LECTURE XII. RANDOM ALLOCATIONS

I shall derive three of the simpler results given by Kolchin, Sevastyanov, and Chistyakov in their book Random allocations, using the methods introduced in the first lecture. One is the exact joint distribution of the numbers of urns containing 0,1,2,... balls when ν balls are distributed at random (uniformly) among k urns. The second is the analogous problem for cycle lengths of random permutations. The third is the Poisson approximation to the distribution of the number of empty urns in the first problem, when the expected proportion of empty urns is small.

In the urn problem, let N_α be the number of urns containing α balls for $\alpha \in \{0,\dots,\nu\}$. We want to compute the

$$p(n) = P\{N = n\} \tag{1}$$

for $n: \{0,\dots,\nu\} \to Z^+$ satisfying

$$\Sigma n_\alpha = k \tag{2}$$

and

$$\Sigma \alpha n_\alpha = \nu. \tag{3}$$

Of course, for n that do not satisfy (2) and (3), p(n) = 0. We shall see that

$$p(n) = \frac{1}{\Pi[(\alpha!)^{n_\alpha} n_\alpha!]} \; \frac{k!\nu!}{k^\nu}. \tag{4}$$

In order to study the distribution of N we define a new random vector N', obtained by removing a randomly selected ball (uniformly distributed over the set of all balls, independent of N) and replacing it in a randomly selected urn (uniformly distributed over the set of all urns, independent of previous choices). Let I be the original number of balls in the urn from which this ball was selected and J the number of balls that are in the urn in which it is replaced after this replacement. Then

(5) $$P^N\{I = i \text{ and } J = j\} = \frac{iN_i}{\nu} \cdot \frac{N'_{j-1}+1}{k} .$$

The reason for the second factor is that we put the ball into an urn which then had j-1 balls and there was one urn that had j-1 balls but now has j. Also

(6) $$N' = N - \delta_I + \delta_{I-1} + \delta_J - \delta_{J-1}$$

where, for $i \in Z^+$, δ_i is the function on Z^+ defined by

(7) $$\delta_{i\alpha} = \begin{cases} 1 & \text{if } \alpha = i \\ 0 & \text{otherwise.} \end{cases}$$

As the first step in proving (4) we observe that it holds in the very special case where one urn contains all the balls, that is

(8) $$n_\alpha = \begin{cases} 1 & \text{if } \alpha = \nu \\ k-1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha \notin \{0,\nu\}. \end{cases}$$

In this case,

(9) $$p(n) = \frac{1}{k^{\nu-1}} = \frac{1}{(k-1)!\nu!} \quad \frac{k!\nu!}{k^\nu} .$$

In addition I shall verify that the p(n), now thought of as defined by (4), satisfy the identity

(10) $$p(n)P\{N'=n'|N=n\} = p(n')P\{N'=n|N=n'\}.$$

Since the sample space is connected in the sense of Lemma I.2, this will imply that the desired probabilities are given by (4). In view of (4), (5), and (6), the identity (10) is equivalent to the condition that, for all n, i, and j,

(11) $$\frac{1}{\Pi[(\alpha!)^{n_\alpha} n_\alpha!]}\, i n_i (n'_{j-1}+1) = \frac{1}{\Pi[(\alpha!)^{n'_\alpha} n'_\alpha!]}\, j n'_j (n_{i-1}+1)$$

where

(12) $$n' = n - \delta_i + \delta_{i-1} + \delta_j - \delta_{j-1}.$$

It is convenient to introduce n", defined by

(13) $$n'' = n - \delta_i + \delta_{i-1} = n' - \delta_j + \delta_{j-1}.$$

Then, inserting a factor on both sides analogous to the products over α, but with n or n' replaced by n" we see that (11) is equivalent to

(14) $$\Pi\left[(\alpha!)^{\delta_{i-1,\alpha}-\delta_{i,\alpha}} \frac{n''_\alpha!}{(n''_\alpha+\delta_{i,\alpha}-\delta_{i-1,\alpha})!}\right]\cdot\frac{i n_i}{n_{i-1}+1}$$
$$= \Pi\left[(\alpha!)^{\delta_{j-1,\alpha}-\delta_{j,\alpha}} \frac{n''_\alpha!}{(n''_\alpha+\delta_{j,\alpha}-\delta_{j-1,\alpha})!}\right]\cdot\frac{j n'_j}{n'_{j-1}+1},$$

which is true, since both sides are equal to 1.

Now let us look at an analogous problem for cycle lengths of random permutations. We recall that any permutation Π of $\{1,\dots,\nu\}$ can be represented as a product of disjoint cycles, and this representation is unique apart from the order in which the cycles are multiplied. A cycle, denoted by a finite sequence $(\beta_1\dots\beta_\ell)$ with $\ell \geq 2$ of distinct elements of $\{1\dots\nu\}$ enclosed in parentheses, is defined as a permutation by

(15) $$(\beta_1\dots\beta_\ell)\beta_i = \beta_{i+1} \quad \text{if} \quad i \in \{1\dots\ell-1\}$$

(16) $$(\beta_1\dots\beta_\ell)\beta_\ell = \beta_1$$

and

(17) $$(\beta_1 \ldots \beta_\ell)\beta = \beta \text{ if } \beta \notin \{\beta_1,\ldots,\beta_\ell\}.$$

Two such sequences determine the same cycle if and only if one is a cyclic permutation of the other. In the representation of a permutation as a product of cycles, we count each element of $\{1,\ldots,n\}$ that is transformed into itself by the permutation as a cycle of length 1. Now for a random permutation Π of $\{1,\ldots,\nu\}$, uniformly distributed over the set of all $\nu!$ such permutations, let N_α be the number of cycles of length α in its decomposition as a product of cycles, for $\alpha \in \{1,\ldots,\nu\}$. We shall see that, for any $n: \{1,\ldots,\nu\} \to Z^+$ such that

(18) $$\sum \alpha n_\alpha = \nu$$

we have

(19) $$p(n) = P\{N = n\} = \frac{1}{\Pi[\alpha^{n_\alpha} n_\alpha !]} .$$

As in several earlier lectures, we construct another random permutation Π' such that (Π,Π') is an exchangeable pair by choosing a random transposition (IJ) (cycle of length two), uniformly distributed over all $\binom{n}{2}$ possible choices, independent of Π, and defining

(20) $$\Pi' = \Pi \circ (IJ).$$

It is not difficult to see how the cycle structure of Π' is related to that of Π. With

(21) $$\pi' = \pi \circ (ij),$$

let us first look at the case where i and j belong to the same cycle of π. There is no essential loss of generality in supposing this cycle to be $(1\ldots\ell)$ and choosing $i = 1$. Then, with $2 \leq j < \ell$,

(22) $$(1\ldots\ell)(1j) = (1\ j{+}1\ldots\ell)(j\ 2\ldots j{-}1).$$

Thus, in this case, the effect of multiplying by a transposition (ij) is to separate the cycle of length ℓ into two cycles, whose lengths are one more than the numbers of elements between i and j in the original cycle of length ℓ. By interchanging the roles of π and π' we see that, if i and j fall into different cycles of π, the effect of multiplying π by (ij) is to join the two cycles into one, whose length is, of course the sum of the lengths of the original cycles.

Now we can write down the transition probabilities for multiplication by a random transposition as in (20). For distinct positive integers a and b,

(23) $$P^{\Pi}\{N'_{a+b} = N_{a+b}-1 \ \& \ N'_a = N_a+1 \ \& \ N'_b = N_b+1 \ \& \text{ other } N'_\alpha = N_\alpha\}$$
$$= \frac{2(a+b)N_{a+b}}{\nu(\nu-1)}$$

and

(24) $$P^{\Pi}\{N'_{a+b} = N_{a+b}+1 \ \& \ N'_a = N_a-1 \ \& \ N'_b = N_b-1 \ \& \text{ other } N'_\alpha = N_\alpha\}$$
$$= \frac{2abN_aN_b}{\nu(\nu-1)} .$$

Also

(25) $$P^{\Pi}\{N'_{2a} = N_{2a}-1 \ \& \ N'_a = N_a+2 \ \& \text{ other } N'_\alpha = N_\alpha\}$$
$$= \frac{2aN_{2a}}{\nu(\nu-1)}$$

and

(26) $$P^{\Pi}\{N'_{2a} = N_{2a}+1 \ \& \ N'_a = N_a-2 \ \& \text{ other } N'_\alpha = N_\alpha\}$$
$$= \frac{a^2N_a(N_a-1)}{\nu(\nu-1)} .$$

The arguments in the four cases are as follows:

(i) For (IJ) to break a cycle of length a+b into two cycles of different lengths a and b, I must fall into one of the $(a+b)N_{a+b}$ places available in cycles of length a+b, and then there remain two positions for j.

(ii) For (IJ) to join two cycles of different lengths a and b, one of I and J must fall into one of the aN_a places available in cycles of length a and the other into one of the bN_b places available in cycles of length b.

(iii) For (IJ) to break a cycle of length 2a into two cycles of length a, I must fall into one of the $2aN_{2a}$ places available in cycles of length 2a and then the position of J is determined.

(iv) For (IJ) to join two cycles of length a, I must fall into one of the aN_a places available in cycles of length a and then J must fall into one of the $a(N_a-1)$ places available in other cycles of length a.

Now we are prepared to prove that the probabilities $p(n) = P\{N = n\}$ are given by (19). First we observe that this is obviously true when all cycle lengths are 1:

$$P\{N_1 = \nu \text{ \& other } N_\alpha = 0\} = \frac{1}{\nu!} . \tag{27}$$

Then it will suffice to verify that the p(n), thought of as defined by the final expression in (19), satisfy the identity (10) with the present interpretation. Because of (23) - (26), these identities are equivalent to the two identities:

$$\frac{1}{\Pi[\alpha^{n_\alpha} n_\alpha!]} \frac{2(a+b)n_{a+b}}{\nu(\nu-1)} = \frac{1}{\Pi[\alpha^{n'_\alpha} n'_\alpha!]} \frac{2abn'_a n'_b}{\nu(\nu-1)} \tag{28}$$

where

$$n'_{a+b} = n_{a+b}-1, \quad n'_a = n_a+1, \quad n'_b = n_b+1 \tag{29}$$

and other $n'_\alpha = n_\alpha$, and

$$\frac{1}{\Pi[\alpha^{n_\alpha} n_\alpha!]} \frac{2an_{2a}}{\nu(\nu-1)} = \frac{1}{\Pi[\alpha^{n'_\alpha} n'_\alpha!]} \frac{a^2 n'_a(n'_a-1)}{\nu(\nu-1)} , \tag{30}$$

where

$$n'_{2a} = n_{2a}-1, \quad n'_a = n_a+2 \tag{31}$$

and other $n'_\alpha = n_\alpha$. Of course (28) is required to hold only for $a \neq b$. Now (28) is equivalent to

(32) $$\frac{ab(n_a+1)(n_b+1)}{(a+b)n_{a+b}} = \Pi\left(\alpha^{n'_\alpha - n_\alpha} \frac{n'_\alpha!}{n_\alpha!}\right),$$

which is readily verified subject to (29), and (30) is equivalent to

(33) $$\frac{a^2(n_a+2)(n_a+1)}{2an_{2a}} = \Pi\left(\alpha^{n'_\alpha - n_\alpha} \frac{n'_\alpha!}{n_\alpha!}\right),$$

which is also true, subject to (31). This completes the proof of (19).

Next let us look at the Poisson approximation to the distribution of N_0, the number of empty urns, in the case where ν is large compared with k. We have

(34) $$P^N\{N'_0 = N_0+1\} = \frac{N_1}{\nu}\left(1 - \frac{N_0+1}{k}\right)$$

and

(35) $$P^N\{N'_0 = N_0-1\} = \left(1 - \frac{N_1}{\nu}\right)\frac{N_0}{k}.$$

These are consequences of (5) and (6) but the reader may find it easier to verify them directly. Applying the identity (I.6) to the antisymmetric function F defined by

(36) $$F(N,N') = f(N_0)\mathscr{I}\{N'_0 = N_0-1\} - f(N'_0)\mathscr{I}\{N_0 = N'_0-1\},$$

we obtain, with the aid of (34) and (35),

(37) $$\begin{aligned} 0 &= \left[E\left(1 - \frac{N_1}{\nu}\right)\frac{N_0}{k} f(N_0) - \frac{N_1}{\nu}\left(1 - \frac{N_0+1}{k}\right)f(N_0+1)\right] \\ &= \left[\frac{1}{k} N_0 f(N_0) - \frac{k+1}{\nu} N_1 f(N_0+1) + \frac{N_0 N_1}{\nu}\left(f(N_0+1) - f(N_0)\right)\right], \end{aligned}$$

where $f\colon Z^+ \to R$ is arbitrary.

As suggested by the resemblance of (37) to (VIII.1), we take f to be the solution of equation (VIII.5), that is

(38) $$\lambda f(w+1) - wf(w) = h(w) - \mathcal{P}_\lambda h,$$

where

(39) $$\lambda = \frac{k+1}{\nu} EN_1 = (k+1)(1 - \frac{1}{k})^{\nu-1},$$

h is for the present an arbitrary bounded function on Z^+ to R, and $\mathcal{P}_\lambda h$ is defined by (VIII.4). Then (37) yields

(40) $$Eh(N_0) = \mathcal{P}_\lambda h + E(\lambda - \frac{k+1}{\nu} N_1)f(N_0+1) + E \frac{N_0 N_1}{\nu} (f(N_0+1) - f(N_0)).$$

We need to bound the remainder in (40), that is, the sum of the last two terms, especially in the case $h = h_A$ given by (VIII.32). Recall the definition of the linear mappings U_λ, V_λ given in (VIII.18) (that is, in our present notation, $U_\lambda h = f$) and (VIII.29). We bound the last term in (40) by using (VIII.42), that is

(41) $$|V_\lambda h_A(w)| \leq \lambda^{-1} \wedge 1.$$

Thus

(42) $$|E \frac{N_0 N_1}{\nu} (f(N_0+1) - f(N_0))| \leq (\lambda^{-1} \wedge 1) E \frac{N_0 N_1}{\nu} .$$

But

(43) $$EN_0 N_1 = k(k-1) \cdot \nu \cdot \frac{1}{k} (1 - \frac{2}{k})^{\nu-1} = \nu(k-1)(1 - \frac{2}{k})^{\nu-1}.$$

Consequently

(44) $$E \frac{N_0 N_1}{\nu} (f(N_0+1) - f(N_0))| \leq (\lambda^{-1} \wedge 1)(k-1)(1 - \frac{2}{k})^{\nu-1}.$$

This is small provided $\frac{\nu}{k}$ is large.

It remains to bound the second term on the right hand side of (40). In order to obtain a bound comparable to the bound obtained in (44) for the third term, a moderately subtle argument will be needed. With the urns numbered, let M_β be the number of balls in the β^{th} urn. Then

$$EN_1 f(N_0+1) = E \sum_{\beta=1}^{k} \mathscr{I}\{M_\beta = 1\} f(N_0+1) \tag{45}$$

$$= E \sum_{\beta=1}^{k} \mathscr{I}\{M_\beta = 1\} E[f(N_0+1)|M_\beta = 1]$$

$$= EN_1 Ef(N_0^*+1),$$

where N_0^* is a random variable whose distribution is the same as the conditional distribution of N_0 given $M_1 = 1$. It follows that for any determination of random variables N_0' and $N_0'^*$ with the same distributions as N_0 and N_0^* on a common probability space we have

$$|E(\lambda - \frac{k+1}{\nu} N_1) f(N_0+1)| \tag{46}$$

$$= \frac{k+1}{\nu} EN_1 |E[f(N_0'+1) - f(N_0'^*+1)]|$$

$$\leq \frac{k+1}{\nu} EN_1 E|N_0'-N_0'^*| \sup|V_\lambda h|$$

when $f = U_\lambda h$, where U_λ and V_λ are defined by (VIII.18) and (VIII.29).

We can define random variables N_0' and $N_0'^*$ with the desired distributions on a common probability space in the following way. Somewhat imprecisely, I shall call these random variables N_0 and N_0^*. Having distributed ν balls among k urns, thus determining N we choose an urn and a ball at random, uniformly distributed over all possible such pairs, independent of N. We throw away the ball that we selected, remove the balls from the urn that was selected, discard that urn, and distribute the balls uniformly among the remaining urns, conditionally uniformly given the earlier choices. We shall see that, with this definition of N_0 and N_0^* on a common probability space (described somewhat informally, I must admit), we have

$$E|N_0 - N_0^*| \leq (2 + \frac{\nu}{k})(1 - \frac{1}{k})^{\nu-1}. \tag{47}$$

First we observe that $N_0^* - N_0$ is at most one, and it can be equal to one only in the ball is selected from an urn containing exactly one ball. Thus

$$E(N_0^* - N_0)_+ \le \frac{1}{\nu} EN_1 = (1 - \frac{1}{k})^{\nu-1}. \tag{48}$$

Now let A be the number of balls in the urn that was selected. Then, unless an empty urn was selected, $N_0 - N_0^*$ is at most equal to the number of balls out of the A redistributed balls that fall into the N_0 empty urns. Thus

$$\begin{aligned} E(N_0 - N_0^*)_+ &\le E\,\frac{N_0}{k} + EA\,\frac{N_0}{k} \le \frac{\nu+k}{k^2} EN_0 \\ &= \frac{\nu+k}{k}(1 - \frac{1}{k})^{\nu}. \end{aligned} \tag{49}$$

Then (47) follows from (48) and (49). Substituting in (46) we obtain

$$|E(\lambda - \frac{k+1}{\nu} N_1) f(N_0+1)| \le (1+ \frac{1}{k})(\nu+2k)(1- \frac{1}{k})^{2(\nu-1)} \sup|V_\lambda h_A|. \tag{50}$$

Finally, (40), (41), (44), and (50) yield

$$|P\{N_0 \in A\} - \mathcal{P}_\lambda h_A| \le (3 + \frac{\nu}{k})(1 - \frac{1}{k})^{\nu-1}[1 \wedge (k(1 - \frac{1}{k})^{\nu-1})]. \tag{51}$$

Note that this approaches zero as $\frac{\nu}{k}$ approaches infinity.

In the first part of this lecture, the method of auxiliary randomization was applied to determine the exact distribution in two simple problems. In both of these cases the results are well known. For the first problem suppose ν balls are distributed uniformly among k urns. The joint distribution of the numbers N_α of urns containing α balls, for $\alpha \in \{0,\dots,\nu\}$ was obtained by applying the identity (10), where (N,N') is an appropriately chosen exchangeable pair. The second problem was to determine the joint distribution of the cycle lengths of a random permutation.

In the second part, starting with (34), the formalism of Lecture VIII, resulting from appropriate specialization of diagram (I.28) was applied to obtain the Poisson approximation to the distribution of the number of empty cells when ν is large compared to k. This method seems to be more powerful than the simpler approach of the first part of the lecture when good approximations are wanted.