LECTURE VII. COUNTING LATIN RECTANGLES

The problem of determining an asymptotic expression for the number $N_{k,n}$ of $k \times n$ Latin rectangles as n approaches infinity was first solved by Erdös and Kaplansky (1946) for the case

(1) $$k = o((\log n)^{\frac{3}{2}}).$$

They proved that, subject to (1),

(2) $$p_{k,n} = \frac{N_{k,n}}{(n!)^k} \sim e^{-\frac{k(k-1)}{2}}.$$

This result was extended to

(3) $$k = o(n^{1/3})$$

by Yamamoto (1951). The case k = 2 is the familiar "problème des rencontres," where the exact solution,

(4) $$p_{2,n} = \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

shows that, in this case, the approximation

(5) $$p_{2,n} \sim e^{-1}$$

given by (2) is extremely good if n is at all large. In this lecture I shall prove Yamamoto's result that, for $k = o(n^{\frac{1}{2}})$,

(6) $$p_{k,n} = e^{-\frac{k(k-1)}{2} + O(\frac{k^3}{n})}.$$

In a later lecture I shall derive a more accurate approximation than (6). These two lectures are based on my 1978 paper in the Journal of Combinatorial Theory, Series A.

A $k \times n$ Latin rectangle $\mathscr{L}$ is a $k \times n$ rectangular array in which the entries are elements of $\{1,\ldots,n\}$ and there are no repetitions in any row or column. If we choose a $k \times n$ rectangular array by taking as the rows independent random permutations of $1,\ldots,n$, then $p_{k,n}$ defined in (2) is the probability that the resulting array is a Latin rectangle. I shall follow Erdös and Kaplansky and Yamamoto in imagining that we are already given a $k \times n$ Latin rectangle $\mathscr{L}$ and choose as $(k+1)$st row a random permutation Π of $\{1,\ldots,n\}$. We shall see that for $k = o(n^{\frac{1}{2}})$ the probability $p^*_{k,n,\mathscr{L}}$ that the resulting configuration is a $(k+1) \times n$ Latin rectangle is given by

$$p^*_{k,n,\mathscr{L}} = e^{-k+O(\frac{k^2}{n})}, \tag{7}$$

which implies (6). This is intuitively plausible since the expected number of coincidences of the new row with the k original rows is k and coincidences in the different columns are nearly independent rare events, so that (7) is the appropriate Poisson approximation. We can express W, the number of coincidences of the $(k+1)$st row with earlier rows, as

$$W = \sum_{i=1}^{n} \mathscr{I}\{\Pi(i) \in S(i)\}, \tag{8}$$

where $S(i)$ is the set of numbers occurring in the i^{th} column of the given $k \times n$ rectangle.

As in earlier lectures, the principal tool will be auxiliary randomization. Let I be uniformly distributed over $\{1,\ldots,n\}$ and let the conditional distribution of J given I be uniform over the complement of $\{I\}$ in $\{1,\ldots,n\}$. Also let

$$\Pi' = \Pi \circ (I,J), \tag{9}$$

where (I,J) is the designated random transposition, so that

$$\Pi'(i) = \begin{cases} \Pi(i) & \text{if } i \notin \{I,J\} \\ \Pi(J) & \text{if } i = I \\ \Pi(I) & \text{if } i = J. \end{cases} \tag{10}$$

The ordered pair (Π,Π') of random permutations is an exchangeable pair in the sense that the joint distribution of (Π,Π') is the same as that of (Π',Π). This can be seen by imagining that we first choose I and J, then $\Pi(i) = \Pi'(i)$ for $i \notin \{I,J\}$ and finally we choose one of the two remaining possibilities for Π and the other for Π'. We define

$$W' = \sum_{i=1}^{n} \mathscr{I}\{\Pi'(i) \in S(i)\} \tag{11}$$

and observe that (W,W') is also an exchangeable pair. It follows by the usual antisymmetry argument that, for any $f\colon \{0,\dots,n\} \to R$,

$$\begin{aligned} 0 &= E[f(W)\mathscr{I}\{W'=W+1\} - f(W')\mathscr{I}\{W=W'+1\}] \\ &= E[f(W)P^{\Pi}\{W'=W+1\} - f(W-1)P^{\Pi}\{W'=W-1\}]. \end{aligned} \tag{12}$$

From this it will follow that

$$Eh(W) = \mathscr{B}_{n,\frac{k}{n}}h + (n-1)E(V_{n,\frac{k}{n}}h(\bar{W}))(\mathscr{I}\{CD^*\} - \mathscr{I}\{DD^*\}) \tag{13}$$

for arbitrary $h\colon \{0,\dots,n\} \to R$, where

$$\mathscr{B}_{n,p}h = \sum_{w=0}^{n} \binom{n}{w} p^w (1-p)^{n-w} h(w), \tag{14}$$

and the other symbols will be defined in the course of the argument. A crude bound for the remainder on the right-hand side of (13) with h the indicator function of zero will yield (7).

In order to approximate the conditional probabilities occurring on the right-hand side of (12) it will be necessary to introduce four events, C, D, C^*, and D^*. The even C is the creation of a coincidence at I and the event D the destruction of a coincidence at I when Π is replaced by Π', and C^* and D^* are the corresponding events at J. More precisely

$$C = \{\Pi(I) \notin S(I)\ \&\ \Pi(J) \in S(I)\}, \tag{15}$$

$$D = \{\Pi(I) \in S(I)\ \&\ \Pi(J) \notin S(I)\}, \tag{16}$$

$$C^* = \{\Pi(J) \notin S(J)\ \&\ \Pi(I) \in S(J)\}, \tag{17}$$

and

$$D^* = \{\Pi(J) \in S(J)\ \&\ \Pi(I) \notin S(J)\}. \tag{18}$$

Then

$$\{W'=W+1\} = C(C^* \cup D^*)^c \cup C^*(C \cup D)^c, \tag{19}$$

and consequently

$$\begin{aligned} P^{\Pi}\{W'=W+1\} &= P^{\Pi}(C)-P^{\Pi}(CC^*)-P^{\Pi}(CD^*)+P^{\Pi}(C^*)-P^{\Pi}(CC^*)-P^{\Pi}(C^*D) \\ &= 2[P^{\Pi}(C)-P^{\Pi}(CC^*)-P^{\Pi}(CD^*)]. \end{aligned} \tag{20}$$

Similarly

$$P^{\Pi}\{W'=W-1\} = 2[P^{\Pi}(D)-P^{\Pi}(CD^*)-P^{\Pi}(DD^*)]. \tag{21}$$

It is not difficult to evaluate $P^{\Pi}(C)$ and $P^{\Pi}(D)$ and to obtain reasonable upper bounds for $P^{\Pi}(CD^*)$ and $P^{\Pi}(DD^*)$. No satisfactory upper bound for $P^{\Pi}(CC^*)$ can be obtained, but fortunately we shall not need it. We have

$$\begin{aligned} P^{\Pi}(C) &= P^{\Pi}\{\pi(I) \notin S(I)\}P^{\Pi}\{\pi(J) \in S(I)|\pi(I) \notin S(I)\} \\ &= \frac{n-W}{n} \cdot \frac{k}{n-1}, \end{aligned} \tag{22}$$

since there are n-W possible values i of I for which $\pi(i) \notin S(i)$ and for any such i there are k of the $\pi(j)$ with $j \neq i$ for which $\pi(j) \in S(i)$. Similarly

$$\begin{aligned} P^{\Pi}(D) &= P^{\Pi}\{\pi(I) \in S(I)\}P^{\Pi}\{\pi(J) \notin S(I)|\pi(I) \in S(I)\} \\ &= \frac{W}{n} \cdot \frac{n-k}{n-1}. \end{aligned} \tag{23}$$

By the definition of C and D* in (15) and (18) we have

$$\begin{aligned} P^{\Pi}(CD^*) &\leq P^{\Pi}\{\pi(J) \in S(J) \ \& \ \pi(J) \in S(I)\} \\ &= P^{\Pi}\{\pi(J) \in S(J)\}P^{\Pi}\{\pi(J) \in S(I)|\pi(J) \in S(J)\} \\ &= \frac{W}{n} \cdot \frac{k-1}{n-1}. \end{aligned} \tag{24}$$

Similarly

$$\begin{aligned} P^{\Pi}(DD^*) &\leq P^{\Pi}\{\pi(I) \in S(I) \ \& \ \pi(J) \in S(J)\} \\ &= \frac{W(W-1)}{n(n-1)}. \end{aligned} \tag{25}$$

Now we can derive (13). Substituting (20) and (21) and then (22) and (23) in (12), we obtain

$$(26)\qquad 0 = \frac{1}{2} E[f(W)P^{\Pi}\{W'=W+1\}-f(W-1)P^{\Pi}\{W'=W-1\}]$$

$$= E[f(W)(P^{\Pi}(C)-P^{\Pi}(CC^*)-P^{\Pi}(CD^*))$$

$$-f(W-1)(P^{\Pi}(D)-P^{\Pi}(CD^*)-P^{\Pi}(DD^*))]$$

$$= E\left[f(W)\left(\frac{k(n-W)}{n(n-1)} -P^{\Pi}(CC^*)-P^{\Pi}(CD^*)\right)\right.$$

$$\left.-f(W-1)\left(\frac{W(n-k)}{n(n-1)} -P^{\Pi}(CD^*)-P^{\Pi}(DD^*)\right)\right].$$

Letting

$$(27)\qquad p = \frac{k}{n},$$

we obtain, after multiplying (26) by n-1,

$$(28)\qquad E[p(n-W)f(W) - (1-p)Wf(W-1)]$$

$$= (n-1)E[f(W)(\mathscr{I}\{CC^*\}+\mathscr{I}\{CD^*\})$$

$$-f(W-1)(\mathscr{I}\{DD^*\}+\mathscr{I}\{CD^*\})]$$

$$= (n-1)E[(f(W)-f(W-1))\mathscr{I}\{CD^*\}$$

$$-(f(W-1)-f(W-2))\mathscr{I}\{DD^*\}].$$

I have used the fact that

$$(29)\qquad Ef(W)\mathscr{I}\{CC^*\} = Ef(W')\mathscr{I}\{DD^*\}$$

$$= Ef(W-2)\mathscr{I}\{DD^*\}.$$

With $\bar{W}$ defined by

$$(30)\qquad \bar{W} = \sum_{i\notin\{I,J\}} \mathscr{I}\{\Pi(i) \in S(i)\}$$

we can rewrite (28) in the form

$$(31)\qquad E[p(n-W)f(W)-(1-p)Wf(W-1)]$$

$$= (n-1)E[f(\bar{W}+1)-f(\bar{W})](\mathscr{I}\{CD^*\}- \mathscr{I}\{DD^*\}).$$

In order to express this in a more directly useful form we introduce the diagram

$$(32)\qquad \mathfrak{F}_0 \underset{U_{n,p}}{\overset{T_{n,p}}{\rightleftarrows}} \mathscr{X}_0 \underset{\iota_0}{\overset{\mathcal{B}_{n,p}}{\rightleftarrows}} R,$$

associated with the binomial distribution for n trials with probability p. Here $\mathscr{X}_0$ is the linear space of all h: $\{0,\dots,n\} \to R$, $\mathfrak{F}_0$ the linear space of all

$f\colon \{0,\ldots,n-1\} \to R$, $\mathcal{B}_{n,p}$ is defined by (14), ι_0 is the appropriate inclusion mapping, and $T_{n,p}\colon \mathfrak{F}_0 \to \mathcal{X}_0$ is defined by

(33) $$(T_{n,p}f)(w) = p(n-w)f(w) - (1-p)wf(w-1).$$

The linear mapping $U_{n,p}\colon \mathcal{X}_0 \to \mathfrak{F}_0$ is defined implicitly by the condition that, for all $w \in \{0,\ldots,n\}$,

(34) $$p(n-w)(U_{n,p}h)(w) - (1-p)w(U_{n,p}h)(w-1) = h(w) - \mathcal{B}_{n,p}h.$$

See the second paragraph below for details. Finally we define $V_{n,p}\colon \mathcal{X}_0 \to \mathcal{G}_0$ where $\mathcal{G}_0$ is the space of all functions on $\{0,\ldots,n-2\} \to R$ by

(35) $$(V_{n,p}h)(w) = (U_{n,p}h)(w+1) - (U_{n,p}h)(w).$$

Then we can rewrite (31) as

(36) $$Eh(W) = \mathcal{B}_{n,p}h + (n-1)E(V_{n,p}h)(\bar{W})(\mathcal{P}\{CD^*\} - \mathcal{P}\{DD^*\})$$

for arbitrary $h\colon \mathcal{X}_0 \to R$, which is (13).

Now it will be possible, using (36), to prove (7), which is equivalent to

(37) $$P\{W=0\} = (1 - \frac{k}{n})^n(1 + O(\frac{k^2}{n})) \quad \text{for } k = o(n^{\frac{1}{2}}).$$

For large w the trivial bound

(38) $$P\{W \geq w\} \leq \binom{n}{w} \frac{k}{n}\frac{k}{n-1} \cdots \frac{k}{n-w+1} = \frac{k^w}{w!}$$

will be useful. Let

(39) $$h_{w_0}(w) = \begin{cases} 1 & \text{if } w = w_0 \\ 0 & \text{if } w \neq w_0. \end{cases}$$

An upper bound for $V_{n,p}h_{w_0}$ will be obtained from an explicit expression for $U_{n,p}h_{w_0}$. This will lead to a proof, using (36), that, for w_0 of the order of k, $E_{h_{w_0}}(W)$ is of the order of $\mathcal{B}_{n,p}h_{w_0}$, after which it will be easy to prove that the second term on the right hand side of (36) is of the order of

$\frac{k^2}{n}\mathcal{B}_{n,p}h_{w_0}$ when h is replaced by h_{w_0} with w_0 of the order of k. Specializing to $w_0 = 0$ we obtain (37). Undoubtedly this part of the proof is more complicated than is necessary.

In order to write down the explicit expression for $U_{n,p}h$, first let us write C_w for the subset $\{0,\dots,n\}$ consisting of all $w' \le w$. Then we can solve (34) explicitly for $U_{n,p}h$, expressing the result in several different forms

$$(40)\qquad (U_{n,p}h)(w) = \sum_{\ell=0}^{w} \frac{w!}{\ell!}\,\frac{(n-(w+1))!}{(n-\ell)!}\,\frac{1}{p}\left(\frac{p}{1-p}\right)^{\ell-w}[h(\ell)-\mathcal{B}_{n,p}h]$$

$$= -\sum_{\ell=w+1}^{n} \frac{w!}{\ell!}\,\frac{(n-(w+1))!}{(n-\ell)!}\,\frac{1}{p}\left(\frac{p}{1-p}\right)^{\ell-w}[h(\ell)-\mathcal{B}_{n,p}h]$$

$$= \sum_{\ell=0}^{n} \frac{w!}{\ell!}\,\frac{(n-(w+1))!}{(n-\ell)!}\,\frac{1}{p}\left(\frac{p}{1-p}\right)^{\ell-w}(\mathcal{I}\{\ell\le w\} - \mathcal{B}_{n,p}h_{C_w})h(\ell).$$

The first form of (40) is readily verified, by induction on w, to be the unique solution of (34). The equivalence of the first two forms reduces, after multiplication by an appropriate factor, to the fact that

$$(41)\qquad \sum_{\ell=0}^{n}\binom{n}{\ell}p^{\ell}(1-p)^{n-\ell}[h(\ell) - \mathcal{B}_{n,p}h] = 0.$$

The third form follows from the first by applying the definition (14) of $\mathcal{B}_{n,p}h$ and interchanging the order of summation. Similar arguments for the Poisson distribution are given in greater detail in Lecture VIII.

The basic identity (36) can be applied to $h = h_{w_0}$ to obtain

$$(42)\qquad Eh_{w_0}(W) - \mathcal{B}_{n,p}h_{w_0}$$

$$= (n-1)E[(V_{n,p}h_{w_0})(W-1)P^{\Pi}(CD^*) - (V_{n,p}h_{w_0})(W-2)P^{\Pi}(DD^*)]$$

$$\le E\Psi_{w_0}(W)$$

by (24) and (25), where

$$(43)\qquad \Psi_{w_0}(w) = \frac{(k-1)w}{n}\left(V_{n,p}h_{w_0}\right)_+(w-1) + \frac{w(w-1)}{n}\left(-V_{n,p}h_{w_0}\right)_+(w-2).$$

We shall see that

$$\mathcal{B}_{n,p}\Psi_{w_0} = O\left(\frac{k^2}{n}\,\mathcal{B}_{n,p}h_{w_0}\right) \tag{44}$$

for w_0 of the order of k. With the aid of the trivial bound (38) and (36), it will follow that, for such w_0

$$Eh_{w_0}(W) = \mathcal{B}_{n,p}h_{w_0}\left(1 + O\left(\frac{k^2}{n}\right)\right) \tag{45}$$

for $k = o(n^{\frac{1}{2}})$, which is the desired result.

In order to write down explicit expressions for $V_{n,p}h_{w_0}$ and Ψ_{w_0} it will be convenient to define

$$A(w) = \begin{cases} -A_1(w) & \text{if} \quad w \le w_0-2 \\ A_2(w) & \text{if} \quad w = w_0-1 \\ -A_3(w) & \text{if} \quad w \ge w_0, \end{cases} \tag{46}$$

where

$$A_1(w) = (w+1)\mathcal{B}_{n,p}h_{C_{w+1}} - (n-w-1)\,\frac{k}{n-k}\,\mathcal{B}_{n,p}hC_w, \tag{47}$$

$$A_2(w) = (w+1)\mathcal{B}_{n,p}h_{C^c_{w+1}} + (n-w-1)\,\frac{k}{n-k}\,\mathcal{B}_{n,p}h_{C_w}, \tag{48}$$

and

$$A_3(w) = -(w+1)\mathcal{B}_{n,p}h_{C^c_{w+1}} + (n-w-1)\,\frac{k}{n-k}\,\mathcal{B}_{n,p}h_{C^c_w}. \tag{49}$$

From the final form of (40) it follows that

$$(U_{n,p}h_{w_0})(w) = \begin{cases} -\dfrac{w!}{w_0!}\,\dfrac{(n-(w+1))!}{(n-w_0)!}\,\dfrac{n}{k}\left(\dfrac{k}{n-k}\right)^{w_0-w}\mathcal{B}_{n,p}h_{C_w} & \text{if } w \le w_0-1 \\[2ex] \dfrac{w!}{w_0!}\,\dfrac{(n-(w\ 1))!}{(n-w_0)!}\,\dfrac{n}{k}\left(\dfrac{k}{n-k}\right)^{w_0-w}\mathcal{B}_{n,p}h_{C^c_w} & \text{if } \quad w \ge w_0. \end{cases} \tag{50}$$

From this and the definition of $V_{n,p}$ in (35) it follows that

$$(V_{n,p}h_{w_0})(w) = \frac{w!}{w_0!}\,\frac{(n-(w+2))!}{(n-w_0)!}\,\frac{n}{k}\left(\frac{k}{n-k}\right)^{w_0-1-w}A(w). \tag{51}$$

It is not difficult to verify that A_1, A_2, and A_3 are positive-valued. In fact

(52) $$A_1(w) = (w+1)\sum_{j=0}^{w+1}\binom{n}{j}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j} - (n-w-1)\frac{k}{n-k}\sum_{i=0}^{w}\binom{n}{i}\left(\frac{k}{n}\right)^i\left(1-\frac{k}{n}\right)^{n-i}$$

$$> \sum_{j=1}^{w+1}\left(\frac{n!(w+1)}{j!(n-j)!} - \frac{n!(n-w-1)}{(j-1)!(n-j+1)!}\right)\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j} > 0.$$

The positivity of A_2 is obvious and the argument for A_3 is similar to (52). Consequently, with Ψ_{w_0} defined by (43),

(53) $$\mathcal{B}_{n,p}\Psi_{w_0} = \frac{k-1}{k}\frac{1}{n-w_0}\binom{n}{w_0}\left(\frac{k}{n}\right)^{w_0}\left(1-\frac{k}{n}\right)^{n-w_0}A_2(w_0)$$

$$+ \sum_{w=2}^{w_0}\frac{w!(n-w)!}{w_0!(n-w_0)!}\frac{1}{k}\left(\frac{k}{n-k}\right)^{w_0+1-w}A_1(w-2)\frac{n!}{w!(n-w)!}\left(\frac{k}{n}\right)^w\left(1-\frac{k}{n}\right)^{n-w}$$

$$+ \sum_{w=w_0+2}^{n}\frac{w!(n-w)!}{w_0!(n-w_0)!}\frac{1}{k}\left(\frac{k}{n-k}\right)^{w_0+1-w}A_3(w-2)\frac{n!}{w!(n-w)!}\left(\frac{k}{n}\right)^w\left(1-\frac{k}{n}\right)^{n-w}$$

$$= \left[\frac{k-1}{k}\frac{A_2(w_0)}{n-w_0} + \frac{1}{n-k}\sum_{w=2}^{w_0}A_1(w-2) + \frac{1}{n-k}\sum_{w=w_0+2}^{n}A_3(w-2)\right]\mathcal{B}_{n,p}h_w.$$

But it follows from the definitions (47) and (49) of A_1 and A_3 that

(54) $$\sum_{w=0}^{w_0-2}A_1(w) < \frac{w_0^2}{2}$$

and

(55) $$\sum_{w=w_0}^{n}A_3(w) \le \sum_{w=0}^{n}A_3(w)$$

$$= \frac{nk}{n-k}\sum_{w=0}^{n}\sum_{j=w+1}^{n}\binom{n}{j}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}$$

$$= \frac{nk}{n-k}\sum_{j=1}^{n}n\binom{n-1}{j-1}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j} = \frac{nk^2}{n-k}.$$

It follows that

(56) $$\mathcal{B}_{n,p}\Psi_{w_0} < \frac{\mathcal{B}_{n,p}h_{w_0}}{n-k}\left[\frac{w_0^2}{2} + \frac{n(k-w_0+1)}{n-w_0} + \frac{nk^2}{n-k}\right].$$

Let us restrict ourselves to $k^2 \le \varepsilon n$ with ε to be specified later.

Then (56) shows that for every positive constant C_1 there exists a positive constant C_2 such that, if $w_0 \leq C_1 k$ then

(57) $$\mathcal{B}_{n,p}\Psi_{w_0} \leq C_2 \frac{k^2}{n} \mathcal{B}_{n,p} h_{w_0}.$$

Now suppose that for a particular positive constant C_n^*, possibly depending on n,

(58) $$E h_{w_0}(W) \leq (1 + C_n^* \frac{k^2}{n}) \mathcal{B}_{n,p} h_{w_0}$$

for all $w_0 \leq C_1 k$. Our aim is to show that then a similar result holds with C_n^* replaced by a smaller number unless it is already fairly small. This will imply (45) for $w_0 \leq C_1 k$. By (42),

(59) $$\begin{aligned} E h_{w_0}(W) &\leq \mathcal{B}_{n,p} h_{w_0} + \sum_{w=0}^{n} \Psi_{w_0}(w) E h_w(W) \\ &\leq \mathcal{B}_{n,p} h_{w_0} + \sum_{w \leq C_1 k} \Psi_{w_0}(w)(1 + C_n^* \frac{k^2}{n}) \mathcal{B}_{n,p} h_w + \sum_{w > C_1 k} \Psi_{w_0}(w) \frac{k^w}{w!} \\ &\leq \mathcal{B}_{n,p} h_{w_0} + (1 + C_n^* \frac{k^2}{n}) C_2 \frac{k^2}{n} \mathcal{B}_{n,p} h_{w_0} + \sum_{w > C_1 k} \Psi_{w_0}(w) \frac{k^w}{w!}. \end{aligned}$$

The second inequality uses (58) and (38), and the final equality uses (57).

The principal task remaining is to bound the final sum in (58). Let C_1 be a constant greater than 1, to be specified later, and let $w^* = [C_1 k]$. We need an upper bound for $A_3(w-2)$ defined by (49). Then for $w \geq w^*+1$

(60) $$\begin{aligned} A_3(w-2) &< k\, \mathcal{B}_{n,p} h_{C^c_{w-2}} \\ &= k \sum_{w'=w-1}^{n} \binom{n}{w'} \left(\frac{k}{n}\right)^{w'} \left(1 - \frac{k}{n}\right)^{n-w'} \\ &< k \frac{C_1}{C_1 - 1} \binom{n}{w-1} \left(\frac{k}{n}\right)^{w-1} \left(1 - \frac{k}{n}\right)^{n-w+1}. \end{aligned}$$

Then it follows from (43), (46)-(49), and (51) that, for $w_0 \leq w^*$

$$(61)\quad \sum_{w>w^*} \Psi_{w_0}(w)\,\frac{k^w}{w!}$$

$$\le \sum_{w>w^*} \frac{1}{n}\,\frac{w!}{w_0!}\,\frac{(n-w)!}{(n-w_0)!}\,\frac{n}{k}\left(\frac{k}{n-k}\right)^{w_0+1-w} A_3(w-2)\,\frac{k^w}{w!}$$

$$= \mathbb{B}_{n,p} h_{w_0} \sum_{w>w^*} \frac{n^n}{(n-k)^{n-w+1}}\,\frac{(n-w)!}{n!}\,A_3(w-2)$$

$$< \mathbb{B}_{n,p} h_{w_0} \sum_{w>w^*} \frac{n^n}{(n-k)^{n-w+1}}\,\frac{(n-w)!}{n!}\,k\,\frac{C_1}{C_1-1}\,\frac{n!}{w!(n-w+1)!}\left(\frac{k}{n}\right)^{w-1}\left(1-\frac{k}{n}\right)^{n-w+1}$$

$$< \mathbb{B}_{n,p} h_{w_0}\,\frac{C_1}{C_1-1} \sum_{w>w^*} \frac{1}{n(n-w+1)}\,\frac{k^w}{w!}$$

$$< \mathbb{B}_{n,p} h_{w_0}\left(\frac{C_1}{C_1-1}\right)^2 \cdot \frac{1}{n} \le \frac{3}{n}\,\mathbb{B}_{n,p} h_{w_0}$$

if we take $C_1 = 3$. Substituting in (59), for $w_0 \le C_1 k$,

$$(62)\qquad Eh_{w_0}(W) \le \left(1 + \frac{3}{n} + C_2\left(1 + \varepsilon\, C_n^*\right)\frac{k^2}{n}\right)\mathbb{B}_{n,p} h_{w_0}$$

$$\le \left(1 + \frac{3}{n} + \left(C_2 + \frac{1}{2}\, C_n^*\right)\frac{k^2}{n}\right)\mathbb{B}_{n,p} h_{w_0},$$

provided we choose $\varepsilon \le \frac{1}{2C_2}$. Recall that this was proved under the assumption that (58) holds. Iterating this, starting with (58) but with C_n^* replaced by $\frac{1}{2}\, C_n^* + (C_2+3)$, we obtain by induction that

$$(63)\qquad Eh_{w_0}(W) \le \left(1 + 2(C_2+3)\,\frac{k^2}{n}\right)\mathbb{B}_{n,p} h_{w_0}.$$

Once we have this upper bound, a similar argument, but much simpler, yields an analogous lower bound. This completes the proof of (45).

In Lecture XI I shall try to present a corrected version of my 1978 paper in the Journal of Combinatorial Theory, Series A. Here I have proved only Yamamoto's result (6). Following Erdös and Kaplansky (1946) and also Yamamoto (1951, 1969), I first studied the probability that $W = 0$, where W is the number of coincidences of a random permutation of $\{1,\ldots,n\}$ with the columns of a given k x n Latin rectangle $\mathscr{L}$. The event $\{W = 0\}$ can be thought of as the event that the result of placing the random permutation below the k x n Latin rectangle $\mathscr{L}$ is a (k+1) x n Latin rectangle. To a first approximation which would be adequate for this lecture, W has a Poisson distribution with parameter k. However, the more accurate binomial approximation is introduced for use in Lecture XI.

It may be helpful to sketch the basic argument in the form that I used in my paper, which has the same structure as the argument used in the first part of Lecture XII on random allocations. In addition to the basic random permutation Π from which W is computed, we introduce another random permutation Π' related to Π by (9), which means that the values of Π at two randomly selected positions, I and J, are interchanged. Then using the fact that (W,W') is an exchangeable pair, where W' is related to Π' as W is to Π, we have

$$\frac{P\{W=w+1\}}{P\{W=w\}} = \frac{P\{W'=w+1|W=w\}}{P\{W'=w|W=w+1\}}. \tag{64}$$

Intuitively it is not hard to convince oneself, as in (15)-(25), that

$$P\{W'=w+1|W=w\} \approx 2\,\frac{n-w}{n}\cdot\frac{k}{n-1} \tag{65}$$

and

$$P\{W'=w|W=w+1\} \approx 2\,\frac{w+1}{n}\cdot\frac{n-k}{n-1} \tag{66}$$

leading to an approximate evaluation of the left-hand side of (64) and thus of the distribution of W. The details of a rigorous proof are fairly complicated but can undoubtedly be kept simpler than the details of the present proof, based on bounding the remainder in (13). Of course the identity (13) is the result of routine application of the basic formalism to the basic problem.

The remainder of this lecture, starting below (36), is devoted to the problem of bounding the remainder in that formula. This requires a careful study of the auxiliary functions and mappings associated with the binomial distribution by the formalism of the bottom row of diagram (I.28).