ON THE PASSAGE OF A RANDOM WALK FROM GENERALIZED BALLS*

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We derive a strong approximation for the first passage time of zero-mean random walks from generalized balls in Euclidean spaces by the corresponding first passage time of an appropriate vector valued Wiener process as the radius of the ball goes to infinity. As consequences we derive a weak invariance principle and some strong laws for the passage time of the walk. Multidimensional extensions of some limit theorems of Robbins and Siegmund on boundary crossing probabilities for sample sums are also formulated, including their last-time result.

1. Introduction.

Let $d \geq 1$ be a fixed integer and consider a sequence $X, X_1, X_2, \ldots$ of independent and identically distributed random vectors with values in $\mathbf{R}^d$. Introduce the corresponding partial sum process, or continuous-time random walk

$$S(t) = \sum_{i=1}^{[t]} X_i, \quad t \geq 0,$$

where $[.]$ denotes integer part and $S(t) = 0$ for $0 \leq t < 1$, and let $h:\mathbf{R}^d \to \mathbf{R}$ be a

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norm on $\mathbb{R}^d$. This means that we assume, throughout, that for any $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, $h(x) \geq 0$, $h(x) = 0$ if and only if $x = (0,\ldots,0) \in \mathbb{R}^d$, $h(x+y) \leq h(x) + h(y)$ and $h(\lambda x) = |\lambda| h(x)$. Set

$$M(u) = M_h(u) = \sup \{h(S(t)): 0 \leq t \leq u\}, \ u \geq 0.$$

One of our objectives in this paper is the study of the first passage time

$$N(t) = N_h(t) = \inf \{u: M(u) > t\} = \inf \{u: h(S(u)) > t\}$$

of the random walk S from the generalized h-ball of radius $t \geq 0$, centered at the origin. The infimum of the empty set is meant as ∞.

In fact N is an extended or generalized renewal process, the asymptotic behaviour of which, as $t \to \infty$, has been investigated by several authors under the assumption that $h(EX) > 0$ for the finite expectation vector $EX \in \mathbb{R}^d$ (see the references in Horváth (1984)). Recently Horváth (1984), (1984a), (1984b) and (1986) derived a number of univariate ($d=1$) and multivariate strong approximation results for N with several consequences and applications, not necessarily assuming that h is a norm, but always assuming that $h(EX) > 0$. In this case N, when appropriately centered, behaves as a constant multiple of a scale-changed standard Brownian motion. While the assumption $h(EX) > 0$ is the natural one in renewal theory, within the framework of standard random walk theory, it is more natural to look at N under the assumption that $EX = (0,\ldots,0)$. (The case $d = 1$ and $h(x) = x$ was investigated by Horváth (1985).

In what follows, together with the assumptions made in the first paragraph, we assume that $EX = (0,\ldots,0)$. If Γ is a positive semidefinite symmetric dxd matrix, then a univariate $\mathbb{R}^d$-valued Gaussian process $W(t) = (W_1(t),\ldots,W_d(t))$, $t \geq 0$, is said to be a d-dimensional Wiener process with covariance matrix Γ if $W(0) = (0,\ldots,0)$ and for any $s, t \geq 0$, $EW(t) = (0,\ldots,0)$ and

$$E\begin{pmatrix} W_1(s)W_1(t) & \cdots & W_1(s)W_d(t) \\ \vdots & & \vdots \\ W_d(s)W(t) & \cdots & W_d(s)W_d(t) \end{pmatrix} = \Gamma \min(s,t).$$

Given such a process, the corresponding analogue of the M process is

$$V(u) = V_h(u) = \sup\{h(W(t)):\ 0 \leq t \leq u\},\ u \geq 0,$$

and the first passage time of our Wiener process W from the generalized h-ball of radius $t \geq 0$ is

$$Z(t) = Z_h(t) = \inf\{u:\ V(u) > t\} = \inf\{u:\ h(W(u)) > t\}.$$

THEOREM 1. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random d-dimensional vectors, centered at expectations and such that the $(2+\delta)$-th moments of each of their components are finite where $\delta > 0$. Suppose that the covariance matrix Γ of X is nonsingular and that h is a norm on $\mathbf{R}^d$. Then without changing its distribution we can redefine the sequence $\{X_n,\ n \geq 1\}$ on a richer probability space $(\Omega, \mathbf{A}, P)$ together with an R^d-valued Wiener process W(t) with covariance matrix Γ and an almost surely finite random variable $t_0 = t_0(\omega)$ such that whenever $t \geq t_0$,

$$Z_h(t-a_t) \leq N_h(t) \leq Z_h(t+a_t)$$

almost surely, where

$$a_t = o(t^{1-2\alpha}(\log\log t)^{1/2-\alpha}) \text{ as } t \to \infty$$

with

$$\alpha = \begin{cases} \min(\frac{\delta}{81d}, \frac{1}{81d}), \text{ in general} \\ \frac{\delta}{2(2+\delta)}, \text{ if the components of X are independent.} \end{cases}$$

Now let D[0,1] be the Skorohod space of functions on [0,1] that are right-continuous and have left-hand limits, endowed with the J_1 topology. The normalized first passage time

$$R_T(t) = N_h(tT)/T^2, \; 0 \leq t \leq 1,$$

is a random element of this space for each $T > 0$, together with the passage time $\{Z_h(t), 0 \leq t \leq 1\}$. Our first corollary is a weak invariance principle.

COROLLARY 1. Under the conditions of Theorem 1 the processes $R_T(.)$ converge weakly in D[0,1] to $Z_h(.)$ as $T \to \infty$.

Clearly, $\Pr\{Z_h(1) \leq u\} = \Pr\{V_h(u) > 1\}$, for any $u > 0$. Hence if the distribution of $V_h(u)$ is known, then we also know the distribution of $Z_h(1)$. Precise knowledge about the distribution of $V_h(u)$ is available if the components of X and hence of W are independent and if $h(x) = ||x|| = \max(|x_1|,\ldots,|x_d|)$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, is the maximum norm or $h(x) = |x| = (|x_1|^2 + \ldots + |x_d|^2)^{1/2}$ is the Euclidean norm. In this case the distribution of $V_{||\,||}(u)$ is known for any $d \geq 1$, while the distribution $V_{|\,|}(u)$, the supremum of the Bessel process, is given by Imhof (1984) for $d = 3$.

In order to formulate the strong laws following from the theorem, we introduce the integrals

$$I(f,C) = \int_1^\infty \frac{f(t)}{t^{3/2}} \exp\{-C \frac{f^2(t)}{t}\} \, dt$$

and

$$J(f,C) = \int_1^\infty \frac{1}{f^2(t)} \exp\{-C \frac{t}{f^2(t)}\} \, dt,$$

where f is a continuous function on $[1,\infty)$ and $C > 0$ is a constant. Whenever f is strictly increasing, f^{-1} will denote its inverse function.

COROLLARY 2. Suppose that the conditions of Theorem 1 hold.

(i) Let ψ be a positive continuous function on $[1,\infty)$ strictly increasing to ∞ and such that $\psi(t)/t^2$ is decreasing. If $I(\psi^{-1},C) < \infty$ for some $C > 0$, then

$$\liminf_{t\to\infty} N_h(t)/\psi(t) > 0 \text{ almost surely,} \tag{1.1}$$

while if $I(\psi^{-1},C) = \infty$ for all $C > 0$, then

$$\liminf_{t\to\infty} N_h(t)/\psi(t) = 0 \text{ almost surely.} \tag{1.2}$$

(ii) Let Ψ be a positive continuous function on $[1,\infty]$ such that $\Psi(t)/t^2$ is strictly increasing. If $J(\Psi^{-1},C) < \infty$ for some $C > 0$, then

$$\limsup_{t\to\infty} N_h(t)/\Psi(t) < \infty \text{ almost surely,} \tag{1.3}$$

while if $J(\Psi^{-1},C) = \infty$ for all $C > 0$, then

$$\limsup_{t\to\infty} N_h(t)/\Psi(t) = \infty \text{ almost surely.} \tag{1.4}$$

The typical ψ function in (1.1) is $\psi(t) = t^2/\log\log t$ and the typical Ψ function in (1.3) is $\Psi(t) = t^2\log\log t$. The interesting feature of (1.1)-(1.4) is that ψ and Ψ do not depend on h. Of course, if we specialize the norm then more precise statements can be made. Let, for example, $h(x) = ||x||$ the maximum norm, and let $T^2 = \max(T_{11},\ldots,T_{dd})$ be the maximal variance of the components, where T_{ik}, $1 \le i, k \le d$, is the i-k entry of Γ. Then the proof of Corollary 2 implies that ("i.o." standing for "infinitely often")

$$\Pr\{N_{\|\ \|}(t) < \psi(t/T) \text{ i.o. as } t \to \infty\} = \begin{cases} 1, & \text{if } I(\psi^{-1}, 1/2) = \infty, \\ 0, & \text{if } I(\psi^{-1}, 1/2) < \infty, \end{cases}$$

and

$$\Pr\{N_{\|\ \|}(t) > \Psi(t/T) \text{ i.o. as } t \to \infty\} = \begin{cases} 1, & \text{if } J(\Psi^{-1}, \pi^2/8) = \infty, \\ 0, & \text{if } J(\Psi^{-1}, \pi^2/8) < \infty, \end{cases}$$

where ψ and Ψ are as in Corollary 2. If we use the classical law of the iterated logarithm and Chung's law of the iterated logarithm (Chung (1948) or Csörgő and Révész (1981) p. 48), then we obtain directly from the theorem that

$$\liminf_{t\to\infty} (\log\log t)\, N_{\|\ \|}(t)/t^2 = 1/(2T^2)$$

and

$$\limsup_{t\to\infty} N_{\|\ \|}(t)/(t^2 \log\log t) = 8/(\pi^2 T^2)$$

almost surely.

Now we turn to the limiting behaviour of probabilities that the walk ever goes out of h-balls with radii determined by some upper class functions. The results here are rather straightforward generalizations of some deep results of Robbins and Siegmund (1970).

THEOREM 2. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random d-dimensional vectors, centered at expectations, such that Γ is finite and let h be a norm on R^d. If g(t) is a continuous function for $t > \min(1,\tau)$, $\tau > 0$, such that $t^{-1/2}g(t)$ is eventually non-decreasing as $t\to\infty$, and $I(g, 1/((2TC_2)) < \infty$, where $C_2 = \max\{h(x): \|x\| = 1\}$, then
$\lim_{m\to\infty} \Pr\{h(S(n)) > m^{1/2} g(n/m) \text{ for some } n > \tau m\} = \Pr\{h(W(t)) > g(t) \text{ for some } t > \tau\}$, where W(t) is an R^d-valued Wiener process with covariance matrix Γ.

Of course, when $h(x) = ||x||$, then $C_2 = 1$, and when $h(x) = |x|$, then $C_2 = d^{1/2}$. When $d = 1$, Theorem 2 reduces to the two-sided version of part (i) of Theorem 2 of Robbins and Siegmund (1970). The corresponding d-dimensional extensions of the two-sided versions of part (ii) of their Theorem 2 and also of their Remark (c) are also simple (cf. the proof in Section 2) and easily formulated. Also, apart from the forms of the limiting distributions, it is easy to formulate the d-dimensional extensions of the two-sided versions of the formulae of Robbins and Siegmund (1970), p.1412, corresponding to $g(t) = at+b$, $a > 0$, $-\infty < b < \infty$. For example, for any $\varepsilon > 0$ let

$$L_h(\varepsilon) = \sup\{n: h(S(n)) \geq n\varepsilon\}$$

be the last time that the normalized walk $S(n)/n$, $n=1,2,\ldots$, is out of an h-ball of radius ε. Then $L_h(\varepsilon)$ is almost surely finite and

$$\lim_{\varepsilon\to 0} \Pr\{\varepsilon^2 L_h(\varepsilon) \geq a\} = \lim_{m\to\infty} \Pr\{\max_{n\geq m} h(S(n)/n) \geq a^{1/2} m^{-1/2}\}$$

$$= \Pr\{V_h(1) \geq a^{1/2}\}.$$

This is the d-dimensional extension of the two-sided version of the last-time result first obtained by Robbins, Siegmund and Wendel (1968). Unfortunately, the limiting distribution is known in the present generality only in the special cases mentioned above. This is of course all the more true for the limiting distribution

$$\Pr\{\sup_{t\geq\tau} (h(W(t)) - g(t)) \geq 0\} = \Pr\{\sup_{0\leq t\leq\tau^{-1}} (h(W(t)) - tg(t^{-1})) \geq 0\}$$

arising in Theorem 2.

The results of Robbins and Siegmund (1970), such as the one-dimensional form of Theorem 2, were motivated by certain sequential statistical procedures introduced and developed by Darling and Robbins in a series of papers and further developed by Robbins and Siegmund (see the references in Robbins (1970)). Most of these procedures formally extend to the d-dimensional case. For example, one can consider sequential power-one Darling-Robbins tests for testing the hypothesis H_0: $EX = E(X^{(1)},\ldots,X^{(d)}) = (0,\ldots,0)$. Since $\Pr\{h(S_n) > a_n \text{ for some } n > m\} \leq \sum_{i=1}^{d} \Pr\{C_2 |\sum_{j=1}^{n} X_j^{(i)}| > a_n \text{ for some } n > m\}$, where a_n is any sequence used by Darling and Robbins for this problem, their iterated logarithm inequalities can be used for controlling the probability of the error of the first kind. However, little is known, for example, about the expected sample size $EN(h;m)$ needed to reject H_0 when it is not true, where $N(h;m)$ is the first passage time $N(h;m) = \inf\{n > m: h(S(n)) > a_n\}$

Under H_0, Theorem 1 and its corollaries hold true of course for the modified first passage time $N_{h,m}(t) = \inf\{u > m: h(S(u)) > t\}$ with the obvious modifications.

2. Proofs.

In what follows we use the symbol $\overset{\mathcal{D}}{=}$ to denote distributional equality and $\xrightarrow{Pr}$ will denote convergence in probability.

We shall frequently use the well-known fact that there exist two constants C_1, $C_2 > 0$ such that

$$C_1||x|| \leq h(x) \leq C_2||x||, \quad x \in \mathbb{R}^d. \tag{2.1}$$

Proof of Theorem 1. Komlós, Major and Tusnády (Theorem 2.6.3 in Csörgő and Révész (1981)), in the special case when the components of X are independent, and Berkes and Philipp (1979), in the general case, proved that under the conditions of the theorem the sequence $\{X_n, n \geq 1\}$ can be redefined without changing its distribution on a richer probability space $(\Omega, \mathcal{A}, P)$ together with an $\mathbb{R}^d$-valued Wiener process $W(t)$ with covariance matrix Γ and an almost surely finite random variable $t_1 = t_1(\omega)$ such that

$$\Delta_1(T) = \sup_{0\leq t\leq T} ||S(t) - W(t)|| \leq C_3 r(T) \text{ if } T \geq t_1 \tag{2.2}$$

where $C_3 > 0$ is a finite constant and $r(T)$ is a deterministic function such that

$$(2.3) \qquad r(T) = o(T^{1/2-\alpha}) \text{ as } T \to \infty,$$

with α as given in the formulation of the theorem. We shall work on this probability space.

Applying the Minkowski inequality first for the supremum norm and then for the h-norm, we obtain via the right side of (2.1) that

$$\Delta_2(T) = \sup_{0 \leq t \leq T} |M(t) - V(t)| \leq C_2 \Delta_1(T),$$

and hence, with $C_4 = C_2 C_3$,

$$(2.4) \qquad \Delta_2(T) \leq C_4\, r(T) \quad \text{if } T \geq t_1.$$

Next we claim that there exist a constant $C_5 > 0$ and an almost surely finite random variable $t_2 = t_2(\omega)$ such that

$$(2.5) \quad \min\,(V(T), M(T)) \geq b(T) = C_5 T^{1/2} (\log\log T)^{-1/2} \quad \text{if } T \geq t_2.$$

Indeed, for any $0 < \varepsilon < 1$ and $1 \leq k \leq d$, by the left side of (2.1),

$$\begin{aligned} V(T) &\geq C_1 \sup_{0 \leq t \leq T} ||W(t)|| \\ &\geq C_1 \sup_{0 \leq t \leq T} |W_k(t)| \\ &\geq (1-\varepsilon)\, C_1\, \pi(T_{kk}/8)^{1/2} T^{1/2} (\log\log T)^{-1/2} \end{aligned}$$

whenever $t \geq t_3^* = t_3^*(\omega)$, where t_3^* is some almost surely finite random variable. This follows from Chung's (1948) law of the iterated logarithm (cf. also Csörgö and Révész (1981) p.48). Also,

$$M(T) \geq C_1 \sup_{0 \leq t \leq T} ||S(t)||$$

$$\geq C_1 \sup_{0 \leq t \leq T} |W(t)| - \Delta_1(T)$$

$$\geq (1-\varepsilon)\ C_1 \pi (T_{kk}/8)^{1/2} T^{1/2} (\log \log T)^{-1/2} - \Delta_1(T)$$

whenever $t \geq t_3^*$. Now (2.5) follows from the last two inequalities and (2.2). Of course, $t_2 \geq t_1$ almost surely.

Now let $\varepsilon > 0$ be a fixed number and let $t_3 > 0$ be a (deterministic) threshold number such that

$$\text{(2.6)} \qquad b(\{(1+\varepsilon)/C_5^2\}\ t^2 \log \log t) > t \text{ if } t \geq t_3.$$

Define $t_o = \max\ (t_2, t_3)$. Then, if $t \geq t_o$,

$$N(t) = \inf\ \{u: M(u) > t\}$$

$$= \inf\ \{u: M(u) > t,\ 0 \leq u \leq \frac{1+\varepsilon}{C_5^2} t^2 \log \log t\}$$

$$\leq \inf\ \{u: V(u) > t + C_4\ r\ (\frac{1+\varepsilon}{C_5^2}\ t^2 \log \log t),\ 0 \leq u \leq \frac{1+\varepsilon}{C_5^2} t^2 \log \log t\}$$

$$= \inf\ \{u: V(u) > t + C_4\ r(\frac{1+\varepsilon}{C_5^2}\ t^2 \log \log t)\}$$

$$= Z(t + C_4\ r(\frac{1+\varepsilon}{C_5^2}\ t^2 \log \log t)),$$

where the first equality is by (2.5) and (2.6), the inequality is by (2.4), and the next equality is again by (2.5) and (2.6). Similarly,

$$N(t) \geq Z(t - C_4 r(\frac{1+\varepsilon}{C_5^2}\ t^2 \log \log t)) \quad \text{if } t \geq t_o,$$

and in view of (2.3) the theorem follows.

Proof of Corollary 1. Let $\rho_T(\cdot,\cdot)$ denote the usual distance that metrizes $D[0,T]$. ($\rho_1(\cdot,\cdot)$ is denoted by $d(\cdot,\cdot)$ in Billingsley (1968)). Since by the scale transformation of the Wiener process

(2.7) $$\{V(t)/T^{1/2}: 0 \le t \le T\} \overset{\mathcal{D}}{=} \{V(u): 0 \le u \le 1\},$$

easy consideration shows that

(2.8) $$\{Z(t)/T^2: 0 \le t \le T\} \overset{\mathcal{D}}{=} \{Z(u): 0 \le u \le 1\}.$$

Therefore, it is sufficient to show that

(2.9) $$T^{-2}\rho_T(N(\cdot),\ Z(\cdot)) \overset{P}{\to} 0 \text{ as } T \to \infty$$

on the probability space $(\Omega,\ \mathbf{A},\ P)$ of Theorem 1.

Theorem 1 implies that for any fixed $0 < \varepsilon < 1$ there is a $T_0 = T_0(\varepsilon) \ge 0$ such that for any $T > T_0$,

$$P\{Z(t-a_T) \le N(t) \le Z(t+a_T),\ T_0 \le t \le T\} > 1-\varepsilon.$$

On the other hand, it is easy to show that

$$T^{-2} \sup_{0\le t\le T_0} N(t) = T^{-2}\, N(T_0) \to 0 \text{ and } T^{-2} \sup_{0\le t\le T_0} Z(t) = T^{-2}Z(T_0) \to 0$$

almost surely as $T \to \infty$. Therefore it is enough to show that

$$T^{-2}\rho_T(U_T^+(\cdot),\ Z(\cdot)) \overset{P}{\to} 0 \text{ and } T^{-2}\rho_T(U_T^-(\cdot),\ Z(\cdot)) \overset{P}{\to} 0$$

as $T \to \infty$, where $U_T^+(t) = Z(t+a_T)$ and $U_T^-(t) = Z(t-a_T)$, which by (2.8) is equivalent to

(2.10) $$\rho_1(Y_T^+(\cdot),\ Z(\cdot)) \overset{P}{\to} 0 \text{ and } \rho_1(Y_T^-(\cdot),\ Z(\cdot)) \overset{P}{\to} 0 \text{ as } T \to \infty,$$

where

$$Y_T^+(t) = Z(t + T^{-1}a_T) \text{ and } U_T^-(t) = Z(t - T^{-1}a_T).$$

To show (2.10) we need one more preliminary. We claim that at any fixed $0 < t^* \leq 1$ the process Z is stochastically left continuous. Indeed, for any $s > 1$ and $\varepsilon > 0$,

$$P\{Z(t^*) - Z(t^*/s) > \varepsilon\} = P\{Z(st^*/s) - Z(t^*/s) > \varepsilon\}$$

$$= P\{s^{-2}(Z(st^*) - Z(t^*)) > \varepsilon\}$$

by (2.8), which by the almost sure right -continuity of Z goes to zero if $s \downarrow 1$.

Now let us introduce the following strictly increasing continuous mapping of [0,1] onto itself:

$$\lambda^-(t) = \begin{cases} t/2, & \text{if } 0 \leq t \leq 2T^{-1}a_T, \\ t-T^{-1}a_T, & \text{if } 2T^{-1}a_T \leq t \leq 1 - T^{-1}, \\ (1+a_T)t - a_T, & \text{if } 1-T^{-1} \leq t \leq 1, \end{cases}$$

where, of course, we assume that T is so large as to make this definition meaningful. We have

$$\rho_1(Y_T^-(\cdot), Z(\cdot)) \leq \sup_{0\leq t\leq 1} |\lambda^-(t) - t| + \sup_{0\leq t\leq 1} |Z(\lambda^-(t)) - Z(t-T^{-1}a_T)|$$

$$\leq T^{-1}a_T + 2Z(T^{-1}a_T) + \sup_{0\leq t\leq 1} (Z((1+a_T)t-a_T) - Z(t-T^{-1}a_T))$$

$$\leq T^{-1}a_T + 2Z(T^{-1}a_T) + (Z(1) - Z(1-T^{-1}-T^{-1}a_T))$$

for Z is a nondecreasing process. Since $T^{-1}a_T \to 0$ as $T \to \infty$, the second term here converges to $Z(0) = 0$ almost surely, while the third term converges to zero in probability due to the stochastic left continuity of Z at one. Hence we have shown that the second relation in (2.10) holds true. Introducing an appropriate λ^+ function, one shows similarly that the first relation in (2.10) also holds true. The corollary is proved.

Proof of Corollary 2. First consider (1.1). As a first step we show that if $I(\psi^{-1},C) < \infty$, then

$$\liminf_{t\to\infty} Z(t)/\psi(t) \geq (2CC_2T)^{-1} \text{ almost surely,} \tag{2.11}$$

where $Z = Z_h$, C_2 is as in (2.1) and $T^2 = \max(T_{11},\ldots,T_{dd})$, Set $K = 2CC_2T$ and introduce the function $f_K(t) = \psi^{-1}(Kt)$. The condition that $I(\psi^{-1},C) < \infty$ implies that $I(f_K,(2C_2T)^{-1}) < \infty$. The classical Erdös- Feller-Kolmogorov-Petrovski upper class integral test then implies that there exists an almost surely finite random variable $u_0 = u_0(\omega)$ such that whenever $u \geq u_0$,

$$\begin{aligned} V(u) &\leq C_2 \sup_{0\leq s\leq u} ||W(s)|| \leq C_2\, T(C_2T)^{-1}\psi^{-1}(Ku) \\ &= \psi^{-1}(Ku). \end{aligned}$$

This is the same as

$$V(\psi(t)/K) \leq t \text{ whenever } t \geq t_4 = \psi(u_0)/K, \tag{2.12}$$

which implies

$$Z(t) \geq \psi(t)/K \text{ whenever } t \geq t_4. \tag{2.13}$$

Hence we proved (2.11).

Now we show that (2.11) and Theorem 1 imply (1.1). Let $0 < \varepsilon < 1$ be given and introduce $\psi_{1/(1-\varepsilon)}(t) = \psi(t/(1-\varepsilon))$. Since $\psi^{-1}_{1/(1-\varepsilon)}(t) = (1-\varepsilon)\psi^{-1}(t)$, $I(\psi^{-1},C) < \infty$ implies that $I(\psi^{-1}_{1/(1-\varepsilon)},C/(1-\varepsilon)^2) < \infty$. Hence by (2.11) $\liminf_{t\to\infty} Z(t)/\psi_{1/(1-\varepsilon)}(t) \geq (1-\varepsilon)^2(2CC_2T)^{-1}$ almost surely.

This means that there exist an almost surely finite random variable $t_5 = t_5(\omega)$ such that $Z(t) \geq (1-\varepsilon)^2(2CC_2T)^{-1}\psi_{1/(1-\varepsilon)}(t)$ whenever $t \geq t_5$. Let $0 < t_6 < \infty$ be such a number that $a_t/t < \varepsilon$ whenever $t \geq t_6$, where a_t is as in Theorem 1. Now if $t \geq \max(t_0, t_6, t_5/(1-\varepsilon))$, then

$$N(t) \geq Z(t-a_t) \geq Z(t(1-\varepsilon)) \geq \frac{(1-\varepsilon)^2}{2CC_2T}\psi_{1/(1-\varepsilon)}(t(1-\varepsilon))$$

$$\geq \frac{(1-\varepsilon)^2}{2CC_2T}\psi(t).$$

Hence we in fact obtained that $I(\psi^{-1},C) < \infty$ implies that

$$\liminf_{t\to\infty} N_h(t)/\psi(t) \geq 1/(2CC_2T) \text{ almost surely.}$$

Now we consider (1.2). Again, as a first step we show that if $I(\psi^{-1},C) = \infty$ for all $C > 0$, then

$$\liminf_{t\to\infty} Z(t)/\psi(t) = 0 \text{ almost surely.} \tag{2.14}$$

Note that whenever $\psi(t) \to \infty$ as $t \to \infty$, a routine application of Blumenthal's zero-one law (Itô and McKean (1965)) implies that the left side of (2.14) is almost surely a constant in $[0,\infty]$. Now if this left side is not less than $1/K > 0$ with some K, then there exists an almost surely finite random variable $t_4 = t_4(\omega)$ such that (2.13) and hence (2.12) holds, which in turn implies

$$C_1 \sup_{0\leq s\leq u} ||W(s)|| \leq V(u) \leq TT^{-1}\psi^{-1}(Ku) = \psi^{-1}(Ku)$$

whenever $u \geq u_0 = \psi^{-1}(Kt_4)$. But then the Erdös-Feller-Kolmogorov-Petrovski test implies that $I(f_K,(2C_1T)^{-1}) < \infty$ which in turn implies that $I(\psi^{-1},C) < \infty$ with $C = K/(2C_1T)$. This proves (2.14).

We show now that the theorem and (2.14) imply (1.2). Let the same arbitrary $0 < \varepsilon < 1$ be given as above. If $I(\psi^{-1},C) = \infty$ for all $C > 0$ then $I(\psi^{-1}{}_{1/(1+\varepsilon)},C) = \infty$ for all $C > 0$. Hence by (2.14),

$$\liminf_{t\to\infty} Z(t)/\psi_{1/(1+\varepsilon)}(t) = 0 \text{ almost surely.}$$

This means that for any $\delta > 0$ there exists a sequence $t_7(\omega) < t_8(\omega) < \ldots$ of random variables converging to ∞ almost surely such that $Z(t_n) < \delta\ \psi_{1/(1+\varepsilon)}(t_n)$ almost surely for all large enough n. For all these n's if $t_n \geq \max(t_0,t_6)$, then

$$\begin{aligned} N(t_n) \leq Z(t_n+a_{t_n}) \leq Z(t_n(1+\varepsilon)) &\leq \delta\ \psi_{1/(1+\varepsilon)}(t_n(1+\varepsilon)) \\ &= \delta\ \psi(t_n), \end{aligned}$$

and hence (1.2) follows.

The proof of (1.3) and (1.4) is completely analogous. The only difference is that instead of the Erdös- Feller-Kolmogorov-Petrovski test we use Chung's (1948) test for the supremum of the modulus of a scalar-valued Wiener process.

<u>Proof of Theorem 2</u>. Following Robbins and Siegmund (1970), for any $0 < \tau < C < \infty$ we have

$$p_m(\tau,c) \leq \Pr\{h(S(n)) \geq m^{1/2}g(n/m) \text{ for some } n \geq \tau\, m\} \leq p_m(\tau,c) + q_m(c),$$

where

$$p_m(\tau,c) = \Pr\{h(S(n)) \geq m^{1/2}g(n/m) \text{ for some } \tau m \leq n \leq cm\}$$

and

$$q_m(c) = \Pr\{h(S(n)) \geq m^{1/2} q(n/m) \text{ for some } n > cm\}.$$

By simple considerations,

$$\lim_{m\to\infty} p_m(\tau,c) = \lim_{m\to\infty} \Pr\{m^{-1/2} \max_{m\tau\leq n\leq cm} (h(S(n)) - g(n/m)) \geq 0\}$$

$$= \lim_{m\to\infty} \Pr\{m^{-1/2} \sup_{\tau\leq t\leq c} (h(S(mt)) - g(t)) \geq 0\}.$$

Since Donsker's invariance principle easily implies that $m^{-1/2}h(S(m.))$ converges weakly in Skorohod's space $D[\tau,c]$ to $h(W(.))$, we obtain

$$\lim_{m\to\infty} p_m(\tau,c) = \Pr\{h(W(t)) \geq g(t) \text{ for some } \tau \leq t \leq c\}$$

exactly as in Robbins and Siegmund (1970). On the other hand, since

$$q_m(c) \leq \Pr\{C_2||S(n)|| \geq m^{1/2} g(n/m) \text{ for some } n > cm\}$$

$$\leq \sum_{k=1}^{d} \Pr\{|S_n^{(k)}| \geq m^{1/2} \frac{1}{TC_2} g(n/m) \quad \text{for some } n > cm\},$$

where

$$S_n^{(k)} = \sum_{j=1}^{n} X_j^{(k)}/T_{kk}^{1/2}, \quad k=1,\ldots,d,$$

it follows from the conditions on g and Lemma 5 of Robbins and Siegmund (1970) (the difficult part of the proof of their Theorem 2) that

$$\liminf_{c\to\infty} \limsup_{m\to\infty} q_m(c) = 0.$$

The theorem now follows by first letting $m\to\infty$ and then letting $c\to\infty$, as in Robbins and Siegmund (1970).

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