induces an action of G on $\mathcal{X} \cap E$.*

PROOF. Let $h \in H$, $g \in G$, with actions $x \to x+b$ and $x \to \ell(x)$, respectively, where $b \in F$ and ℓ is a linear function. Then the action of ghg^{-1} is $x \to x + \ell(b)$. Since $\ell(b) \in F$ by Assumption 13.2.3, $ghg^{-1} \in H$. This shows that K is a group and that H is normal in K (Sections 5.9, 7.6). Let $x = x_1 + x_2$ with $x_1 \in E$ and $x_2 \in F$ be the unique decomposition of $x \in R^n$ and define $\pi : R^n \to E$ by $\pi(x) = x_1$. Then π is a maximal invariant under H, with range E. If Hx is an arbitrary H-orbit ($x \in \mathcal{X}$) and if $g \in G$, then $gHx = Hgx$ (since H is normal) so that g transforms the H-orbit of x into the H-orbit of gx. Hence, G acts on $\mathcal{X}/H$, therefore on $\mathcal{X} \cap E$. □

As a result of Assumption 13.2.3 and Lemma 13.2.4, a maximal invariant under K, and its distribution, can be obtained in steps by applying H first, then G. If p is a probability density on $\mathcal{X}$ with respect to Lebesgue measure, then the distribution on $\mathcal{X}/H$ is represented by a density on $\mathcal{X} \cap E$ (with respect to Lebesgue measure) proportional to $p^*(x) \equiv \int p(x + b)db$, $x \in \mathcal{X} \cap E$, where the integration is over $b \in F$. Then in the second stage we have G acting on $\mathcal{X} \cap E$ with probability density $p^*(x)$. Thus, the conditions for the validity of (13.1.2) need to be investigated only for linear transformations.

13.2.5. REMARK. It may be possible to replace G by a smaller group (i.e., a proper subgroup) that produces the same transformations of $\mathcal{X} \cap E$ as G does. Let G_0 be the subgroup of G that leaves every point of $\mathcal{X} \cap E$ fixed. It is a normal subgroup of G, and G/G_0 is a group under which $\mathcal{X} \cap E$ undergoes the same transformations as under G. (The new group G/G_0 also acts **effectively**, meaning that for every member of the group other than e there is at least one point that does not remain fixed). Unless $G_0 = \{e\}$, G/G_0 is smaller than G. As an example, suppose that $\mathcal{X} = R^n$, $\dim E = n_1$, $\dim F = n_2$ with $n_1, n_2 > 0$ and $n_1 + n_2 = n$. Choose a basis of R^n with the first n_1 basis vectors spanning E, the last n_2 spanning F. If C is an $n \times n$ matrix, partition it as $C = ((C_{ij}))$, $i, j = 1, 2$, with $C_{ii} : n_i \times n_i$. Then the elements of G are represented by matrices C for which $C_{12} = 0$ in order that G leave F invariant. Let the matrices C of G satisfy the further requirement that $C_{11} \in O(n_1)$ (but $C_{22} \in GL(n)$ and $C_{21} \in M(n_2, n_1)$ are unrestricted). Then G is a proper subgroup of $GL(n)$. Identify R^n/F with E, then the action of G on E is represented by the matrices C_{11}, i.e., the action of $O(n_1)$ on R^{n_1}. This has a local cross section at every point x except at $x = 0$. However, the isotropy subgroup at such x, say G_0, is not compact since it consists of all C with $C_{11} = I_{n_1}$ but C_{21} and C_{22} unrestricted. Thus, we seem to have failed in our attempt to obtain a local cross section with compact isotropy subgroup. The situation is remedied by observing that G_0 leaves every $x \in E$ fixed. Then G/G_0 is a smaller group (in fact, isomorphic to $O(n_1)$) that produces the same transformations of E, and the isotropy subgroup at $x \neq 0$ is now trivial, therefore compact. □

If G is a group that acts by linear transformations of $\mathcal{X} \subset R^n$, then we shall say that $\mathcal{X}$ is a **linear G-space**. The additional condition guaranteeing the existence everywhere of a local cross section with compact isotropy subgroup is that the space be a **Cartan G-space**. This notion, defined below, was introduced by Palais (1961). An explanation for the name can be found in that paper. In the definition $\mathcal{X}$ may be any l.c. space on which a l.c. group G acts continuously. Recall the definition, equation (2.3.2), of the symbol $((A, B))$ for any two subsets A, B of $\mathcal{X}$.

13.2.6. DEFINITION. *A subset A of $\mathcal{X}$ is called* **thin** *if* $((A, A))$ *has compact closure.*

13.2.7. DEFINITION. *$\mathcal{X}$ is called a* **Cartan G-space** *if every $x \in \mathcal{X}$ has a thin neighborhood.*

It is an immediate consequence of Definitions 13.2.6 and 13.2.7 that in a Cartan G-space every isotropy subgroup G_x is necessarily compact. Palais (1961) showed for Lie groups G that in a G-space where every G_x is compact the existence of a slice at each x is equivalent to $\mathcal{X}$ being a Cartan G-space. Using this result the following theorem was proved by Wijsman (1966):

13.2.8. THEOREM. *Let $\mathcal{X}$ be an open subset of R^n and G a closed Lie subgroup of $GL(n)$ acting on $\mathcal{X}$ by linear transformations. If $\mathcal{X}$ is a Cartan G-space (so that G_x is compact at each $x \in \mathcal{X}$), then there is an invariant set $\mathcal{X}_0 \subset \mathcal{X}$, open in R^n, with $\lambda(\mathcal{X} - \mathcal{X}_0) = 0$ ($\lambda = n$-dimensional Lebesgue measure) such that at each point of $\mathcal{X}_0$ there is a local cross section. Hence, (13.1.2) is valid.*

13.2.9. REMARK. The local cross section of Theorem 13.2.8 may be chosen *flat* at each x, i.e., a subset of a translate of a linear subspace. This fact, however, will not be used. □

13.2.10. REMARK. In some statistical applications $\mathcal{X}$ is a product of identical copies of a G-space where G consists of translations and linear transformations. The existence everywhere (after removal of a set of Lebesgue measure 0) of a local cross section with compact (even trivial) isotropy subgroup can then be shown more directly, without using Lemma 5 in Wijsman (1966). This is done in Theorem 7.1 of Wijsman (1972). (But note that the proof of Lemma 7.1 in that paper can be replaced by an appeal to Theorem 1.1.3 in Palais (1961).) □

Some useful sufficient conditions for a G-space to be a Cartan G-space will be presented in Section 13.4.

13.3. Method of proper action. An account of this approach is given by Andersson (1982). The main assumption under which (13.1.2) will be derived is that G act properly on $\mathcal{X}$ (Definition 2.3.6). More precisely, the following assumption will be made.

13.3.1. ASSUMPTION. *The group G is l.c. and acts properly on a l.c. and σ-compact space $\mathcal{X}$.*

13.3.2. THEOREM. *Let Assumption 13.3.1 be satisfied and let p_1, p_2 be two probability densities on $\mathcal{X}$ with respect to a measure λ that is relatively invariant with multiplier χ. Let $T = t(X)$ be a maximal invariant under the action of G and let μ_G be any left Haar measure on G. Then (13.1.2) is valid.*

The proof of Theorem 13.3.2 will be given informally since for some of the details the reader will be referred to the literature. The starting point is Section 7.3, where the group H acts properly on $\mathcal{X}$ to the right and β is left Haar measure on H. We can copy the results of that section by changing H to G acting on the *left* of $\mathcal{X}$ and changing β to *right* Haar measure ν_G on G. The multiplier χ of Theorem 7.3.3 is now a left multiplier since G acts on the left of $\mathcal{X}$, and Δ_r^H in Theorem 7.3.3 is to be replaced by Δ_ℓ^G, which will simply be written Δ_ℓ. This change should also be incorporated in Definition 7.3.4 of quotient measure. Then Theorem 7.3.3 says that if μ is a measure on $\mathcal{X}$ that is relatively invariant with multiplier Δ_ℓ, then there is a unique measure $\mu^\flat = \mu/\nu_G$ on $\mathcal{X}/G$ defined by (7.3.9). If equations (7.3.2) and (7.3.3) are taken into account, then (7.3.9) reads (from right to left)

$$\int f d\mu = \int \mu^\flat(dz) \int f(gx)\nu_G(dg) \tag{13.3.1}$$

in which on the right-hand side x is any point in $\mathcal{X}$ for which $\pi(x) = z$, where $\pi : \mathcal{X} \to \mathcal{X}/G$ is the orbit projection. Equation (13.3.1) is valid for $f \in \mathcal{K}(\mathcal{X})$, and therefore for any μ-integrable f. With help of (7.1.10) and Proposition 7.1.5 we can rewrite (13.3.1) as

$$\int f d\mu = \int \mu^\flat(dz) \int f(gx)\Delta_\ell(g)\mu_G(dg), \quad \pi(x) = z. \tag{13.3.2}$$

Equation (13.3.2) is valid for any μ that is relatively invariant with multiplier Δ_ℓ. However, what is needed is an equation similar to (13.3.2) but valid, instead, for the given measure λ that has multiplier χ. The key to the required transition is the following lemma, which is a special case of Lemme 1 in Appendice I of Bourbaki (1963).

13.3.3. LEMMA. *Under Assumption 13.3.1 there exists on $\mathcal{X}$ a real valued and continuous function $F \geq 0$ such that (a) F is not identically 0 on any G-orbit; and (b) for every compact $K \subset \mathcal{X}/G$, $\pi^{-1}(K) \cap \operatorname{supp} F$ is compact.*

The main elements that go into the proof of Lemma 13.3.3 will be sketched later in this section. First, however, we show how the lemma is used. Let ρ be an arbitrary continuous homomorphism $G \to R_+^*$ and define the function F^ρ on $\mathcal{X}$ by

$$F^\rho(x) = \int F(gx)\rho(g^{-1})\nu_G(dg). \tag{13.3.3}$$

It should be shown first that the right-hand side of (13.3.3) is well defined. We shall show that for each x the integrand as a function of g is in $\mathcal{K}(G)$. Also, the continuity of the integral as a function of x will be needed. Both properties will follow by allowing x to be variable within a compact neighborhood V of a fixed point, say $x_0 \in \mathcal{X}$. Then $\pi(V) = K$ is a compact subset of $\mathcal{X}/G$ and $\pi^{-1}(K) \cap \operatorname{supp} F = W$, say, is compact by (b) of Lemma 13.3.3. Thus, if $x \in V$, then $F(gx) = 0$ unless $gx \in W$. Using the notation (2.3.2), we have that with $x \in V$ the integrand in (13.3.3) is 0 unless $g \in ((V, W)) = C$, say. Then C is compact by Proposition 2.3.8. The continuity of the integrand as a function of g is immediate by the continuity of F and ρ. The continuity of the integral as a function of x follows, as in the proof of Lemma 6.5.6(i), by observing that with g in (13.3.3) restricted to C the convergence $F(gx) \to F(gx_0)$ as $x \to x_0$ is uniform in g.

A simple computation, using the right-invariance of ν_G, shows that

$$F^\rho(gx) = \rho(g)F^\rho(x), \quad g \in G. \tag{13.3.4}$$

Also, $F^\rho > 0$ everywhere by virtue of (a) of Lemma 13.3.3. Define now the measure λ_ρ by

$$\lambda_\rho(dx) = (1/F^\rho(x))\mu(dx). \tag{13.3.5}$$

A direct computation, using (13.3.4) and the modulus Δ_ℓ of μ, shows that λ_ρ is relatively invariant with multiplier Δ_ℓ/ρ. Therefore, since λ has multiplier χ, we can make $\lambda_\rho = \lambda$ by choosing

$$\rho = \Delta_\ell/\chi. \tag{13.3.6}$$

Now write (13.3.2) with f replaced by f/F^ρ. On the left-hand side we get $\int f d\lambda$. On the right-hand side the inner integral becomes

$$\int \frac{f(gx)}{F^\rho(gx)}\Delta_\ell(g)\mu_G(dg) = \frac{1}{F^\rho(x)}\int f(gx)\chi(g)\mu_G(dg), \tag{13.3.7}$$

using (13.3.5) and (13.3.6). Next, replace f by fp, where f is bounded, measurable, and invariant, and p is a probability density with respect to λ. The result is

$$\int fpd\lambda = \int f(z)\mu^\flat(dz)(F^\rho(x))^{-1}\int p(gx)\chi(g)\mu_G(dg) \tag{13.3.8}$$

and it follows that the density of the maximal invariant T with respect to the quotient measure $\mu^\flat$ is

$$p^T(z) = (F^\rho(x))^{-1}\int p(gx)\chi(g)\mu_G(dg), \quad z = \pi(x), \tag{13.3.9}$$

where the right-hand side depends on x only through $\pi(x)$ (i.e., the right-hand side is invariant under $x \to g_1x$, $g_1 \in G$). If (13.3.9) is written down for two such p's, then in the ratio the factor in front of the integrals cancels and (13.1.2) is obtained.

The proof of Lemma 13.3.3 involves several new concepts that rightfully belong in Section 2.2 but were not needed until now. A cover $\mathcal{F}$ (Section 2.2) of a topological space is called **locally finite** if every point has a neighborhood that meets only a finite number of members of $\mathcal{F}$. A cover $\mathcal{F}$ has a **locally finite refinement** if there is a locally finite cover $\mathcal{F}_1$ every member of which is contained in a member of $\mathcal{F}$. Now suppose there is given on a topological space $\mathcal{X}$ a family $\{f_i : i \in I\}$ of real valued nonnegative functions, where I is an arbitrary (not necessarily countable) index set. For $x \in \mathcal{X}$ define $f(x) = \sum_{i \in I} f_i(x)$. Then $f(x)$ need not be finite for there may be an infinite number of terms in the sum (even uncountably so). However, this cannot happen if the family of supports of the f_i is locally finite for then there can only be a finite number of positive terms in the sum. A **continuous partition of unity** on a topological space $\mathcal{X}$ is a family $\{f_i : i \in I\}$ with each f_i continuous and ≥ 0, the family $\{\operatorname{supp} f_i : i \in I\}$ is a locally finite cover of $\mathcal{X}$, and $\sum_{i \in I} f_i(x) = 1$ for every $x \in \mathcal{X}$. A partition of unity $\{f_i\}$ is said to be **subordinate** to a given cover $\mathcal{F}$ if $\{\operatorname{supp} f_i\}$ is a locally finite refinement of $\mathcal{F}$. A topological space is called **paracompact** if it is Hausdorff and every open cover has a locally finite open refinement.

It is proved as part of Theorem 5 in Bourbaki (1966b), I, §9.10, that a space that is both l.c. and σ-compact is paracompact. Under our Assumption 13.3.1 the space $\mathcal{X}/G$ is l.c. by Theorem 2.3.13(a). The σ-compactness of $\mathcal{X}$ implies that $\mathcal{X}/G$ is σ-compact since if $\mathcal{X} \subset \cup K_i$ (countable union) with K_i compact, then $\mathcal{X}/G \subset \cup \pi(K_i)$, and $\pi(K_i)$ is compact by the continuity of π. Hence, by Theorem 5 of Bourbaki (1966b) quoted above, $\mathcal{X}/G$ is paracompact. Furthermore, it is shown as a combination of Propositions 3 and 4 in Bourbaki (1966b), IX, §4, that for a given locally finite open cover $\mathcal{F}$ of a paracompact space there is a continuous partition of unity subordinate to $\mathcal{F}$.

We shall sketch now the main elements of the proof of Lemma 13.3.3. For every point $z \in \mathcal{X}/G$ select a point $x_z \in \pi^{-1}(z)$. Since $\mathcal{X}$ is l.c. there exists $f_z \in \mathcal{K}_+(\mathcal{X})$ such that $f_z(x_z) = 1$ (Lemma 6.3.2). Let V_z be the open set, with compact support, on which $f_z > 0$ and define $W_z = \pi(V_z)$, so that $W_z \subset \mathcal{X}/G$ is open since π is an open mapping. Since $\mathcal{X}/G$ is paracompact, the open cover $\{W_z : z \in \mathcal{X}/G\}$ has a locally finite open refinement $\{U_i : i \in I\}$ and there is a continuous partition of unity $\{g_i : i \in I\}$ subordinate to $\{U_i\}$. For each function g_i on $\mathcal{X}/G$ there is a corresponding invariant function $g_i \circ \pi$ on $\mathcal{X}$. For each $i \in I$ choose $z_i \in \mathcal{X}/G$ such that $U_i \subset W_{z_i}$ and define $F_i = (g_i \circ \pi)f_{z_i}$. Then $F_i \in \mathcal{K}(\mathcal{X})$ (because this is true for f_{z_i} and $g_i \circ \pi$ is continuous) and $\operatorname{supp} F_i \subset \pi^{-1}(U_i)$ (because this is true for $g_i \circ \pi$). From the latter property it follows that $\{\operatorname{supp} F_i : i \in I\}$ is locally finite so that we can define $F = \sum_{i \in I} F_i$. Then $F \geq 0$, finite, and continuous. For each $z \in \mathcal{X}/G$, since $\sum_i g_i(z) = 1$, there is $i \in I$ (depending on z) such that $g_i(z) > 0$. Since $z \in U_i \subset W_{z_i}$, the orbit $\pi^{-1}(z)$ meets V_{z_i} so that there is on this orbit a point, say y, at which f_{z_i} is positive. Hence, $F_i(y) > 0$ so that $F(y) > 0$. This establishes (a) of Lemma 13.3.3. Part (b) can also be verified. For the details see Bourbaki (1963), proof of Appendice I, Lemme 1.

13.4. Comparison of the two methods. Sufficient conditions for the ratio-of-integrals representation. In Bourbaki (1966b) III, §4.4, Proposition 7 states that if a l.c. group G acts continuously on a Hausdorff space $\mathcal{X}$, then the action is proper if and only if for every $x, y \in \mathcal{X}$ there are neighborhoods V_x, V_y such that $((V_x, V_y))$ has compact closure (notation (2.3.2)). Taking $x = y$, then by Definitions 13.2.6 and 13.2.7 we see that if G acts properly on $\mathcal{X}$, then $\mathcal{X}$ is a Cartan G-space. It is not known whether the converse is true. Conceivably, the condition of proper action could be stronger than the Cartan condition. However, in all examples tried so far either both conditions are met or neither is. This is true in particular for the examples that follow.

In many statistical applications it is possible to obtain a maximal invariant in steps. If at each step the Cartan condition is satisfied, then the formula (13.1.2) holds. A special case of this was used in Section 13.2 with an affine group by applying first the translations, then the linear transformations. If the method of proper action is used, then it is again true that only the proper action at each step has to be verified. This is proved in Wijsman (1985), where several examples can be found. With either method it may be necessary to verify at the intermediate stage(s) that the maximal invariant is topological.

Below we list a variety of situations where both the Cartan condition and proper action hold. In most cases proofs will be omitted, but some can be found, e.g., in Wijsman (1985). It is hoped that the needs of most statistical applications can be met by combining several of these general rules and special cases. It is assumed throughout, without further mention, that $\mathcal{X}$, $\mathcal{X}_1$, etc., are l.c. spaces and G, G_1, etc., l.c. groups. We shall write $(\mathcal{X}, G) \in C$ to mean that $\mathcal{X}$ is a Cartan G-space; similarly, $(\mathcal{X}, G) \in P$ means that G acts properly on $\mathcal{X}$; and $(\mathcal{X}, G) \in CP$ means that both is true.

13.4.1. If G is compact, then $(\mathcal{X}, G) \in CP$. □

13.4.2. $(G, G) \in CP$. □

13.4.3. If $(\mathcal{X}_1, G) \in C$ or P, and G acts continuously on $\mathcal{X}_2$, then $(\mathcal{X}_1 \times \mathcal{X}_2, G) \in C$ or P. □

13.4.4. If $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ with $G = \mathcal{X}_1 = R^m$ for some $m \geq 1$, and G acts on $\mathcal{X}_1$ by translations and acts trivially on $\mathcal{X}_2$, then $(\mathcal{X}, G) \in CP$. This is a special case of 13.4.3, with the help of 13.4.2. □

13.4.5. Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, $G = G_1 \times G_2$, and $(\mathcal{X}_i, G_i) \in C$ or P, $i = 1, 2$. Then $(\mathcal{X}, G) \in C$ or P. □

13.4.6. If $(\mathcal{X}, G) \in C$ or P and G_1 is a closed subgroup of G, then $(\mathcal{X}, G_1) \in C$ or P. □

13.4.7. Let $G = GL(p)$, $\mathcal{X} =$ all $p \times n$ matrices X of rank p and the action $X \to CX$ (matrix multiplication), $C \in G$. Then $(\mathcal{X}, G) \in CP$. For $n = p$ this is a special case of 13.4.2. In order to show it for $n > p$ consider all sets J_i of p distinct integers taken from $\{1, \ldots, n\}$, $i = 1, \ldots, \binom{n}{p}$, and write $\mathcal{X} = \bigcup_i \mathcal{X}_i$, where $\mathcal{X}_i$ consists of all $X \in \mathcal{X}$ whose columns x_j with $j \in J_i$ are linearly independent. Then $\mathcal{X}_i$ is open and invariant, and $(\mathcal{X}_i, G) \in CP$ by 13.4.2 and 13.4.3. □

13.4.8. Let positive integers n, p, r be fixed, with $n \geq p + r$, and let L be a fixed $r \times n$ matrix of rank r. Let $\mathcal{X}$ be all $p \times n$

matrices X of rank p such that the row spaces of X and L have only 0 in common. Let G consist of matrices $C \in GL(p)$ and all matrices $B \in M(p,r)$, with the action on $\mathfrak{X}$ defined by $X \to C(X+BL)$. Then $(\mathfrak{X}, G) \in CP$. This can be shown, after an orthogonal transformation of the row space of X, by a combination of 13.4.4, 13.4.6, and 13.4.7.

As a special case take $r = 1$, so $n \geq p+1$, and $L = [1, \dots, 1]$, and write x_j for the jth column of X. Then the action is $x_j \to C(x_j + b)$, $j = 1, \dots, n$, where C runs through $GL(p)$ (or a closed subgroup) and b through all $p \times 1$ vectors. For this case the Cartan property was proved in Wijsman (1972, Theorem 7.1). Another special case is the canonical form of MANOVA. If $L = [I_r, 0] : r \times n$, then the action is $x_j \to C(x_j + b_j)$, $j = 1, \dots, r$, and $x_j \to Cx_j$, $j = r+1, \dots, n$. □

13.4.9. Let $\mathfrak{X} = PD(p)$, $G = GL(p)$, and the action $S \to CSC'$, $S \in \mathfrak{X}$, $C \in G$. Then $(\mathfrak{X}, G) \in CP$. This is the kind of action that frequently occurs in problems with covariance matrices. □

13.4.10. Example. Let $(\mathfrak{X}, G)$ be as in Section 9.4 so that $G = GL(p)$ and $\mathfrak{X} = \mathfrak{X}_1 \times \mathfrak{X}_2$ with $\mathfrak{X}_1 = PD(p)$ and $\mathfrak{X}_2 = M(q,p)$. The action is given by (9.4.1). This is essentially the situation in Sinha and Sarkar (1984) and is also relevant to MANOVA. Since $(\mathfrak{X}_1, G) \in CP$ by 13.4.9 and G acts continuously on $\mathfrak{X}_2$, $(\mathfrak{X}, G) \in CP$ by 13.4.3. Thus, (13.1.2) applies. The material in Sections 9.3 and 9.4 also shows that the maximal invariant $US^{-1}U'$ ($= Q$ of (9.4.2)) is topological. (The homeomorphism with $\mathfrak{X}/G$ is even an analytic diffeomorphism.) □

13.4.11. Example (canonical correlations). Partition $X : p \times n$ into two matrices $X_i : p_i \times n$ ($i = 1, 2$, $p_1 + p_2 = p \leq n$) with X_1 the first p_1 and X_2 the last p_2 rows of X. Assume that X is of rank p. Let $\mathfrak{X}$ be all such X and let $G = G_1 \times G_2 \times G_3$, where $G_i = GL(p_i)$, $i = 1, 2$, $G_3 = O(n)$, and the action of G on $\mathfrak{X}$ is given by $X_i \to C_i X_i$ for $C_i \in G_i$, $i = 1, 2$, and $X \to X\Gamma'$ for $\Gamma \in G_3$. For $i = 1, 2$, $(\mathfrak{X}_i, G_i) \in CP$ by 13.4.7, hence $(\mathfrak{X}_1 \times \mathfrak{X}_2, G_1 \times G_2) \in CP$ by 13.4.5. Finally, $(\mathfrak{X}, G) \in CP$ by 13.4.1 applied to $G = G_3$ and $\mathfrak{X} = (\mathfrak{X}_1 \times \mathfrak{X}_2)/(G_1 \times G_2)$. Hence, (13.1.2) is valid. A maximal invariant is the set of canonical correlations. That this maximal invariant is topological can be seen by first applying G_3, producing a maximal invariant $S = XX'$ that is topological by Section 9.2, Case 1. The rest follows from Section 10.5.

The proof of $(\mathcal{X}, G) \in CP$ given above proceeded by first applying $G_1 \times G_2 = G'$, say, and then G_3. An explicit expression for a topological maximal invariant at the first step was not needed because G_3 is compact and therefore 13.4.1 applies at the second step in any case. We could also have reversed the order: if G_3 is applied first then we obtain at the first step the topological maximal invariant $S = XX' \in PD(p) = \mathcal{X}'$, say. Then at the second stage the action is $S \to CSC$, $C \in G'$. Since G' is a closed subgroup of $GL(p)$, we may apply 13.4.6 and 13.4.9 to conclude $(\mathcal{X}', G') \in CP$. Therefore, $(\mathcal{X}, G) \in CP$ as found before. □