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COMPARING COHERENT SYSTEMS

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It is a well known engineering principle that "redundancy at the component level is more effective than redundancy at the system level." Here, redundancy simply means components are connected in parallel and the principle results from comparing the systems obtained when this parallel protocol is applied both at the component and systems levels. It is shown in this paper that if parallel or series protocols are ruled out, corresponding versions of the above principle are not possible. This question is examined both in structural as well as in reliability (stochastic) terms.

1. Introduction. Let $S = \{0, 1, \ldots, m\}$ denote the set of all possible states of both the system and its components, and let $C = \{1, \ldots, n\}$ be the component set. The vector $\mathbf{x} = (x_1, \ldots, x_n) \in S^n$ represents the situation where components $1, \ldots, n$ are in states $x_1, \ldots, x_n$ respectively. In particular we write $\mathbf{k} = (k, \ldots, k)$ for $k \in S$.

The state of the system is a function of the component state vector $\mathbf{x} \in S^n$. A function $\phi: S^n \to S$ is called a multistate system structure (MSS) of order n provided it is nondecreasing, i.e. $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ whenever $x_i \leq y_i$ for all $i \in C$ ($\mathbf{x} \leq \mathbf{y}$).

We also use throughout the paper the following notational convention.

Notation 1.1. For $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathcal{R}^n$, $i = 1, \ldots, k$ and $\psi: \mathcal{R}^k \to \mathcal{R}$ we let

$$\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k) = (\psi(x_{11}, x_{21}, \ldots, x_{k1}), \ldots, \psi(x_{1n}, x_{2n}, \ldots, x_{kn})) \in \mathbf{R}^n. \tag{1.1}$$

Note that ϕ is an MSS of order n if and only if

$$\phi(\max_{1 \leq i \leq k} \mathbf{x}_i) \geq \max_{1 \leq i \leq k} \phi(\mathbf{x}_i) \text{ for all } \mathbf{x}_1, \ldots, \mathbf{x}_k \in S^n \text{ and } k \geq 2, \tag{1.2}$$

or equivalently

$$\phi(\min_{1 \leq i \leq k} \mathbf{x}_i) \leq \min_{1 \leq i \leq k} \phi(\mathbf{x}_i) \text{ for all } \mathbf{x}_1, \ldots, \mathbf{x}_k \in S^n \text{ and } k \geq 2, \tag{1.3}$$

where $\max_{1 \leq i \leq k} \mathbf{x}_i (\min_{1 \leq i \leq k} \mathbf{x}_i)$ is the vector of coordinatewise maximums (minimums). Inequality (1.2) expresses mathematically a well known engineering principle that states that "redundancy at the component level is more effective than redundancy at the system level", and (1.3) expresses a related dual principle. These principles are presented in their simplest form in Barlow and Proschan (1975).

We recall that the MSS of order k defined by $\psi(\mathbf{x}) = \max_{1 \leq i \leq k} x_i$ ($\psi(x) = \min_{1 \leq i \leq k} x_i$) for $\mathbf{x} \in S^k$ is called a parallel (series) system and note that using (1.1) the principle expressed by (1.2) ((1.3)) can be rewritten as follows. We express it in this form for ease in describing our subsequent results.

Principle 1.2. If ϕ is an MSS of order n and ψ is a parallel (series) system of order k, then the MSS of order $k \times n$ defined by

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(1.4) $$\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) \quad \text{for } \mathbf{x}_1, \ldots, \mathbf{x}_k \in S^n$$

is uniformly better (worse) than the MSS of order $k \times n$ defined by

(1.5) $$\psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)) \quad \text{for } \mathbf{x}_1, \ldots, \mathbf{x}_k \in S^n$$

In this paper, we will consider the question of which of the two MSS's of order $k \times n$ defined in (1.4) and (1.5) for general ϕ and ψ is uniformly better. As an example to better visualize the two competing alternatives, assume that

$$\phi(x_1,x_2,x_3) = \min\{x_1,\max\{x_2,x_3\}\}$$

and

$$\psi(y_1,y_2,y_3,y_4) = \max\{y_1,\min\{y_1,y_2,y_3\}\}$$

for $x_i,x_j \in \{0,1\}$, $i = 1,2,3$, $j = 1,2,3,4$. Since ϕ and ψ can be represented respectively as

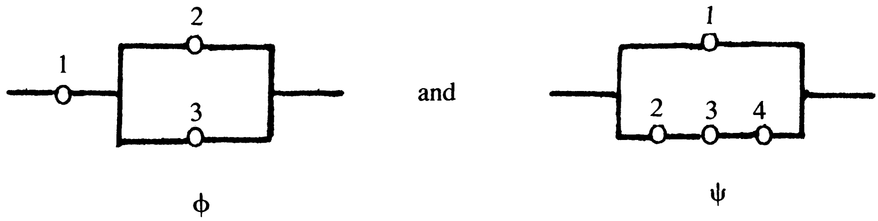

FIGURE 1.1.

the two alternatives are to build either the system illustrated in Figure 1.2 or in Figure 1.3.

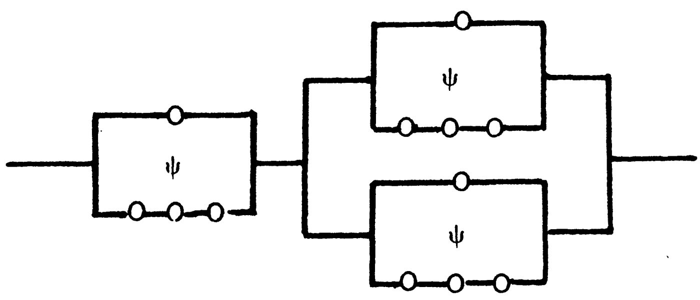

FIGURE 1.2.

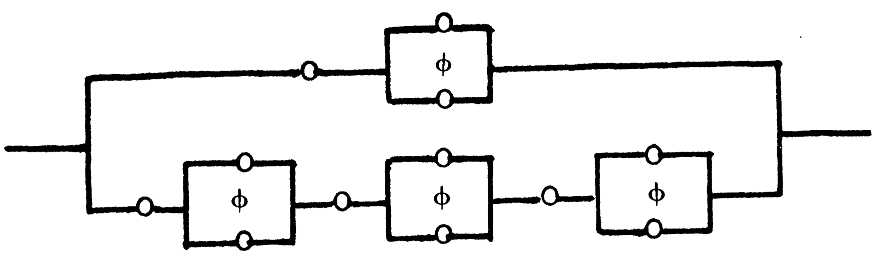

FIGURE 1.3.

The solution to this problem in the binary setting, i.e. when $S = \{0,1\}$ is given in Section 2. This is that if series and parallel systems are ruled out, neither of the resulting systems is uniformly better than the other. This is our main result which is given by Lemma 2.1. An interesting consequence of this result is given in Theorem 2.2.

In section 3 we consider the problem in the multistate setting and a weaker result is given in Proposition 3.1. An example is given to show that this cannot be improved upon in general but, if the specialized type of MSS of Barlow and Wu (1978) is considered, a direct analog of the binary result is obtainable. This is given in Proposition 3.2.

Finally in Section 4 we comment on the possibility of obtaining stochastic versions of the results given in the previous section. It is shown that even in the binary case only weak results can be achieved.

2. Binary System Structures. In this section we consider the binary setting where $S = \{0,1\}$ in which case an MSS is called a binary system structure (BSS). We assume that any BSS ϕ of order n considered here is coherent in the sense that for each $i \in C$ there is $\mathbf{x} \in \{0,1\}^n$ such that

$$(2.1) \qquad \phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) < \phi(x_1, \ldots, x_{i-1}, 1 x_{i+1}, \ldots, x_n).$$

We also recall from (3.6) of Chapter 1 of Barlow and Proschan (1975) that if ϕ is a coherent BSS of order n, then ϕ has a representation of the form

$$\phi(\mathbf{x}) = \min_{i \leqslant j \leqslant k} \max_{i \in K_j} x_i$$

where $\bigcup_{j=1}^{k} K_j = C$ and for all $i \neq j$, K_i is not a subset of K_j. These sets are called the min cut sets of ϕ and we refer the reader to Barlow and Proschan (1975) for properties of min cut sets and related notions.

Our main result will be a consequence of the following lemma.

LEMMA 2.1. *Let ϕ and ψ be coherent BSS's of orders $n \geqslant 2$ and $k \geqslant 2$, respectively.* (1) *If ϕ is not a parallel system and ψ is not a series system then there exist $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\{0,1\}^n$ such that*

$$\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) > \psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)).$$

(2) *If ϕ is not a series system and ψ is not a parallel system then there exist $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\{0,1\}^n$ such that*

$$\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) < \psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)).$$

Proof. 1) We will construct $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\{0,1\}^n$ such that the desired inequality holds. Since ψ is not series, $k \geqslant 2$, and ψ is coherent we can find a min cut set K^ψ which contains at least two elements. Furthermore since ϕ is not parallel, $n \geqslant 2$, and ϕ is coherent there are at least two different min cut sets of ϕ; call them K_1^ϕ and K_2^ϕ. Now for each $i \in K^\psi$, choose K_1^ϕ or K_2^ϕ and construct $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \{0,1\}^n$ defining $x_{ij} = 0$ if $j \in K^\phi$ (where K^ϕ is whichever one of K_1^ϕ or K_2^ϕ was chosen) and $x_{ij} = 1$ otherwise. Also construct $\mathbf{x}_i$ for $i \in K^\psi$ so that not all of them are associated with only one of K_1^ϕ or K_2^ϕ. For $i \notin K^\psi$ define $\mathbf{x}_i = \mathbf{1} = (1, \ldots, 1)$. Thus $(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)) \in \{0,1\}^k$ has zeros for all the components $i \in K^\psi$ so that $\psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)) = 0$. On the other hand $x_{ij} = 1$ for all $i \notin K^\psi$ so that $A = \{j\colon \psi(x_{1j}, x_{2j}, \ldots, x_{kj}) = 0\} = \{j;\, x_{ij} = 0 \text{ for all } i \in K^\psi\} = K_1^\phi \bigcap K_2^\phi$. But since K_1^ϕ and K_2^ϕ are different min cut sets, $K_1^\phi \bigcap K_2^\phi$ must be strictly contained in K_1^ϕ and K_2^ϕ. Thus A does not contain any min cut set of ϕ. Consequently $\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) = 1$.

2) The second part of the lemma is proven similarly. □

The main result now follows easily.

THEOREM 2.2. *Let ϕ and ψ be coherent BSS's of orders $n \geqslant 2$ and $k \geqslant 2$* respectively. Then

$$\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) = \psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k))$$

for all $\mathbf{x}_i \in \{0,1\}^n$, $i = 1, \dots, k$, *if and only if* ϕ *and* ψ *are both parallel or both series.*

Proof. If the equality holds, it follows from Lemma 2.1 that: (i) either ϕ is parallel or ψ is series; and (ii) either ϕ is series or ψ is parallel. Combining (i) and (ii) we have that either ϕ and ψ are series or ϕ and ψ are parallel. Necessity of the equality follows immediately. □

Note 2.3. In proving Theorem 2.2 we used the contrapositive form of the two statements in Lemma 2.1. These results are that under the assumptions of the lemma: (i) If

$$\phi(\psi(\mathbf{x}_1, \dots, \mathbf{x}_k) \leq \psi(\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)) \text{ for all } \mathbf{x}_1, \dots, \mathbf{x}_k \in \{0,1\}^n,$$

then either ϕ is parallel or ψ is series. (ii) If

$$\phi(\psi(\mathbf{x}_1, \dots, \mathbf{x}_k) \geq \psi(\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)) \text{ for all } \mathbf{x}_1, \dots, \mathbf{x}_k \in \{0,1\}^n,$$

then either ϕ is series or ψ is parallel.

It is easy to show that the converses of (i) and (ii) above also hold.

As a special case the result of Theorem 2.4 of Chapter 1 of Barlow and Proschan (1975) follows from Note 2.3.

3. Multistate System Structures. We now examine the extent to which the results in the previous section can be generalized to the case of multistate system structures. Any MSS ϕ considered in this section will further satisfy the following two condtions: (i) $\phi(\mathbf{k}) = k$ for all $k \in S$; and (ii) for each $i \in C$ and $j \geq 1$ there exists $\mathbf{x} \in S^n$ such that

$$\phi(x_1, \dots, x_{i-1}, j-1, x_{i+1}, \dots, x_n) < \phi(x_1, \dots, x_{i-1}, j, x_{i-1}, \dots, x_n).$$

These will be called coherent MSS's of order n. This last concept coincides with the middle and most reasonable multistate concept of coherence discussed in Griffith (1980).

A full generalization of Theorem 2.2 is not possible in the multi-state case even under fairly strong conditions. We give however some weaker results and an instructive counterexample.

The first result is in the spirit of the remarks in Note 2.3.

PROPOSITION 3.1. *Let* $\phi(\psi)$ *be a coherent MSS of order* n (k). Then

$$\phi(\psi(\mathbf{x}_1, \dots, \mathbf{x}_k)) \leq \psi(\phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_k)) \text{ for all } \mathbf{x}_1, \dots, \mathbf{x}_k \in S^n$$

and all coherent MSS ψ (ϕ) *of order* $k(n)$, *if and only if* $\phi(\psi)$ *is a parallel (series) MSS.*

Proof. The "if" part is straightforward. For the "only if" let $\psi(\mathbf{x}) = \max_{1 \leq i \leq k} x_i$ for $\mathbf{x} \in S^k$. Obviously for this choice of ψ the reverse inequality holds. Thus

$$\phi(\max(\mathbf{x}_i, \dots, \mathbf{x}_k)) = \max_{1 \leq i \leq k} \phi(\mathbf{x}_i).$$

By the same proof as that of Proposition 2.2 of Griffith (1980) the result follows. The proof of the dual result is similar. □

The following example shows that the generalization of Theorem 2.2 (and Note 2.3) is false even under stronger coherence assumptions.

Example 3.2. Let ϕ and ψ be identical MSS's defined as follows:

$$\phi(0,0) = \phi(0,1) = \phi(1,0) = \phi(0,2) = \phi(2,0) = 0,$$

$$\phi(1,1) = 1 \text{ and } \phi(1,2) = \phi(2,1) = \phi(2,2) = 2.$$

Then it is not hard to see that $\phi(\psi(\mathbf{x}_1, \mathbf{x}_2) = \psi(\phi(\mathbf{x}_1), \phi(\mathbf{x}_2))$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \{0,1,2\}^2$. Moreover ϕ and ψ are coherent and even satisfy the strong coherence assumption of Griffith (1980). However neither ϕ nor ψ are either series or parallel.

If we consider the more restrictive multistate system structures proposed by Barlow and Wu (1978) we can obtain an extension of Theorem 2.2. An MMS ϕ of order n is of the type proposed by Barlow and Wu (1978) (BW-MSS) if it is of the form

$$\phi(\mathbf{x}) = \min_{1 \leq j \leq k} \max_{i \in K_j} x_i \quad \text{for } \mathbf{x} \in S^n,$$

where $\bigcup_{j=1}^{k} K_j = C$ and for $i \neq j$, K_i is not a subset of K_j. These functions are a particular subfamily of the coherent MSS's. Moreover for x_i binary, ϕ is a coherent BSS with min cut sets $K_1, \ldots, K_k$.

PROPOSITION 3.3. *Let ϕ and ψ be BW-MSS's of order $n \geq 2$ and $k \geq 2$, respectively. Then*

$$\phi(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k)) = \psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k)) \quad \text{for all } \mathbf{x}_1, \ldots, \mathbf{x}_k \in S^n$$

if and only if ϕ and ψ are both parallel or both series. The result remains true if equality holds for all $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \{k_1, k_2\}^n$, where $0 \leq k_1 < k_2 \leq m$.

Proof. We need only show the result for the weaker assumption. As mentioned above ϕ and ψ reduce to coherent BSS's when restricted to $\{0,1\}^n$ and $\{0,1\}^k$, respectively. We consider $f(x) = (k_2 - k_1)^{-1}(x - k_1)$ so that when $x \in \{k_1, k_2\}$, $f(x) \in \{0,1\}$.

To prove sufficiency note that

$$\begin{aligned} \phi(\psi(f(\mathbf{x}_1), \ldots, f(\mathbf{x}_k))) &= \phi(f(\psi(\mathbf{x}_1, \ldots, \mathbf{x}_k))) \\ = f(\phi(\psi(f(\mathbf{x}_1, \ldots, \mathbf{x}_k))) &= f(\psi(\phi(\mathbf{x}_1), \ldots, \phi(\mathbf{x}_k))) \\ = \psi(f(\phi(\mathbf{x}_1)), \ldots, f(\phi(\mathbf{x}_k))) &= \psi(\phi(f(\mathbf{x}_1)), \ldots, \phi(f(\mathbf{x}_k))). \end{aligned}$$

Hence,

$$\phi(\psi(\mathbf{y}_1, \ldots, \mathbf{y}_k)) = \psi(\phi(\mathbf{y}_1), \ldots, \phi(\mathbf{y}_k)) \quad \text{for all } \mathbf{y}_1, \ldots, \mathbf{y}_k \in \{0,1\}^n,$$

and the result follows from Theorem 2.2.

Note 3.4. By similar methods, analogs of Lemma 2.1 and Note 2.3 can also be given for BW-MSS's.

4. Further Remarks. Stochastic versions of the results of the previous sections do not necessarily hold even in the binary setting. However, an analog of Proposition 3.1 can be obtained. We consider only the binary case although similar results hold in the multistate case.

We let ϕ and ψ be coherent BSS's of orders n and k respectively, and compare the reliability functions,

$$h_{\phi(\psi)}(\mathbf{P}_1, \ldots, \mathbf{P}_k) = E\,\phi(\psi(\mathbf{X}_1, \ldots, \mathbf{X}_k))$$

and

$$h_{\psi(\phi)}(\mathbf{P}_1, \ldots, \mathbf{P}_k) = E\,\psi(\phi(\mathbf{X}_1), \ldots, \phi(\mathbf{X}_k)),$$

of the two competing coherent BSS's of order $k \times n$ defined in (1.4), (1.5). Here $\mathbf{X}_1 = (X_{i1}, \ldots, X_{in})$ for $i = 1, \ldots, k$ are independent random vectors of independent binary random variables, and $\mathbf{P}_i = (p_{i1}, \ldots, p_{in})$ for $i = 1, \ldots, k$ are defined by $p_{ij} = P\{X_{ij} = 1\}$.

We also let $h_\phi(p_1, \ldots, p_n) = E\,\phi(X_1, \ldots, X_n)$ $(h_\psi(q_1, \ldots, q_k) = \mathrm{E}\,\psi(Y_1, \ldots, Y_k))$ when $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_k)$ are independent binary random variables and $p_i = P\{X_i = 1\}$ for $i = 1, \ldots, n$ $(q_j = P\{Y_j = 1\}$ for $j = 1, \ldots, k)$.

PROPOSITION 4.1. (1) *If ϕ is a parallel (series) BSS, then*

$$h_{\phi(\psi)}(\mathbf{P}_1, \ldots, \mathbf{P}_k) = h_\phi(h_\psi(p_{11}, \ldots, p_{k1}), \ldots, h_\psi(p_{1n}, \ldots, p_{kn})) \tag{4.1}$$

$$\leq (\geq) h_\psi(h_\phi(p_{11}, \dots, p_{1n}), \dots, h_\phi(p_{k1}, \dots, p_{kn}))$$
$$= h_{\psi(\phi)}(\mathbf{P}_1, \dots, \mathbf{P}_k)$$

for all $\mathbf{P}_1, \dots, \mathbf{P}_k \in [0,1]^n$ *and any coherent BSS* ψ. (2) *Conversely if for any coherent BSS* ψ *and some* $\mathbf{P}_1, \dots, \mathbf{P}_k \in (0,1)^n$, *inequality (4.1) holds, then* ϕ *is a parallel (series) BSS.*

Proof. (1) Follows from Proposition 3.1 by taking expectations. (2) Taking $\psi(\mathbf{x}) = \max_{1 \leq i \leq k} x_i$ for $\mathbf{x} \in \{0,1\}^k$, we have

$$h_{\phi(\psi)}(\mathbf{P}_1, \dots, \mathbf{P}_k) - h_{\psi(\phi)}(\mathbf{P}_1, \dots, \mathbf{P}_k)$$
$$= E[\phi(\psi(\mathbf{X}_1, \dots, \mathbf{X}_k)) - \psi(\phi(\mathbf{X}_1), \dots, \phi(\mathbf{X}_k))]$$
$$= E[\phi(\max_{1 \leq i \leq k} \mathbf{X}_i) - \max_{1 \leq i \leq k} \phi(\mathbf{X}_1)] \leq 0.$$

Hence,

$$\phi(\max_{1 \leq i \leq k} \mathbf{x}_i) = \max_{1 \leq i \leq k} \phi(\mathbf{x}_i) \quad \text{for all } \mathbf{x}_1, \dots, \mathbf{x}_k \in \{0,1\}^n$$

and from Theorem 2.3, Chapter 1 of Barlow and Proschan (1975) ϕ must be a parallel BSS.

The dual statement is proved similarly. □

It is easy to check that ϕ and ψ are both parallel or series BSS's if and only if we have equality in (4.1) for all $\mathbf{P}_1, \dots, \mathbf{P}_k \in [0,1]^n$. It is not true however that if equality holds in (4.1) for some $\mathbf{P}_1, \dots, \mathbf{P}_k \in (0,1)^n$ then ϕ and ψ are both parallel or series BSS's. An example of this last fact can be constructed by simply taking ϕ and ψ identical, but neither being a parallel or series BSS, and taking $\mathbf{P}_1, \dots, \mathbf{P}_k \in (0,1)^n$ $(k=n)$ such that $p_{ij} = p_{ji}$ for all $i,j = 1, \dots, n$. It is obvious that this construction provides equality in (4.1). It is also easy to show that if equality holds in (4.1) for some $\mathbf{P}_1, \dots, \mathbf{P}_k \in (0,1)^n$ and either ϕ or ψ is parallel (series) then so is ψ or ϕ.

REFERENCES

Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, New York.

Barlow, R. E. and Wu, A. (1978). Coherent Systems with Multistate Components, *Math. Oper. Res.* *3* 275–281.

Griffith, W. E. (1980). Multistate Reliability Models, *J. Appl. Prob.* *17* 735–744.