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ON TP_2 AND LOG-CONCAVITY

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Inter-relations between the TP_2 property and log-concavity of density functions have been investigated. The general results are then applied to noncentral chi-square density functions and beta density functions.

1. Results on Density Functions

Definition 1. A function $f\colon \mathcal{R}^2 \to \mathcal{R}$ is said to be TP_2 (Karlin (1968)) if, for $x_1 < x_2$, $y_1 < y_2$

$$f(x_1,y_2)f(x_2,y_1) \leqslant f(x_1,y_1)f(x_2,y_2). \tag{1.1}$$

We shall say that $1/f$ is TP_2, if (1.1) holds for f with the inequality reversed.

Let X be a positive random variable having the p.d.f. $f(\cdot,\theta,\lambda)$ with respect to Lebesgue measure; $\theta > 0$, $\lambda \geqslant 0$.

Definition 2. The p.d.f. $f(x,\theta,\lambda)$ is said to have the reproductive property (RP) in θ, if there exists a distribution function $G(\cdot,s)$ on $\mathcal{R}^+$ $(s > 0)$ such that

$$\textstyle\int_0^x f(x-y,\theta,\lambda)\, G(dy,s) = f(x,\theta+s,\lambda). \tag{1.2}$$

THEOREM 1. *Suppose $f(x,\theta,\lambda)$ has the RP in θ. Then* (i) $f(x,\theta,\lambda)$ TP_2 *in* $(x,\lambda) \to 1/f(x,\theta,\lambda)$ TP_2 *in* (θ,λ), (ii) $f(x,\theta,\lambda)$ TP_2 *in* $(x,\theta) \to f(x,\theta,\lambda)$ *log-concave in* θ.

Proof. (i) For $0 < x_1 < x_2$, $\lambda_1 < \lambda_2$ we have

$$f(x_2,\theta,\lambda_1)f(x_1,\theta,\lambda_2) \leqslant f(x_2,\theta,\lambda_2)f(x_1,\theta,\lambda_1). \tag{1.3}$$

Write $x_1 = x_2 - y$. Integrating (1.3) with respect to $G(dy,s)$ we get

$$f(x_2,\theta,\lambda_1)f(x_2,\theta+s,\lambda_2) \leqslant f(x_2,\theta,\lambda_2)f(x_2,\theta+s,\lambda_1), \tag{1.4}$$

which shows that $1/f(x,\theta,\lambda)$ is TP_2 in (θ,λ).

(ii) For $0 < x_1 < x_2$, $\theta_1 < \theta_2$, we have

$$f(x_1,\theta_2,\lambda)f(x_2,\theta_1,\lambda) \leqslant f(x_2,\theta_2,\lambda)f(x_1,\theta_1,\lambda). \tag{1.5}$$

Write $x_1 = x_2 - y$. Integrating (1.5) with respect to $G(dy,s)$ we get

$$f(x_2,\theta_2+s,\lambda)f(x_2,\theta_1,\lambda) \leqslant f(x_2,\theta_2,\lambda)f(x_2,\theta_1+s,\lambda), \tag{1.6}$$

which shows that $f(x,\theta,\lambda)$ is log-concave in θ. □

Definition 3. The p.d.f. $f(x,\theta,\lambda)$ is said to have the mixture property (MP) in (θ,λ) if there exists a non-negative random variable K with the distribution $H(\cdot,\tau)$ with $\tau > 0$ such that

$$\textstyle\int_0^\infty f(x,\theta+k,\lambda)\, H(dk,\tau) = f(x,\theta,\lambda+\tau). \tag{1.7}$$

Suppose H in Definition 3 possesses a density function h with respect to a σ-finite measure ν.

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THEOREM 2. *Suppose $f(x,\theta,\lambda)$ has the MP in (θ,λ). Then* (i) $f(x,\theta,0)$ TP$_2$, $h(k,\tau)$ TP$_2$ $\rightarrow f(x,\theta,\tau)$ TP$_2$ *in* (x,τ), (ii) $1/f(x,\theta,\lambda)$ TP$_2$ *in* (θ,λ) $\rightarrow f(x,\theta,\lambda)$ *log-concave in* λ, (iii) $f(x,\theta,\lambda)$ TP$_2$ *in* $(x,\theta) \rightarrow f(x,\theta,\lambda)$ TP$_2$ *in* (x,λ), (iv) $f(x,\theta,\lambda)$ *log-concave in* $\theta \rightarrow f(x,\theta,\lambda)$ *log-concave in* λ.

Proof: (i) This follows from Karlin (1968, p. 17). (ii) For $\theta_1 \leqslant \theta_2$, $\lambda_1 < \lambda_2$,

$$f(x,\theta_2,\lambda_2)f(x,\theta_1,\lambda_1) \leqslant f(x,\theta_2,\lambda_1)f(x,\theta_1,\lambda_2). \tag{1.8}$$

Write $\theta_2 = \theta_1 + k$. Integrating (1.8) with respect to $H(dk,\tau)$ we get

$$f(x,\theta_1,\lambda_2+\tau)f(x,\theta_1,\lambda_1) \leqslant f(x,\theta_1,\lambda_1+\tau)f(x,\theta_1,\lambda_2). \tag{1.9}$$

which shows that $f(x,\theta,\lambda)$ is log-concave in λ.

(iii) For $0 < x_1 < x_2$, $\theta_1 \leqslant \theta_2$ we have

$$f(x_2,\theta_1,\lambda)f(x_1,\theta_2,\lambda) \leqslant f(x_2,\theta_2,\lambda)f(x_1,\theta_1,\lambda). \tag{1.10}$$

Write $\theta_2 = \theta_1 + k$. Integrating (1.10) with respect to $H(dk,\tau)$ we get

$$f(x_2,\theta_1,\lambda)f(x_1,\theta_1,\lambda+\tau) \leqslant f(x_2,\theta_1,\lambda+\tau)f(x_1,\theta_1,\lambda), \tag{1.11}$$

which shows that $f(x,\theta,\lambda)$ is TP$_2$ in (x,λ).

(iv) For $\theta_1 \leqslant \theta_2$, $0 \leqslant s$ we have

$$f(x,\theta_2+s,\lambda)f(x,\theta_1,\lambda) \leqslant f(x,\theta_2,\lambda)f(x,\theta_1+s,\lambda). \tag{1.12}$$

Write $\theta_2 = \theta_1 + k$ and integrate (1.12) with respect to $H(dk,\tau_1)\,H(ds,\tau_2)$. Then we get

$$f(x,\theta_1,\lambda+\tau_1+\tau_2)f(x,\theta_1,\lambda) \leqslant f(x,\theta_1,\lambda+\tau_1)f(x,\theta_1,\lambda+\tau_2). \tag{1.13}$$

The above shows that $f(x,\theta,\lambda)$ is log-concave in λ. □

THEOREM 3. *Suppose $f(x,\theta,\lambda)$ is log-concave in x. Then*

$$f(x,\theta,\lambda) \textit{ has the RP in } \theta \rightarrow f(x,\theta,\lambda) \textit{ is } \mathrm{TP}_2 \textit{ in } (x,\theta). \tag{1.14}$$

Proof. For $0 < x_1 < x_2$, $0 < y$ we have

$$f(x_1,\theta,\lambda)f(x_2+y,\theta,\lambda) \leqslant f(x_2,\theta,\lambda)f(x_1+y,\theta,\lambda). \tag{1.15}$$

Write $x_1 = x_2 - z$. Integrating (1.15) with respect to $G(dz,s)$ we get

$$f(x_2,\theta+s,\lambda)f(x_2+y,\theta,\lambda) \leqslant f(x_2,\theta,\lambda)f(x_2+y,\theta+s,\lambda), \tag{1.16}$$

which shows that $f(x,\theta,\lambda)$ is TP$_2$ in (x,θ). □

Define $C(x,\theta,\lambda)$ by

$$f(x,\theta,\lambda)\,C(x,\theta,\lambda) = f(\lambda,\theta,x), \tag{1.17}$$

LEMMA 1. *If both $f(x,\theta,\lambda)$ and $C(x,\theta,\lambda)$ are log-concave in λ, then $f(x,\theta,\lambda)$ is log-concave in x.*

The above lemma is a well-known fact; see Das Gupta (1976, 1980).

Combining the above results, we get the following:

THEOREM 4. Suppose the following conditions hold: (a) $f(x,\theta,\lambda)$ has the RP in θ, as defined in (1.2), (b) $f(x,\theta,\lambda)$ has the MP in (θ,λ), as defined in (1.7), (c) $C(x,\theta,\lambda)$, as defined in (1.17), is log-concave in λ. Then the following are equivalent: (i) $f(x,\theta,\lambda)$ is TP$_2$ in (x,λ), (ii) $1/f(x,\theta,\lambda)$ is TP$_2$ in (θ,λ), (iii) $f(x,\theta,\lambda)$ is log-concave in λ, (iv) $f(x,\theta,\lambda)$ is log-concave in x, (v) $f(x,\theta,\lambda)$ is TP$_2$ in (x,θ), (vi) $f(x,\theta,\lambda)$ is log-concave in θ.

Moreover, all the above results (i)–(vi), under the conditions (a)–(c), are implied by the condition (d) $f(x,\theta,0)$ is TP$_2$, $h(k,\tau)$ is TP$_2$.

2. Application to Noncentral Chi-Square Distribution. Suppose $f(x,\theta,\lambda)$ is the p.d.f. of the noncentral chi-square distribution with θ degrees of freedom and the noncentrality parameter λ. Then (1.2) holds with $G(\cdot,s)$ as the distribution of χ^2_s. Moreover, (1.7) holds if H is taken such that $K/2$ is distributed as Poisson with mean $\tau/2$. With this specification of h, condition (d) of Theorem 4 obtains. It can also be seen that $C(x,\theta,\lambda)$, as defined in (1.17), is given by

$$C(x,\theta,\lambda) = (\lambda/x)^{\theta/2-1}, \tag{2.1}$$

which is log-concave in λ if $\theta/2 \geqslant 1$. Hence (i)–(iii) of Theorem 4 hold when $\theta > 0$, and (iv)–(vi) hold when $\theta \geqslant 2$. It can be seen easily that $f(x,\theta,0)$ is log-concave in x when $\theta \geqslant 2$; also $f(x,\theta,0)$ is TP_2 in (x,θ), and $f(x,\theta,0)$ is log-concave in θ. Ghosh (1973) gave an alternative proof of the TP_2 property of $f(x,\theta,\lambda)$ in (x,θ) when $\theta > 2$. Karlin (1968) proved that $f(x,\theta,\lambda)$ is log-concave in x when $\theta > 2$.

Remark. The chain of arguments used in the above theorems can be used also for discrete random variables after minor modifications.

3. Results on C.D.F.'s. Let X be a positive r.v. with the p.d.f. $f(x,\theta,\lambda)$ with respect to Lebesgue measure. The c.d.f. of X is given by

$$F(C,\theta,\lambda) \equiv P[X \leqslant C] \equiv 1 - \bar{F}(C,\theta,\lambda). \tag{3.1}$$

LEMMA 2. (a) *If $f(x,\theta,\lambda)$ satisfies (1.2), then so does $F(x,\theta,\lambda)$.* (b) *If $f(x,\theta,\lambda)$ satisfies (1.7), then so does $F(x,\theta,\lambda)$*ς (c) *If $f(x,\theta,\lambda)$ is* TP_2 *in (x,θ)(or, in (x,λ)), then $F(x,\theta,\lambda)$ is also* TP_2 *in (x,θ) (or, in (x,λ)).* (d) *If $f(x,\theta,\lambda)$ is log-concave in x, then $F(x,\theta,\lambda)$ is also log-concave in x. The above results (b)–(d) also hold for $\bar{F}$.*

Proof. The results (a) and (b) are trivial. The results (c) follows from Karlin's (1968) theorem and the fact that the indicator function of the set $(-\infty,C]$ is TP_2 in (x,C). The result (d) follows from Prekopa's Theorem; see Das Gupta (1976, 1980). □

Remark. If f or F satisfies RP, then it trivially follows that $F(c,\theta,\lambda)$ is decreasing in θ; this fact also follows from the condition that $F(c,\theta,\lambda)$ is TP_2 in (c,θ).

THEOREM 5. (a) *Suppose $f(x,\theta,\lambda)$ or $F(x,\theta,\lambda)$ satisfies the RP in θ, as given in (1.2). Then* (i) $F(x,\theta,\lambda)$ TP_2 *in* $(x,\lambda) \to 1/F(x,\theta,\lambda)$ TP_2 *in* (θ,λ). (ii) $F(x,\theta,\lambda)$ TP_2 *in* $(x,\theta) \to F(x,\theta,\lambda)$ *log-concave in* θ. (iii) $F(x,\theta,\lambda)$ *log-concave in* $x \to F(x,\theta,\lambda)$ TP_2 *in* (x,θ).

(b) *If $f(x,\theta,\lambda)$ or $F(x,\theta,\lambda)$ satisfies the MP in (θ,λ) as given in (1.7), then* (i) $1/F(x,\theta,\lambda)$ TP_2 *in* $(\theta,\lambda) \to f(x,\theta,\lambda)$ *log-concave in* λ. (ii) $F(x,\theta,\lambda)$ TP_2 *in* $(x,\theta) \to F(x,\theta,\lambda)$ TP_2 *in* (x,λ). (iii) $F(x,\theta,\lambda)$ log-concave in $\theta \to F(x,\theta,\lambda)$ *log-concave in* λ. *The above results in (a) and (b) also hold if F is replaced by $\bar{F}$.*

This theorem can be proved following the proofs of Theorems 1 and 2. However, we need to note some additional facts in order to prove the results for $\bar{F}$. If $F(x,\theta,\lambda)$ satisfies (1.2), we get

$$\textstyle\int_0^c \bar{F}(c-y,\theta,\lambda)\, G(dy,s) = \bar{F}(c,\theta + s,\lambda) - \bar{G}(c,s). \tag{3.2}$$

So, in order for Theorem 5(a)(i) to hold for $\bar{F}$ we must have $\bar{F}(c,\theta,\lambda)$ increasing in λ; but this is implied by the condition that $\bar{F}(c,\theta,\lambda)$ is TP_2 in (c,λ). Also, for Theorem 5(a)(ii) to hold for $\bar{F}$ we need $\bar{F}(c,\theta,\lambda)$ increasing in θ; again this is implied by the condition that $\bar{F}(c,\theta,\lambda)$ is TP_2 in (c,θ).

4. Application to Chi-Square Distribution. (a) If $F(\cdot,\theta,\lambda)$ is the c.d.f. of the chi-square distribution with θ degrees of freedom and noncentrality parameter λ, then the results (i)–(iii) of Theorem 4 hold for F or $\bar{F}$ in place of f, and (iv)–(vi) hold for F or $\bar{F}$ in place of f when $\theta \geq 2$.

(b) If $f(\cdot,\theta,\lambda)$ is the p.d.f. of χ^2_θ, then it is well-known that $f(x,\theta,0)$ is TP_2. Hence, following the proof of Theorem 1(ii) it can be shown that both $F(c,\theta,0)$ and $\bar{F}(c,\theta,0)$ are log-concave in $\theta > 0$.

(c) It follows from Lemma 2 and the subsequent remark that both $F(c,\theta,\lambda)$ and $\bar{F}(c,\theta,\lambda)$ are log-concave in c when $\theta \geq 2$. However, a stronger result can be obtained when $\lambda = 0$ by appealing to Prekopa's Theorem.

Suppose $X \sim \chi^2_\theta$, and let f^* be the p.d.f. of $Y = \log X$. Then $f^*(y,\theta)$ is log-concave in y for $\theta > 0$. Using Prekopa's Theorem, we get

$$F^*(\alpha_1 d_1 + \alpha_2 d_2, \theta) \geq [F^*(d_1,\theta)]^{\alpha_1} [F^*(d_2,\theta)]^{\alpha_2}, \tag{4.1}$$

for any d_1, d_2 and $0 \leq \alpha_1, \alpha_2 \leq 1$, $\alpha_1 + \alpha_2 = 1$, where F^* is the c.d.f. corresponding to f^*. The above inequality is equivalent to

$$F(c_1^{\alpha_1} c_2^{\alpha_2}, \theta) \geq [F(c_1, \theta)]^{\alpha_1} [F(c_2, \theta)]^{\alpha_2} \tag{4.2}$$

for any positive c_1, c_2, where F is the c.d.f. of χ^2_θ. Thus from the "arithmetic mean geometric mean" inequality we get

$$F(\alpha_1 c_1 + \alpha_2 c_2, \theta) \geq [F(c_1, \theta)]^{\alpha_1} [F(c_2, \theta)]^{\alpha_2}, \tag{4.3}$$

which shows that $F(c,\theta)$ is log-concave in c for $\theta > 0$. Incidently (4.2) also holds for $\bar{F}$ in place of F.

5. More Results on p.d.f.'s. Let X be a positive random variable with the p.d.f. $f(\cdot,\theta)$ with respect to Lebesgue measure.

Definition 4. The density $f(\cdot,\theta)$ is said to have the restricted reproductive property (RRP) in θ, if there exists a positive r.v. Y with the distribution $G(\cdot,\theta,\delta)$ such that

$$\int_0^x f(x-y, \theta)\, G(dy, \theta, \delta) = f(x, \theta + \delta). \tag{5.1}$$

THEOREM 6. *Suppose the following conditions hold:* (a) $f(x,\theta)$ *is* TP_2, (b) f *satisfies the RRP, as given in (5.1),* (c) $G(\cdot, \theta,\delta)$, *as given in Definition 4, is stochastic decreasing in* θ. *Then* $\bar{F}(x,\theta)$ *is log-concave in* θ.

Proof. It follows from (a) that $\bar{F}(c,\theta)$ is TP_2. For $0 < c_1 < c_2$, $\theta_1 < \theta_2$ we have

$$\bar{F}(c_2, \theta_1)\bar{F}(c_1, \theta_2) \leq \bar{F}(c_2, \theta_2)\bar{F}(c_1, \theta_1). \tag{5.2}$$

Write $c_2 = c_1 + y$. Integrating (5.2) with respect to $G(dy, \theta_2, \delta)$ we get

$$\begin{aligned} \bar{F}(c_2, \theta_1)\bar{F}(c_2, \theta_2 + \delta) &\leq \bar{F}(c_2, \theta_2)\int_0^\infty \bar{F}(c_2-y_1, \theta_1)\, G(dy, \theta_1, \delta) \\ &\leq F(c_2, \theta_2)\int_0^\infty \bar{F}(c_2-y, \theta_1)\, G(dy, \theta_1, \delta) \\ &= \bar{F}(c_2, \theta_2)\bar{F}(c_2, \theta_1 + \delta). \end{aligned} \tag{5.3}$$

6. Application to Beta Distribution. Suppose $U \sim \beta_{m,n}$ and $V \sim \beta_{\delta,m+n}$ are independently distributed. Then $UV \sim \beta_{m+\delta,n}$. Write $X = -\log U$, $Y = -\log V$, and $\theta = m$. Let $f(\cdot, \theta)$ be the density of X and $G(\cdot, \theta, \delta)$ be the c.d.f. of Y. Then the conditions (a)–(c) of Theorem 5 hold. Hence $P[U \leq c]$ is log-concave in m.

Remark. Some of the above results relating to chi-square distribution are given in the Ph.D. dissertation of Sarkar. Furthermore, following the ideas of Das Gupta and Perlman (1974), Sarkar (1982) has shown that $\chi^2_{m,\alpha}$ is log-concave in $m > 0$, where $P[\chi^2_m > \chi^2_{m,\alpha}] = \alpha$.

Remark. The only basic result relating the TP_2 properties and log-concavity available in the literature is the following (Karlin (1968)): A positive-valued function g is log-concave iff. $g(x-y)$ is TP_2 in (x,y). This result follows easily from the developments in Theorem 1; one has to consider a special G in Definition 2 which assigns the entire probability mass to a positive number.

Remark. The reproductive property (RP) stated in Definition 2 looks similar to the semigroup property introduced by Proschan and Sethuraman (1977). The semigroup property of a TP_2 function $f(\theta, x)$ is defined as follows:

$$f(\theta_1 + \theta_2, y) = \int f(\theta_1, x) f(\theta_2, y-x) d\nu(x),$$

In the above, (i) $\mathcal{X} = \mathcal{R}$, $\theta \in \Theta \subset \mathcal{R}$ is an interval, or (ii) $\mathcal{X} = \{\ldots, -1, 0, 1, 2 \ldots\}$. Θ is an interval or an interval of integers, and ν is some measure on $\mathcal{X}$. Proschan and Sethuraman (1977) have shown that

$$\psi(\theta_1, \ldots, \theta_n) = \int \pi_{i=1}^{n} f(\theta_i, x_i) \phi(x_1 x_2, \ldots, x_n) \pi_{i=1}^{n} d\mu(x_i)$$

is Schur-concex whenever ϕ is Schur-convex, μ being the Lebesgue measure for case (i) and the counting measure for case (ii). Some related results are given in the book by Marshall and Olkin (1979).

The RP is slightly more general than the above semigroup property, and its use, as illustrated in our main results, is also different.

Remark. It should be noted that $C(x,\theta,\lambda)$ in (1.17) is defined only for those (x,θ,λ) for which $f(x,\theta,\lambda) > 0$.

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